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The Exponential Accuracy of Fourier and Tchebyshev Differencing Methods

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The Exponential Accuracy of Fourier and Tchebyshev Differencing Methods

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Abstract

It is shown that when differencing analytic functions using the pseudospectral Fourier or Tchebyshev methods, the error committed decays to zero at an exponential rate.

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Introduction

The pseudospectral differencing methods, involve the exact differentation of interpolants which are based on different sets of selected points; each method is usually named after the base functions used to expand such interpolants.

We discuss the pseudospectral Fourier and Tchebyshev differencing methods — the two most extensively used among all of the above, e.g., the survey of Gottlieb, Hussaini and Orszag [5] and the references therein. This stems from the possibility of implementing the FFT in these cases: one can efficiently travel between the ´physical´ and ´phase´ spaces, making the (global) pseudospectral calculations in these two cases, almost as economical as the (local) finite difference ones. The definitive advantage of the former lies, however, in their remarkable accuracy properties, which is the topic of this paper.

As is well known, the pseudospectral differencing of (sufficiently) smooth functions, enjoy ´infinite´ order of accuracy; that is, measured w.r.t. the inverse number of selected points, the error committed is bounded by <u>any fixed polynomial</u> order (e.g., Kreiss and Oliger [8] for the Fourier case, and a different detailed study of Canuto and Quarteroni [1], which includes, among others, the Tchebyshev case).

Here we show, that if the function under consideration is further assumed to be analytic, then the asymptotic decay rate of the error with either the Fourier or Tchebyshev differencing is, in fact, <u>exponential</u>. This should be compared with the <u>polynomial</u> decay rate obtained by finite difference/finite element differencing methods.

We start in Section 2, discussing the Fourier differencing of smooth functions: following [8], we first derive the aliasing relation, which implies 'infinite' order of accuracy in this case. In Section 3, we show the exponential decay rate of the error, with Fourier differencing of analytic functions. Tchebyshev differencing method is likewise treated in Section 4: after putting the aliasing relation in an identical form to the one obtained in the Fourier case, the various error estimates follow along the same lines.

Similar to our treatment of the stability question in [15, Part II], we emphasize here the central role played by the aliasing relations, from which we derive all the results below. Thanks to these aliasing relations, the error decay behavior is 'essentially' due to the corresponding decay of either the Fourier or Tchebyshev coefficients; an exponential decay of the latter is widely known in the analytic case. Also, by considering the Fourier/Tchebyshev coefficients, the above derivation may still offer an exponential decay rate of fractional order in non-analytic, smooth cases (e.g., standard cut-off functions).

In closing, we would like to point out that the above results are intimately related to Bernstein's theorem, regarding the exponential convergence of best polynomial approximations. Specifically, given an analytic function, Bernstein's proof verifies the exponential convergence of its truncated Tchebyshev series expansion, e.g., [11, Section 6]. Using the Gauss-Tchebyshev rule to compute that expansion's coefficients, we are then led to the Tchebyshev interpolant; the further error inferred by such discretization (which is exactly an aliasing error), is known to be also exponentially small, e.g., [3, p. 239]. In other words, we conclude that the above Tchebyshev interpolent — so called near minimax polynomial, approximates a given analytic function within an exponentially decaying error. In fact, the results below indicate that given an analytic function, both the Fourier and Tchebyshev interpolants approximate the function and its derivatives, within an exponential accuracy. Indeed, these results manifest themselves for example, in the global error behavior of pseudospectrally solved PDE´s, e.g., [5,6,13].

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2. Fourier Differencing of Smooth Functions

Let $w(x)$ be a 2π-periodic function, whose values, $w_\nu = w(x_\nu)$, are assumed known at the $2N$ equidistant grid points $x_\nu=\nu h$, $h=\frac{\pi}{N}$, $\nu=0,1,\cdots 2N-1$. The (pseudospectral) Fourier differencing of such function, refers to differentation of the trigonometric interpolant of these grid values: one constructs the trigonometric interpolant[1]

$$\tilde{w}(x) = \tilde{w}(x;N) = \sum_{p=-N}^{N}{}'' \hat{w}_p e^{ipx}, \qquad \hat{w}_p = \frac{1}{N}\cdot \sum_{\nu=0}^{2N-1} w_\nu e^{-ip\nu h}, \tag{2.1}$$

(1) (double) primed summation indicates halving first (and last) terms.

and use its derivative

$$\frac{d\tilde{w}}{dx}(x_\nu) = \sum_{p=-N}^{N}{}'' ip\hat{w}_p e^{ipx_\nu}$$

to approximate the 'true' value, $\frac{dw}{dx}(x=x_\nu)$.

