NASA
Reference
Publication
1089

June 1982

A Formulation of Rotor-Airframe Coupling for Design Analysis of Vibrations of Helicopter Airframes

Raymond G. Kvaternik
and William C. Walton, Jr.

NASA

FEDD DOCUMENT

Note that this document bears the label "FEDD," an acronym for "FOR EARLY DOMESTIC DISSEMINATION." The FEDD label is affixed to documents that may contain information having high commercial potential.

The FEDD concept was developed as a result of the desire to maintain U.S. leadership in world trade markets and encourage a favorable balance of trade. Since the availability of tax-supported U.S. technology to foreign business interests could represent an unearned benefit, research results that may have high commercial potential are being distributed to U.S. industry in advance of general release.

The recipient of this report must treat the information it contains according to the conditions of the FEDD label on the front cover.

NASA
Reference
Publication
1089

1982

A Formulation of Rotor-Airframe Coupling for Design Analysis of Vibrations of Helicopter Airframes

Raymond G. Kvaternik
and William C. Walton, Jr.
Langley Research Center
Hampton, Virginia

National Aeronautics
and Space Administration

Scientific and Technical
Information Branch

CONTENTS

SUMMARY

The NASA Langley Research Center has been conducting research to establish foundations for adequate representation and treatment of the airframe structure in design analyses of helicopter vibrations. This paper presents a body of formulations to couple a finite-element analysis model of the airframe to a rotor analysis model and calculate airframe vibration levels. The rotor is represented by a general set of linear differential equations with periodic coefficients, and the connections between the rotor and airframe are specified through general linear equations of constraint. Background is provided relating to development of rotor and airframe models, and as an aid to structural engineers, the origins of linearized rotor equations are reviewed. Coupling equations are derived and then applied to combine the rotor and airframe equations into one set of linear differential equations governing vibrations of the rotor-airframe system. These equations are solved by the harmonic balance method to yield the system steady-state vibrations. A key feature of the solution process is to represent the airframe in terms of forced responses calculated at harmonics of the rotor rotational frequency. A method based on matrix partitioning is presented for quick recalculations of vibrations in design studies when only relatively few airframe members are varied. All relations are presented in forms suitable for direct computer implementation. During a discussion of the method, it is argued that a properly linearized rotor model should be as good for coupled rotor-airframe vibration analysis as the underlying nonlinear model, and representing the airframe by harmonic forced responses is recommended for airframe structural design work.

INTRODUCTION

Helicopters are prone to vibrations arising from the cyclic nature of the rotor actions. The vibrations normally pervade both the rotor and the airframe and can seriously degrade both service life and ride qualities. Vibrations also frequently limit the maximum speed in forward flight. In U.S. civil and military helicopters, it is usual to incorporate devices solely to attenuate vibrations. These devices have added significant weight and complexity.

U.S. helicopter companies have rarely relied on analysis during design in their efforts to limit vibration. With only a few exceptions, helicopters have been designed to performance requirements with past experience with vibrations taken into account, and new problems with vibration have been solved during flight tests and operation. There is now a recognized need to account for vibrations during the analytical phases of design. The advent of modern methods of computer analysis has provided the opportunity to achieve such a capability.

Four technical factors influencing vibrations of a helicopter should be recognized: (1) vibratory loads induced by the rotor actions, (2) response of the rotor, (3) coupling of the rotor and airframe, and (4) response of the airframe. The greatest excitation of vibrations is caused by cyclic loads on the rotor blades due to their interaction with the airstream. The dynamic characteristics of the rotor and airframe and the coupling of these two systems determine the manner in which the helicopter responds to this excitation. Thus, approaches to establish reliable methods for vibration analysis should address all four factors.

Among the analysis methods now employed by industry applicable to helicopter vibrations, two categories can be distinguished: (1) methods for analysis of airframe behavior and (2) methods for analysis of rotor behavior. For the nonrotating airframe components, the NASTRAN® computer code (ref. 1), which embodies the finite-element method for structural analysis, has become a standard tool used throughout the helicopter industry for design calculations of internal structural loads and checks on airframe vibrations. For rotating components, there has been extensive work on formulating equations of motion of rotors and devising computer solutions of the equations. (See, for example, refs. 2 to 6.) These categories of methods have evolved during the past 25 years since high-speed digital computers were introduced. The methods for rotors and airframes have generally been developed separately. To realize the ideal of a trusted vibration analysis, continued work is required to improve airframe analysis methods and rotor analysis methods separately. In addition, there is a requirement for general methods for combining mathematical models of the rotor and the airframe to calculate the vibrations of a helicopter as a single system.

The literature on practical methods for calculating vibrations of a helicopter as a single system has been sparse until quite recently when a number of papers have appeared. The earlier papers come from Gerstenberger and Wood (ref. 7) and Novak (ref. 8). In their analyses, a key feature is representation of the airframe by impedances at the rotor attachment points. In a later paper, Staley and Sciarra (ref. 9) also applied this approach. Reference 10 describes a vibration analysis of a helicopter design notable for the exposure of the treatment of the airframe. Here the airframe is also represented by impedances. The work of these earlier investigators includes correlations of analytical results with measured flight vibration data. References 7 to 10 provide a technical basis on which to formulate a general method of vibration analysis suitable for airframe structural design work. Three other recent papers (refs. 11 to 13) have specifically addressed practical methods for calculating helicopter vibrations. In contrast to references 7 to 10, they use a modal representation of the airframe, which is a significant alternative line of development.

It is recognized that this discussion excludes a number of existing computer simulations of the helicopter in flight (see, for example, refs. 3, 4, 6, and 14). Such simulations, of course, incorporate representations of both the rotor and the airframe and the connections between the two and thus theoretically could be applied to calculate vibrations. However, there is little note in the literature of their use to calculate airframe vibrations. These simulations have been applied mainly to development of flight controls, to checks on the stability of rotors, and to calculation of blade vibratory loads. As a rule, the current simulations incorporate only cursory, if any, treatment of the airframe elasticity, and they would be cumbersome to use directly for airframe structural design work.

References 15 to 19 contain relevant subsidiary analysis procedures and solutions of vibrations of simplified rotor-airframe systems. Reference 15 is particularly interesting since it is evidently the first paper to pose the problem of generalizing the harmonic balance method and is evidently the only attempt to solve this problem prior to the present paper. The harmonic balance method has been widely used to solve for steady-state vibrations of particular rotating systems, but a perfectly general algorithm is needed to systematize analysis of rotor-airframe vibrations. Also, as a matter of interest, the reader may compare the present paper with reference 16 with regard to the assumptions underlying the equations which express the coupling of the rotor to the airframe.

The present paper is an outcome of recent efforts at the NASA Langley Research Center to establish foundations for adequate representation and treatment of the airframe structure in design analysis of helicopter vibrations. The subject addressed is the problem of coupling a general finite-element model of the airframe system to rotor mathematical models. Presented are a body of formulations and discussions intended as a basis of both engineering and computational theory for efficiently implementing such couplings and calculating the airframe vibration levels. Emphasis is placed on analysis of vibrations during industrial design of airframe components. Rotor analysis is discussed, but with the limited purpose of ensuring correct inclusion of rotor effects in the airframe analysis.

The presentation consists of a text and a number of appendices. The text describes the salient features of the proposed method. The appendices provide background on equations which appear in the text and discuss subsidiary topics. The text covers the following topics:

1. The general form of the equations for the three components of a linear coupled rotor-airframe vibration model: linearized rotor model, finite-element airframe model, and linearized interface model

2. The steps for combining the component equations to form the coupled equations

3. The steps for solution of the coupled equations by the harmonic balance method, including reduction of the number of airframe degrees of freedom in the harmonic balance equations by use of airframe harmonic forced responses

4. A block diagram indicating the basic sequence of tasks for calculating airframe vibrations

5. The plausibility and utility of a linear rotor-airframe model

6. Computational advantages of the method

The appendices discuss the following:

1. The three component models

2. An alternative representation of the rotor by impedances

3. Subsidiary computations

4. A method for quick recalculation of airframe vibrations in design studies when only relatively few airframe members are varied

5. Additional block diagrams indicating the sequence of tasks for calculating airframe vibrations including steps for reanalysis and for representation of the rotor by impedances

PREPARATORY REMARKS

The Helicopter Vibration Problem

The system treated is a flexible helicopter in steady flight as sketched in figure 1. The vehicle is viewed as stationary and the air as moving. For convenience

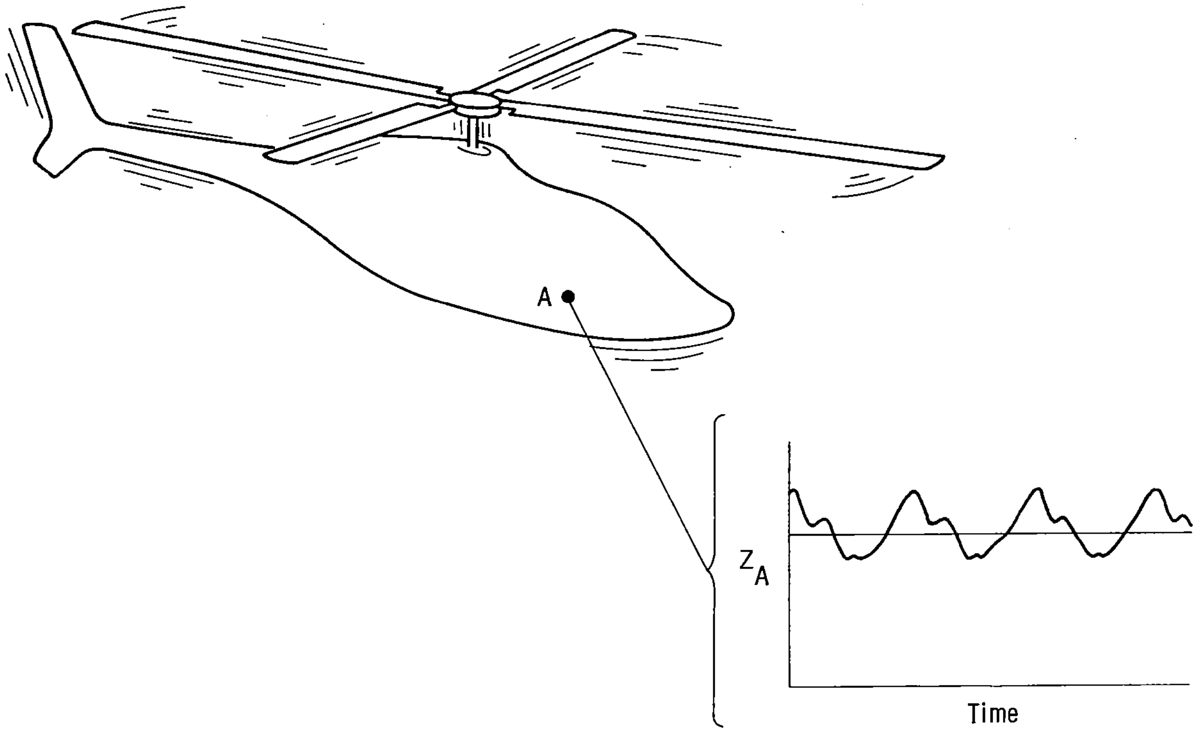


Figure 1.- Steady-state vibrations of a flexible helicopter in steady flight.

of discussion, it is assumed that there is only one rotor and that the required anti-torque is provided by constant external forces. A turning rotor interacting with the airstream gives rise to vibrations which normally pervade both the rotor and the airframe. In steady flight, the vibration amplitudes take on a periodic character, as indicated in the figure.

Steady Flight

Examples of steady flight conditions are hover, straight and level flight, and circling with constant speed, bank angle, and altitude. Nonsteady conditions occurring during maneuvers such as vertical ascents, pull-ups, and push-overs can be realistically treated under steady flight assumptions if, as usual, the time to execute the maneuver encompasses many rotor revolutions.

Rotor and Airframe Systems

Figure 2 illustrates the helicopter components which must be considered in order to represent the interface between the rotating and nonrotating parts. The portion of the helicopter containing all those parts which in the analysis are considered to be rotating is called the rotor system. The remaining portion of the helicopter is called the airframe system.

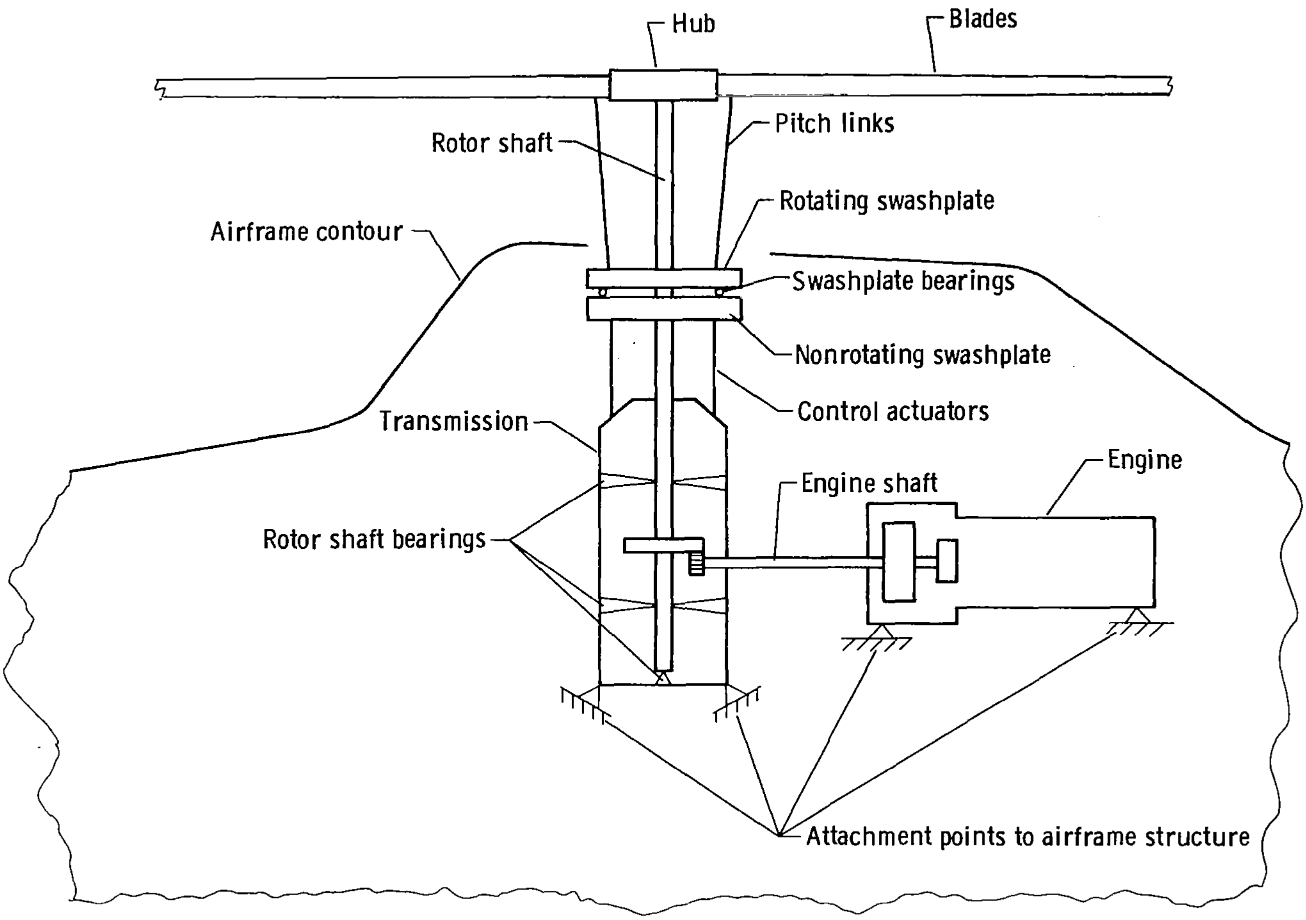
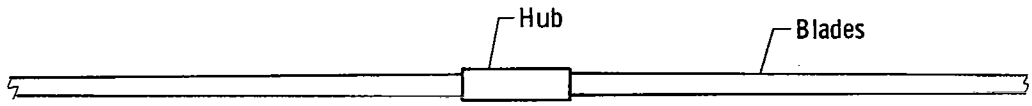


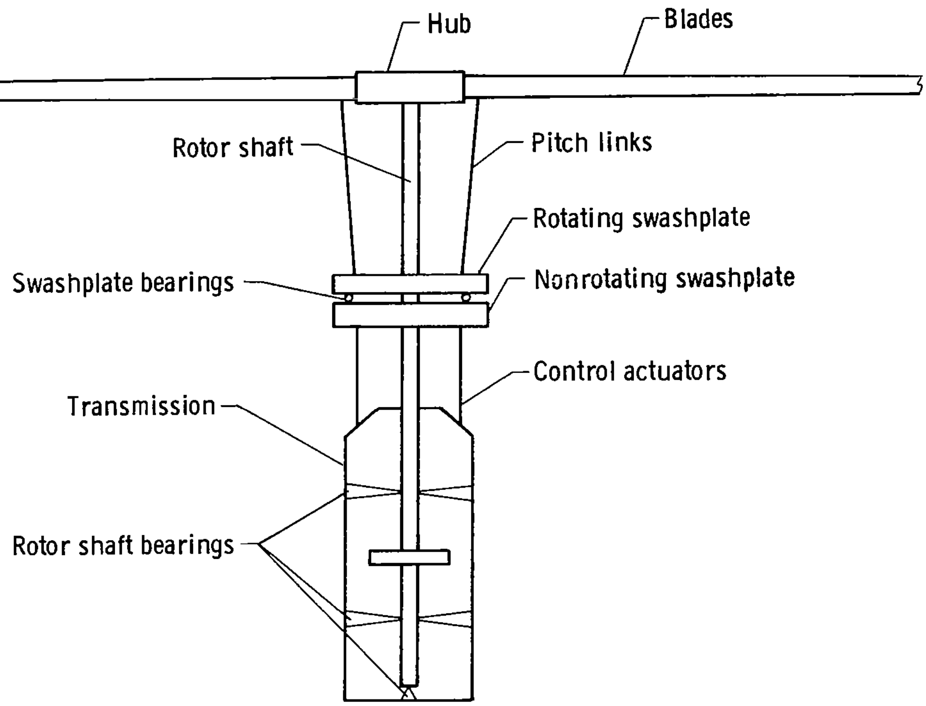
Figure 2.- Components of a helicopter which typically must be considered in defining the interface between the rotor system and the airframe system.

Figure 3 illustrates two ways in which rotor systems could be defined for analysis. In figure 3(a), the rotor system is limited to blades and hub. In this case, some parts of the airframe system must be considered to be nonrotating even though certain of these parts such as the rotor shaft, pitch links, and rotating swashplate actually do rotate. Thus, this point of view implicitly incorporates a fictitious bearing connecting the hypothetically stationary rotor shaft and the rotating hub. The torque needed to maintain the rotational speed of the rotor is implicitly assumed to be externally applied at the hub bearing. Effects in the rotor system resulting from rotation of components not included in the rotor system can be accounted for by specifying an impedance at the hub. A characteristic of the point of view of figure 3(a) is that only one load path leading from the rotor system to the airframe system, namely, that provided by the shaft, is taken into account. This point of view of the rotor system has been the one predominantly adopted in analysis of helicopter vibrations.

In figure 3(b), the rotor system includes the blades, hub, rotor shaft, pitch links, rotating swashplate, nonrotating swashplate, control actuators, and transmission. The engine and engine shaft, which are included in the airframe system, are



(a) System consisting of blades and hub.



(b) System consisting of blades and hub, rotor shaft, pitch links, rotating swashplate, nonrotating swashplate, control actuators, and transmission.

Figure 3.- Two illustrative definitions of the rotor system.

considered to be nonrotating. Thus, the point of view of figure 3(b) implicitly incorporates a fictitious bearing in the engine shaft. The torque needed to maintain the rotational speed of the rotor is implicitly assumed to be externally applied at this fictitious bearing. The impedance effects of rotating components in the engine which are not included in the rotor system can be accounted for as an impedance at the fictitious bearing. The point of view of figure 3(b) allows additional load paths between the rotor and the airframe.

In this paper, the restriction is made that the rotor system may include only those rotating parts having rotational speeds which are integer multiples of the rotor speed. This would generally rule out direct inclusion of the engine and engine shaft (fig. 2) as rotating parts, since they usually operate at a noninteger multiple of the rotor speed. Effects of engine rotation can be indirectly included as an impedance, as mentioned.

EQUATIONS OF MOTION OF COUPLED ROTOR-AIRFRAME SYSTEM

Three elements are needed for forming the equations of motion of a coupled rotor-airframe system: (1) the rotor equations of motion, (2) the airframe equations of motion, and (3) the coupling equations. General linearized equations for these three elements are derived in appendices A, B, and C. In this section, these equations are briefly described, and then linear equations of motion are derived for a general coupled rotor-airframe system.

Rotor Equations of Motion

The linearized rotor equations derived in appendix A are presented in virtual work form as

$$\int_0^t \delta W_r \, dt = \int_0^t \delta\{DVR\}^T \Big\{\{QRO\} + [MR]\{D\ddot{V}R\} + [CR]\{D\dot{V}R\} + [KR]\{DVR\}\Big\} \, dt = 0 \qquad (1)$$

Equation (1) is a variational equation equivalent to a set of linear ordinary differential equations of second order. The independent variable is the time t. In equation (1) the matrices [MR], [CR], [KR], and {QRO} are known matrices which characterize the rotor system. The elements of the vector {DVR} are unknown variables which determine the vibratory motions of the rotor. These variables are often called generalized coordinates. However, for visualization they may be thought of as representing small displacements from a trim solution. The known matrices are all periodic functions of time with period $2\pi/\Omega$ where Ω is the rotor rotational frequency. The matrices [MR], [CR], and [KR] represent the inertial, damping, and stiffness properties of the rotor including contributions from both mechanical and aerodynamic models. The vector {QRO} comes from the rotor trim solution and is the vector of external forces required to maintain the conditions assumed in calculating the trim solution. Fourier series forms for the known matrices in equation (1) are defined as follows:

$$[MR] = [MR0C] + [MR1S] \sin \Omega t + [MR1C] \cos \Omega t + [MR2S] \sin 2\Omega t + \ldots \qquad (2a)$$

$$[CR] = [CR0C] + [CR1S] \sin \Omega t + [CR1C] \cos \Omega t + [CR2S] \sin 2\Omega t + \ldots \qquad (2b)$$

$$[KR] = [KR0C] + [KR1S] \sin \Omega t + [KR1C] \cos \Omega t + [KR2S] \sin 2\Omega t + \ldots \qquad (2c)$$

$$\{QRO\} = \{QRO0C\} + \{QRO1S\} \sin \Omega t + \{QRO1C\} \cos \Omega t + \{QRO2S\} \sin 2\Omega t + \ldots \qquad (2d)$$

The coefficient matrices appearing in these series, such as [MR1S] and {QRO1S}, are constant.

