NASA CR-172,432

NASA Contractor Report 172432

ICASE REPORT NO. 84-37

NASA-CR-172432
19840025042

ICASE

THE TIME EVOLUTION OF SPECTRAL DISCRETIZATIONS OF HYPERBOLIC SYSTEMS

Liviu Lustman

Contract Nos. NAS1-17070, NAS1-17261

August 1984

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA
National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

THE TIME EVOLUTION OF SPECTRAL DISCRETIZATIONS
OF HYPERBOLIC SYSTEMS

Liviu Lustman
Systems and Applied Sciences Corporation

Abstract

A Chebyshev collocation spectral method, applied to hyperbolic systems is considered, particularly for those initial boundary value problems which possess only solutions tending to zero at large times. It is shown that the numerical solutions of the system will also vanish at infinity, if and only if, the numerical solution of a scalar equation of the same type does. This result is then generalized for other spectral approximations.

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17261 while the author was in residence at the University of California, Los Angeles. Additional support was received under NASA Contract No. NAS1-17070 while in residence at the Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23665.

In [1] the authors studied the spectrum of the Chebyshev collocation operator for the heat equation, proving that it shares certain properties with the spectrum of the corresponding differential operator. Here we present similar results for spectral discretization of hyperbolic systems.

Assume that the differential system, under given boundary conditions has only solutions which decay in time to zero. To test whether the numerical solutions behave in the same way, it is sufficient to check the corresponding scalar problem with zero inflow:

$$(1) \qquad \begin{cases} u_t = u_x & \text{in } |x| < 1,\ t > 0 \\ u = 0 & \text{at } x = 1 \end{cases} .$$

If all the numerical solutions of (1) decay, so will the solutions of the general system, provided that the spectral approximation contains an even or odd forcing polynomial (to be subsequently defined).

In order to prove this statement we require the following two lemmas:

Lemma 1. Let $p(x)$ be a real Hurwitz polynomial (i.e., having only roots with negative real parts), and let ε be a complex number, less than one in magnitude. Then the polynomial

$$q(x) = p(x) + \varepsilon p(-x)$$

is also Hurwitz.

<u>Proof.</u> Let us first recapitulate a few facts about positive pairs. Two polynomials $g(x)$ and $h(x)$ are said to form a positive pair if:

a. they have positive coefficients and real, negative and distinct roots

b. their degrees are equal, or degree(h) = degree(g) -1

c. their roots interlace:

$$0 > u_1 > v_1 > u_2 > v_2 \qquad \text{etc.},$$

where u_i are the roots of g and v_i the roots of h. The basic theorem states that the polynomial $f(x)$ is Hurwitz, if and only if,

$$f(x) = g(x^2) + xh(x^2)$$

with g and h forming a positive pair.
Without loss of generality, let p have real positive coefficients; write p using a positive pair as:

$$p(x) = g(x^2) + xh(x^2).$$

The polynomial q is Hurwitz, if and only if, $Q(x) = |q(x)|^2$ is.
For Q we have the expression:

$$\begin{aligned} Q(x) &= \big(p(x) + \varepsilon p(-x)\big)\ \big(p(x) + \varepsilon^* p(-x)\big) \\ &= (1 + 2\ \mathrm{Re}\ \varepsilon + |\varepsilon|^2)g^2(x^2) + (1 - 2\ \mathrm{Re}\ \varepsilon + |\varepsilon|^2)x^2\ h^2(x^2) \\ &\qquad + x\big(2(1 - |\varepsilon|^2)g(x^2)h(x^2)\big) \end{aligned}$$

and all its coefficients are positive. Thus we must prove that:

$$G(x) = (1 + 2\mathrm{Re}\ \varepsilon + |\varepsilon|^2)g^2(x) + (1 - 2\mathrm{Re}\ \varepsilon + |\varepsilon|^2)xh^2(x)$$

and

$$H(x) = 2(1 - |\varepsilon|^2)g(x)h(x)$$

form a positive pair. After simplifying positive factors, it is enough to show that

$$G^+(x) = A^2\ g^2(x) + xh^2(x)$$

and

$$H^+(x) = g(x)h(x)$$

form a positive pair, where $A^2 = (1 + 2\mathrm{Re}\ \varepsilon + |\varepsilon|^2)/(1 - 2\mathrm{Re}\ \varepsilon + |\varepsilon|^2)$. As g and h form a positive pair and A is positive, the polynomial $Ag(x^2) + xh(x^2)$ is Hurwitz, and so is its square:

$$\left(Ag(x^2) + xh(x^2)\right)^2 = A^2\ g^2(x^2) + x^2\ h^2(x) + x\left(2Ag(x^2)h(x^2)\right).$$

By examining the last formula we recognize G^+ and H^+, which form a positive pair, as required.