In order to examine the error we commit by such approximation, it is convenient to work with Sobolev space W^s, defined for integral orders s,

$$(2.2) \qquad W^s \equiv W_2^s = \{w(x) \mid \|w\|^2_{W^s} = \sum_{k=0}^{s} \left\|\frac{d^{(k)}w}{dx^k}\right\|^2_{L^2[0,2\pi]} < \infty\},$$

and extended by interpolation for fractional orders. Thanks to Plancherel's formula, W^s is isometrically isomorphic to H^s: assuming $w(x)$ admits a formal Fourier expansion

$$(2.3a) \qquad w(x) \sim \sum_{p=-\infty}^{\infty} \hat{w}(p)e^{ipx}, \qquad \hat{w}(p) = \frac{1}{2\pi}\cdot\int_0^{2\pi} w(\xi)e^{-ip\xi}d\xi;$$

then we can equally work with H^s, s real, which consists of those functions, $w(x)$, having a finite Sobolev norm of order s,

$$(2.3b) \qquad H^s = \{w(x) \mid \|w\|^2_{H^s} = \sum_{p=-\infty}^{\infty} (1+|p|)^{2s}|\hat{w}(p)|^2 < \infty\}.$$

The following lemma, relating the Fourier coefficients of $w(x)$,

$$(2.4) \qquad \hat{w}(p) = \frac{1}{2\pi}\cdot\int_{\xi=0}^{2\pi} w(\xi)e^{-ip\xi}d\xi, \qquad -\infty < p < \infty$$

with those of its trigonometric interpolant, $\tilde{w}(x;N)$,

$$\hat{w}_p = \frac{1}{N} \sum_{\nu=0}^{2N-1} w(x_\nu)e^{-ip\nu h}, \qquad -N \le p \le N, \tag{2.5}$$

is in the heart of our discussion (e.g., Kreiss and Oliger [8]).

Lemma 2.1. (Aliasing).

Assume $w(x)$ *is in* H^s, $s > 1/2$.[2] *Then the following equality holds*

$$\hat{w}_p = \sum_{k=-\infty}^{\infty} \hat{w}(p+2kN), \qquad -N \le p \le N. \tag{2.6}$$

Proof: It is well known that the Fourier expansion in (2.3a) converges in this case, e.g., [16, Chapter II]; inserting that expansion, evaluated at $x = x_\nu$, into (2.5), we obtain

$$\hat{w}_p = \frac{1}{N} \sum_{\nu=0}^{2N-1} w(x_\nu)e^{-ip\nu h} = \frac{1}{N} \sum_{\nu=0}^{2N-1} \Big[\sum_{q=-\infty}^{\infty} \hat{w}(q)e^{iqx_\nu}\Big]e^{-ip\nu h}. \tag{2.7}$$

By Cauchy-Schwartz inequality the inner summation is absolutely convergent

$$\sum_{q=-\infty}^{\infty} |\hat{w}(q)| \le \Big[\sum_{q=1}^{\infty} \frac{2}{(1+q)^{2s}}\Big]^{\frac{1}{2}} \cdot \|w\|_{H^s}, \tag{2.8}$$

hence summations on the right of (2.7) can be interchanged. By so doing, the desired result follows

---

(2) This smoothness assumption on $w(x)$ can be relaxed.

$$\hat{w}_p = \sum_{q=-\infty}^{\infty} \hat{w}(q)\cdot\frac{1}{N}\cdot\sum_{\nu=0}^{2N-1} e^{i\nu(q-p)h} = \sum_{k=-\infty}^{\infty} \hat{w}(p+2kN),$$

noting that the second summation in the middle term vanishes unless $q - p = 0(\mathrm{mod}\ 2N)$, i.e., $q = p + 2kN$.

Equipped with the aliasing lemma, we now may turn to estimate the error between $w(x)$ and its equidistant interpolant $\tilde{w}(x)$: rewriting

$$w(x) = \Big[\sum_{|p|\le N}{}'' + \sum_{|p|\ge N}{}''\Big]\hat{w}(p)e^{ipx},$$

and, with the help of (2.6),

$$\tilde{w}(x) = \sum_{|p|\le N}{}''\hat{w}(p)e^{ipx} + \sum_{|p|\le N}{}''\Big[\sum_{k\ne 0}\hat{w}(p+2kN)\Big]e^{ipx},$$

the difference $w(x) - \tilde{w}(x)$ is readily verified to equal

$$w(x) - \tilde{w}(x) = -\sum_{|p|\le N}{}''\Big[\sum_{k\ne 0}\hat{w}(p+2kN)\Big]e^{ipx} + \sum_{|p|\ge N}{}''\hat{w}(p)e^{ipx}. \tag{2.9}$$

The first summation on the right represents <u>aliasing</u> of the higher modes with the lower ones, $|p| \le N$, while the second summation consists of the <u>truncated</u> higher mode, $|p| \ge N$. A quantitative study of both terms gives us (compare e.g., Kreiss and Oliger [9], Pasciak [12])

Lemma 2.2. <u>(Error Estimate)</u>.