As discussed in appendix A, equation (1) is partitioned for use in coupling the rotor to the airframe:

$$\int_0^t \delta W_r \, dt = \int_0^t \delta \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix}^T \left\{ \begin{Bmatrix} \{QBO\} \\ \{QHO\} \end{Bmatrix} + \begin{bmatrix} [MR11][MR12] \\ [MR21][MR22] \end{bmatrix} \begin{Bmatrix} \{\ddot{DVB}\} \\ \{\ddot{DVH}\} \end{Bmatrix} \right.$$

$$\left. + \begin{bmatrix} [CR11][CR12] \\ [CR21][CR22] \end{bmatrix} \begin{Bmatrix} \{\dot{DVB}\} \\ \{\dot{DVH}\} \end{Bmatrix} + \begin{bmatrix} [KR11][KR12] \\ [KR21][KR22] \end{bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} \right\} dt = 0 \tag{3}$$

The submatrix {DVH} contains the rotor variables which appear explicitly in relations expressing connections of the rotor to the airframe. Of course, each of the submatrices appearing in equation (3) has a Fourier series representation analogous to equations (2). For example,

$$[MR11] = [MR0C11] + [MR1S11] \sin \Omega t + [MR1C11] \cos \Omega t + \ldots \tag{4a}$$

$$\{QBO\} = \{QBO0C\} + \{QBO1S\} \sin \Omega t + \{QBO1C\} \cos \Omega t + \ldots \tag{4b}$$

$$\{QHO\} = \{QHO0C\} + \{QHO1S\} \sin \Omega t + \{QHO1C\} \cos \Omega t + \ldots \tag{4c}$$

In this paper, it is assumed that the linearized rotor equations of motion are specified at the outset of computation. The specification is made by

1. Providing the coefficient matrices in equations (2)

2. Providing the rotor rotational frequency Ω

3. Identifying the variables appearing in {DVH} so that the partitions indicated in equation (3) can be made.

Airframe Equations of Motion

As discussed in appendix B, the airframe equations of motion are based on a typical airframe finite-element model. Formulated in terms of virtual work, the equations take the form,

$$\int_0^t \delta W_a \, dt = \int_0^t \delta\{\bar{Z}\}^T \left\{ [\overline{MA}]\{\ddot{\bar{Z}}\} + [\overline{CA}]\{\dot{\bar{Z}}\} + [\overline{KA}]\{\bar{Z}\} + \{\bar{L}\} \right\} dt = 0 \tag{5}$$

The elements of the vector $\{\bar{Z}\}$ are unknown variables which determine the displacements of the airframe finite-element model. The matrices $[\overline{MA}]$, $[\overline{CA}]$, and $[\overline{KA}]$ represent the mass, damping, and stiffness matrices of the model. These are known constant matrices. The vector $\{\bar{L}\}$ in equation (5) represents the loads directly applied to the airframe from sources which are external to both the airframe system and the rotor mechanical system. It is emphasized that $\{\bar{L}\}$ does not include loads imposed upon the airframe from the rotor system through mechanical connections. These rotor loads are treated as internal loads. Types of practical loads which can contribute to $\{\bar{L}\}$ include

1. Constant aerodynamic and gravitational loads

2. Oscillatory aerodynamic loads generated by the interaction of the rotor system with the airstream and impinging directly on the airframe

3. Oscillatory test loads

The following Fourier series form encompassing these types of loads is assumed for $\{\bar{L}\}$:

$$\{\bar{L}\} = \{\overline{L0C}\} + \{\overline{L1S}\} \sin \Omega t + \{\overline{L1C}\} \cos \Omega t + \ldots \tag{6}$$

The coefficient vectors in this series are constant. These vectors, with the exception of $\{\overline{L0C}\}$, represent forces oscillating about zero mean and are assumed to be prescribed at the outset of computation. It turns out that it is not necessary to know the distribution of the mean loads acting on the airframe to calculate the steady-state vibration levels. However, it may be convenient for some purposes to designate static forces acting on the airframe. In this paper, such known forces are represented by $\{\overline{L0C}\}$. Note that there is no requirement for the static forces represented by $\{\overline{L0C}\}$ to balance the mean rotor forces represented in appendix A by {QHTR}. Equilibrium of the mean rotor forces is explained subsequently.

As noted in appendix B, a partitioned form of equation (5) is introduced to facilitate coupling the airframe to the rotor:

$$\int_0^t \delta W_a \, dt = \int_0^t \delta \begin{Bmatrix} \{\overline{ZH}\} \\ \{\overline{ZS}\} \end{Bmatrix}^T \left\{ \begin{bmatrix} [\overline{MA11}][\overline{MA12}] \\ [\overline{MA21}][\overline{MA22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\overline{ZH}}\} \\ \{\ddot{\overline{ZS}}\} \end{Bmatrix} + \begin{bmatrix} [\overline{CA11}][\overline{CA12}] \\ [\overline{CA21}][\overline{CA22}] \end{bmatrix} \begin{Bmatrix} \{\dot{\overline{ZH}}\} \\ \{\dot{\overline{ZS}}\} \end{Bmatrix} \right.$$

$$\left. + \begin{bmatrix} [\overline{KA11}][\overline{KA12}] \\ [\overline{KA21}][\overline{KA22}] \end{bmatrix} \begin{Bmatrix} \{\overline{ZH}\} \\ \{\overline{ZS}\} \end{Bmatrix} + \begin{Bmatrix} \{\overline{LH}\} \\ \{\overline{LS}\} \end{Bmatrix} \right\} dt = 0 \tag{7}$$

In equation (7), the vector $\{\overline{ZH}\}$ contains the airframe variables which appear explicitly in relations expressing connections of the rotor system to the airframe system. The following Fourier series forms are specified for the partitions of $\{\bar{L}\}$ appearing in equation (7):

$$\{\overline{LH}\} = \{\overline{LH0C}\} + \{\overline{LH1S}\} \sin \Omega t + \{\overline{LH1C}\} \cos \Omega t + \ldots \tag{8a}$$

$$\{\overline{LS}\} = \{\overline{LS0C}\} + \{\overline{LS1S}\} \sin \Omega t + \{\overline{LS1C}\} \cos \Omega t + \ldots \tag{8b}$$

Coupling Equations

The general coupling equations developed in appendix C are

$$\{DVH\} = [\overline{TH}]\{\overline{ZH}\} \tag{9a}$$

$$[\overline{TC}]\{\overline{ZH}\} = \{0\} \tag{9b}$$

where $[\overline{TH}]$ and $[\overline{TC}]$ are known constant matrices. As noted in appendix C, the rows in both $[\overline{TH}]$ and $[\overline{TC}]$ are linearly independent. The first equation explicitly expresses the rotor interface variables $\{DVH\}$ in terms of the airframe interface variables, $\{\overline{ZH}\}$. The second equation expresses any constraints on the airframe interface variables implied by the assumptions made in modeling the rotor system. If no such constraints are implied, then equation (9b) is absent.

Application of Constraints to Airframe Interface Variables

To derive the coupled equations, the constraints represented by equation (9b) must first be applied to the airframe interface variables $\{\overline{ZH}\}$. This modifies both the first coupling equation (9a) and the airframe equation (7) and eliminates equation (9b).

Modification of first coupling equation.- When constraints are applied, some of the variables in $\{\overline{ZH}\}$ are no longer independent variables. To distinguish dependent and independent variables, equation (9b) is written in partitioned form as

$$\left[[\overline{TCD}] \;\vdots\; [\overline{TCI}] \right] \left\{ \begin{matrix} \{\overline{ZHD}\} \\ \{\overline{ZHI}\} \end{matrix} \right\} = \{0\} \tag{10}$$

where $[\overline{TCD}]$ is assumed to be a nonsingular square matrix. The vector $\{\overline{ZHD}\}$, which is associated with the submatrix $[\overline{TCD}]$, represents the dependent variables. The vector $\{\overline{ZHI}\}$, which is associated with the submatrix $[\overline{TCI}]$, represents the independent variables. Since the rows of $[\overline{TC}]$ are linearly independent, at least one nonsingular submatrix always exists. However, in general, elements of the vector $\{\overline{ZH}\}$ must be renumbered and the columns of the matrix $[\overline{TC}]$ must be reordered to position such a submatrix to the left as has been done in equation (10).

Equation (10) yields the following equation for elimination of the dependent variables:

$$\begin{Bmatrix} \{\overline{ZHD}\} \\ \{\overline{ZHI}\} \end{Bmatrix} = \begin{bmatrix} -[\overline{TCD}]^{-1}[\overline{TCI}] \\ [I] \end{bmatrix} \{\overline{ZHI}\} \tag{11}$$

The right side of equation (9a) is renumbered and reordered as was done in obtaining equation (10) from equation (9b):

$$\{DVH\} = \left[[\overline{THD}] \;\vdots\; [\overline{THI}] \right] \begin{Bmatrix} \{\overline{ZHD}\} \\ \{\overline{ZHI}\} \end{Bmatrix} \tag{12}$$

Substitution of equation (11) into equation (12) yields the modified coupling equation

$$\{DVH\} = [TH]\{\overline{ZHI}\} \tag{13}$$

where

$$[TH] = \left[[\overline{THI}] - [\overline{THD}][\overline{TCD}]^{-1}[\overline{TCI}] \right] \tag{14}$$

Modification of airframe equations.- As discussed in appendix B, the application of linear constraints to finite-element models is straightforward and in the NASTRAN finite-element code has been rendered systematic. Nevertheless, the formal steps for incorporating constraints in a finite-element model are stated here.

Equation (11) can be rewritten by reordering rows as follows:

$$\{\overline{ZH}\} = [\overline{TI}]\{\overline{ZHI}\} \tag{15}$$

where $[\overline{TI}]$ is the reordered coefficient matrix. Substitution of this equation into equation (7) produces the following modified form of the airframe equations:

$$\int_0^t \delta W_a \, dt = \int_0^t \delta \begin{Bmatrix} \{\overline{ZHI}\} \\ \{\overline{ZS}\} \end{Bmatrix}^T \left\{ \begin{bmatrix} [\overline{TI}]^T[\overline{MA11}][\overline{TI}] & [\overline{TI}]^T[\overline{MA12}] \\ [\overline{MA21}][\overline{TI}] & [\overline{MA22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\overline{ZHI}}\} \\ \{\ddot{\overline{ZS}}\} \end{Bmatrix} \right.$$

$$+ \begin{bmatrix} [\overline{TI}]^T[\overline{CA11}][\overline{TI}] & [\overline{TI}]^T[\overline{CA12}] \\ [\overline{CA21}][\overline{TI}] & [\overline{CA22}] \end{bmatrix} \begin{Bmatrix} \{\dot{\overline{ZHI}}\} \\ \{\dot{\overline{ZS}}\} \end{Bmatrix}$$

$$\left. + \begin{bmatrix} [\overline{TI}]^T[\overline{KA11}][\overline{TI}] & [\overline{TI}]^T[\overline{KA12}] \\ [\overline{KA21}][\overline{TI}] & [\overline{KA22}] \end{bmatrix} \begin{Bmatrix} \{\overline{ZHI}\} \\ \{\overline{ZS}\} \end{Bmatrix} + \begin{Bmatrix} [\overline{TI}]^T\{\overline{LH}\} \\ \{\overline{LS}\} \end{Bmatrix} \right\} dt = 0 \qquad (16)$$

To distinguish between the variables of the original and modified finite-element models, the variables of equation (16) are renamed:

$$\begin{Bmatrix} \{\overline{ZHI}\} \\ \{\overline{ZS}\} \end{Bmatrix} = \begin{Bmatrix} \{ZH\} \\ \{ZS\} \end{Bmatrix} = \{Z\} \qquad (17)$$

And equation (16) is rewritten as

$$\int_0^t \delta W_a \, dt = \int_0^t \delta\{Z\}^T \{F\} \, dt = 0 \qquad (18)$$

where

$$\{F\} = [MA]\{\ddot{Z}\} + [CA]\{\dot{Z}\} + [KA]\{Z\} + \{L\} \qquad (19)$$

The matrices [MA], [CA], [KA], and {L} are defined by comparing equation (19) with equation (16). The numbers of variables in {Z}, {ZH}, and {ZS} are denoted by NZ, NZH, and NZS.

The following partitioned form of equation (18) is introduced:

$$\int_0^t \delta W_a \, dt = \int_0^t \delta \begin{Bmatrix} \{ZH\} \\ \{ZS\} \end{Bmatrix}^T \begin{Bmatrix} \{FH\} \\ \{FS\} \end{Bmatrix} dt = 0 \qquad (20)$$

where

$$\begin{Bmatrix} \{FH\} \\ \{FS\} \end{Bmatrix} = \{F\} = \begin{bmatrix} [MA11][MA12] \\ [MA21][MA22] \end{bmatrix} \begin{Bmatrix} \{\ddot{Z}H\} \\ \{\ddot{Z}S\} \end{Bmatrix} + \begin{bmatrix} [CA11][CA12] \\ [CA21][CA22] \end{bmatrix} \begin{Bmatrix} \{\dot{Z}H\} \\ \{\dot{Z}S\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [KA11][KA12] \\ [KA21][KA22] \end{bmatrix} \begin{Bmatrix} \{ZH\} \\ \{ZS\} \end{Bmatrix} + \begin{Bmatrix} \{LH\} \\ \{LS\} \end{Bmatrix} \tag{21}$$

Formulas for the submatrices in equation (21) are readily discerned by comparison with equations (16) and (17). For example,

$$[MA11] = [\overline{TI}]^T[\overline{MA11}][\overline{TI}] \tag{22a}$$

$$\{LH\} = [\overline{TI}]^T\{\overline{LH}\} \tag{22b}$$

Consistent with equations (7), (8), (21), and (22), Fourier series forms are defined for $\{L\}$, $\{LH\}$, and $\{LS\}$:

$$\{L\} = \{L0C\} + \{L1S\} \sin \Omega t + \{L1C\} \cos \Omega t + \ldots \tag{23a}$$

$$\{LH\} = \{LH0C\} + \{LH1S\} \sin \Omega t + \{LH1C\} \cos \Omega t + \ldots \tag{23b}$$

$$\{LS\} = \{LS0C\} + \{LS1S\} \sin \Omega t + \{LS1C\} \cos \Omega t + \ldots \tag{23c}$$

Definition of an Airframe Trim Solution

As explained in appendix A, definition of a rotor trim solution provides a basis for linearization of the rotor equations. The airframe finite-element model directly produces linear equations of motion, so there is no requirement for such a linearization step. However, definition of an airframe trim solution facilitates somewhat the derivations shown in this paper. The airframe trim solution is defined to be the vector $\{Z\}$ satisfying the following conditions:

1. The variables in $\{ZH\}$ are constrained to be zero:

$$\{ZH\} = \{0\} \tag{24a}$$

2. All the variables in $\{Z\}$ are periodic with period $2\pi/\Omega$:

$$\{Z(t+2\pi/\Omega)\} = \{Z(t)\} \tag{24b}$$

3. The generalized forces in the vector $\{FS\}$ given by equation (21) are zero:

$$\{FS\} = \{0\} \tag{24c}$$

Equation (24c) expresses that the defined airframe trim solution equilibrates all the airframe generalized forces $\{F\}$ except for the forces $\{FH\}$ corresponding to the variables involved in expressing mechanical connections between the rotor and airframe systems. A solution of the finite-element equations meeting the stated conditions always exists.

Notation and partitioning corresponding to the airframe trim solution are indicated by the following equation:

$$\{Z\}_{trim} \equiv \{ZO\} = \begin{Bmatrix} \{ZHO\} \\ \{ZSO\} \end{Bmatrix} \tag{25}$$

where in view of equation (24a),

$$\{ZHO\} = \{0\} \tag{26}$$

The generalized forces corresponding to the airframe trim solution are indicated by

$$\{F\}_{trim} \equiv \{FO\} = \begin{Bmatrix} \{FHO\} \\ \{FSO\} \end{Bmatrix} \tag{27}$$

where in view of equation (24c),

$$\{FSO\} = \{0\} \tag{28}$$

The vector partition $\{ZSO\}$ which constitutes the unspecified elements of the trim solution vector $\{ZO\}$ is determined by the following differential equation obtained from equation (21):

$$[MA22]\{\ddot{ZSO}\} + [CA22]\{\dot{ZSO}\} + [KA22]\{ZSO\} + \{LS\} = \{0\} \tag{29}$$

The generalized forces in the vector {FO} may be interpreted as the external forces necessary, in addition to the specified external forces {L}, to maintain the airframe variables {Z} at the specified trim values {ZO}. The vector {FHO}, which constitutes the unspecified elements of {FO}, is given explicitly by

$$\{FHO\} = [MA12]\{Z\ddot{S}O\} + [CA12]\{Z\dot{S}O\} + [KA12]\{ZSO\} + \{LH\} \tag{30}$$

which follows from equation (21). Note that the forces {FHO} correspond to the variables {ZH}, which are the airframe variables directly involved in expressing the mechanical connections between the rotor and the airframe systems. Conforming with equation (24b), {ZSO} can be written in Fourier series form as

$$\{ZSO\} = \{ZSO0C\} + \{ZSO1S\} \sin \Omega t + \{ZSO1C\} \cos \Omega t + \ldots \tag{31}$$

and it follows that {FHO} has the Fourier series form,

$$\{FHO\} = \{FHO0C\} + \{FHO1S\} \sin \Omega t + \{FHO1C\} \cos \Omega t + \ldots \tag{32}$$

Procedures for computing the airframe trim solution are given in appendices D and E.

Airframe Equations of Motion Varied From Trim

The vector of airframe variables {Z} is expressed as

$$\{Z\} = \{ZO\} + \{DZ\} \tag{33}$$

where the elements of the vector {DZ} represent variations from trim. This equation is analogous to equation (A18a) which was used to linearize the rotor equations. As in preceding steps, {DZ} is partitioned as

$$\{DZ\} = \begin{Bmatrix} \{DZH\} \\ \{DZS\} \end{Bmatrix} \tag{34}$$

From equations (17), (25), and (26), equation (33) may be written as two equations:

$$\{ZH\} = \{DZH\} \tag{35a}$$

$$\{ZS\} = \{ZSO\} + \{DZS\} \tag{35b}$$

From equation (33), the virtual displacements $\delta\{Z\}$ are given by

$$\delta\{Z\} = \delta\{DZ\} \tag{36}$$

By using equations (18), (19), (33), and (36) the virtual work equation of an airframe finite-element model may be expressed in the form,

$$\int_0^t \delta W_a \, dt = \int_0^t \delta\{DZ\}^T \Big\{\{FO\} + [MA]\{\ddot{DZ}\} + [CA]\{\dot{DZ}\} + [KA]\{DZ\}\Big\} \, dt = 0 \tag{37}$$

where

$$\{FO\} = [MA]\{\ddot{ZO}\} + [CA]\{\dot{ZO}\} + [KA]\{ZO\} + \{L\} \tag{38}$$

Equation (37) is equivalent to the differential equations,

$$\{FO\} + [MA]\{\ddot{DZ}\} + [CA]\{\dot{DZ}\} + [KA]\{DZ\} = \{0\} \tag{39}$$

The following partitioned forms of equations (37) and (39) are noted for subsequent use in coupling the airframe to the rotor:

$$\begin{aligned}\int_0^t \delta W_a \, dt = \int_0^t \delta \begin{Bmatrix}\{DZH\} \\ \{DZS\}\end{Bmatrix}^T \Bigg\{ \begin{Bmatrix}\{FHO\} \\ \{0\}\end{Bmatrix} + \begin{bmatrix}[MA11][MA12] \\ [MA21][MA22]\end{bmatrix} \begin{Bmatrix}\{\ddot{DZH}\} \\ \{\ddot{DZS}\}\end{Bmatrix} + \begin{bmatrix}[CA11][CA12] \\ [CA21][CA22]\end{bmatrix} \begin{Bmatrix}\{\dot{DZH}\} \\ \{\dot{DZS}\}\end{Bmatrix} \\ + \begin{bmatrix}[KA11][KA12] \\ [KA21][KA22]\end{bmatrix} \begin{Bmatrix}\{DZH\} \\ \{DZS\}\end{Bmatrix} \Bigg\} \, dt = 0\end{aligned} \tag{40}$$

$$\begin{Bmatrix}\{FHO\} \\ \{0\}\end{Bmatrix} + \begin{bmatrix}[MA11][MA12] \\ [MA21][MA22]\end{bmatrix} \begin{Bmatrix}\{\ddot{DZH}\} \\ \{\ddot{DZS}\}\end{Bmatrix} + \begin{bmatrix}[CA11][CA12] \\ [CA21][CA22]\end{bmatrix} \begin{Bmatrix}\{\dot{DZH}\} \\ \{\dot{DZS}\}\end{Bmatrix} + \begin{bmatrix}[KA11][KA12] \\ [KA21][KA22]\end{bmatrix} \begin{Bmatrix}\{DZH\} \\ \{DZS\}\end{Bmatrix} = \begin{Bmatrix}\{0\} \\ \{0\}\end{Bmatrix} \tag{41}$$

Coupling of Rotor and Airframe Equations

Virtual work for the rotor system and the airframe system is additive. Therefore the virtual work equation for the two systems is given by

$$\int_0^t \delta W \, dt = \int_0^t (\delta W_r + \delta W_a) \, dt = 0 \tag{42}$$

Using the partitioned forms which have been established in equations (3) and (40) yields for equation (42)

$$\int_0^t \delta W \, dt = \int_0^t \delta \begin{Bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH\} \\ \{DZS\} \end{Bmatrix} \end{Bmatrix}^T \left\{ \begin{Bmatrix} \begin{Bmatrix} \{QBO\} \\ \{QHO\} \end{Bmatrix} \\ \begin{Bmatrix} \{FHO\} \\ \{0\} \end{Bmatrix} \end{Bmatrix} + \begin{bmatrix} \begin{bmatrix} [MR11][MR12] \\ [MR21][MR22] \end{bmatrix} & \\ & \begin{bmatrix} [MA11][MA12] \\ [MA21][MA22] \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \{D\ddot{V}B\} \\ \{D\ddot{V}H\} \end{Bmatrix} \\ \begin{Bmatrix} \{D\ddot{Z}H\} \\ \{D\ddot{Z}S\} \end{Bmatrix} \end{Bmatrix} \right.$$

$$+ \begin{bmatrix} \begin{bmatrix} [CR11][CR12] \\ [CR21][CR22] \end{bmatrix} & \\ & \begin{bmatrix} [CA11][CA12] \\ [CA21][CA22] \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \{D\dot{V}B\} \\ \{D\dot{V}H\} \end{Bmatrix} \\ \begin{Bmatrix} \{D\dot{Z}H\} \\ \{D\dot{Z}S\} \end{Bmatrix} \end{Bmatrix}$$

$$\left. + \begin{bmatrix} \begin{bmatrix} [KR11][KR12] \\ [KR21][KR22] \end{bmatrix} & \\ & \begin{bmatrix} [KA11][KA12] \\ [KA21][KA22] \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH\} \\ \{DZS\} \end{Bmatrix} \end{Bmatrix} \right\} dt = 0 \tag{43}$$

The block diagonal form of the three coefficient matrices appearing in equation (43) indicates that the rotor equations and the airframe equations are still uncoupled. Substituting equations (17) and (35a) into equation (13) yields

$$\{DVH\} = [TH]\{DZH\} \tag{44}$$

Coupling of the rotor and airframe equations in equation (43) is effected by using equation (44) to eliminate the variables {DVH} as independent variables. Formally this is accomplished by substituting the transformation,

$$
\left\{\begin{matrix}\left\{\begin{matrix}\{DVB\}\\\{DVH\}\end{matrix}\right\}\\ \\ \left\{\begin{matrix}\{DZH\}\\\{DZS\}\end{matrix}\right\}\end{matrix}\right\} = \begin{bmatrix}[I]\,[0]\,[0]\\ [0][TH][0]\\ \\ [0]\,[I]\,[0]\\ [0]\,[0]\,[I]\end{bmatrix}\left\{\begin{matrix}\{DVB\}\\\{DZH\}\\\{DZS\}\end{matrix}\right\} \tag{45}
$$

into equation (43). This gives the following form for the virtual work equation of the coupled system in terms of the independent variables:

$$
\begin{aligned}
\int_0^t \delta W\,dt = \int_0^t \delta\left\{\begin{matrix}\{DVB\}\\\{DZH\}\\\{DZS\}\end{matrix}\right\}^T & \left\{\begin{bmatrix}[MR11] & [MR12][TH] & [0]\\ [TH]^T[MR21] & \begin{matrix}[TH]^T[MR22][TH]\\+\,[MA11]\end{matrix} & [MA12]\\ [0] & [MA21] & [MA22]\end{bmatrix}\left\{\begin{matrix}\{\ddot{DVB}\}\\\{\ddot{DZH}\}\\\{\ddot{DZS}\}\end{matrix}\right\}\right.\\
&+\begin{bmatrix}[CR11] & [CR12][TH] & [0]\\ [TH]^T[CR21] & \begin{matrix}[TH]^T[CR22][TH]\\+\,[CA11]\end{matrix} & [CA12]\\ [0] & [CA21] & [CA22]\end{bmatrix}\left\{\begin{matrix}\{\dot{DVB}\}\\\{\dot{DZH}\}\\\{\dot{DZS}\}\end{matrix}\right\}\\
&+\begin{bmatrix}[KR11] & [KR12][TH] & [0]\\ [TH]^T[KR21] & \begin{matrix}[TH]^T[KR22][TH]\\+\,[KA11]\end{matrix} & [KA12]\\ [0] & [KA21] & [KA22]\end{bmatrix}\left\{\begin{matrix}\{DVB\}\\\{DZH\}\\\{DZS\}\end{matrix}\right\}\\
&\left.+\left\{\begin{matrix}\{QBO\}\\ [TH]^T\{QHO\}+\{FHO\}\\ \{0\}\end{matrix}\right\}\right\}dt = 0
\end{aligned} \tag{46}
$$

The virtual work equation must be satisfied for any variations of independent variables. Therefore, the following differential equation is obtained:

$$\begin{bmatrix} [MR11] & [MR12][TH] & [0] \\ [TH]^T[MR21] & \begin{matrix}[TH]^T[MR22][TH] \\ +\ [MA11]\end{matrix} & [MA12] \\ [0] & [MA21] & [MA22] \end{bmatrix} \begin{Bmatrix} \{D\ddot{V}B\} \\ \{D\ddot{Z}H\} \\ \{D\ddot{Z}S\} \end{Bmatrix} + \begin{bmatrix} [CR11] & [CR12][TH] & [0] \\ [TH]^T[CR21] & \begin{matrix}[TH]^T[CR22][TH] \\ +\ [CA11]\end{matrix} & [CA12] \\ [0] & [CA21] & [CA22] \end{bmatrix} \begin{Bmatrix} \{D\dot{V}B\} \\ \{D\dot{Z}H\} \\ \{D\dot{Z}S\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [KR11] & [KR12][TH] & [0] \\ [TH]^T[KR21] & \begin{matrix}[TH]^T[KR22][TH] \\ +\ [KA11]\end{matrix} & [KA12] \\ [0] & [KA21] & [KA22] \end{bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DZH\} \\ \{DZS\} \end{Bmatrix} = - \begin{Bmatrix} \{QBO\} \\ [TH]^T\{QHO\} + \{FHO\} \\ \{0\} \end{Bmatrix} \tag{47}$$

Equations (46) and (47) are the virtual work equations and the Lagrange equations of motion, respectively, of the coupled rotor-airframe system.