Lemma 2. Let $\phi(x)$, $B(x)$ be polynomials of degree M. Then the solution of

$$(2) \qquad \begin{cases} u_t = \pm u_x + \alpha(t)B(x) & \text{in } t > 0, \ |x| < \infty \\ u(x,t=0) = \phi(x) \end{cases}$$

is

$$(3) \qquad u(x,t) = \phi(x \pm t) + \int_0^t \alpha(\theta)B(x \pm \theta)d\theta.$$

Proof. Take the Laplace transform

$$f(t) \longrightarrow \overline{f}(\tau) = \int_0^\infty e^{-\tau t} f(t)dt.$$

Then:

$$\tau\overline{u} - \phi = \pm \frac{\partial \overline{u}}{\partial x} + \overline{\alpha}B \ .$$

Since ϕ and B are polynomials in x one may invert formally the operators $\left(\tau \pm \frac{\partial}{\partial x}\right)$ to obtain:

$$\overline{u}(x,\tau) = \sum_{k=0}^{\infty} \frac{(\pm 1)^k \, B^{(k)}(x)}{\tau^{k+1}} \overline{\alpha}(\tau) + \sum_{k=0}^{\infty} \frac{(\pm 1)^k \, \phi^{(k)}(x)}{\tau^{k+1}}$$

$$= \sum \overline{\alpha}(\tau) \frac{B^{(k)}(x)}{k!} (\pm 1)^k \frac{k!}{\tau^{k+1}} + \sum_{k=0}^{\infty} \frac{\phi^{(k)}(x)}{k!} (\pm 1)^k \frac{k!}{\tau^{k+1}} \ .$$

As $\frac{k!}{\tau^{k+1}}$ is the Laplace transform of t^k, we recognize the transform of the Taylor series for $B(x \pm t)$, $\phi(x \pm t)$. We can write:

$$\overline{u}(x,\tau) = \overline{\alpha}\ \overline{B(x \pm t)} + \overline{\phi(x \pm t)} \tag{4}$$

and finally obtain (3) by interpreting the product above as the transform of a convolution. Remark that the solution $u(x,t)$ is itself a polynomial of degree M in x, with time dependent coefficients.

Consider now the first order hyperbolic system:

$$(5) \quad \begin{cases} \left.\begin{aligned} &\text{a.} \quad u_t = u_x \\ &\text{b.} \quad v_t = -v_x \end{aligned}\right\} & \text{in } |x| < 1,\ t > 0 \\ \text{c.} \quad u = Rv & \text{at } x = 1 \\ \text{d.} \quad v = Lu & \text{at } x = -1 \ . \end{cases}$$

Here u, v are column vectors of dimensions m and n respectively. These variables are coupled only in the boundary conditions via the two constant matrices L and R, of dimensions $m \times n$ and $n \times m$.

We shall study this system and its spectral discretizations to prove that if all the solutions of (5) tend to zero for large times, so do the numerical solutions.

First, remark that there is decay in time, if and only if, both matrices LR and RL are contractions, i.e.,

$$\|LRx\| < \|x\|, \quad \|RLx\| < \|x\|$$

for all $x \neq 0$, in some suitable norm. This is obvious if L and R are interpreted as reflection coefficients, since u and v are transported without change on characteristics between $x = 1$ and $x = -1$.