<u>Assume</u> $w(x)$ <u>is in</u> H^s, $s > \frac{1}{2}$. <u>Then for any real</u> σ, $0 \le \sigma \le s$, <u>we have</u>

(2.10) $$\|w(x)-\tilde{w}(x;N)\|_{H^\sigma} \leq \left(1+2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\right)^{\frac{1}{2}}\cdot\|w\|_{H^s}\cdot\left(\frac{1}{N}\right)^{s-\sigma}.$$

Proof. Starting with (2.9), then by definition

(2.11) $$\|w(x) - \tilde{w}(x;h)\|^2_{H^\sigma} = \sum_{|p|\leq N}{}'' (1+|p|)^{2\sigma}\left|\sum_{k\neq 0}\hat{w}(p+2kN)\right|^2 + \sum_{|p|\geq N}{}'' (1+|p|)^{2\sigma}|\hat{w}(p)|^2 .$$

Cauchy-Schwartz inequality implies

$$\left|\sum_{k\neq 0}\hat{w}(p+2kN)\right|^2 \leq \sum_{k\neq 0}(1+|p+2kN|)^{2s}\cdot|\hat{w}(p+2kN)|^2\cdot\sum_{k\neq 0}(1+|p+2kN|)^{-2s},$$

with the second summation not exceedng a value of

$$\sum_{k\neq 0}(1+|p+2kN|)^{-2s} \leq 2N^{-2s}\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}, \qquad |p| \leq N.$$

Inserted into (2.11), we find that the aliasing part of the error given in the first term on the right, is bounded by

$$2N^{-2s}\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\cdot\sum_{|p|\leq N} N^{2\sigma}\sum_{k\neq 0}(1+|p+2kN|)^{2s}\ |\hat{w}(p+2kN)|^2$$
$$\leq 2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\cdot\left(\frac{1}{N}\right)^{2(s-\sigma)}\cdot\|w\|^2_s ;$$

The truncation error, given in the second term on the right of (2.11), is equally found to be bounded by

$$\sum_{|p|\geq N}'' N^{2(\sigma-s)}\cdot(1+|p|)^{2s}\ |\hat{w}(p)|^2 \leq (\tfrac{1}{N})^{2(s-\sigma)}\cdot\|w\|_s^2 \ .$$

Added together, the last two estimates yield (2.10).

Remark 1. Observe that requiring $w(x)$ to have more than "one-half" bounded derivative enable us to control the aliasing part of the error; apart from that restriction, there is an error in decay in any Sobolev norm weaker than that of $w(x)$, which is equally due to aliasing and truncation errors.

Remark 2. The aliasing relation (2.5) for the zeroth mode $p = 0$, implies that the trapezoidal rule is highly accurate for the integration of smooth periodic functions (Davis and Rabinowitz [3]): indeed, the error committed in this case is solely due to aliasing

$$\frac{1}{N}\cdot\sum_{\nu=0}^{2N}{}'' w(x_\nu) - \frac{1}{2\pi}\cdot\int_0^{2\pi} w(\xi)d\xi = \sum_{k\neq 0}\hat{w}(2kN).$$

This allows us to replace the H^σ-norm, measuring the error on the left of (2.10), with its more applicable discrete counterpart (Gottlieb et.al. [5]):

$$|||w(x)-\tilde{w}(x;N|||_{H^\sigma}^2 = \sum_{k=0}^{\sigma}\frac{1}{2N}\cdot\sum_{\nu=0}^{2N}{}''\left[\frac{d^{(k)}w}{dx^k}(x_\nu) - \frac{d^{(k)}\tilde{w}}{dx^k}(x_\nu;N)\right]^2 , \quad \sigma \text{ integral.}$$