HARMONIC BALANCE EQUATIONS

Expansion of the Unknowns in Terms of Rotor Harmonics

The equations of motion of a coupled rotor-airframe system (eq. (47)) are linear differential equations with time-dependent coefficients. The coefficients are periodic with period $2\pi/\Omega$. The most general periodic solution of these equations representing the most general steady-state vibration may be expressed in the following Fourier series form:

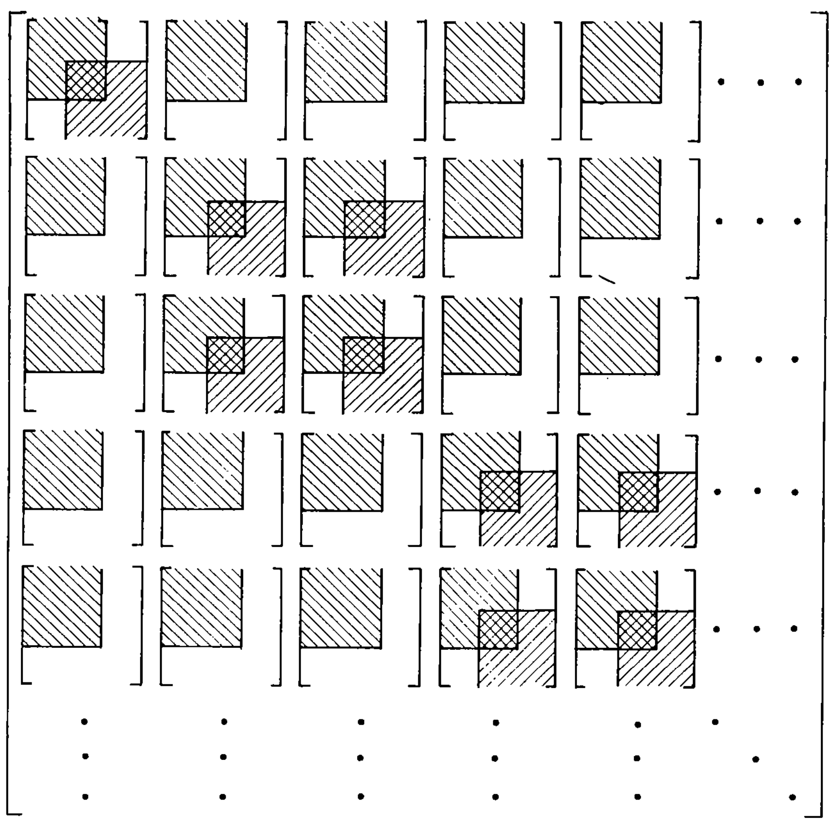
$$\begin{Bmatrix} \{DVB\} \\ \{DZH\} \\ \{DZS\} \end{Bmatrix} = \begin{Bmatrix} \{DVB0C\} \\ \{DZH0C\} \\ \{DZS0C\} \end{Bmatrix} + \begin{Bmatrix} \{DVB1S\} \\ \{DZH1S\} \\ \{DZS1S\} \end{Bmatrix} \sin \Omega t + \begin{Bmatrix} \{DVB1C\} \\ \{DZH1C\} \\ \{DZS1C\} \end{Bmatrix} \cos \Omega t + \begin{Bmatrix} \{DVB2S\} \\ \{DZH2S\} \\ \{DZS2S\} \end{Bmatrix} \sin 2\Omega t$$

$$+ \begin{Bmatrix} \{DVB2C\} \\ \{DZH2C\} \\ \{DZS2C\} \end{Bmatrix} \cos 2\Omega t + \ldots + \begin{Bmatrix} \{DVBNS\} \\ \{DZHNS\} \\ \{DZSNS\} \end{Bmatrix} \sin N\Omega t + \begin{Bmatrix} \{DVBNC\} \\ \{DZHNC\} \\ \{DZSNC\} \end{Bmatrix} \cos N\Omega t + \ldots \tag{48}$$

Following the usual terminology, the first term on the right side in equation (48) is referred to as the zeroth harmonic term, the second term as the first harmonic sine term, the third term as the first harmonic cosine term, the fourth term as the second harmonic sine term, and so forth. When speaking generally, reference is made to the Nth harmonic sine and cosine terms. Of course, for N = 0, there is only a cosine term, referred to as the constant, or zeroth harmonic, term.

Reduction of the Virtual Work Equations to Algebraic Equations

The virtual work equations (eq. (46)) are reduced to a set of algebraic equations known as harmonic balance equations by substituting equation (48) into equation (46) and carrying out the indicated integration over one rotor revolution. That is, the upper limit t is set equal to $2\pi/\Omega$. The requirement that the resulting equation be satisfied for independent variations of the unknowns leads to a set of simultaneous linear algebraic equations with the following matrix form:

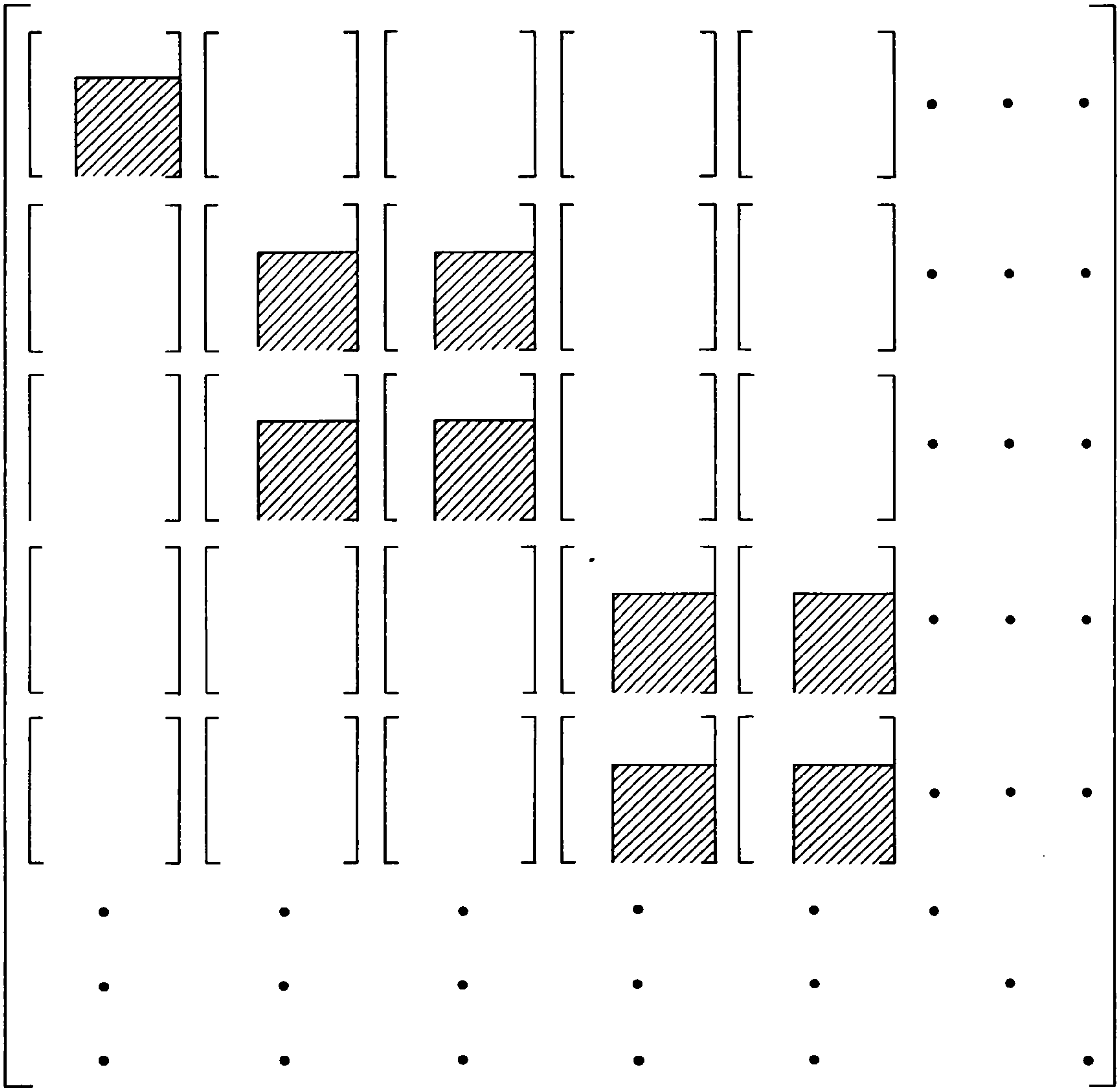


$$
\left\{\begin{matrix} \left\{\begin{matrix}\{DVB0C\}\\ \{DZH0C\}\\ \{DZS0C\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{DVB1S\}\\ \{DZH1S\}\\ \{DZS1S\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{DVB1C\}\\ \{DZH1C\}\\ \{DZS1C\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{DVB2S\}\\ \{DZH2S\}\\ \{DZS2S\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{DVB2C\}\\ \{DZH2C\}\\ \{DZS2C\}\end{matrix}\right\} \\ \vdots \end{matrix}\right\}
= -
\left\{\begin{matrix} \left\{\begin{matrix}\{QB00C\}\\ [TH]^T\{QH00C\} + \{FH00C\}\\ \{0\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{QB01S\}\\ [TH]^T\{QH01S\} + \{FH01S\}\\ \{0\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{QB01C\}\\ [TH]^T\{QH01C\} + \{FH01C\}\\ \{0\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{QB02S\}\\ [TH]^T\{QH02S\} + \{FH02S\}\\ \{0\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{QB02C\}\\ [TH]^T\{QH02C\} + \{FH02C\}\\ \{0\}\end{matrix}\right\} \\ \vdots \end{matrix}\right\}
\tag{49}
$$

The coefficient matrix in equation (49) is a known constant matrix. Computation of this matrix is developed in the next section. The submatrices denoted by hatching slanting downward to the right arise from the rotor equations; submatrices denoted by hatching slanting upward to the right arise from the airframe equations. In the crosshatched areas, contributions from the rotor equations and the airframe equations are combined by matrix addition. Blank areas are null. As can be seen, the vector on the right side is comprised of contributions from the rotor and airframe trim forces. The reader is reminded that the force vector {FHOOC} is not necessarily known because $\{\overline{\text{LOC}}\}$ in equation (6) is not necessarily known.

Computation of the Coefficient Matrix of the Harmonic Balance Equations

Airframe contributions.- The airframe contributions to the coefficient matrix of equation (49) cluster along the main diagonal of the matrix as indicated by sketch A. Except for the zeroth harmonic contributions, these airframe contributions form a



Sketch A.- Form of coefficient matrix of harmonic balance equations with rotor contributions deleted.

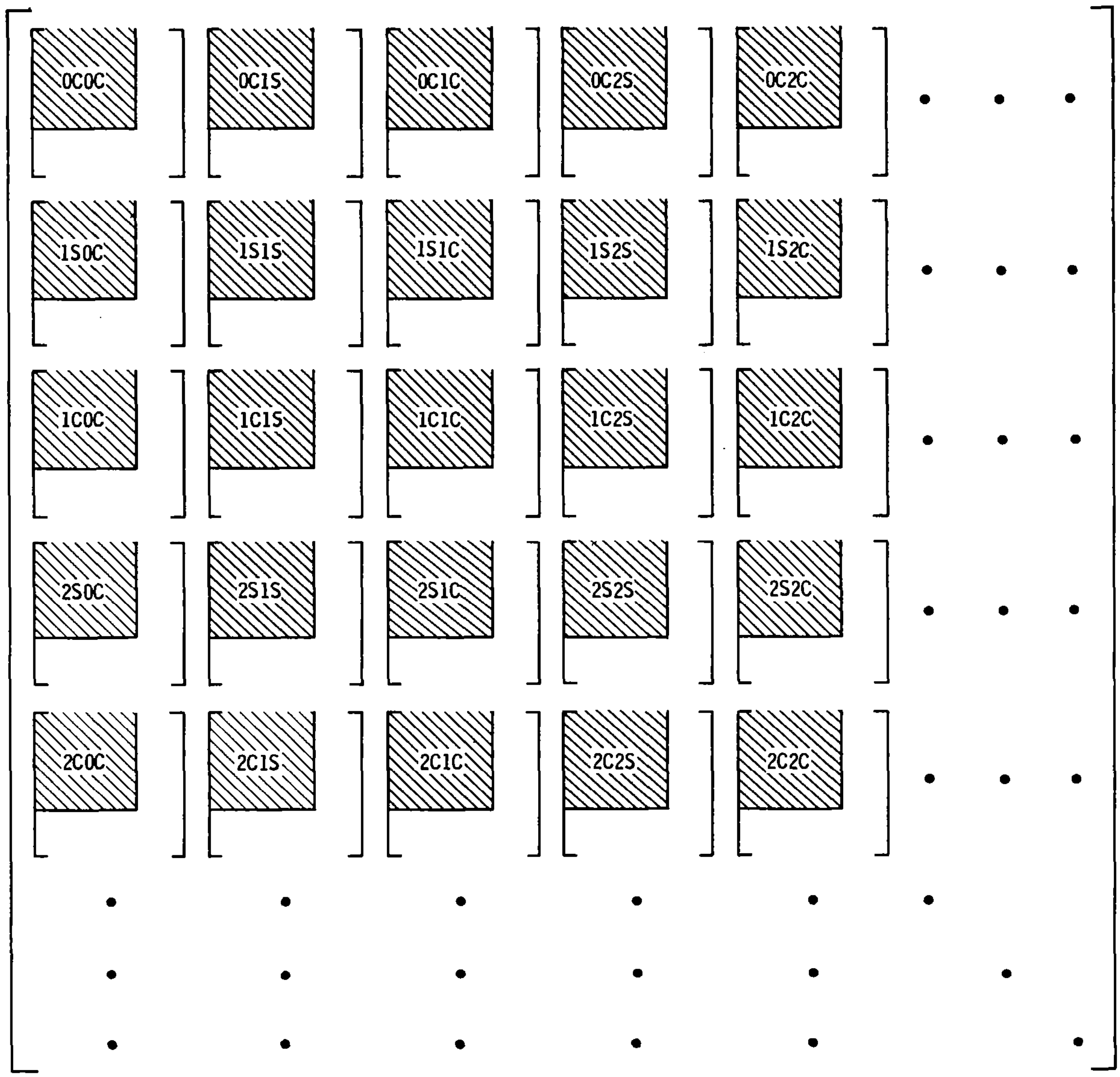
pattern of uncoupled blocks of four corresponding to the harmonics. Each block of four couples the sine and cosine contributions at a given harmonic. The zeroth harmonic contributions from the airframe form a single uncoupled block. Expressions for the zeroth harmonic block and the Nth harmonic block of four are given by the following equations:

$$\left[\quad\quad\right] = \left[\begin{matrix}[KA11][KA12]\\ [KA21][KA22]\end{matrix}\right] \tag{50a}$$

$$\begin{bmatrix}\left[\quad\right] & \left[\quad\right]\\ \left[\quad\right] & \left[\quad\right]\end{bmatrix} = \begin{bmatrix}\left[\begin{bmatrix}[KA11][KA12]\\ [KA21][KA22]\end{bmatrix} - (N\Omega)^2\begin{bmatrix}[MA11][MA12]\\ [MA21][MA22]\end{bmatrix}\right] & \left[-N\Omega\begin{bmatrix}[CA11][CA12]\\ [CA21][CA22]\end{bmatrix}\right]\\ \left[+N\Omega\begin{bmatrix}[CA11][CA12]\\ [CA21][CA22]\end{bmatrix}\right] & \left[\begin{bmatrix}[KA11][KA12]\\ [KA21][KA22]\end{bmatrix} - (N\Omega)^2\begin{bmatrix}[MA11][MA12]\\ [MA21][MA22]\end{bmatrix}\right]\end{bmatrix} \tag{50b}$$

The need may arise to introduce forces acting on the airframe which are specified in terms of impedances. For example, the traditional method of representing frequency-independent structural damping leads to forces of this type. Also giving rise to such forces are impedance representations of vibration control devices and drive system and engine effects. Such forces are introduced through a simple modification of equations (50), as explained in appendix F.

Rotor contributions.- The rotor contributions to the coefficient matrix of equation (49) form an array of blocks as indicated in sketch B. As the sketch shows, a four-character indexing system $X_1X_2X_3X_4$ is used to identify the positions of the rotor blocks in the coefficient matrix. In this system, the first character is an



Sketch B.- Form of coefficient matrix of harmonic balance equations with airframe contributions deleted.

integer indicating row harmonic, the second character is the letter S or C indicating sine or cosine, the third character is an integer indicating column harmonic, and the fourth character is the letter S or C indicating sine or cosine. The following rule has been derived for computing these rotor blocks:

$$\boxed{X_1X_2X_3X_4} = \begin{bmatrix} [I] & [0] \\ [0] & [TH] \end{bmatrix}^T \begin{bmatrix} DX_1X_2X_3X_4 \end{bmatrix} \begin{bmatrix} [I] & [0] \\ [0] & [TH] \end{bmatrix}$$

$$= \begin{bmatrix} [I] & [0] \\ [0] & [TH] \end{bmatrix}^T \begin{bmatrix} [DX_1X_2X_3X_4 11] & [DX_1X_2X_3X_4 12] \\ [DX_1X_2X_3X_4 21] & [DX_1X_2X_3X_4 22] \end{bmatrix} \begin{bmatrix} [I] & [0] \\ [0] & [TH] \end{bmatrix}$$

$$= \begin{bmatrix} [DX_1X_2X_3X_4 11] & [DX_1X_2X_3X_4 12][TH] \\ [TH]^T[DX_1X_2X_3X_4 21] & [TH]^T[DX_1X_2X_3X_4 22][TH] \end{bmatrix} \tag{51}$$

where the matrix $[DX_1X_2X_3X_4]$ is computed according to

$$\begin{aligned} [DX_1X_2X_3X_4] = {} & (UOK)[KROC] + \frac{1}{2}(ULK)[KRLP] + \frac{1}{2}(UHK)[KRHP] \\ & + X_3\Omega\Big((UOC)[CROC] + \frac{1}{2}(ULC)[CRLQ] + \frac{1}{2}(UHC)[CRHQ]\Big) \\ & - (X_3\Omega)^2\Big((UOM)[MROC] + \frac{1}{2}(ULM)[MRLP] + \frac{1}{2}(UHM)[MRHP]\Big) \end{aligned} \tag{52}$$

The matrices in equation (52) are the coefficient matrices of the harmonic expansions of [MR], [CR], and [KR] given in equations (2a) to (2c). Nine integer parameters appear in equation (52): UOK, ULK, UHK, UOC, ULC, UHC, UOM, ULM, and UHM. Four indices also appear: P, Q, L, and H. These items are computed by an algorithm shown in appendix G. The submatrices $[DX_1X_2X_3X_4 11]$, $[DX_1X_2X_3X_4 12]$, $[DX_1X_2X_3X_4 21]$, and $[DX_1X_2X_3X_4 22]$ appearing in equation (51) are obtained by partitioning the matrix $[DX_1X_2X_3X_4]$ corresponding to the partitions which are indicated in equation (3).