Next we introduce the spectral approximation for system (5). Define the Chebyshev points

$$x_j = \cos(\pi j/N) \qquad 0 \leq j \leq N$$

and collocate at the interior points $x_1, x_2 \cdots, x_{N-1}$. This means that the numerical solutions u_N, v_N are polynomials in x of degree $< N$, subject to the boundary conditions (5c–d), which also satisfy the differential equations (5a–b) at $x = x_j$, $0 < j < N$. Since the polynomial $T_N'(x)$ vanishes at exactly these points and is of degree $N - 1$, we find that the spectral solutions satisfy:

$$(6) \quad \begin{cases} \text{a.} \quad \dfrac{\partial u_N}{\partial t} = \dfrac{\partial u_N}{\partial x} + \alpha(t)T_N'(x) \\[2ex] \text{b.} \quad \dfrac{\partial v_N}{\partial t} = -\dfrac{\partial v_N}{\partial x} + \beta(t)T_N'(x) \\[2ex] \text{c.} \quad u_N = Rv_N \qquad \text{at} \quad x = 1 \\[2ex] \text{d.} \quad v_N = Lu_N \qquad \text{at} \quad x = -1 \end{cases}$$

(we assume that the time evolution is not discretized). Using Lemma 2 we can solve (6a-b) in the form:

$$\overline{\phi(1+t)} + \bar{\alpha}\ \overline{T'_N(1+t)} = R\left(\overline{\psi(1-t)} + \bar{\beta}\ \overline{T'_N(1-t)}\right)$$

$$\overline{\psi(-1-t)} + \bar{\beta}\ \overline{T'_N(-1-t)} = L\left(\overline{\phi(-1+t)} + \bar{\alpha}\ \overline{T'_N(-1+t)}\right)$$

or

$$\begin{bmatrix} I_m\ \overline{T'_N(1+t)} & -R\ \overline{T'_N(1-t)} \\ -L\ \overline{T'_N(-1+t)} & I_n\ \overline{T'_N(-1-t)} \end{bmatrix} \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} \overset{\equiv}{\text{def}}\ A \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix} = \begin{bmatrix} R\overline{\psi(1-t)} - \overline{\phi(1+t)} \\ L\overline{\phi(-1+t)} + \overline{\phi(-1-t)} \end{bmatrix} \overset{\equiv}{\text{def}}\ r\ .$$

(I_m and I_n are identity matrices of the corresponding dimensions.) Since A and r are rational functions of τ (as transforms of polynomials) it is clear that $\bar{\alpha}$ and $\bar{\beta}$ will be rational. This means that the original functions α,β must be of the form:

$$\sum_k e^{\lambda_k t}\ p_k(t) \tag{7}$$

for some polynomials p_k; moreover the exponents λ_k are precisely the poles of $\bar{\alpha}$ and $\bar{\beta}$. These poles may be values where $\det |A| = 0$, or poles of the right-hand-side. However, we have the expansion:

$$A = \frac{1}{\tau^N}(A_0 + A_1\ \tau + \cdots)$$

$$r = \frac{1}{\tau^M}(r_0 + r_1 \tau + \cdots)$$

with $M \leq N$, since the degree of the coefficients is exactly $N - 1$, while the degree of r is $< N$. Thus $\tau = 0$, which is the only pole of the right-hand-side cannot be a pole of $\bar{\alpha}$ or $\bar{\beta}$ and we only have to discuss the zeros of $\det |A|$. Finally, since a function of the form (7) decays, if and only if, $\mathrm{Re}\lambda_k < 0$ for all k, we are left to prove that the determinant of A vanishes only in the open left half of the complex plane.

Introduce the polynomials of degree $N - 1$ $p(\tau)$ and $q(\tau)$, defined by:

$$\overline{T_N'(1+t)} = \frac{1}{\tau^N} p(\tau)$$

$$\overline{T_N'(1-t)} = \frac{1}{\tau^N} q(\tau) .$$

It is easily checked that $q(\tau) = p(-\tau)(-1)^{N-1}$; moreover T_N' is even or odd, satisfying

(8) $$T_N'(1) = (-1)^{N-1} T_N'(-1) .$$

Using these symmetry properties, $\det |A| = 0$ implies that

(9) $$\det \begin{vmatrix} I_m p(\tau) & -Rp(-\tau) \\ -Lp(-\tau) & I_n p(\tau) \end{vmatrix} = 0 .$$