Returning to our original question, we find — choosing $\sigma = 1$ in Lemma 2.1 — that the error in Fourier differencing does not exceed

$$\left\|\frac{dw}{dx}(x) - \frac{d\tilde{w}}{dx}(x;N)\right\| \leq \text{Const}\cdot\|w\|_{H^s}\cdot(\tfrac{1}{N})^{s-1} , \tag{2.12}$$

for arbitrary real s, $s > 1$. The norm on the left refers, of course, to the $H^0 = L^2$ norm of the error, with a uniform Constant = 2 on the right. It can be replaced, in fact, by any other reasonable (possibly discrete) norm: for example, Sobolev's inequality implies for the somewhat more applicative maximum norm

$$\underset{0\leq\nu\leq 2N-1}{\mathrm{Max}} \left|\frac{dw}{dx}(x_\nu) - \frac{d\tilde{w}}{dx}(x_\nu;N)\right| \leq \mathrm{Const}_\sigma \cdot \|w\|_{H^s} \cdot \left(\frac{1}{N}\right)^{s-\sigma}, \qquad s > \sigma > \frac{3}{2}.$$

Consider now a sufficiently smooth 2π-periodic function $w(x)$. Differencing such function by local methods, such as finite difference or finite element methods, leads to an error bound of the type (2.12) with a finite, fixed[(3)] degree, polynomial decay; the latter is usually identified with the accuracy order of the differencing method. With this terminology in mind, the (global) Fourier differencing method was thus shown to be "infinitely" order accurate: the discretization error decays faster then any fixed degree polynomial rate, e.g., [1-2], [4-7], [14-15]. It is worth emphasizing that phrasing the error estimate (2.12) as "infinite" order of accuracy, is limited on both accounts:

1. Consider a sufficiently smooth function $w(x)$ in H^s, $s \gg 1$. The error's order of magnitude for a given Fourier differencing of such functions, may be difficult to calculate: an a' priori knowledge regarding the size of the factors $\|w\|_{H^k}$, $k \leq s$, is required in this case.

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(3)That is, independent of N.

2. Assume $w(x)$ is a C^{∞}- function. One cannot detect the exact asymptotic decay rate, according to the error estimate (2.12): because of its factor dependence on the power s — when s increases so does $\|w\|_{H^s}$ — one may not conclude, for example, an exponential convergence rate simply by placing arbitrarily large powers s, since the optimal s depends of course (usually in unknown manner) on N.

3. Fourier Differencing of Analytic Functions

In this section, we show that the Fourier differencing of 2π-periodic analytic functions, admits an exponentially decaying error; furthermore, in some cases, the error's order of magnitude may be calculated as well.

To this end, assume

$$-\eta_0 < \mathrm{Im} z < \eta_0 \tag{3.1a}$$

to be the strip of analyticity where $w(z)$ admits the absolutely convergent expansion

$$w(z) = \sum_{p=-\infty}^{\infty} \hat{w}(p)e^{ipz}, \qquad |\mathrm{Im} z| \leq \dot{\eta} < \eta_0. \tag{3.1b}$$

Denoting

$$M(\eta) = \operatorname*{Max}_{|\mathrm{Im} z|\leq\eta} |w(z)|, \tag{3.2}$$

we may now state

Theorem 3.1.

<u>Assume</u> $w(x)$ <u>is</u> 2π<u>-periodic analytic, with analyticity strip of width</u> $2\eta_0$. <u>Then for any</u> η, $0 < \eta < \eta_0$, <u>we have</u>

$$\left\| \frac{dw}{dx}(x) - \frac{d\tilde{w}}{dx}(x;N) \right\| \leq 4M(\eta)\left(\frac{ctgh(N\eta)}{e^{2\eta}-1}\right)^{\frac{1}{2}} \cdot Ne^{-N\eta}. \tag{3.3}$$

<u>Proof.</u> Making the change of variables, $\zeta = e^{iz}$, then $v(\zeta) = w(z=-iLog\zeta)$ admits the power series expansion

$$v(\zeta) = w(z=-iLog\zeta) = \sum_{p=-\infty}^{\infty} \hat{w}(p)\zeta^p. \tag{3.4}$$

By the periodic analyticity of $w(z)$ in the strip $|Imz|<\eta_0$, $v(\zeta)$ is found to be single valued analytic in the corresponding annulus $e^{-\eta_0}<\zeta<e^{\eta_0}$, whose Laurent's expansion is given in (3.4):

$$\hat{w}(q) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{v(\zeta)}{\zeta^{q+1}}\, d\zeta, \qquad e^{-\eta_0} < r < e^{\eta_0}. \tag{3.5}$$

To estimate the error of Fourier differencing in this case, we employ (2.11) with $\sigma = 1$, obtaining

$$\|w(x)-\tilde{w}(x;N)\|^2_{H^1} \leq N^2 \cdot \sum_{|p|\leq N}{}'' \left| \sum_{k\neq 0} \hat{w}(p+2kN)\right|^2 + \sum_{|p|\geq N}{}''(1+|p|)^2|\hat{w}(p)|^2 \tag{3.6}$$