By way of illustration, consider evaluation of the submatrix [D2S2C]. In this case, $X_1 = 2$, $X_2 = S$, $X_3 = 2$, and $X_4 = C$. Then from the algorithm of appendix G, the indices and parameters appearing in equation (52) are

$$L = 0 \qquad H = 4 \qquad P = S \qquad Q = C \tag{53a}$$

and

$$\left.\begin{array}{lll} UHK = 1 & UHC = 1 & UHM = 1 \\ ULK = 0 & ULC = 0 & ULM = 0 \\ UOK = 0 & UOC = -1 & UOM = 0 \end{array}\right\} \tag{53b}$$

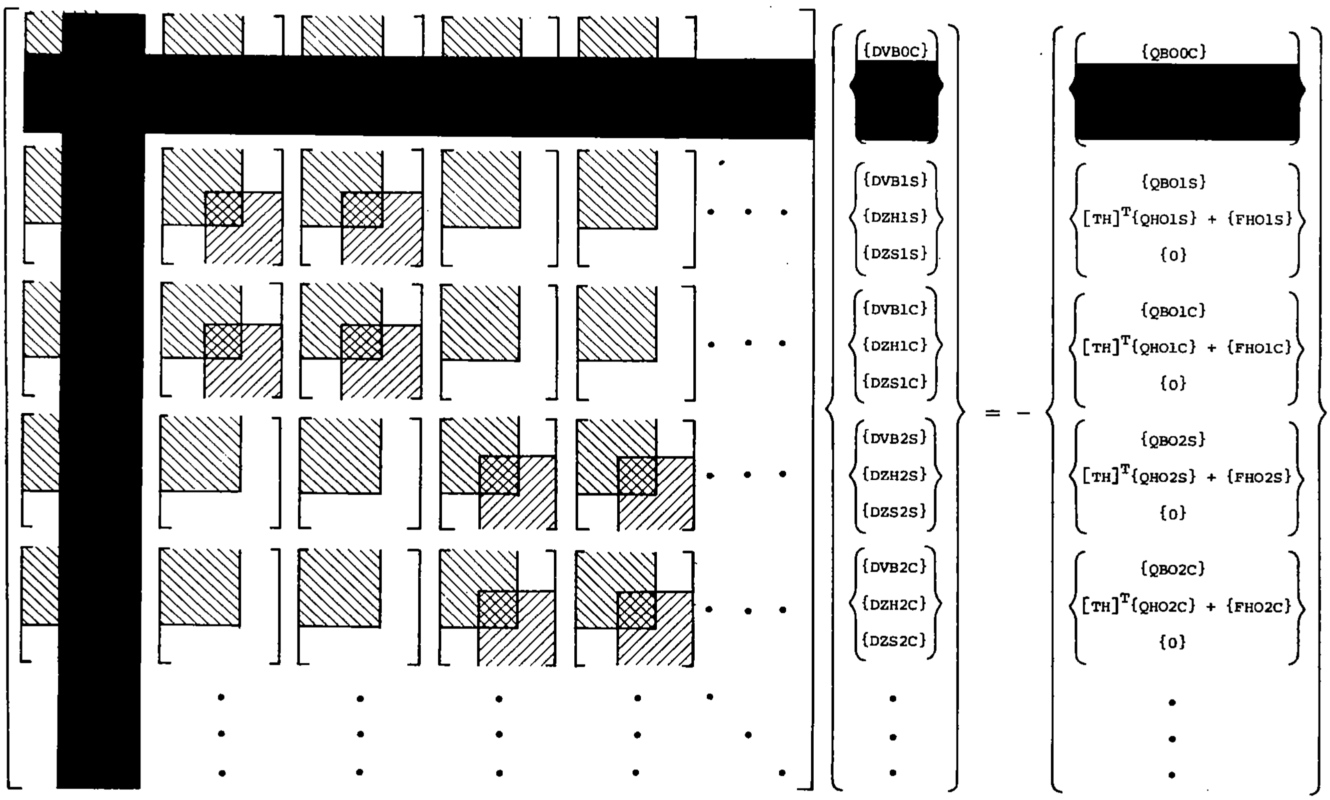
Substituting these quantities into equation (52) yields

$$[D2S2C] = \frac{1}{2}[KR4S] + 2\Omega\left(-[CR0C] + \frac{1}{2}[CR4C]\right) - (2\Omega)^2\,\frac{1}{2}[MR4S] \tag{54}$$

Treatment of Airframe Static Forces

As already noted, to compute the airframe vibrations there is no requirement to account for external static forces on the airframe. However, if desired, static external forces may be designated to act on the airframe, and such forces are incorporated in the vector $\{\overline{LOC}\}$, as discussed in conjunction with equation (6). Such designated static forces are accounted for in the airframe trim solution and are eventually reflected in the vector {FHOOC} which is defined by equation (32) and appears on the right side of equation (49). As previously noted, there is no requirement for the static forces $\{\overline{LOC}\}$ to balance the mean rotor forces represented by {QHTR}. As subsequently seen, specification of $\{\overline{LOC}\}$ has no effect on the computed vibration levels. The only effect is that the vibratory responses are superimposed on a static deformation.

Of course, for a helicopter in steady flight, the static rotor and airframe forces must balance. This balance is accounted for by suppressing the mean airframe displacements {DZHOC} and {DZSOC}. Thus, the corresponding equations and coefficients (rows and columns) are deleted from equation (49), as indicated by the darkened bands in the following equation:



(55)

Note that the quantities $[TH]^T\{QHOOC\}$ and $\{FHOOC\}$ no longer appear in the equations and consequently do not affect the results. If the coupled rotor-airframe system admits rigid-body displacements, the removal of the indicated equations and coefficients excludes the mean (static) components of such displacements as otherwise indeterminate components of the solution. It is very important to recognize that oscillatory ($N \neq 0$) rigid-body responses are not excluded.

The assumption that $\{DZHOC\}$ and $\{DZSOC\}$ vanish is subject to physical interpretation. It is equivalent to applying additional static external forces (forces of constraint) at the rotor-airframe connection points. These forces may be presumed to null the mean displacements $\{DZHOC\}$. Then the equations in equation (49) corresponding to the variables $\{DZSOC\}$ uncouple with zero right sides; so these variables vanish without further addition of external forces. The total forces which must be applied at the rotor-airframe interface to suppress the mean interface displacements $\{DZHOC\}$ are designated by $\{FHTR\}$ and are given by

$$
\{FHTR\} = -[TH]^T\{QHOOC\}
- \Big[\; [\quad] \; [\quad] \; [\quad] \; [\quad] \; [\quad] \cdots \Big]
\left\{ \begin{array}{c}
\left\{ \begin{array}{c} \{DVBOC\} \\ \{0\} \\ \{0\} \end{array} \right\} \\
\left\{ \begin{array}{c} \{DVB1S\} \\ \{DZH1S\} \\ \{DZS1S\} \end{array} \right\} \\
\left\{ \begin{array}{c} \{DVB1C\} \\ \{DZH1C\} \\ \{DZS1C\} \end{array} \right\} \\
\left\{ \begin{array}{c} \{DVB2S\} \\ \{DZH2S\} \\ \{DZS2S\} \end{array} \right\} \\
\left\{ \begin{array}{c} \{DVB2C\} \\ \{DZH2C\} \\ \{DZS2C\} \end{array} \right\} \\
\vdots
\end{array} \right\}
\qquad (56)
$$

The row matrix appearing in this equation is comprised of those rows of the coefficient matrix of equation (49) corresponding to {DZH0C}. Airframe contributions, denoted by hatching slanting upward to the right, have been retained for convenience of computation. However, because of the zeros appearing in the column matrix, these airframe contributions have no effect on the result. Consistent with the basic assumption that the rotor displaces only slightly from the rotor trim solution, one can expect that {FHTR} (which should be interpreted as including any forces represented by {FH0C}) nearly cancels the mean forces (represented by $[TH]^T\{QHTR\}$) specified in computing the rotor system trim solution. The cancellation may not be exact because in the final solution of the harmonic balance equations, the rotor interface variables may undergo oscillatory displacements, whereas these variables are assumed to be zero in the trim solution. These additional oscillatory components of displacement can result in additional increments of mean forces generated by the rotor.

Alternative Representation of the Rotor System by Impedances

The rotor contributions to the harmonic balance equations (eq. (49)) have been derived analytically from the linearized equations of motion of the rotor system by using the assumed harmonic series given in equation (48). Alternatively, it has been suggested in the literature (see, for example, ref. 8) that the rotor contributions to the harmonic balance equations can be obtained by representing the rotor with impedances. The impedances would be obtained from numerical integration of the nonlinear equations of motion of the rotor system. This alternative method of deriving harmonic balance equations is discussed in appendix H.

Definite Forms of the Harmonic Balance Equations

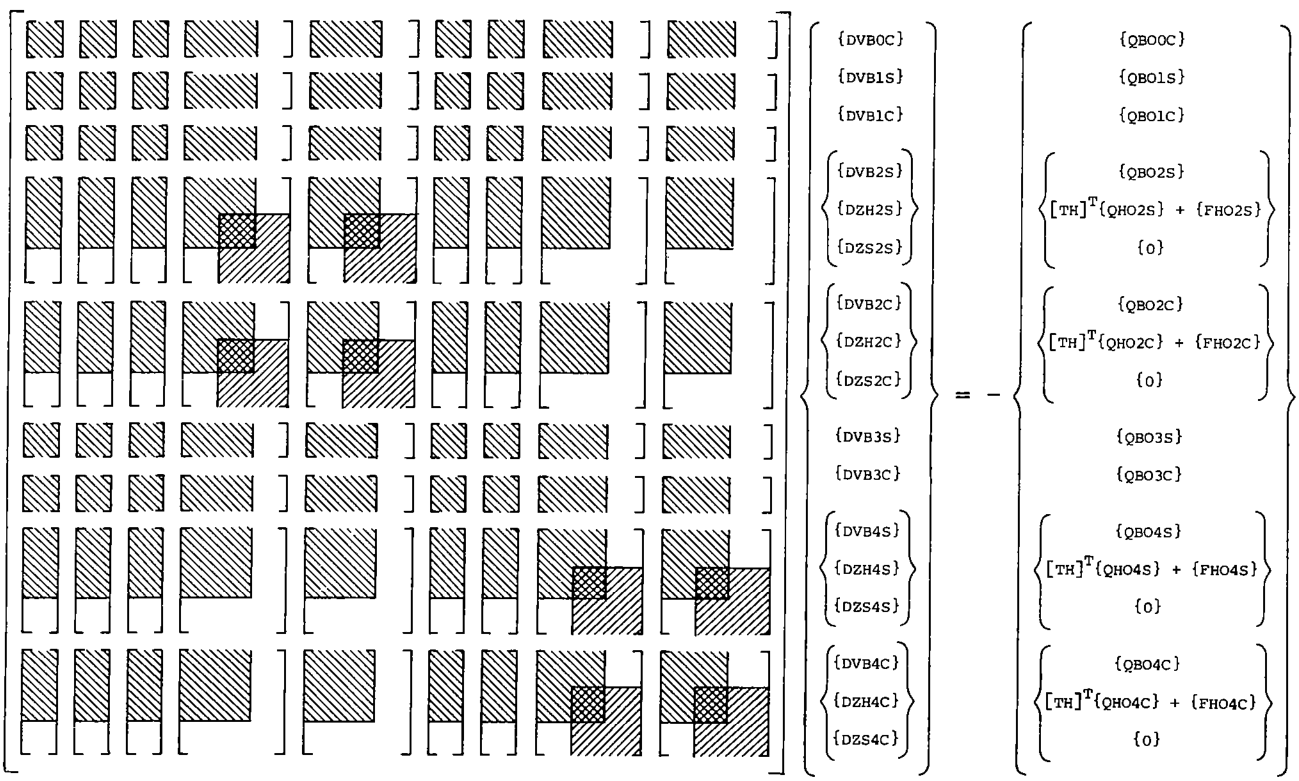
The harmonic balance equations (eq. (49)) constitute an infinite set of equations with an infinite number of unknowns. To effect a solution in practice, it is necessary to truncate the general Fourier series for the assumed steady-state solution given in equation (48). Also, in equation (48), some variables corresponding to the remaining harmonics may be specified to be zero or to have some designated nonzero value. The following equation is an example of an assumed definite form of equation (48) which might be appropriate for a helicopter with a two-bladed rotor:

$$\begin{Bmatrix} \{DVB\} \\ \{DZH\} \\ \{DZS\} \end{Bmatrix} = \begin{Bmatrix} \{DVB0C\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \begin{Bmatrix} \{DVB1S\} \\ \{0\} \\ \{0\} \end{Bmatrix} \sin \Omega t + \begin{Bmatrix} \{DVB1C\} \\ \{0\} \\ \{0\} \end{Bmatrix} \cos \Omega t + \begin{Bmatrix} \{DVB2S\} \\ \{DZH2S\} \\ \{DZS2S\} \end{Bmatrix} \sin 2\Omega t + \begin{Bmatrix} \{DVB2C\} \\ \{DZH2C\} \\ \{DZS2C\} \end{Bmatrix} \cos 2\Omega t$$
$$+ \begin{Bmatrix} \{DVB3S\} \\ \{0\} \\ \{0\} \end{Bmatrix} \sin 3\Omega t + \begin{Bmatrix} \{DVB3C\} \\ \{0\} \\ \{0\} \end{Bmatrix} \cos 3\Omega t + \begin{Bmatrix} \{DVB4S\} \\ \{DZH4S\} \\ \{DZS4S\} \end{Bmatrix} \sin 4\Omega t + \begin{Bmatrix} \{DVB4C\} \\ \{DZH4C\} \\ \{DZS4C\} \end{Bmatrix} \cos 4\Omega t \qquad (57)$$

In this example, all variables corresponding to harmonics greater than four have been set equal to zero. The following variables corresponding to harmonics less than four have also been set equal to zero: {DZH0C}, {DZS0C}, {DZH1S}, {DZS1S}, {DZH1C}, {DZS1C}, {DZH3S}, {DZS3S}, {DZH3C}, and {DZS3C}. Specification of zero values for the variables {DZH0C} and {DZS0C} may be interpreted as a suppression of static rigid-body displacements. Specification of zero values for the remaining variables in the list is consistent with the fact that a two-bladed rotor (with perfect balance and track) does not exert forces on the airframe at the odd-numbered harmonics.

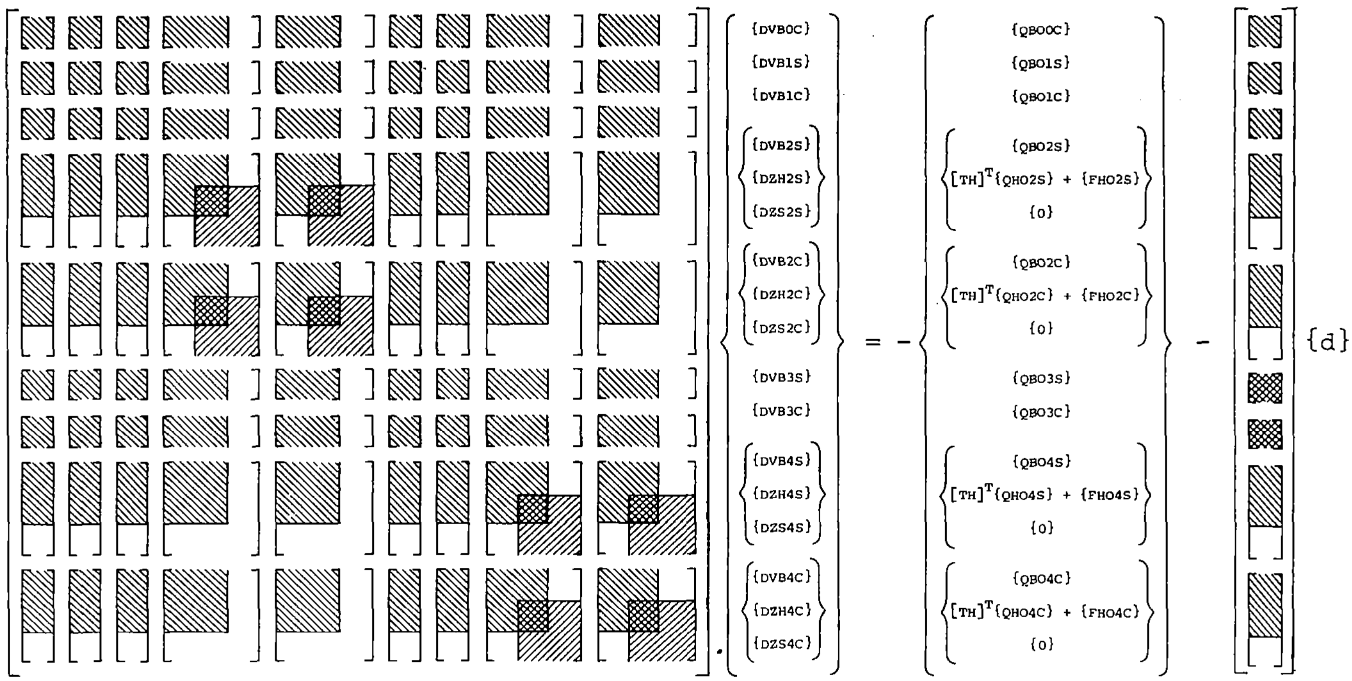
In general, dropping all terms beyond a given harmonic involves some approximation. Practically speaking, the ordinarily considered vibratory loadings of helicopter airframes decrease very sharply with increasing harmonic number. Therefore, useful solutions can be expected with few harmonics included. Suppressing variables corresponding to harmonics less than the designated maximum might involve additional approximations. However, such specifications are usually based on advance knowledge that the specific rotor system does not produce responses with certain harmonics.

When a definite form is specified for equation (48), the harmonic balance equations (eq. (49)) reduce to a finite number of simultaneous equations with an equal number of unknowns. These equations are obtained from equation (49) by setting any specified variables to their designated values and then deleting the equations corresponding to the specified variables. This may be seen by substituting equation (48) into equation (46), setting the specified variables to their designated values, and restricting the virtual displacements to forms consistent with the designated values of the variables. For the example case characterized by equation (57), equation (49) reduces to the following matrix form:



(58)

The finite coefficient matrix is obtained from the infinite coefficient matrix of equation (49) by deleting rows and columns corresponding to the specified variables. In the example, all specified variables were designated zero. If any of these specified variables are designated to have nonzero values, known columns are produced which may be transposed to the right. The coefficient matrix is unaffected. Say, for example, that some of the variables in the vector {DZH3S} of equation (57) were set to nonzero values instead of zero with other specifications of variables in the equation unchanged. That is, suppose {DZH3S} = {d}. Then, the only change to equation (58) is an additional column on the right:



(59)

Comments on Instability and Resonance

The most general solution to differential equation (47) may be expressed as the sum of any particular solution and the complementary solution. The steady-state solution expressed by equation (48) is a particular solution. The complementary solution is defined to be the most general solution of the homogeneous equations formed when the vector on the right side of equation (47) is nulled. The complementary solution may be stable or unstable. The computational procedures of this paper, based on the harmonic balance equations (eq. (49)), are not explicitly affected by the stability characteristics of the complementary solution. A steady-state solution will be generated by the procedures whether or not the complementary solution is stable.

In vibration analysis, an important consideration is resonance. Resonance is indicated when the particular solution increases without bound as some combination of system parameters is approached. Resonance conditions are reflected in the harmonic balance solution.

REDUCTION OF HARMONIC BALANCE EQUATIONS USING HARMONIC FORCED RESPONSES OF THE AIRFRAME

The Requirement and the Approach To Reduce the Number of Airframe Degrees of Freedom

The matrices representing airframe contributions to the harmonic balance equations (denoted by hatching slanting upward to the right in eq. (49)) may be large. Typical engineering finite-element models of helicopter airframes have several thousand degrees of freedom. For practical vibration analysis, the number of airframe degrees of freedom must be reduced. In this paper, this reduction is brought about by representing the airframe by forced responses calculated at harmonics of the rotor rotational frequency. As subsequently shown, the use of these "harmonic forced responses" eliminates from equation (49) all airframe variables except those explicitly involved in expressing the mechanical connections between the rotor and the airframe. The elimination introduces no additional approximations.

For convenience of discussion, subsequent developments refer to the indefinite form of the harmonic balance equations given by equation (49). From the preceding discussion of definite forms, it is straightforward to trace the consequences of specifying a definite form of these equations.

Harmonic Forced Responses

In this paper, harmonic forced responses for the Nth harmonic ($N \neq 0$) are defined to be a set of column vectors assembled in matrix form as indicated in sketch C:

$$\begin{bmatrix} [UZHIN] & [UZHON] \\ [UZSIN] & [UZSON] \\ \hdashline -[UZHON] & [UZHIN] \\ -[UZSON] & [UZSIN] \end{bmatrix}$$

Sketch C.- Matrix of harmonic forced responses for $N \neq 0$.

These forced responses satisfy

$$\left[\begin{array}{c:c} \begin{bmatrix}[KA11] & [KA12]\\ [KA21] & [KA22]\end{bmatrix} - (N\Omega)^2\begin{bmatrix}[MA11] & [MA12]\\ [MA21] & [MA22]\end{bmatrix} & -N\Omega\begin{bmatrix}[CA11] & [CA12]\\ [CA21] & [CA22]\end{bmatrix} \\ \hdashline +N\Omega\begin{bmatrix}[CA11] & [CA12]\\ [CA21] & [CA22]\end{bmatrix} & \begin{bmatrix}[KA11] & [KA12]\\ [KA21] & [KA22]\end{bmatrix} - (N\Omega)^2\begin{bmatrix}[MA11] & [MA12]\\ [MA21] & [MA22]\end{bmatrix}\end{array}\right]\left[\begin{array}{cc} [UZHIN] & [UZHON] \\ [UZSIN] & [UZSON] \\ \hdashline -[UZHON] & [UZHIN] \\ -[UZSON] & [UZSIN]\end{array}\right] = \left[\begin{array}{cc} [FUIN] & [FUON] \\ [0] & [0] \\ \hdashline -[FUON] & [FUIN] \\ [0] & [0]\end{array}\right] \tag{60}$$

where [UZHON] and [UZSON] represent out-of-phase responses and [UZHIN] and [UZSIN] represent in-phase responses. The square submatrices [FUIN] and [FUON] may be any two matrices for which

$$\begin{bmatrix}[FUIN] & [FUON] \\ -[FUON] & [FUIN]\end{bmatrix}$$

is nonsingular. Note that this condition is satisfied if either [FUIN] or [FUON] is nonsingular. It should be recognized that the definition of harmonic forced responses herein is a generalization of the ordinary definition of forced response at the frequency Ω. The matrix of harmonic forced responses shown in sketch C is of order 2(NZ) by 2(NZH). Note that only NZH column vectors of this matrix need be computed because of repetition in the submatrices comprising the coefficient matrix in equation (60).

Two commonly used special cases of this general form of the harmonic forced responses are the responses to unit imposed harmonic forces and to unit imposed harmonic displacements. The first case is obtained by setting [FUIN] = [I] and [FUON] = [0] in equation (60). The second case is obtained by setting [UZHIN] = [I] and [UZHON] = [0]. This can be brought about by a special choice of the forces [FUIN] and [FUON]. Appendix I discusses this case of responses to unit imposed displacements.

In the case of unit imposed forces, the matrix

$$\begin{bmatrix}[UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN]\end{bmatrix}$$

may be identified as the matrix of mobilities of the airframe corresponding to the interface degrees of freedom. In the case of unit imposed displacements, the matrix

$$\begin{bmatrix} [FUIN] & [FUON] \\ -[FUON] & [FUIN] \end{bmatrix}$$

may be identified as the matrix of impedances of the airframe corresponding to the interface degrees of freedom.

Harmonic forced responses for the zeroth harmonic are defined to be a set of column vectors assembled in the matrix form indicated in sketch D:

$$\begin{bmatrix} [UZHIO] \\ [UZSIO] \end{bmatrix}$$

Sketch D.- Matrix of harmonic forced responses for zeroth harmonic.

These zeroth harmonic forced responses satisfy

$$\begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} \begin{bmatrix} [UZHIO] \\ [UZSIO] \end{bmatrix} = \begin{bmatrix} [FUIO] \\ [0] \end{bmatrix} \tag{61}$$

where the submatrix $[FUIO]$ appearing on the right may be any nonsingular matrix. The matrix shown in sketch D is of order NZ by NZH. The special case of unit imposed forces is obtained by setting $[FUIO] = [I]$, and the case of unit imposed displacements is obtained by setting $[UZHIO] = [I]$.

Computation of Harmonic Forced Responses

The computation of airframe harmonic forced responses resulting from imposed harmonic forces is a standard operation in general-purpose finite-element computer codes such as NASTRAN. Thus, the airframe forced responses defined by equation (60) may be computed directly. Although it is not necessary for the procedures of this paper, there may be some conceptual advantages to utilization of the special forms of the harmonic forced responses corresponding to unit imposed forces and unit imposed displacements. As discussed earlier, responses to unit imposed forces are computed using equation (60) after setting $[FUIN] = [I]$ and $[FUON] = [0]$. Direct computation of responses which result from unit imposed displacements $\left([UZHIN] = [I], [UZHON] = [0]\right)$ is not so convenient with available codes. However, a simple formula for computing responses to unit imposed displacements from any solutions of equation (60) is discussed in appendix I.