Then, the matrix identity

$$\begin{pmatrix} I_m p(\tau) & -Rp(-\tau) \\ -Lp(-\tau) & I_n p(\tau) \end{pmatrix} \begin{pmatrix} I_m p(\tau) & Rp(-\tau) \\ 0 & I_n p(\tau) \end{pmatrix} = \begin{pmatrix} I_m p^2(\tau) & 0 \\ -Lp(\tau)p(-\tau) & I_n p^2(\tau) - LRp^2(-\tau) \end{pmatrix}$$

shows that the solutions of (9) may be found among the solutions of $p(\tau) = 0$ and the solutions of

$$\det |p^2(\tau)I_n - p^2(-\tau)LR| = 0.$$

This last equation simplifies to:

$$p(\tau) \pm \lambda p(-\tau) = 0 \tag{10}$$

where λ is any eigenvalue of LR. We know that $|\lambda| < 1$, since LR is a contraction. Moreover, in [2] it was shown that the polynomial $p(\tau)$ is Hurwitz. Using Lemma 1, we conclude that the solutions τ of (10) have negative real parts, and α and β tend to zero at large time, decreasing exponentially. This is sufficient to make all solutions of (6) vanish as $t -> \infty$. Indeed, let u and v solve (5) with polynomials of degree $< N$ as initial values. Then the differences $u - u_N$, $v - v_N$ satisfy the equation (2) with $\phi = 0$, and the explicit formula (3) shows they decay; since u and v also decay, by assumption, our assertion is proved.

The proof we presented hinges on two facts:

a. $p(\tau)$ is Hurwitz - exactly the required property for the scalar equation (1).

b. formula (8) - a symmetry property of T'_N.

Many other spectral approximations are possible - using different sets of orthogonal polynomials, or Galerkin and τ-methods instead of collocation. All of them will replace (5) by an equation similar to (6), the only difference being that T'_N must be changed to another polynomial (e.g., T_N for the τ- method). This polynomial is known explicitly for any given spectral method and we shall call it the forcing polynomial, since it appears as an inhomogenous term in (6). If the forcing polynomial has the correct symmetry and the corresponding scalar problem has only decaying solutions, the method of proof readily extends to the general spectral method. We should also remark at this point that the most usual Chebyshev discretization for $u_t = u_x$, collocating at the interior points and at the outflow, has the forcing polynomial $(x+1)T'_N(x)$, which is neither odd nor even, and thus is not covered by our theory.

In conclusion, we summarize our results in the following

Theorem. Let the solutions of (5) decay to zero in time. Consider a spectral approximation for (5) possessing an even or odd forcing polynomial, such that the numerical solutions obtained by this method applied to (1) tend to zero as time increases. Then the numerical solution corresponding to the system (5) also tends to zero as time increases.

REFERENCES

[1] D. Gottlieb and L. Lustman, The spectrum of the Chebyshev collocation operator for the heat equation, SIAM J. Numer. Anal., 20, no 5, (1983), pp. 909-921.

[2] D. Gottlieb and L. Lustman, The DuFort-Frankel Chebyshev method for parabolic initial boundary value problems, Comput. Fluids, 11, no 2, (1983), pp. 107-120.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-172432 ICASE Report No. 84-37		

4. Title and Subtitle

The Time Evolution of Spectral Discretizations of Hyperbolic Systems

5. Report Date

August 1984

6. Performing Organization Code

7. Author(s)

Liviu Lustman

8. Performing Organization Report No.

84-37

9. Performing Organization Name and Address

Institute for Computer Applications in Science and Engineering
Mail Stop 132C, NASA Langley Research Center
Hampton, VA 23665

10. Work Unit No.

11. Contract or Grant No.

NAS1-17261
NAS1-17070

12. Sponsoring Agency Name and Address

National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered

Contractor Report

14. Sponsoring Agency Code

505-31-83-01

15. Supplementary Notes

Langley Technical Monitor: R. H. Tolson
Final Report

16. Abstract

A Chebyshev collocation spectral method, applied to hyperbolic systems is considered, particularly for those initial boundary value problems which possess only solutions tending to zero at large times. It is shown that the numerical solutions of the system will also vanish at infinity, if and only if, the numerical solution of a scalar equation of the same type does. This result is then generalized for other spectral approximations.

17. Key Words (Suggested by Author(s))

spectral methods
hyperbolic systems
eigenvalues
Hurwitz polynomials

18. Distribution Statement

64 Numerical Analysis

Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	13	A02