Using (3.5), we sum the aliased amplitudes

$$\left[\sum_{k<0} + \sum_{k>0}\right]\hat{w}(p+2kN) = \frac{1}{2\pi i}\left[\int_{|\zeta|=r} \frac{v(\zeta)d\zeta}{\zeta^{p+1}(\zeta^{2N}-1)} + \int_{|\zeta|=r^{-1}} \frac{v(\zeta)d\zeta}{\zeta^{p+1}(\zeta^{-2N}-1)}\right], \quad r = e^{\eta}>1,$$

so that the first term on the right of (3.6) does not exceed a value of

$$(3.7a) \qquad 4N^2 \frac{M^2(\eta)}{(e^{2N\eta}-1)^2} \cdot \sum_{|p|\leq N}'' e^{2\eta p} \leq 4M^2(\eta)\frac{\mathrm{ctgh}(N\eta)}{e^{2\eta}-1} N^2 e^{-2N\eta}.$$

The truncation contribution to the error in the second term on the right of (3.6), does not exceed[(4)]

$$(3.7b) \qquad 4M^2(\eta)\Big[\sum_{p\geq N}' (1+p^2)e^{-2\eta p} + \sum_{p\leq -N}' (1+p^2)e^{2\eta p}\Big] \leq 8\, \frac{M^2(\eta)}{e^{2\eta}-1} N^2 e^{-2N\eta}.$$

Adding the last two bounds, yield (3.3).

Remark 3. According to the above derivation, the asymptotic dependence of the overall error on N, is due to equal size contributions of both the aliasing and truncation parts. However, one can do better with regard to the truncation error: indeed, let us denote

$$M_k(\eta) = e^{2k\eta} \cdot \sum_{j=0}^{k} \max_{|\zeta|=e^{\eta}} |v^{(j)}(\zeta)|;$$

then by invoking the relation

$$q\hat{w}(q) = \frac{1}{2\pi i} \cdot \int_{|\zeta|=r} \frac{dv}{d\zeta}(\zeta)\zeta^{-q}d\zeta,$$

[(4)]We assume N is sufficiently large, $N > (e^{2\eta}-1)^{-1}$.

the truncation contribution in (3.2b) is, in fact, found to be bounded by

$$2M_1^2(\eta)\Big[\sum_{p\geq N}{}' e^{-2\eta p} + \sum_{p\leq -N}{}' e^{2\eta p}\Big] \leq 4\,\frac{M_1^2(\eta)}{e^{2\eta}-1}\, e^{-2N\eta}\ .$$

Compared with the truncation estimate in (3.7b), we see that the loss of the N^2-factor is regained here.

Remark 4. Estimate (3.3) shows that the error with Fourier differencing of an analytic function $w(x)$, decays exponentially w.r.t. its asymptotic dependence on N; furthermore, equipped with a bound on $w(x)$ when moved into the complex plane, one can estimte the size of the error in this case, using the somewhat more aesthetic upper bound

$$\left\| \frac{dw}{dx}(x) - \frac{d\tilde{w}}{dx}(x;N)\right\| \leq \frac{4M(\eta)}{\sinh(\eta)} N e^{-N\eta} \tag{3.8}$$

Remark 5. The exponential convergence follows for derivatives higher than one: with the usual loss of a factor of N for each derivative, we obtain

$$\| w(x)-\tilde{w}(x;N)\|_{H^\sigma} \leq \mathrm{Const}_\sigma\cdot\frac{M(\eta)}{\sinh(\eta)}\, N^\sigma e^{-N\eta}. \tag{3.9}$$

The preferable discrete estimates, follow along the lines of an earlier remark, or alternatively, using Sobolev inequality to implement L^∞ error estimates.

4. Tchebyshev Differencing - The Non-Periodic Case

In the non-periodic case, the Tchebyshev differencing is usually advocated, e.g., [1-2], [4-6], [10], [13-14]. Let $w(x)$ be defined for $-1 \leq x \leq 1$, and assume its values $w_\nu = w(x_\nu)$ are known at the $N+1$ gridpoints $x_\nu=\cos(\nu h)$, $h = \frac{\pi}{N}$, $\nu = 0,1,\cdots N$. The (pseudospectral) Tchebyshev differencing of such function refers to differentiations of the polynomial interpolant of these gridvalues: one constructs the polynomial interpolant

$$(4.1) \qquad \tilde{w}_T(x) = \tilde{w}_T(x,N) = \sum_{p=0}^{N}{}'' \hat{w}_p T_p(x), \qquad \hat{w}_p = \frac{2}{N} \cdot \sum_{\nu=0}^{N}{}'' w_\nu T_p(x_\nu),$$

in terms of Tchebyshev polynomials $T_p(x) = \cos[p(\cos^{-1}x)]$, and use its derivative

$$\frac{d\tilde{w}}{dx}(x_\nu) = \sum_{p=0}^{N}{}'' \hat{w}_p \frac{dT_p}{dx}(x=x_\nu)$$

to approximate the 'true' value, $\frac{dw}{dx}(x=x_\nu)$. The latter summation, can be translated into standard cosine FFT-like summation using a single two-step recurssion formula, see [4-6]; thus Tchebyshev differencing admits a fast efficient implementation.