In design studies, the airframe contributions to the harmonic balance equations often must be recomputed many times to study the effects of varying structural members and masses or varying impedances of vibration control devices. Appendix E discusses a procedure to recompute the harmonic forced responses and the airframe trim solution which saves computing effort when only a small number of items are varied.

Reduction of the Equations

The reduction of the harmonic balance equations is accomplished by substitutions of the following types:

$$\begin{Bmatrix} \{DVBOC\} \\ \{DZHOC\} \\ \{DZSOC\} \end{Bmatrix} = \begin{bmatrix} [I] & \\ & [UZHIO] \\ & [UZSIO] \end{bmatrix} \begin{Bmatrix} \{DVBOC\} \\ \{QZHOC\} \end{Bmatrix} \tag{62a}$$

$$\begin{Bmatrix} \{DVBNS\} \\ \{DZHNS\} \\ \{DZSNS\} \\ --- \\ \{DVBNC\} \\ \{DZHNC\} \\ \{DZSNC\} \end{Bmatrix} = \left[\begin{array}{cc|cc} [I] & & [0] & \\ & [UZHIN] & & [UZHON] \\ & [UZSIN] & & [UZSON] \\ \hline [0] & & [I] & \\ & -[UZHON] & & [UZHIN] \\ & -[UZSON] & & [UZSIN] \end{array}\right] \begin{Bmatrix} \{DVBNS\} \\ \{QZHNS\} \\ --- \\ \{DVBNC\} \\ \{QZHNC\} \end{Bmatrix} \tag{62b}$$

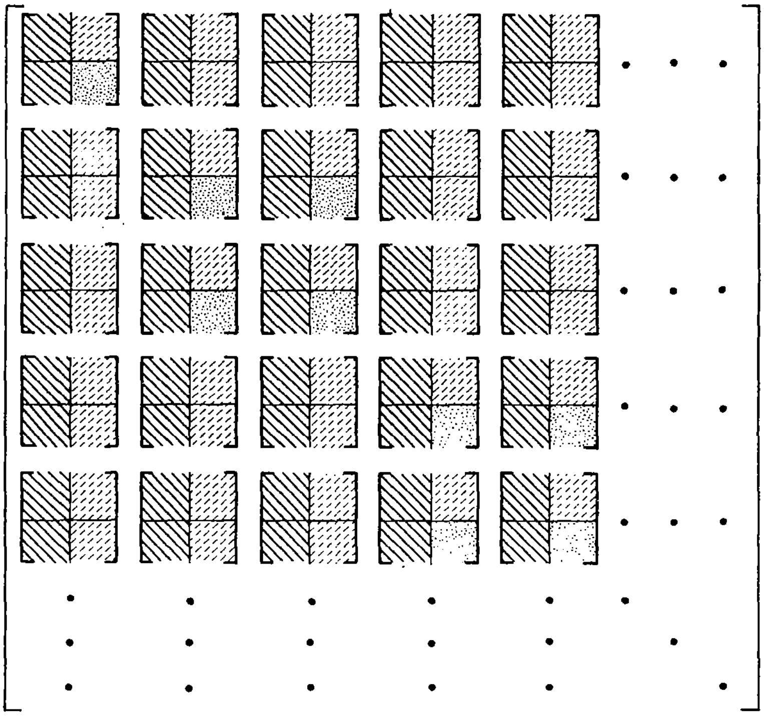
These equations may be regarded as transformations of the airframe coordinates by which the airframe coordinates are expressed in terms of relatively few new variables represented by $\{QZHOC\}$, $\{QZHNS\}$, and $\{QZHNC\}$. Collectively, the transformation equations corresponding to the individual harmonics may be expressed as a single transformation equation which takes the matrix form

$$
\begin{Bmatrix}
\begin{Bmatrix} \{DVB0C\} \\ \{DZH0C\} \\ \{DZS0C\} \end{Bmatrix} \\
\begin{Bmatrix} \{DVB1S\} \\ \{DZH1S\} \\ \{DZS1S\} \end{Bmatrix} \\
\begin{Bmatrix} \{DVB1C\} \\ \{DZH1C\} \\ \{DZS1C\} \end{Bmatrix} \\
\begin{Bmatrix} \{DVB2S\} \\ \{DZH2S\} \\ \{DZS2S\} \end{Bmatrix} \\
\begin{Bmatrix} \{DVB2C\} \\ \{DZH2C\} \\ \{DZS2C\} \end{Bmatrix} \\
\vdots
\end{Bmatrix}
=
\begin{bmatrix}
\begin{bmatrix} [I] & \\ & [UZHI0] \\ & [UZSI0] \end{bmatrix} & & & \\
& \left[\begin{array}{cc|cc} [I] & & [0] & \\ & [UZHI1] & & [UZHO1] \\ & [UZSI1] & & [UZSO1] \\ \hline [0] & & [I] & \\ & -[UZHO1] & & [UZHI1] \\ & -[UZSO1] & & [UZSI1] \end{array}\right] & & \\
& & \left[\begin{array}{cc|cc} [I] & & [0] & \\ & [UZHI2] & & [UZHO2] \\ & [UZSI2] & & [UZSO2] \\ \hline [0] & & [I] & \\ & -[UZHO2] & & [UZHI2] \\ & -[UZSO2] & & [UZSI2] \end{array}\right] & \\
& & & \ddots
\end{bmatrix}
\begin{Bmatrix} \{DVB0C\} \\ \{QZH0C\} \\ \{DVB1S\} \\ \{QZH1S\} \\ \{DVB1C\} \\ \{QZH1C\} \\ \{DVB2S\} \\ \{QZH2S\} \\ \{DVB2C\} \\ \{QZH2C\} \\ \vdots \end{Bmatrix}
\tag{63}
$$

Substituting equation (63) in equation (49) leads to equations having the matrix form

$$
\begin{bmatrix}
[\quad] & [\quad] & [\quad] & [\quad] & [\quad] & \cdots \\
\multicolumn{6}{c}{\text{[redacted]}} \\
[\quad] & [\quad] & [\quad] & [\quad] & [\quad] & \cdots \\
\multicolumn{6}{c}{\text{[redacted]}} \\
[\quad] & [\quad] & [\quad] & [\quad] & [\quad] & \cdots \\
\multicolumn{6}{c}{\text{[redacted]}} \\
[\quad] & [\quad] & [\quad] & [\quad] & [\quad] & \cdots \\
\multicolumn{6}{c}{\text{[redacted]}} \\
[\quad] & [\quad] & [\quad] & [\quad] & [\quad] & \cdots \\
\multicolumn{6}{c}{\text{[redacted]}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{Bmatrix} \{DVB0C\} \\ \{QZH0C\} \\ \{DVB1S\} \\ \{QZH1S\} \\ \{DVB1C\} \\ \{QZH1C\} \\ \{DVB2S\} \\ \{QZH2S\} \\ \{DVB2C\} \\ \{QZH2C\} \\ \vdots \end{Bmatrix}
= -
\begin{Bmatrix}
\begin{Bmatrix} \{QB00C\} \\ [TH]^T\{QH00C\} + \{FH00C\} \\ \text{[redacted]} \end{Bmatrix} \\
\begin{Bmatrix} \{QB01S\} \\ [TH]^T\{QH01S\} + \{FH01S\} \\ \text{[redacted]} \end{Bmatrix} \\
\begin{Bmatrix} \{QB01C\} \\ [TH]^T\{QH01C\} + \{FH01C\} \\ \text{[redacted]} \end{Bmatrix} \\
\begin{Bmatrix} \{QB02S\} \\ [TH]^T\{QH02S\} + \{FH02S\} \\ \text{[redacted]} \end{Bmatrix} \\
\begin{Bmatrix} \{QB02C\} \\ [TH]^T\{QH02C\} + \{FH02C\} \\ \text{[redacted]} \end{Bmatrix} \\
\vdots
\end{Bmatrix}
\tag{64}
$$

The equations corresponding to the darkened bands are identically satisfied. Therefore, equation (64) reduces to a matrix equation of the form,



$$
\begin{Bmatrix} \begin{Bmatrix} \{DVB0C\} \\ \{QZH0C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DVB1S\} \\ \{QZH1S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DVB1C\} \\ \{QZH1C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DVB2S\} \\ \{QZH2S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DVB2C\} \\ \{QZH2C\} \end{Bmatrix} \\ \vdots \end{Bmatrix} = - \begin{Bmatrix} \begin{Bmatrix} \{QB0OC\} \\ [TH]^T\{QH0OC\} + \{FH0OC\} \end{Bmatrix} \\ \begin{Bmatrix} \{QB01S\} \\ [TH]^T\{QH01S\} + \{FH01S\} \end{Bmatrix} \\ \begin{Bmatrix} \{QB01C\} \\ [TH]^T\{QH01C\} + \{FH01C\} \end{Bmatrix} \\ \begin{Bmatrix} \{QB02S\} \\ [TH]^T\{QH02S\} + \{FH02S\} \end{Bmatrix} \\ \begin{Bmatrix} \{QB02C\} \\ [TH]^T\{QH02C\} + \{FH02C\} \end{Bmatrix} \\ \vdots \end{Bmatrix} \tag{65}
$$

The coefficient matrix in equation (65) is shown as an array of blocks each partitioned into four submatrices. The difference in shading of the submatrices is for convenience in tracking rearrangements of the coefficient matrix.

Employing the four-character indexing system defined in a preceding section, the typical block in equation (65) is designated $[EX_1X_2X_3X_4]$. Symbols for the partitions are defined by

$$
\left[\begin{array}{c|c} & \\ \hline & \end{array}\right] = \begin{bmatrix} EX_1X_2X_3X_4 \end{bmatrix} = \begin{bmatrix} [EX_1X_2X_3X_4 11] & [EX_1X_2X_3X_4 12] \\ [EX_1X_2X_3X_4 21] & [EX_1X_2X_3X_4 22] \end{bmatrix} \tag{66}
$$

The four submatrices in a typical block are given by

$$[EX_1X_2X_3X_4 11] = [DX_1X_2X_3X_4 11] \tag{67a}$$

$$[EX_1X_2X_3X_4 21] = [TH]^T[DX_1X_2X_3X_4 21] \tag{67b}$$

$$[EX_1X_2X_3X_412] = [DX_1X_2X_3X_412][TH][UZHIX_3]$$

$$- \alpha[DX_1X_2X_3P12][TH][UZHOX_3] \tag{67c}$$

$$[EX_1X_2X_3X_422] = [TH]^T[DX_1X_2X_3X_422][TH][UZHIX_3]$$

$$- \alpha[TH]^T[DX_1X_2X_3P22][TH][UZHOX_3]$$

$$+ \beta[FUIX_3] + \gamma[FUOX_3] \tag{67d}$$

All matrices appearing on the right sides in equations (67) have been discussed in preceding sections. They arise from the rotor contributions to the harmonic balance equations, from the coupling equations, and from the airframe harmonic responses. The three integer parameters α, β, and γ and one index P are computed by an algorithm shown in appendix J.

By way of illustration, consider evaluation of the submatrix comprising the block [E2S2C]. In this case, $X_1 = 2$, $X_2 = S$, $X_3 = 2$, and $X_4 = C$. From the algorithm of appendix J, the index and the parameters appearing in equations (67) are given by

$$P = S \tag{68a}$$

and

$$\alpha = -1 \qquad \beta = 0 \qquad \gamma = 1 \tag{68b}$$

Substituting these quantities into equations (67) yields

$$[E2S2C11] = [D2S2C11] \tag{69a}$$

$$[E2S2C21] = [TH]^T[D2S2C21] \tag{69b}$$

$$[E2S2C12] = [D2S2C12][TH][UZHI2] + [D2S2S12][TH][UZHO2] \tag{69c}$$

$$[E2S2C22] = [TH]^T[D2S2C22][TH][UZHI2]$$

$$+ [TH]^T[D2S2S22][TH][UZHO2] + [FUO2] \tag{69d}$$

As has been discussed, it may be necessary in design studies to vary the airframe contributions to the harmonic balance equations (eq. (49)). Of course, this results in changes in the coefficient matrix of equation (65), and a new solution of this equation is required. Appendix K discusses a procedure for this reanalysis which can save computing effort if the submatrices $[EX_1X_2X_3X_411]$ are relatively large compared with the submatrices $[EX_1X_2X_3X_422]$. Some additional comments on solution of equation (65) of a purely computational nature relating to order and bandedness of the coefficient matrix are given in appendix L.

RECOVERY OF AIRFRAME RESPONSES

The total airframe displacements, denoted by the vector {Z}, are given by equation (33):

$$\{Z\} = \{ZO\} + \{DZ\}$$

In terms of the partitions defined in equations (17), (25), and (34), this equation can be written as

$$\begin{Bmatrix} \{ZH\} \\ \{ZS\} \end{Bmatrix} = \begin{Bmatrix} \{ZHO\} \\ \{ZSO\} \end{Bmatrix} + \begin{Bmatrix} \{DZH\} \\ \{DZS\} \end{Bmatrix}$$

$$= \begin{Bmatrix} \{0\} \\ \{ZSO\} \end{Bmatrix} + \begin{Bmatrix} \{DZH\} \\ \{DZS\} \end{Bmatrix} \tag{70}$$

where {ZHO} has been set equal to zero as in equation (26). The displacements {ZSO} are obtained from the airframe trim solution as previously discussed and can be written in the Fourier series form shown in equation (31). The harmonic solutions for the displacements {DZH} and {DZS} are obtained by substituting the solution of equation (65) into equation (63). These results can be combined to obtain the airframe displacements and the corresponding velocities and accelerations as

$$\begin{Bmatrix} \{ZH\} \\ \{ZS\} \end{Bmatrix} = \begin{Bmatrix} \{DZHOC\} \\ \{ZSOOC\} + \{DZSOC\} \end{Bmatrix} + \begin{Bmatrix} \{DZH1S\} \\ \{ZSO1S\} + \{DZS1S\} \end{Bmatrix} \sin \Omega t + \begin{Bmatrix} \{DZH1C\} \\ \{ZSO1C\} + \{DZS1C\} \end{Bmatrix} \cos \Omega t$$

$$+ \begin{Bmatrix} \{DZH2S\} \\ \{ZSO2S\} + \{DZS2S\} \end{Bmatrix} \sin 2\Omega t + \begin{Bmatrix} \{DZH2C\} \\ \{ZSO2C\} + \{DZS2C\} \end{Bmatrix} \cos 2\Omega t + \ldots \tag{71a}$$

$$\begin{Bmatrix} \{\dot{ZH}\} \\ \{\dot{ZS}\} \end{Bmatrix} = \Omega \begin{Bmatrix} \{DZH1S\} \\ \{ZSO1S\} + \{DZS1S\} \end{Bmatrix} \cos \Omega t - \Omega \begin{Bmatrix} \{DZH1C\} \\ \{ZSO1C\} + \{DZS1C\} \end{Bmatrix} \sin \Omega t$$

$$+ 2\Omega \begin{Bmatrix} \{DZH2S\} \\ \{ZSO2S\} + \{DZS2S\} \end{Bmatrix} \cos 2\Omega t - 2\Omega \begin{Bmatrix} \{DZH2C\} \\ \{ZSO2C\} + \{DZS2C\} \end{Bmatrix} \sin 2\Omega t + \ldots \quad (71b)$$

$$\begin{Bmatrix} \{\ddot{ZH}\} \\ \{\ddot{ZS}\} \end{Bmatrix} = -(\Omega)^2 \begin{Bmatrix} \{DZH1S\} \\ \{ZSO1S\} + \{DZS1S\} \end{Bmatrix} \sin \Omega t - (\Omega)^2 \begin{Bmatrix} \{DZH1C\} \\ \{ZSO1C\} + \{DZS1C\} \end{Bmatrix} \cos \Omega t$$

$$- (2\Omega)^2 \begin{Bmatrix} \{DZH2S\} \\ \{ZSO2S\} + \{DZS2S\} \end{Bmatrix} \sin 2\Omega t - (2\Omega)^2 \begin{Bmatrix} \{DZH2C\} \\ \{ZSO2C\} + \{DZS2C\} \end{Bmatrix} \cos 2\Omega t - \ldots \quad (71c)$$

If internal member loads are desired, the computed displacements given in equation (71a) can be introduced as static displacements imposed on the finite-element model of the airframe. For typical finite-element codes, this is a straightforward operation. Alternatively, a finite-element code can generate a matrix which, when multiplied by {Z}, yields desired internal loads.

The harmonics of the resultant forces acting on the airframe at the interface with the rotor are often of interest. These forces are designated by {FZH0C}, {FZH1S}, {FZH1C}, {FZH2S}, {FZH2C}, and so forth. The forces can be recovered by substituting the solution of equation (65) into the right side of

$$\begin{Bmatrix} \{FZH0C\} \\ \{FZH1S\} \\ \{FZH1C\} \\ \{FZH2S\} \\ \{FZH2C\} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{bmatrix} [F0C0C] & & & & & & \\ & [F1S1S] & [F1S1C] & & & & \\ & [F1C1S] & [F1C1C] & & & & \\ & & & [F2S2S] & [F2S2C] & & \\ & & & [F2C2S] & [F2C2C] & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \cdot \end{bmatrix} \begin{Bmatrix} \{QZH0C\} \\ \{QZH1S\} \\ \{QZH1C\} \\ \{QZH2S\} \\ \{QZH2C\} \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \quad (72)$$

where the indicated submatrices, designated $[FX_1X_2X_3X_4]$, are given by

$$[FX_1X_2X_3X_4] = \beta[FUIX_3] + \gamma[FUOX_3] \tag{73}$$

The four-character indexing system and the matrices and parameters appearing on the right side of equation (73) are the same as those defined for equation (67d).

BASIC COMPUTING SEQUENCE

With the equations which have been developed, various computing sequences can be formed for calculating airframe vibrations. A basic computing sequence is outlined here. Two additional sequences are described in appendix M. These incorporate steps for reanalysis by the method of appendix E and for representation of the rotor by the impedance method of appendix H.

Figure 4 shows a block diagram indicating the basic sequence of tasks for calculating airframe vibrations. Blocks indicating tasks to be performed are arranged in three columns to distinguish work done by rotor analysts, work done by airframe analysts, and joint work. Supplementary notes follow, keyed by number to the individual blocks:

1. <u>Specify the linearized rotor model</u>: The specification is made by providing the coefficient matrices in the Fourier series expansions of [MR], [CR], [KR], and {QRO} represented by equations (2).

2. <u>Identify rotor-airframe connection points</u>: The rotor and airframe analysts agree on an arrangement of discrete points at which to designate connections. The displacements and rotations at these points needed to express connectivity are identified. This defines the vector {c} discussed in conjunction with equations (C2).

3. <u>Partition the rotor equations to isolate interface variables</u>: Among the rotor variables {DVR}, a subset {DVH} is identified as the variables which explicitly appear in expressions characterizing connections of the rotor model to the airframe model (eq. (C2a)). The matrices specifying the linear rotor model are partitioned accordingly. See discussion of equation (3) and following.

4. <u>Form rotor contributions to the coupling equations</u>: The matrices [THR] and [GER] are formed. See equations (C2a) and (C11a) and related discussion.

5. <u>Identify harmonics in assumed solution</u>: The steady-state solution given by equation (48) is specialized to a definite form agreed on by the rotor and airframe analysts. See equation (57) for an example of a definite form.

6. <u>Specify loads impinging directly on airframe</u>: The distribution of external oscillating loads impinging directly on the airframe must be described to enable the airframe analysts to define the vector $\{\bar{L}\}$ appearing in equation (5). Static components may be included, but it is not necessary to include them. Loads transmitted from the rotor to the airframe through the rotor-airframe connections are not included in this specification.

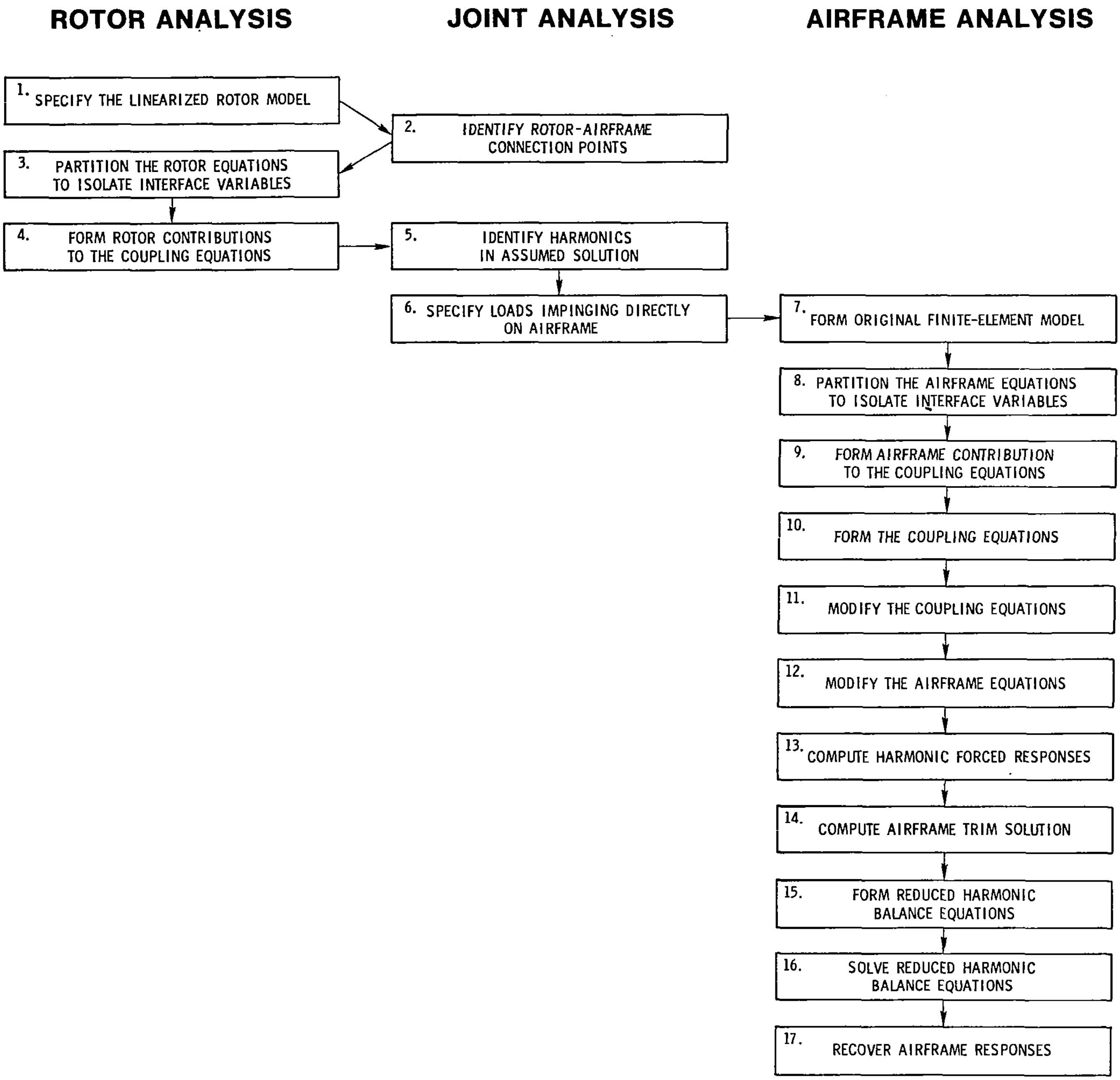


Figure 4.- Block diagram indicating the basic sequence of tasks for calculating vibrations.

7. Form original finite-element model: A finite-element code is employed to generate the matrices $[\overline{MA}]$, $[\overline{CA}]$, and $[\overline{KA}]$, appearing in equation (5). Also generated are the load coefficients $\{\overline{L0C}\}$, $\{\overline{L1S}\}$, $\{\overline{L1C}\}$, ..., appearing in equation (6).

8. Partition the airframe equations to isolate the interface variables: Among the airframe variables $\{\overline{Z}\}$, a subset $\{\overline{ZH}\}$ is identified as the variables which explicitly appear in the expressions characterizing connections of the airframe model

to the rotor model (eq. (C2b)). The matrices characterizing the airframe model are partitioned accordingly. See discussion of equation (7) and following.

9. Form airframe contribution to the coupling equations: The matrix [THA] is formed. See equation (C2b) and related discussion.

10. Form the coupling equations: Compute [T1] from equation (C11a). Compute [T2] and [T3] from equation (C11b). Compute $[\overline{TH}]$ from equation (C14a). Compute $[\overline{TC}]$ from equation (C14b). This establishes the coupling equations (9a) and (9b).

11. Modify the coupling equations: Identify the submatrices $[\overline{TCD}]$ and $[\overline{TCI}]$ associated with the matrix $[\overline{TC}]$. See equation (10) and related discussion. Identify the submatrices $[\overline{THD}]$ and $[\overline{THI}]$ associated with the matrix $[\overline{TH}]$. See equation (12) and related discussion. Compute the matrix [TH] from equation (14). The coupling equations are thus reduced to a single equation, equation (13).

12. Modify the airframe equations: The matrix $[\overline{TI}]$ in equation (15) is formed from equation (11), as discussed in the text. Operating with $[\overline{TI}]$ on the partitioned form of the original finite-element equations, as shown by equation (16), establishes the modified form of the airframe equations (eq. (21)). Note that this task has been to some extent routinized in the NASTRAN code through the feature called multipoint constraints.

13. Compute harmonic forced responses: Equation (60) is solved for each nonzero harmonic of interest (see task 5) and the results are assembled as shown in sketch C. For any zeroth harmonic responses, equation (61) is solved and the results assembled as shown in sketch D.

14. Compute airframe trim solution: Equation (D3) is solved for {ZHFNS}, {ZSFNS}, {ZHFNC}, and {ZSFNC}. The vectors representing trim displacements and forces are computed from equations (D6) and (D8). For the zeroth harmonic equations (D9b) to (D9d) are used.

15. Form reduced harmonic balance equations: The matrices $[DX_1X_2X_3X_4]$ are computed according to equation (52) and the algorithm of appendix G. Equations (67) and the algorithm of appendix J are used to compute the coefficient matrix of equation (65). In this process, the only rows and columns generated are those corresponding to retained harmonics (see task 5).

16. Solve reduced harmonic balance equations: Equation (65) is solved.

17. Recover airframe responses: Equations (63) and (71) are used. If interface forces are desired, use equations (72) and (73).

DISCUSSION OF THE METHOD

Two features of the method of this paper seem to have an important bearing on its potential utility: (1) use of linearized equations to represent the rotor and (2) use of harmonic forced responses to represent the airframe. Comments on the implications of these features are offered in the next two sections.

Potential Utility of a Linear Rotor-Airframe Model

With very few exceptions, airframe structural design analysis has always been based on linear equations, and this linearity considerably simplifies the analysis. It will obviously ease the introduction of vibration analysis into airframe design work if rotor models are also linear. A linear rotor model is thus argued to have important advantages, but the question remains whether such a model can be adequate.

The editor of a recent issue of "Vertica" devoted to coupled rotor-fuselage dynamics (ref. 20) has commented that even simplified linear models are plausible for helicopter vibration analysis: "... evidence available in the literature seems to indicate that for aeroelastic response calculations, or the vibration problem of coupled rotor/fuselage systems a much simpler linear formulation might be adequate ..." The following four reports of correlations of analysis with flight test data also bear on the potential of a linear rotor-airframe model:

1. Reference 8 reports correlations for a tandem rotor helicopter with three-bladed rotors. The analysis was based on a rotor-airframe model incorporating linear representations of both the rotor and the airframe. The correlations presented are reproduced in figure 5.

2. Reference 10 reports correlations for a compound helicopter with a four-bladed hingeless rotor. Plots indicative of the correlations are reproduced in figure 6. Nonlinear rotor equations were used directly in the analysis. Therefore, these results cannot be cited with complete logic in support of the adequacy of a linear rotor-airframe model. The report can, however, be used to evaluate the potential of a carefully formed NASTRAN finite-element model, in this case a relatively simple model, to represent the airframe response to the major vibration excitations found in flight.

3. Reference 9 reports a correlation for a tandem rotor helicopter with four-bladed rotors. The relevant results presented are reproduced in figure 7. The analysis was based on a linear rotor-airframe model; however, the linear rotor model was very crude. Specifically, only the trim force term $\{QRO\}$ was retained in the linearized rotor equations (eq. (A24)) - the mass, damping, and stiffness terms being, in effect, ignored.

4. Reference 21 reports a correlation performed for a helicopter with a two-bladed teetering rotor. In this case, the analysis did not incorporate a model of the rotor system. The procedure was to measure the flight vibratory accelerations at the rotor hub and then to impose the measured values of acceleration on a NASTRAN finite-element model of the airframe. The response of the airframe thus calculated was compared with the response measured in flight. Typical results for the major responses are shown in figure 8.

These correlations appear encouraging with regard to the adequacy of linear models for coupled rotor-airframe vibration analyses.

It should be kept in mind that the gross vibratory forces exerted by the rotor on the airframe are represented by the vector $\{QRO\}$ and that this vector would not, in general, be computed by linear theory. The vector $\{QRO\}$ comes from the trim solution, and the trim solution, as defined herein, is a solution (by exact or approximate methods) of the underlying nonlinear rotor equations with the constraint that the rotor-airframe interface points are fixed. In future structural design work, evaluations of {QRO} are likely to be strongly influenced by empirical considerations, as currently are other airframe design loads such as maneuver loads. The remaining terms in the linearized rotor equations (mass, damping, stiffness) are increments of the gross rotor forces resulting from displacements of the rotor from the trim solution. It is a tenet of design to avoid resonant conditions, and if such conditions are avoided, the displacements from trim should be small. Thus, it appears reasonable to expect that a rotor model linearized as discussed in appendix A will be nearly as good for vibration prediction as the underlying nonlinear model.

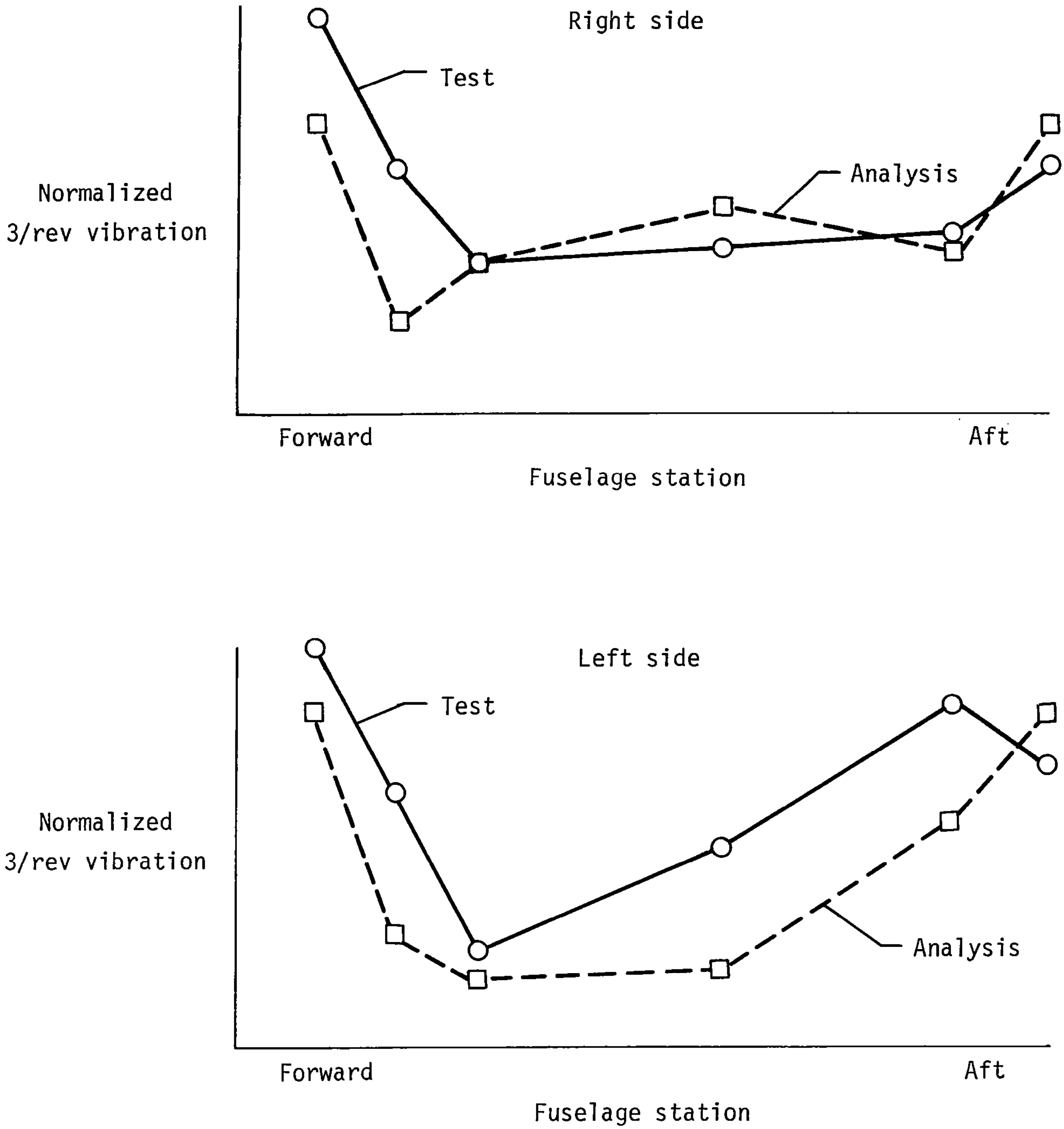


Figure 5.- Correlation of flight test data with a coupled rotor-airframe analysis for 3-per-revolution (3/rev) vibration of a three-bladed tandem-rotor helicopter in high-speed level flight. (From ref. 8.)

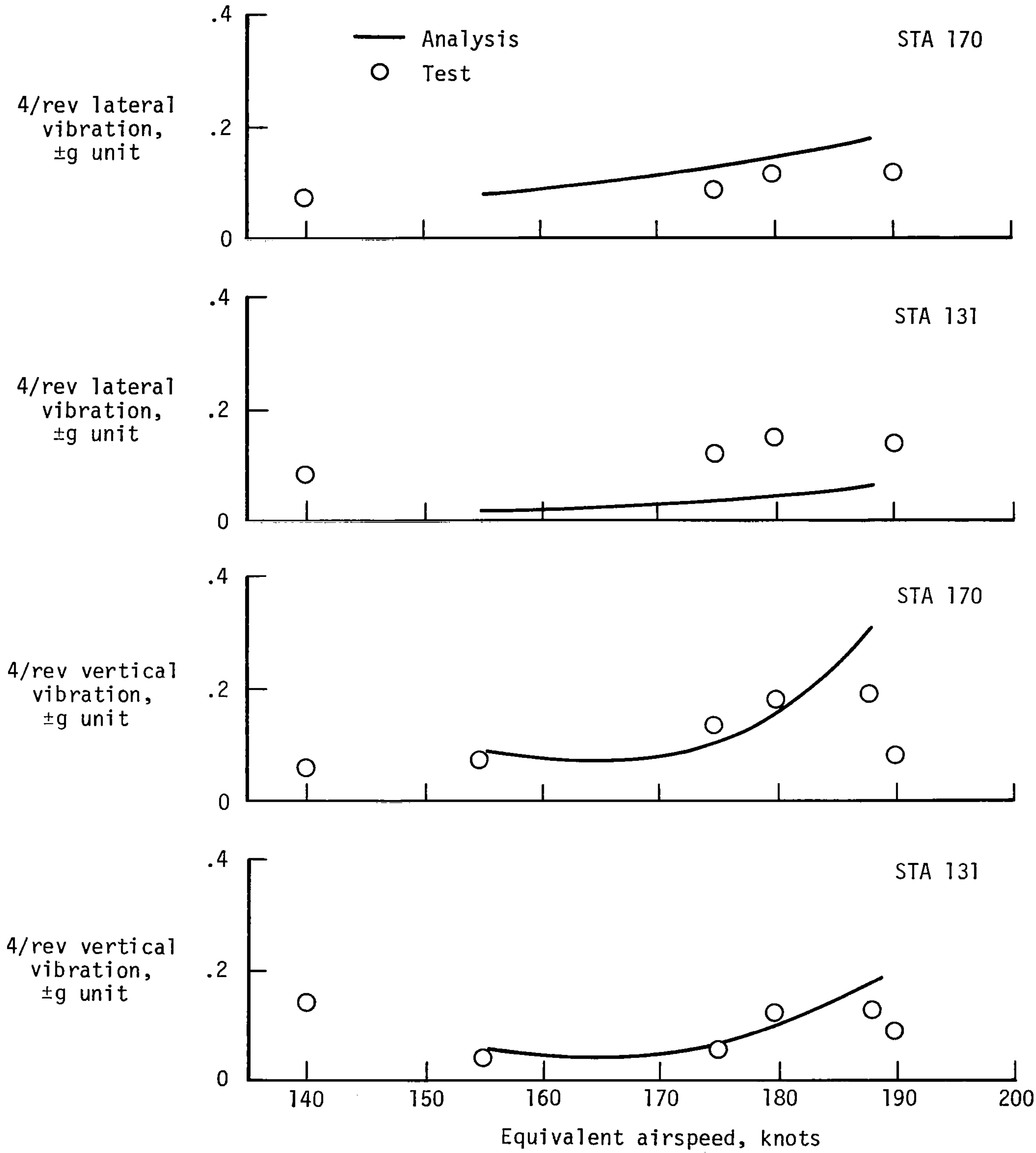


Figure 6.- Comparison of computed coupled rotor-airframe vibrations with measured data for compound helicopter with a four-bladed hingeless rotor. Vertical and lateral 4-per-revolution (4/rev) vibrations at the forward (STA 131) and aft (STA 170) crew stations as a function of airspeed. (From ref. 10.)

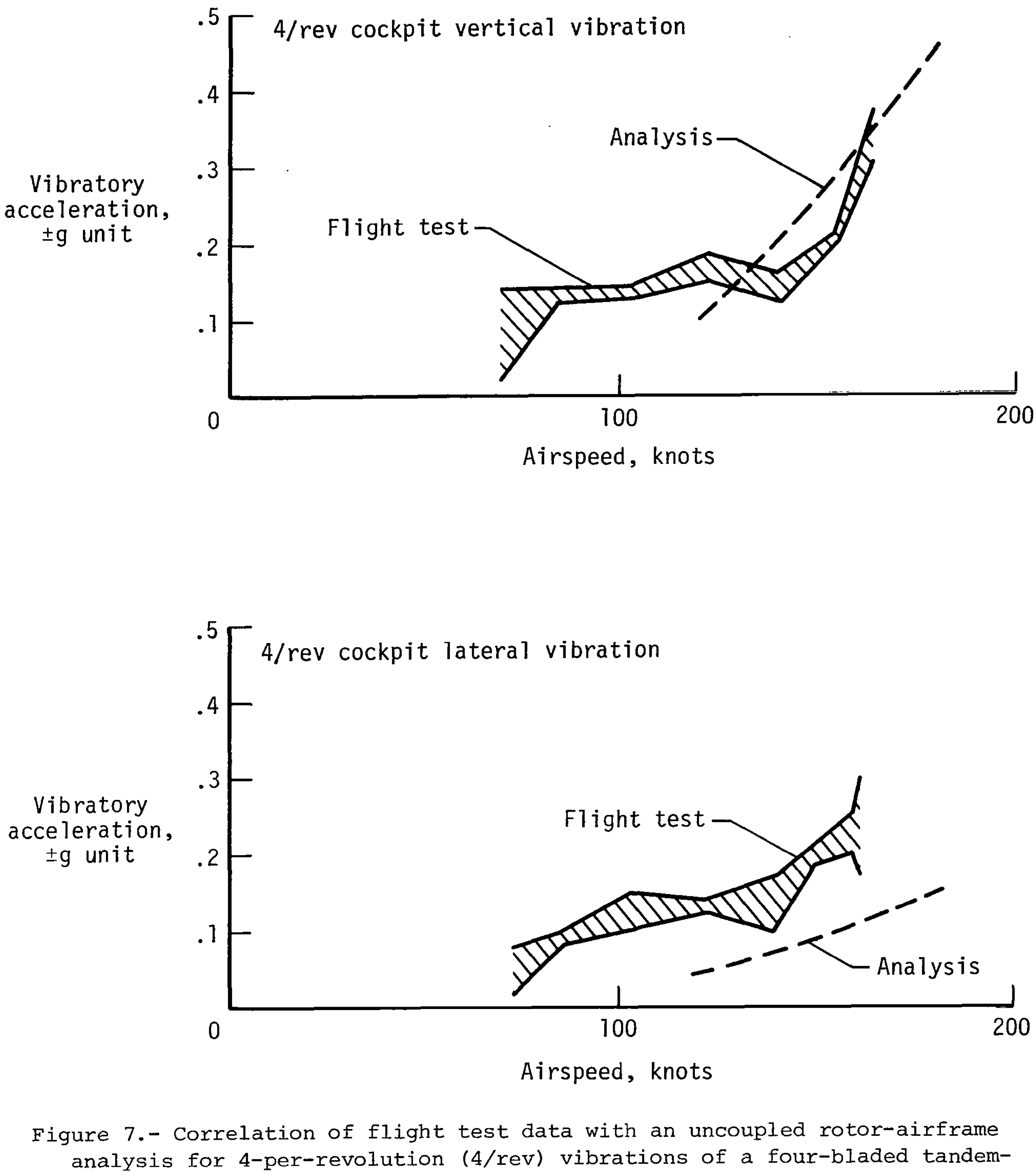


Figure 7.- Correlation of flight test data with an uncoupled rotor-airframe analysis for 4-per-revolution (4/rev) vibrations of a four-bladed tandem-rotor helicopter. (From ref. 9.)

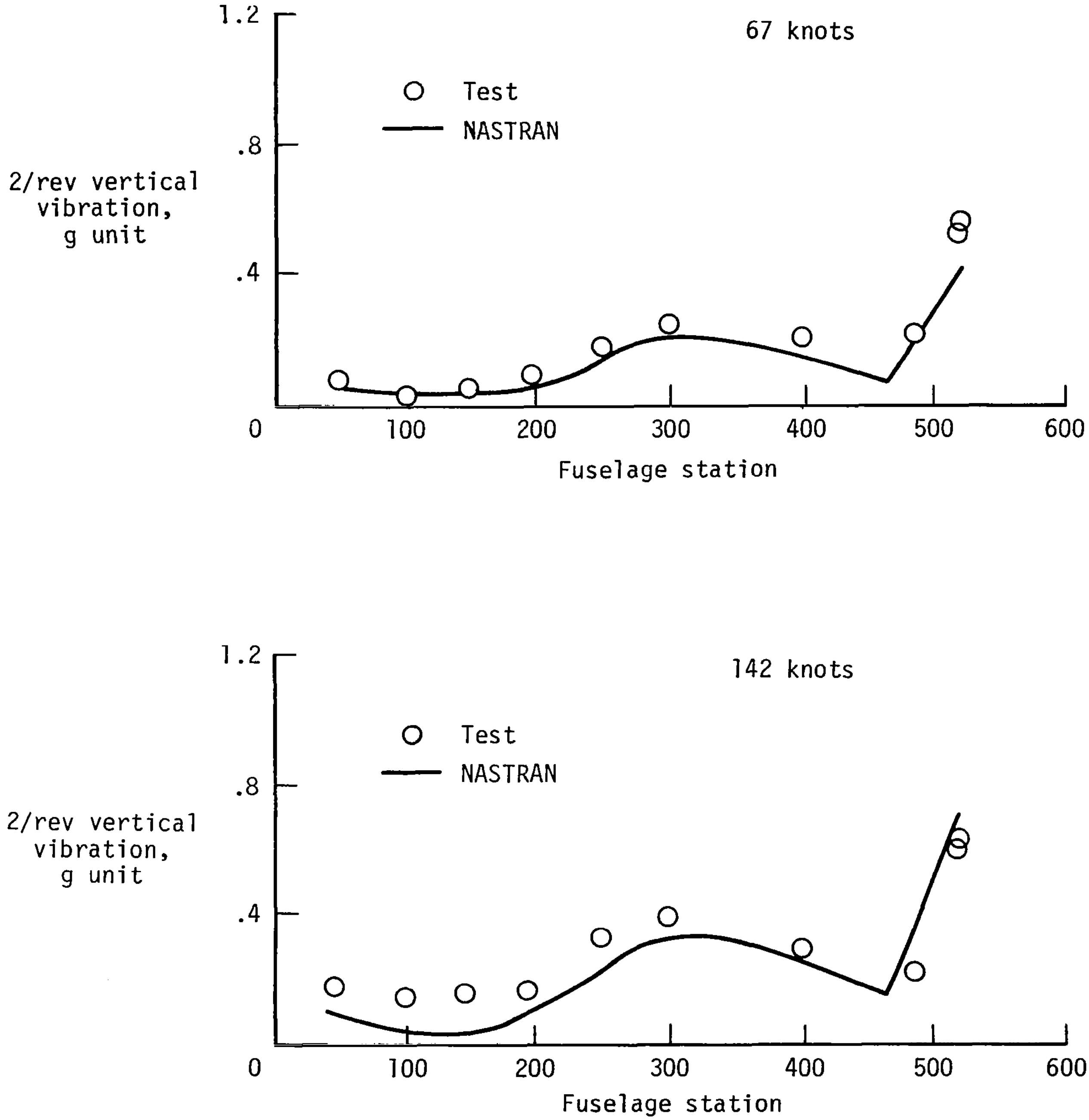


Figure 8.- Comparison of flight test data at two flight speeds with vibration responses of the airframe calculated using measured hub accelerations for 2-per-revolution (2/rev) vertical vibration of a two-bladed teetering rotor helicopter. (From ref. 21.)

Comparison of the Harmonic Forced Response Approach With the Modal Approach for Reducing the Number of Degrees of Freedom of the Airframe Finite-Element Model

In an analysis of helicopter vibrations based on a finite-element model of the airframe, one must be prepared to reduce the number of degrees of freedom of the finite-element model. Two approaches are currently recognized for making this reduction and still preserving the essence of the finite-element model: (1) representing the airframe by forced responses calculated at a few frequencies corresponding to the rotor harmonics of interest, as proposed in this paper, and (2) representing the airframe by superposition of a few of the natural modes of vibration. Whichever approach is used, the data (forced responses or modes) needed to represent the airframe with relatively few degrees of freedom are calculated by using the finite-element model separate from the rotor.

Modal representations are classic and widely taught. Such representations can be used for reducing the number of degrees of freedom when calculating any of the linear structural responses which are of interest in practical flight dynamics. This includes problems of aeroelastic stability and of transient response as well as the present problem of steady-state vibrations. This broad applicability has caused the modal representation of the airframe to be the choice of developers of computer simulations of the helicopter in flight.