To measure the error in this case, one usually employs the appropriately weighted Tchebyshev norm

$$\|w\|_T^2 = \int_{-1}^{1} \frac{w^2(x)}{(1-x^2)^{1/2}}\, dx$$

and the corresponding weighted spaces under the W_T^s norm, s integral,

$$(4.2) \qquad W_T^s = \{w(x) \mid \|w\|^2_{W_T^s} = \sum_{k=0}^{s} \left\|\frac{d^{(k)}w}{dx^k}\right\|^2_T < \infty\};$$

Tchebyshev spaces W_T^s of factional order s are suitably interpreted by interpolation.

We have found it more convenient, however, to work below within the spaces H_T^s, s real: assuming $w(x)$ admits a formal Tchebyshev expansion

$$(4.3) \qquad w(x) \sim \sum_{p=0}^{\infty}{}' \hat{w}(p)T_p(x), \quad \hat{w}(p) = \frac{2}{\pi}\cdot\int_{-1}^{1} \frac{w(\xi)T_p(\xi)}{(1-\xi^2)^{1/2}}\, d\xi,$$

then, in complete analogy with (2.3b), we introduce

$$(4.4) \qquad H_T^s = \{w(x) \mid \|w\|^2_{H_T^s} = \sum_{p=0}^{\infty} (1+p)^{2s}|\hat{w}(p)|^2 < \infty\}.$$

Unlike the Fourier case (endowed with the usual Euclidean weighting), W_T^s and H_T^s are not equivalent unless $s = 0$, in which case they are in fact isometrically isomorphic by the Tchebyshev transform

$$(4.5) \qquad \|w\|^2_{H_T^0} = \frac{2}{\pi}\|w\|^2_{W_T^0} .$$

Making use of the inverse inequalities of Canuto and Quarteroni [1], will enable us, later on, to recover the H_T^s-estimates derived below, within the more standard W_T^s-spaces. We begin with the aliasing relation in this case, which reads (e.g. Gottlieb [4], Reyna [14])

Lemma 4.1. (<u>Aliasing</u>).

<u>Assume</u> $w(x)$ <u>is in</u> H_T^s, $s > \frac{1}{2}$. <u>Then the following equality holds</u>

$$\hat{w}_p = \hat{w}(p) + \sum_{k=1}^{\infty} [\hat{w}(-p+2kN) + \hat{w}(p+2kN)], \qquad 0 \leq p \leq N. \tag{4.6}$$

<u>Proof</u>. Inserting the Tchebyshev expansion in (4.3) evaluated at $x = x_\nu$, into the discrete Tchebyshev coefficient in (4.1), we find

$$\hat{w}_p = \frac{2}{N} \sum_{\nu=0}^{N}{}'' \Big[\sum_{q=0}^{\infty}{}'' \hat{w}(q) T_q(x_\nu) \Big] T_p(x_\nu) = \frac{2}{N} \sum_{q=0}^{\infty}{}'' \hat{w}(q) \Big[\sum_{\nu=0}^{N}{}'' T_q(x_\nu) T_p(x_\nu) \Big].$$

To calculate the inner summation on the right, we employ the identity $2T_q(x)T_q(x) = T_{p+q}(x) + T_{|p-q|}(x)$, while noting that $\frac{1}{N} \sum_{\nu=0}^{N}{}'' T_j(x_\nu)$ vanishes, unless $j = 0 (\mathrm{mod}\ 2N)$ in which case it equals one. Hence, we end up with

$$\hat{w}_p = \sum_{q=0}^{\infty}{}'' \hat{w}(q) \Big[\delta_{qp} + \delta_{q0} \cdot \delta_{p0} + \sum_{k=1}^{\infty} \delta_{q,2kN \pm p} \Big]$$

and (4.6) follows.

Let us define $T_{-p}(x) = T_p(x)$ so that $\hat{w}(-p) = \hat{w}(p)$. Tchebyshev expansion (4.3) takes now the Fourier-like symmetric form

$$w(x) \sim \frac{1}{2} \cdot \sum_{p=-\infty}^{\infty} \hat{w}(p) T_p(x) \tag{4.7}$$

with an aliasing formula, <u>identical</u> to the one we had before in Lemma 2.1:

$$\hat{w}_p = \sum_{k=-\infty}^{\infty} \hat{w}(p+2kN). \tag{4.8}$$

Hence, we can equally conclude the corresponding error estimate, which we quote from Lemma 2.2.

Lemma 4.2. (Error Estimate)

Assume $w(x)$ is in H_T^s, $s > \frac{1}{2}$. Then for any real σ, $0 \le \sigma \le s$, we have

$$\|w(x) - \tilde{w}_T(x;N)\|_{H_T^\sigma} \le 2\Big(1+2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\Big)^{\frac{1}{2}}\cdot\|w\|_{H_T^s}\cdot\Big(\frac{1}{N}\Big)^{s-\sigma}. \tag{4.9}$$

Setting $\sigma = 0$ in (4.9) gives us, in view of (4.5)

$$\|w(x) - \tilde{w}_T(x;N)\|_{W_T^0} \le \Big[2\pi\Big(1+2\cdot\sum_{k=1}^{\infty}(2k-1)^{-2s}\Big)\Big]^{\frac{1}{2}}\cdot\|w\|_{H_T^s}\cdot\Big(\frac{1}{N}\Big)^{s}. \tag{4.10}$$

Using the inverse inequality [1, Lemma 2.1], one can 'raise' the Sobolev norm on the left of (4.10), obtaining (for details see Canuto and Quarteroni [1, Theorem 3.1], Maday and Quarteroni [10])

Corollary 4.3. (Error Estimate).

Assume $w(x)$ is in W_T^s, $s > \frac{1}{2}$. Then for any real σ, $0 \le \sigma \le s$, we have

$$\|w(x) - \tilde{w}_T(x;N)\|_{W_T^\sigma} \le \mathrm{Const}_s\cdot\|w\|_{W_T^s}\cdot\Big(\frac{1}{N}\Big)^{s-2\sigma}. \tag{4.11}$$

Thus, each derivative infers a lost of N^2 factor in this case, rather than the usual factor N associated with the Fourier differencing.

Remark 6. According to Y. Maday (private communication), the factor dependence on the right of (4.11) is factorial, $\mathrm{Const}_s \sim s!$.

We turn now to consider the case where $w(x)$ is analytic in the interval $[-1,1]$. To this end, we employ Bernstein's regularity ellipse, E_r, with foci ± 1 and with sum of its semiaxis equals r, e.g., [11, Section 6]. Denoting

$$(4.12) \qquad M_T(r) = \max_{z \in E_r} |w(z)|,$$

we may now state

Theorem 4.4

Assume $w(x)$ is analytic in $[-1,1]$, having a regularity ellipse whose sum of its semiaxis equals $r_0 = e^{\eta_0} > 1$. Then for any η, $0 < \eta < \eta_0$, we have

$$(4.13) \qquad \| w(x) - \tilde{w}_T(x;N) \|_{H^1_T} \le 8M(\eta)\left(\frac{\operatorname{ctgh}(N\eta)}{e^{2\eta}-1}\right)^{\frac{1}{2}} \cdot Ne^{-N\eta} .$$

Proof. The transformation, $\frac{\zeta+\zeta^{-1}}{2} = z$, takes the regularity ellipse E_{r_0} in the z-plane, into the annulus $r_0^{-1} < |\zeta| < r_0$ in the ζ-plane. Hence, $v(\zeta) = 2w(z = \frac{\zeta+\zeta^{-1}}{2})$ admits the power series expansion

$$(4.14) \qquad v(\zeta) = 2w\left(z = \frac{\zeta+\zeta^{-1}}{2}\right) = \sum_{p=-\infty}^{\infty} \hat{w}(p)\zeta^p, \qquad r_0^{-1} < |\zeta| < r_0 = e^{\eta_0} .$$

Indeed, upon setting $\zeta = e^{i\theta}$ and recalling that $\hat{w}(-p) = \hat{w}(p)$, the above expansion clearly describes the real interval $[-1,1]$,

$$w(z=\cos\theta) = \sum_{p=0}^{\infty}{}'' \hat{w}(p)\cos(p\theta).$$

For the Laurent's expansion given in (4.14), we then find

$$\hat{w}(q) = \frac{1}{2\pi i}\cdot \int_{|\zeta|=r} \frac{v(\zeta)}{\zeta^{q+1}}d\zeta, \qquad e^{-\eta_0} < r < e^{\eta_0}. \tag{4.15}$$

Comparing (4.15) and (3.5), we end up with the same Cauchy integral formulae for the amplitudes in both the Fourier and Tchebyshev expansions; coupled with the identical aliasing relations, (4.13) follows along the lines of Theorem 3.1.