The generality and familiarity of the modal representation is thus acknowledged. However, because of the following three advantages, the authors believe that representing the airframe system by harmonic forced responses is preferable for vibration analyses done to support design of airframe structures:

1. The analyst can take complete advantage of the fact that the important frequencies of excitation are usually known ahead of time and are normally few in number. Only the forced responses corresponding to excitation frequencies at the important harmonics of the rotor rotational frequency need be included in the representation. Often it is known in advance that only a single frequency (namely, the so-called blade passage frequency which equals the rotor rotational frequency multiplied by the number of blades) is important in the loads transmitted from the rotor to the airframe. Thus, the analyst has a reliable measure of the adequacy of the reduced-degree-of-freedom representation of the airframe. With a modal representation, foreknowledge of the important excitation frequencies cannot be comparably utilized.

2. The computational requirements for reductions based on forced responses should be acceptable at all stages of airframe structural design work, including the advanced stages when the airframe finite-element model becomes large. This is questionable for the modal approach. When computational requirements are the concern, the key difference between the two types of reduction is that forced responses are generated by solving large sets of simultaneous equations, whereas natural modes are generated by solving large eigenvalue problems. Thus, solving for the forced responses can be roughly compared with the solutions which are now carried out in sizing the airframe for traditional static design loads. Currently the only procedures considered reliable to solve for the airframe natural modes pose much more severe computational requirements.

3. A reduction based on forced responses lends itself to a relatively quick and simple procedure for recomputations when a few selected members of the airframe finite-element model are varied in design studies. This advantage also arises from the way forced responses are generated, that is, by solution of simultaneous equations. As appendix E explains, this allows recomputation by matrix partitioning, a well-known technique used in static analyses. This procedure does not result in any additional approximations. The only procedures which have been developed for comparable recomputations of natural modes are very involved and do result in additional approximations.

CONCLUDING REMARKS

A linear formulation of rotor-airframe coupling intended for airframe structural design work has been presented. The airframe is represented by a general finite-element model, and the rotor is represented by a set of general linear differential equations with periodic coefficients. Coupling of the rotor to the airframe is specified through general linear equations.

Background relating to representation of the airframe is provided by a synopsis of finite-element modeling as currently practiced in the U.S. helicopter industry. This is supplemented by noting the steps required to incorporate into an airframe model special effects which are presented in terms of impedances. Examples of these special effects include frequency-independent structural damping, gyroscopic forces from rotating engine components, and vibration control devices.

Background on the rotor representation is discussed to facilitate use of this paper by airframe designers. Shown is the process by which linear differential equations with periodic coefficients are derived from the usual engineering analysis models of rotors. These underlying models are normally expressed in terms of nonlinear differential equations, which are linearized by assuming small displacements from a steady flight trim condition. Also discussed is an alternative representation of the rotor suggested in the literature in which the rotor effects are presented in terms of impedances at the rotor mounts.

The coupling equations are derived by assuming that the rotor system and the airframe system are connected at an arbitrary number of discrete points arranged in an arbitrary manner. The derivation is general and, in particular, accounts for any constraints imposed on the airframe model because of assumptions underlying the rotor model. As a supplement, two useful approximate forms of the coupling equations are shown. An illustrative example is worked out, along with the derivation of the coupling equations, and a logical division of responsibilities (between rotor and airframe analysts) for formulation of the coupling equations is suggested.

The coupling equations are applied to combine the rotor and airframe equations into one set of differential equations for the rotor-airframe system. This is accomplished with the displacement method, a customary method for joining components in conjunction with the finite-element method of analysis. Factors relating to equilibrium of static components of trim forces are addressed.

The basis for solving the resulting differential equations to yield the steady-state vibrations is the well-known harmonic balance method in which the system steady-state vibratory responses are assumed to involve only a limited number of frequencies at integer multiples (harmonics) of the rotor rotational frequency. The paper gives an explicit algorithm for generating the harmonic balance equations of a linearized rotor-airframe system.

The number of airframe degrees of freedom is reduced by representing the airframe by harmonic forced responses. These responses are calculated only at the rotor harmonics of interest.

A method is presented for quick recalculation of vibrations when only relatively few members of the airframe finite-element model are identified as candidates for variation, as often happens in design studies of helicopter vibrations. Constant portions of the equations are separated by simple matrix partitioning and solved only once; this procedure is well-known in static analysis. Representation of the airframe by harmonic forced responses lends itself particularly well to this simple way of organizing calculations for rapid calculation of the effects of design changes.

Explicit relations required to generate the rotor-airframe equations and solve for the system vibratory responses are presented in a form suitable for direct computer implementation. Block diagrams illustrating computating sequences are included and discussed.

Finally, during a discussion of the method, two subjects are addressed: the potential utility of linearized rotor-airframe models for practical analysis of helicopter vibrations and the merits of representing the airframe by harmonic forced responses. It is argued that a properly linearized model should be as good for calculating airframe vibrations as the underlying nonlinear model. The procedure of representing the airframe by harmonic forced responses is compared with the alternative procedure of representing the airframe by superimposing a few of the natural modes of vibration. Representation of the airframe by harmonic forced responses is recommended for airframe structural design work.

Langley Research Center
National Aeronautics and Space Administration
Hampton, VA 23665
March 2, 1982

APPENDIX A

ROTOR SYSTEM EQUATIONS OF MOTION

This appendix discusses the relationship of linearized rotor equations to fundamental engineering formulations of the rotor system in terms of nonlinear equations.

Fundamental Equations of Motion

Engineering analyses of the dynamics of helicopter rotors generally start with idealizations which represent the rotor mechanical system in terms of a finite number of independent generalized coordinates. These coordinates uniquely determine the position of any particle of the system with reference to inertial space. Classical techniques of structural mechanics make it possible to derive equations which determine the generalized coordinates as functions of time and thereby define the motions of the system. These equations are generally nonlinear. The equations allow for arbitrary initial conditions and arbitrary external forces. The analyst completes the formulation of the fundamental nonlinear equations by specializing the arbitrary forces in the equations to represent the forces produced by the interactions of the rotor with the airstream.

Generalized coordinates.- In this paper, the symbols for the generalized coordinates are v_j, where $j = 1, 2, 3, \ldots, NV$. The collection of all generalized coordinates is denoted by the vector $\{v\}$. As an example, for the rotor blade depicted in figure A1, a vector of generalized coordinates is

$$\{v\} = \begin{Bmatrix} \xi \\ \beta \\ x_h \\ y_h \\ z_h \\ \theta \\ \psi \end{Bmatrix} \tag{A1}$$

The coordinates appearing in equation (A1) are

x_h, y_h, z_h fore-and-aft, lateral, and vertical hub displacements with respect to inertial axes X_I, Y_I, and Z_I

ψ rotation angle of the blade

θ rigid-body pitch angle

β rigid-body flap angle

ξ single variable multiplying $\eta(r)$, an assigned function of the blade radial position r, to describe displacements $w(r,t)$ along the length of the blade due to elastic deformation

As this example illustrates, a generalized coordinate may be defined so that it is subject to a direct physical interpretation, or the definition may be abstract. In figure A1, the generalized coordinates x_h and ψ are in the first category. The generalized coordinate ξ is an example of an abstract coordinate. Units assigned to a generalized coordinate are usually consistent with its definition, such as length for the displacement x_h. Often a generalized coordinate is regarded as nondimensional, such as the rotation angle ψ or the coordinate ξ.

Form of equations.- The fundamental equations of motion for the rotor system may be presented in the following two equivalent forms:

$$\int_0^t \delta W_r \, dt = \int_0^t \sum_{j=1}^{NV} \delta v_j \, Q_j \, dt = 0 \tag{A2}$$

$$Q_j = 0 \qquad (j = 1, 2, \ldots, NV) \tag{A3}$$

The first form expresses vanishing of a quantity known as virtual work (δW_r), and the second expresses balance of forces. The expressions Q_j appearing in these equations are usually derived by the methods of Lagrange, and this paper assumes that equations (A3) are Lagrange equations. The Lagrange equations are the more familiar of the two forms. The virtual work form is included here because it facilitates derivation of the coupling of rotor and airframe equations and the introduction of constraints. The quantities δv_j appearing in equation (A2) are virtual displacements.

Generalized forces.- The quantity multiplying an increment of a generalized coordinate to calculate work is customarily called a generalized force. The quantities Q_j in equations (A2) and (A3) conform to this definition. It is common and quite useful to think of the jth Lagrange equation,

$$Q_j = 0 \tag{A4}$$

as expressing equilibrium of generalized forces. In special cases, an equation such as equation (A4) corresponds to equilibrium of forces or torques. For example, when a generalized coordinate v_j corresponds to a translation, the expression Q_j may be interpreted as a force acting in the direction of translation. For the example of figure A1, the expression Q_3 would correspond to a force in the fore-and-aft direction, and the equation,

$$Q_3 = 0 \tag{A5}$$

would correspond to equilibrium of external and reaction forces in the fore-and-aft direction. Similarly, if the generalized coordinate v_j corresponds to a rotation about some axis, then the expression Q_j may be interpreted as a torque about that axis. For the example of figure A1, Q_7 would correspond to a torque about the shaft axis, and the equation,

$$Q_7 = 0 \tag{A6}$$

would correspond to equilibrium of applied and reaction torques.

Relation of generalized forces to generalized coordinates.- Terms in the expressions Q_j arise from two sources: the formulation of the equations of motion of the rotor mechanical system and the formulation of the external forces produced by the interactions of the rotor with the airstream. For practical purposes, it may be assumed that terms arising from description of the rotor mechanical system can be expressed as nonlinear functions of the generalized coordinates, their time derivatives, and time explicitly. The terms arising from description of the aerodynamic forces can for many purposes be expressed in the same manner. It is reasonable to expect that rotor models used to account for vibrations in airframe design work in the future will be relatively simple with the aerodynamic terms so described. This renders the generalized forces Q_j to be functions of the generalized coordinates, their time derivatives, and time explicitly. Any explicit dependence on time is of a special character as indicated in subsequent comments on periodicity. Such a relationship between the generalized forces and the generalized coordinates is assumed in this paper. It is further assumed that time derivatives of the generalized coordinates higher than the second do not appear in the expressions Q_j. Ruling out time derivatives higher than the second poses little practical restriction because usual mathematical descriptions of rotor systems, including the aerodynamics, are encompassed by the first and second time derivatives of the generalized coordinates. Variables with higher time derivatives may be appropriate physically to represent certain control system effects in the rotor system equations. However, for linearized control system equations, it is straightforward in principle to reduce such a description to an equivalent description where the highest derivative does not exceed the second. Ordinarily, the number of such variables with higher derivatives is quite small and, consequently, little computational effort is needed to carry out this reduction.

Prescribed coordinates.- Frequently in applying the fundamental rotor equations, it is convenient to designate a specific generalized coordinate v_k to be a prescribed function of time. Then, in all the expressions Q_j the variable v_k is assigned the prescribed form and $\dot{v}_k$ and $\ddot{v}_k$ are obtained by differentiating v_k. For any coordinate v_k thus prescribed, there can be no virtual displacement. Consequently, the kth term is deleted from the summation of virtual work (eq. (A2)) and the equation,

$$Q_k = 0 \tag{A7}$$

which is associated with the generalized coordinate v_k is removed from the Lagrange equations (eq. (A3)). The expression Q_k, no longer appearing in the Lagrange equations, may be interpreted as the generalized force required to constrain v_k to the prescribed behavior in time.

Categorization of generalized coordinates.- It is convenient at this point to define four categories of generalized coordinates symbolized by the following partitions of the vector {v}:

$$\{v\} = \begin{Bmatrix} \{VB\} \\ \{VH\} \\ \{VC\} \\ VT \end{Bmatrix} \equiv \begin{Bmatrix} \begin{Bmatrix} VR \end{Bmatrix} \\ \{VC\} \\ VT \end{Bmatrix} \tag{A8}$$

The four categories are defined as follows:

1. Because of the assumption that there is a single rotor, VT is a single variable representing the shaft rotation angle.

2. The vector {VC} represents the variables which are prescribed as functions of time to establish steady-state trim.

3. The vector {VH} represents variables which explicitly appear in expressions characterizing mechanical connections of the rotor system to the airframe system.

4. The vector {VB} represents the remaining generalized coordinates of the rotor system.

The vector {VR} represents the elements of {VH} and {VB} collectively. The numbers of variables in {VB}, {VH}, {VC}, and {VR} are denoted by NVB, NVH, NVC, and NVR. The vector of all generalized forces, denoted by {Q}, is partitioned in a corresponding manner:

$$\{Q\} = \begin{Bmatrix} \{QB\} \\ \{QH\} \\ \{QC\} \\ QT \end{Bmatrix} \equiv \begin{Bmatrix} \begin{Bmatrix} QR \end{Bmatrix} \\ \{QC\} \\ QT \end{Bmatrix} \tag{A9}$$

The shaft rotation angle VT has been shown separated from the variables in {VH}. Recall that {VH} contains the variables explicitly appearing in expressions of connections between the rotor and airframe systems. This separation arises naturally in derivations where displacements of the shaft rotation axis are referred to a coordinate system fixed in inertial space. In the literature, it is common to reference the shaft axis displacements in this manner, but it is also common to reference the shaft axis displacements to a coordinate system rotating with the rotor. In the latter instance, the separation of VT from the variables in {VH} does not directly emerge, but it is straightforward to perform a transformation of variables which brings about the separation.

If there are multiple rotors, the uncoupled equations of motion of the free-body rotors may be regarded as a single set of equations. Then the single variables VT and QT are replaced by vectors containing the appropriate number of independent rotation angles and torques. This generalization can be accommodated with only minor modifications to the procedures of this paper as long as all rotors are turning at the same rotational frequency. Of course, there is a prominent case of multiple rotors turning at different frequencies: the helicopter with a single main rotor and a relatively small tail rotor to provide antitorque. However, in this case, it is a reasonable practice to calculate the airframe vibrations induced by the rotors with each rotor treated separately. In multiple-rotor helicopter designs to date where the rotors are of comparable size (notably tandem helicopters), the rotors turn at the same frequency.

Specification of rotor rotational frequency.- The rotational frequency of the rotor is introduced by prescribing the rotation angle VT in the form

$$VT = \Omega t + \phi \tag{A10}$$

where the constant Ω is the rotational frequency and the constant ϕ is the phase angle determining the azimuthal positions of the rotor blades at time equal to zero. The equation,

$$Q_{NV} = 0 \tag{A11}$$

is removed from the Lagrange equations (eq. (A3)) and the expression Q_{NV} may be interpreted as the torque required to maintain the rotation at the prescribed frequency.

Comment on periodicity of the nonlinear equations.- Once VT is specified as in equation (A10), the equations of motion of a rotating system have an important property of periodicity. That is, whenever the coordinates v_j are all periodic with period $2\pi/\Omega$, then the expressions Q_j are all periodic with the same period. Stated simply, the generalized forces are periodic whenever the generalized coordinates are periodic.

The Trim Condition

General remarks.- To maintain a specified steady-state flight condition, the average forces and moments acting on the rotor must be in equilibrium with the average forces and moments acting on the airframe. Such a state of equilibrium is called a trim condition. It is important to keep in mind that a trim solution does not generally imply constant values of the rotor variables. Some of the rotor variables are usually varying (cyclically). In the analysis of rotor dynamics, the rotor equations are commonly linearized by assuming that the rotor motions depart only slightly from the trim condition; this assumption has been adopted in this paper. To represent the trim condition, the generalized forces {QH} must be maintained with appropriate mean values which will be designated {QHTR}. To obtain the mean values requires physical interpretation of the coordinates {VH}, knowledge of the rotor-airframe connection, and knowledge of the forces, such as gravitational, inertial, and aerodynamic, acting on the airframe. Extensive analysis may be required to make the specification of {QHTR}. In this paper, however, it is assumed that the specification has been made.

Definition of trim solution.- A necessary step in imposing the required mean values of {QH} is to compute a trim solution. A trim solution is defined herein to be any vector {v} meeting the following conditions:

1. The variables in {VH} are constrained to be zero:

$$\{VH\} = 0 \tag{A12a}$$

2. All the variables are periodic in time with period $2\pi/\Omega$:

$$\left\{v\left(t+\frac{2\pi}{\Omega}\right)\right\} = \{v(t)\} \tag{A12b}$$

3. The mean values of the generalized forces {QH} are

$$\{QH\}_{mean} = \{QHTR\} \tag{A12c}$$

Corresponding to any steady flight condition within the operating envelope, there should exist an exact solution of the rotor fundamental equations of motion meeting the above definition of a trim solution.

The partitioned vector {v} corresponding to a trim solution is given by

$$\{v\}_{trim} \equiv \{VO\} = \left\{\begin{matrix} \{VBO\} \\ \{VHO\} \\ \{VCO\} \\ \Omega t + \phi \end{matrix}\right\} \equiv \left\{\begin{matrix} \left\{ VRO \right\} \\ \\ \{VCO\} \\ \Omega t + \phi \end{matrix}\right\} \tag{A13}$$

where, in view of equation (A12a),

$$\{VHO\} = \{0\} \tag{A14}$$

The generalized forces corresponding to trim are given by

$$\{Q\}_{trim} \equiv \{QO\} = \begin{Bmatrix} \{QBO\} \\ \{QHO\} \\ \{QCO\} \\ T \end{Bmatrix} \equiv \begin{Bmatrix} \begin{Bmatrix} QRO \end{Bmatrix} \\ \{QCO\} \\ T \end{Bmatrix} \tag{A15}$$

where T is rotor torque and, consistent with the requirement of equation (A12c),

$$\{QHO\}_{mean} = \{QHTR\} \tag{A16}$$

Generation of trim solutions.- Various strategies are used to compute trim solutions (see, for example, refs. 3 and 4), and an approximate trim solution is commonly used in place of the exact trim solution. For instance, trim solutions are sometimes obtained by suppressing certain degrees of freedom. For the articulated rotor blade of figure A1, for example, one might assume that the blade deformations have a negligible effect on the trim solution and compute the trim solution with the appropriate degree of freedom (ξ) constrained.

Variables prescribed for trim.- In arriving at a trim solution, some variables (previously denoted collectively by the vector {VC}) are normally prescribed as functions of time that are not varied in subsequent calculations. For example, in figure A1, the blade rigid-body pitch angle θ might be prescribed to have the form,

$$\theta(t) = \theta_o + \theta_{1c} \cos \psi + \theta_{1s} \sin \psi \tag{A17}$$

where θ_o is the collective pitch angle and θ_{1c} and θ_{1s} are the longitudinal and lateral cyclic pitch angles imposed by the flight control system. Equations corresponding to the variables {VC} are deleted from the Lagrange equations (eq. (A3)), the expressions in {QC} being interpreted as the generalized control forces required to maintain the specified trim condition.

Linearization of the Rotor Equations

The rotor equations of motion are linearized by assuming that vibratory motions are small variations (perturbations) from the trim solution. This assumption is reflected in the equations

$$\{VR\} = \{VRO\} + \{DVR\} \tag{A18a}$$

$$\{VC\} = \{VCO\} \tag{A18b}$$

where the elements of the vector {DVR} represent the small variations from trim. The generalized coordinates {VC} have been prescribed to establish trim and are not varied. As previously discussed, the rotor rotation angle VT is also prescribed and is given by equation (A10). Usual practice is to substitute equation (A18a) for the vector of generalized coordinates {VR} into the generalized force expressions Q_j and to neglect higher order terms in the variables {DVR} occurring in the resulting expressions. The following linearized approximation for {QR} is then obtained:

$$\{QR\} = \{QRO\} + [MR]\{D\ddot{V}R\} + [CR]\{D\dot{V}R\} + [KR]\{DVR\} \tag{A19}$$

The elements of the square matrices [MR], [CR], and [KR] are given by

$$MR_{ij} = \left(\frac{\partial Q_i}{\partial \ddot{v}_j}\right)_{trim} \tag{A20a}$$

$$CR_{ij} = \left(\frac{\partial Q_i}{\partial \dot{v}_j}\right)_{trim} \tag{A20b}$$

$$KR_{ij} = \left(\frac{\partial Q_i}{\partial v_j}\right)_{trim} \tag{A20c}$$

$$(i,j = 1,\ 2,\ \ldots,\ NVR)$$

The subscript trim appearing in equations (A20) indicates that the derivatives are evaluated for {v} = {VO}. From the preceding discussion of the periodic character of the generalized forces Q_j, it can be shown that the matrices characterizing equation (A19) are periodic with period $2\pi/\Omega$ and therefore may be expressed as Fourier series as follows:

$$[MR] = [MR0C] + [MR1S] \sin \Omega t + [MR1C] \cos \Omega t + [MR2S] \sin 2\Omega t + \ldots \tag{A21a}$$

$$[CR] = [CR0C] + [CR1S] \sin \Omega t + [CR1C] \cos \Omega t + [CR2S] \sin 2\Omega t + \ldots \tag{A21b}$$

$$[KR] = [KR0C] + [KR1S] \sin \Omega t + [KR1C] \cos \Omega t + [KR2S] \sin 2\Omega t + \ldots \tag{A21c}$$

$$\{QRO\} = \{QRO0C\} + \{QRO1S\} \sin \Omega t + \{QRO1C\} \cos \Omega t + \{QRO2S\} \sin 2\Omega t + \ldots \tag{A21d}$$

The coefficient matrices appearing in these equations, such as [MR1S] and {QRO1S}, are constant.

From equations (A18) and (A10), the virtual displacements δ\{VR\}, δ\{VC\}, and δ(VT) are given by

$$\delta\{VR\} = \delta\{DVR\} \tag{A22a}$$

$$\delta\{VC\} = \{0\} \tag{A22b}$$

$$\delta(VT) = 0 \tag{A22c}$$

Substitution of equations (A8), (A9), (A18), (A19), and (A22) into equation (A2) yields

$$\int_0^t \delta W_r \, dt = \int_0^t \delta\{DVR\}^T \{QR\} \, dt$$

$$= \int_0^t \delta\{DVR\}^T \Big\{ \{QRO\} + [MR]\{D\ddot{V}R\} + [CR]\{D\dot{V}R\} + [KR]\{DVR\} \Big\} \, dt = 0 \tag{A23}$$

from which it follows that

$$\{QRO\} + [MR]\{D\ddot{V}R\} + [CR]\{D\dot{V}R\} + [KR]\{DVR\} = \{0\} \tag{A24}$$

The following partitioned forms of equations (A23) and (A24) are defined for use in coupling the rotor to the airframe:

$$\int_0^t \delta W_r \, dt = \int_0^t \delta \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix}^T \left\{ \begin{Bmatrix} \{QBO\} \\ \{QHO\} \end{Bmatrix} + \begin{bmatrix} [MR11][MR12] \\ [MR21][MR22] \end{bmatrix} \begin{Bmatrix} \{D\ddot{V}B\} \\ \{D\ddot{V}H\} \end{Bmatrix} \right.$$

$$\left. + \begin{bmatrix} [CR11][CR12] \\ [CR21][CR22] \end{bmatrix} \begin{Bmatrix} \{D\dot{V}B\} \\ \{D\dot{V}H\} \end{Bmatrix} + \begin{bmatrix} [KR11][KR12] \\ [KR21][KR22] \end{bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} \right\} dt = 0 \tag{A25}$$

$$\begin{Bmatrix} \{QBO\} \\ \{QHO\} \end{Bmatrix} + \begin{bmatrix} [MR11][MR12] \\ [MR21][MR22] \end{bmatrix} \begin{Bmatrix} \{D\ddot{V}B\} \\ \{D\ddot{V}H\} \end{Bmatrix}$$

$$+ \begin{bmatrix} [CR11][CR12] \\ [CR21][CR22] \end{bmatrix} \begin{Bmatrix} \{D\dot{V}B\} \\ \{D\dot{V}H\} \end{Bmatrix} + \begin{bmatrix} [KR11][KR12] \\ [KR21][KR22] \end{bmatrix} \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{A26}$$

where

$$\{DVR\} = \begin{Bmatrix} \{DVB\} \\ \{DVH\} \end{Bmatrix} \tag{A27}$$

Each of the submatrices appearing in these equations has a Fourier series representation similar to equations (A21). For example,

$$[MR11] = [MR0C11] + [MR1S11] \sin \Omega t + [MR1C11] \cos \Omega t + \ldots \tag{A28a}$$

$$\{QBO\} = \{QBO0C\} + \{QBO1S\} \sin \Omega t + \{QBO1C\} \cos \Omega t + \ldots \tag{A28b}$$

$$\{QHO\} = \{QHO0C\} + \{QHO1S\} \sin \Omega t + \{QHO1C\} \cos \Omega t + \ldots \tag{A28c}$$

Equations (A23) and (A24) are the linearized versions of the virtual work equation and the Lagrange equations given by equations (A2) and (A3). As noted in the main text, it is assumed that these equations are specified at the outset of the computations. The specification is made by providing the coefficient matrices in equations (A21). It is also necessary to specify the variables appearing in {DVH} so that the partitions indicated in equations (A25) and (A26) can be made.