Remark 7. As before, the factor $\left(\frac{\mathrm{ctgh}(N\eta)}{e^{2\eta}-1}\right)^{\frac{1}{2}}$ on the right of (4.13), can be replaced by the more aesthetic bound of $\frac{1}{\sinh(\eta)}$, yielding

$$\|w(x) - \tilde{w}(x;N)\|_{H^1_T} \leq 8\frac{M(\eta)}{\sinh(\eta)}Ne^{-N\eta}. \tag{4.16}$$

Next, an exponential error estimate in terms of the Sobolev norm W^1_T can be derived: with the loss of an additional factor of N in the spirit of an earlier remark, we then find

Corollary 4.5.

Assume $w(x)$ is analytic in $[-1,1]$. Then we have

$$\left\|\frac{dw}{dx}(x) - \frac{d\tilde{w}_T}{dx}(x;N)\right\|_T \leq \mathrm{Const}\cdot\frac{M(\eta)}{\sinh(\eta)}N^2e^{-N\eta}, \qquad 0 < \eta < \eta_0. \tag{4.17}$$

Higher derivatives can be estimated in a similar manner; in particular, since $L^\infty \subset H^{1/2+\varepsilon} \subset W_T^1$, a discrete maximum estimate follows.

Corollary 4.6.

Assume $w(x)$ is analytic in $[-1,1]$. Then we have

$$(4.18) \qquad \max_{0\le\nu\le 2N-1} \left|\frac{dw}{dx}(x_\nu) - \frac{d\tilde{w}}{dx}(x_\nu;N)\right| \le \text{Const}\cdot\frac{M(\eta)}{\sinh(\eta)} N^4 e^{-N\eta}, \qquad 0 < \eta < \eta_0.$$

The fourth power of N on the right of (4.17) can be improved to be $N^{5/2+\varepsilon}$, by imbedding $W^{1,\infty}$ in $H_T^{5/2+\varepsilon}$.

References

[1] C. Canuto and A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces, Math. Comp., 38 (1982), pp. 67-86.

[2] C. Canuto and A. Quarteroni, Error estimates for spectral and pseudospectral approximations of hyperbolic equations, SIAM J. Numer. Anal., 19 (1982), pp. 629-642.

[3] P. Davis and P. Rabinowitz, Methods of Numerical Integration, Academic Press, New York, 1975.

[4] D. Gottlieb, The stability of pseudospectral Chebyshev methods, Math. Comp., 36 (1981), pp. 107-118.

[5] D. Gottlieb, M. Hussaini and S. Orszag, Theory and applications of spectral methods, Spectral Methods for Partial Differential Equations, R. Voigt, D. Gottlieb and M. Hussaini, eds., (Proceedings of the 1982 NASA Langley Research Center Symposium), SIAM, Philadelphia, 1984, pp. 1-54.

[6] D. Gottlieb and S. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1977.

[7] H. Kreiss and J. Oliger, Comparison of accurate methods for the integration of hyperbolic equations, Tellus, 24 (1972), pp. 199-215.

[8] H. Kreiss and J. Oliger, Methods for the Approximate Solution of Time Dependent Problems, GARP Publications Series No. 10, 1973.

[9] H. Kreiss and J. Oliger, Stability of the Fourier method, SIAM, J. Numer. Anal., 16 (1979), pp. 421-433.

[10] Y. Maday and A. Quarteroni, Legendre and Chebyshev spectral approximations of Burgers' equation, Numer. Math., 37 (1981), pp. 321-332.

[11] G. Meinardus, Approximation of Functions: Theory and Numerical Methods, Springer, Berlin, 1967.

[12] J. Pasciak, Spectral and pseudospectral methods for advection equations, Math. Comp., 35 (1980), pp. 1081-1092,

[13] A. Patera, A spectral element method for fluid dynamics: laminar flow in a channel expansion, J. Comput. Phys., 54 (1984), pp. 468-488.

[14] L. Reyna, Stability of Tchebyshev collocation, Courant Institute of Mathematical Sciences, Preprint.

[15] E. Tadmor, Finite-difference, spectral and Galerkin methods for time-dependent problems, ICASE Report No. 83-22, NASA CR-172149, NASA Langley Research Center, Hampton, VA, 1983.

[16] A. Zygmund, Trigonometric Series, Vol. 1, Cambridge University Press, Cambridge, 1968.

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16. Abstract

It is shown that when differencing analytic functions using the pseudospectral Fourier or Tchebyshev methods, the error committed decays to zero at an exponential rate.

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