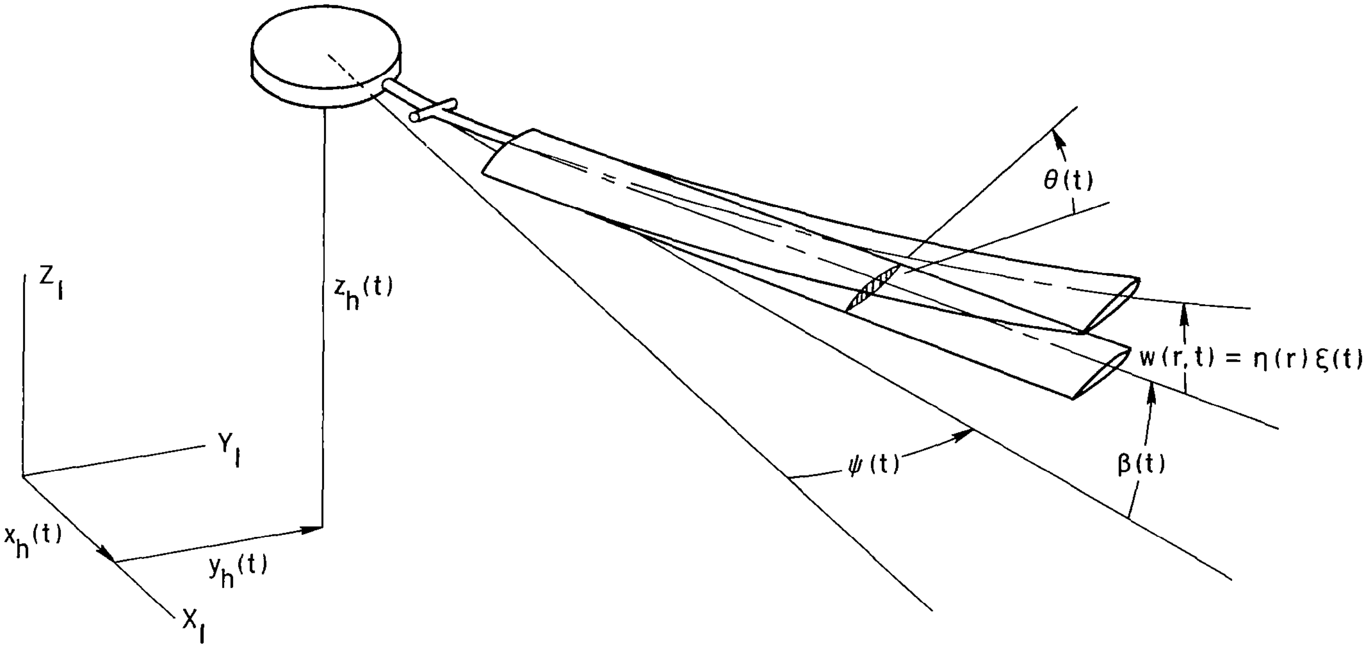


Figure A1.- Example generalized coordinates.

APPENDIX B

AIRFRAME SYSTEM EQUATIONS OF MOTION

This appendix discusses the form of the equations of motion of the airframe system. It is assumed that the equations are based on a typical finite-element method of analysis. Finite-element methods are used extensively for practical analysis of both static and dynamic behavior of flight structures (for commentary, see ref. 22). In particular, the NASTRAN computer code (ref. 1) for finite-element analysis is used by most U.S. helicopter airframe manufacturers.

Finite-Element Models

Finite-element analysis models are assemblies of structural elements connected at discrete points called nodes. Figure B1 and table B1 illustrate a simplified finite-element representation of a helicopter airframe. For convenience of illustration, the assembly in the figure is drawn as two-dimensional. However, the typical assembly would be three-dimensional and may involve many more elements and nodes. Structural elements making up these models may include elastic elements in the form of rods, shear panels, membrane panels, beam segments, and plate panels, as well as concentrated masses and general extended rigid bodies.

Table B1, which is to be associated with figure B1, is a representative tabulation of information defining the geometry (i.e., topology) of a three-dimensional finite-element model. The following comments on the table identify aspects of finite-element models which bear on coupling the rotor system to the airframe system:

1. The numbers in the first column of table B1 identify node points where structural elements are joined.

2. The numbers in the second column designate coordinate systems to which node displacements and rotations are referenced. Figure B1 shows two such coordinate systems.

3. The numbers in the third column designate variables associated with each node. Up to six variables may be defined. The variables are interpreted as displacements and rotations with reference to the indicated node coordinate system. The numbers 1, 2, and 3 indicate displacements along the X, Y, and Z coordinate axes, respectively, and the numbers 4, 5, and 6 indicate the corresponding rotations about these axes. A designator missing from this column does not mean that there is a constraint on the model. A missing designator means that it is not necessary to define the variable in order to formulate the finite-element model.

4. The designators in the fourth column indicate externally applied forces or couples in the direction associated with the designator.

5. The designators in the fifth column indicate inertial forces or couples arising from mass elements incorporated in the model.

6. The designators in the sixth column indicate suppression of a variable by an external constraint posing a restriction on the motions of the model. The example is a free-body structure; therefore no designators appear in this column.

7. The designators in the last column indicate variables which have specified relations to other variables and are therefore not independent. The footnotes to the table explain the internal constraints for the example.

With the exception of the constrained variables designated in the last two columns, the variables designated in the third column are independent and may be viewed as the generalized coordinates of the airframe finite-element model.

General Form of the Equations of Motion

The generalized coordinates of the airframe finite-element model are denoted by $\bar{Z}_j$, where $j = 1, 2, 3, \ldots, \overline{NZ}$, or collectively by the vector $\{\bar{Z}\}$. As discussed, the variables $\bar{Z}_j$ may usually be interpreted as displacements or rotations at nodes. As done in appendix A for the rotor system, the airframe system equations of motion are first presented in general virtual work and differential equation forms which are equivalent:

$$\int_0^t \delta W_a \, dt = \int_0^t \sum_{j=1}^{\overline{NZ}} \delta \bar{Z}_j \, \bar{F}_j \, dt = 0 \tag{B1}$$

$$\bar{F}_j = 0 \qquad (j = 1, 2, \ldots, \overline{NZ}) \tag{B2}$$

The symbols $\bar{F}_j$, which are denoted collectively by the vector $\{\bar{F}\}$, represent the generalized force expressions for the airframe. Consistent with the interpretation of the generalized coordinates, the generalized forces may usually be thought of as net forces and couples at nodes. For the prevalent finite-element methods of analysis, the generalized force expressions have the form,

$$\{\bar{F}\} = [\overline{MA}]\{\ddot{\bar{Z}}\} + [\overline{CA}]\{\dot{\bar{Z}}\} + [\overline{KA}]\{\bar{Z}\} + \{\bar{L}\} \tag{B3}$$

Insertion of equation (B3) into the matrix forms of equations (B1) and (B2) gives the following forms for the virtual work equation and the differential equations of the airframe finite-element model:

$$\int_0^t \delta W_a \, dt = \int_0^t \delta\{\bar{Z}\}^T \left\{[\overline{MA}]\{\ddot{\bar{Z}}\} + [\overline{CA}]\{\dot{\bar{Z}}\} + [\overline{KA}]\{\bar{Z}\} + \{\bar{L}\}\right\} dt = 0 \tag{B4}$$

$$[\overline{MA}]\{\ddot{\bar{Z}}\} + [\overline{CA}]\{\dot{\bar{Z}}\} + [\overline{KA}]\{\bar{Z}\} + \{\bar{L}\} = \{0\} \tag{B5}$$

The square matrices $[\overline{MA}]$, $[\overline{CA}]$, and $[\overline{KA}]$ are constant, real, and symmetric and are either positive definite or positive semidefinite. They are the mass, damping, and stiffness matrices. These matrices may be quite large, since the number of generalized coordinates may range up to several thousand; but the matrices are generally sparsely populated. The vector $\{\bar{L}\}$ represents the external forces acting on the airframe. These forces in general may be arbitrary functions of time.

The following partitioned form of equation (B4) is defined for use in coupling the rotor to the airframe:

$$\int_0^t \delta W_a \, dt = \int_0^t \delta \begin{Bmatrix} \{\overline{ZH}\} \\ \{\overline{ZS}\} \end{Bmatrix}^T \left\{ \begin{bmatrix} [\overline{MA11}][\overline{MA12}] \\ [\overline{MA21}][\overline{MA22}] \end{bmatrix} \begin{Bmatrix} \{\ddot{\overline{ZH}}\} \\ \{\ddot{\overline{ZS}}\} \end{Bmatrix} + \begin{bmatrix} [\overline{CA11}][\overline{CA12}] \\ [\overline{CA21}][\overline{CA22}] \end{bmatrix} \begin{Bmatrix} \{\dot{\overline{ZH}}\} \\ \{\dot{\overline{ZS}}\} \end{Bmatrix} \right.$$

$$\left. + \begin{bmatrix} [\overline{KA11}][\overline{KA12}] \\ [\overline{KA21}][\overline{KA22}] \end{bmatrix} \begin{Bmatrix} \{\overline{ZH}\} \\ \{\overline{ZS}\} \end{Bmatrix} + \begin{Bmatrix} \{\overline{LH}\} \\ \{\overline{LS}\} \end{Bmatrix} \right\} dt = 0 \tag{B6}$$

where

$$\{\bar{Z}\} = \begin{Bmatrix} \{\overline{ZH}\} \\ \{\overline{ZS}\} \end{Bmatrix} \tag{B7a}$$

$$\{\bar{L}\} = \begin{Bmatrix} \{\overline{LH}\} \\ \{\overline{LS}\} \end{Bmatrix} \tag{B7b}$$

The vector $\{\overline{ZH}\}$ represents variables which explicitly appear in expressions characterizing mechanical connections of the rotor system to the airframe system. The vector $\{\overline{ZS}\}$ represents the remaining variables of the airframe finite-element model. The numbers of elements in $\{\overline{ZH}\}$ and $\{\overline{ZS}\}$ are denoted by $\overline{NZH}$ and $\overline{NZS}$.

Comment on Application of Linear Constraints to the Airframe Finite-Element Model Implied by Assumptions Made in Forming the Rotor Model

As discussed in the text and in appendix C, coupling the rotor model to the airframe model may imply additional linear constraints on the airframe finite-element model. Such constraints take the form

$$[\bar{C}]\{\bar{Z}\} = \{0\} \tag{B8}$$

Note that equation (9b) is a special case of this equation. Application of such constraints is a standard procedure in finite-element modeling of structures. For the NASTRAN finite-element code, in particular, inclusion of linear constraints is systematic as long as the analyst can identify a nonsingular square submatrix of $[\bar{C}]$ of order equal to the number of rows of $[\bar{C}]$ (see eq. (9b) and related discussion). If the rows of $[\bar{C}]$ are linearly independent as in equation (9b), at least one such submatrix always exists.

With regard to the practical problem of identifying an appropriate nonsingular square submatrix, the following conditions are expected when coupling rotor models to airframe models:

1. The number of equations of constraint and the number of variables involved are not large.

2. The coefficients of the equations normally can be presented in algebraic form.

3. The engineer is normally working with a complete physical interpretation of the equations.

Under these circumstances, the required identification of a nonsingular square submatrix should not prove difficult. It is noted, however, that on strictly numerical grounds, that is, when the matrix $[\bar{C}]$ is presented in numerical form only, systematic identification of such a nonsingular square submatrix may be difficult.

APPENDIX B

TABLE B1.- REPRESENTATIVE INFORMATION DEFINING GEOMETRY OF A THREE-DIMENSIONAL FINITE-ELEMENT MODEL

Node point	Coordinate system	Node point degrees of freedom (a)	External forces (a)	Inertial forces (a)	External constraints (a)	Internal constraints (a)
1	1	123	-	123	-	-
2	1	123	3	-	-	-
3	1	123	-	123	-	-
4	1	123	-	-	-	-
5	1	123	3	123	-	-
6	1	123	-	123	-	-
7	1	123	-	123	-	-
8	1	123	3	123	-	-
9	1	123	-	123	-	-
10	1	123	-	123	-	-
11	1	123	-	123	-	-
12	1	123	-	123	-	-
13	1	123	-	123	-	-
14	1	123456	-	-	-	[b]123456
15	1	123456	123456	123456	-	-
16	1	123456	-	-	-	[b]123456
17	1	123456	-	123456	-	-
18	1	123456	-	-	-	[b]123456
19	1	123	-	123	-	[c]123
20	1	123456	-	123456	-	-
21	1	123	-	123	-	[c]123
22	1	123	-	123	-	-
23	1	123	-	123	-	-
24	1	123	3	-	-	-
25	1	123	-	-	-	-
26	1	123	-	-	-	-
27	1	123	3	123	-	-
28	1	123	-	123	-	-
29	1	123	-	123	-	-
30	1	123	3	-	-	-
31	1	123	-	-	-	-
32	1	123	-	-	-	-
33	1	123	3	123	-	-
34	1	123	-	123	-	-
35	1	123	-	123	-	-
36	2	123	-	123	-	-
37	2	123456	-	123456	-	[d]1
38	2	123456	2	123456	-	[d]1
39	2	123456	2	123456	-	[d]1

[a]The numbers 1, 2, and 3 designate displacements along X, Y, and Z coordinate axes; 4, 5, and 6 designate rotations about these axes.

[b]Related to degrees of freedom of transmission.

[c]Related to degrees of freedom of engine.

[d]Axial deformation of vertical tail beam suppressed.

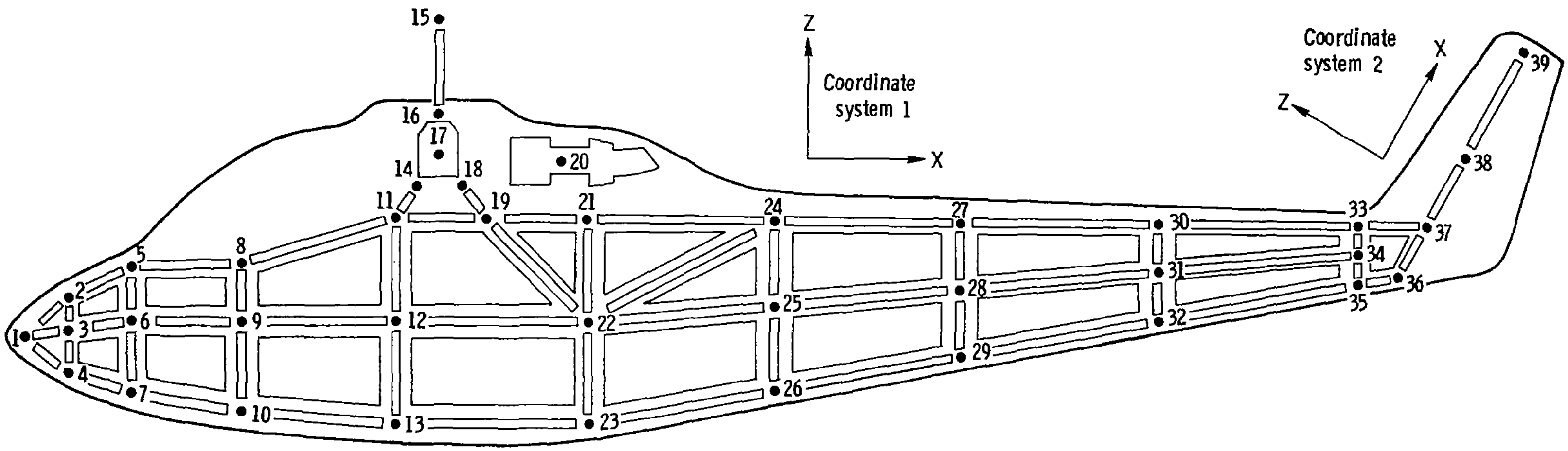


Figure B1.- Simplified finite-element representation of a helicopter airframe.

APPENDIX C

COUPLING EQUATIONS

Connection Points

It is assumed that the rotor system and the airframe system are joined at a finite number of discrete points. For illustration, figure C1 shows a simplified coupled system. In this two-dimensional example, there are two connection points denoted by the open circles. Connection points may, but do not necessarily, coincide with nodes of the airframe finite-element model, which are denoted in the drawing by solid circles. Associated with each of the connection points, variables representing up to three displacements and three rotations are defined as described for node points in appendix B in the section on finite-element models. The displacements and rotations, for all connection points, are denoted by c_j, where $j = 1, 2, \ldots, NI$, or collectively by the vector $\{c\}$.

Implied Constraints

Before coupling equations are established, it is important to recognize that because of different assumptions in the modeling of the rotor system and the modeling of the airframe system, connecting the two models together may imply constraints on either of the systems. Three different situations can occur:

1. A constraint is implied on the airframe model, as would occur in the example of figure C1 if the transmission were modeled as rigid.

2. No constraints are implied, as would occur in the example if the transmission were modeled as flexible.

3. A constraint is implied on the rotor model.

Situations (1) and (2) are addressed by this appendix. Situation (3) is unlikely and is not addressed. However, the required modifications to the computational procedures to account for this situation are straightforward.

Basic Relations

According to the definition of $\{VH\}$ and $\{\overline{ZH}\}$ as vectors of generalized coordinates, it is always possible to establish relations connecting $\{VH\}$ to $\{c\}$ and $\{\overline{ZH}\}$ to $\{c\}$. These relations are represented by the following two equations:

$$\{f_r\} = \{c\} \tag{C1a}$$

$$\{f_a\} = \{c\} \tag{C1b}$$

where the elements of the vector $\{f_r\}$ are functions of the variables VH_j ($j = 1, 2, \ldots, NVH$), and the elements of the vector $\{f_a\}$ are functions of the variables $\overline{ZH}_j$ ($j = 1, 2, \ldots, \overline{NZH}$). The functions in $\{f_r\}$ and $\{f_a\}$ are generally nonlinear.

The rotor displacements have been assumed to vary only slightly from a trim solution, as indicated by equation (A18a). Because the variables in {VHO} for a trim solution are by definition equal to zero, the variables in {VH} are approximated by the small variation variables {DVH}. Consistent with usual assumptions in finite-element analysis, the variables in $\{\overline{ZH}\}$ are assumed to be small. It follows that equations (C1) can be approximated by linear forms indicated by

$$[THR]\{DVH\} = \{c\} \tag{C2a}$$

$$[THA]\{\overline{ZH}\} = \{c\} \tag{C2b}$$

The matrices [THR] and [THA] are constant matrices. For practical interface arrangements, the matrix [THR] has linearly independent columns and the matrix [THA] has linearly independent rows. This implies that the number of rows in [THR] is equal to or greater than the number of columns and that the number of columns in [THA] is equal to or greater than the number of rows.

Figure C2 illustrates specification of the basic relations. In this two-dimensional example, which shows the rotor and airframe models separated for pictorial clarity, {VH} is a three-element vector where VH_1 and VH_2 are hub displacements and VH_3 is the hub pitching rotation. In this case {c} is a four-element vector representing displacements at the two connection points. With the rotor shaft and transmission assumed to be rigid, equation (C1a) becomes

$$\begin{Bmatrix} VH_1 - \ell_1 \sin VH_3 + \ell_2(1 - \cos VH_3) \\ VH_2 + \ell_2 \sin VH_3 + \ell_1(1 - \cos VH_3) \\ VH_1 - \ell_1 \sin VH_3 - \ell_2(1 - \cos VH_3) \\ VH_2 - \ell_2 \sin VH_3 + \ell_1(1 - \cos VH_3) \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \tag{C3}$$

Assumption of small values for the variables on the left of equation (C3) leads to a linear form corresponding to equation (C2a):

$$\begin{bmatrix} 1 & 0 & -\ell_1 \\ 0 & 1 & \ell_2 \\ 1 & 0 & -\ell_1 \\ 0 & 1 & -\ell_2 \end{bmatrix} \begin{Bmatrix} DVH_1 \\ DVH_2 \\ DVH_3 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \tag{C4}$$

To illustrate another way for specifying the basic relations, equation (C1b) is determined by linear interpolation:

$$\left.\begin{aligned}
\frac{a}{\ell_3}\,\overline{ZH}_1 + \frac{\ell_3 - a}{\ell_3}\,\overline{ZH}_3 &= c_1 \\
\frac{a}{\ell_3}\,\overline{ZH}_2 + \frac{\ell_3 - a}{\ell_3}\,\overline{ZH}_4 &= c_2 \\
\frac{b}{\ell_4}\,\overline{ZH}_3 + \frac{\ell_4 - b}{\ell_4}\,\overline{ZH}_5 &= c_3 \\
\frac{b}{\ell_4}\,\overline{ZH}_4 + \frac{\ell_4 - b}{\ell_4}\,\overline{ZH}_6 &= c_4
\end{aligned}\right\} \tag{C5}$$

Equations (C5) are already linear and therefore no linearization step is required. The equations take the following matrix form corresponding to equation (C2b):

$$\begin{bmatrix}
\frac{a}{\ell_3} & 0 & 1 - \frac{a}{\ell_2} & 0 & 0 & 0 \\
0 & \frac{a}{\ell_3} & 0 & 1 - \frac{a}{\ell_3} & 0 & 0 \\
0 & 0 & \frac{b}{\ell_4} & 0 & 1 - \frac{b}{\ell_4} & 0 \\
0 & 0 & 0 & \frac{b}{\ell_4} & 0 & 1 - \frac{b}{\ell_4}
\end{bmatrix}
\begin{Bmatrix}
\overline{ZH}_1 \\ \overline{ZH}_2 \\ \overline{ZH}_3 \\ \overline{ZH}_4 \\ \overline{ZH}_5 \\ \overline{ZH}_6
\end{Bmatrix}
=
\begin{Bmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4
\end{Bmatrix} \tag{C6}$$

Derivation of Coupling Equations

The basic relations (eqs. (C2)) are first written in the form

$$\Big[[THR] \;\vdots\; -[THA] \Big] \begin{Bmatrix} \{DVH\} \\ \{\overline{ZH}\} \end{Bmatrix} = \{0\} \tag{C7}$$

For example, equations (C4) and (C6) written in this form are

$$\left[\begin{bmatrix} 1 & 0 & -\ell_1 \\ 0 & 1 & \ell_2 \\ 1 & 0 & -\ell_1 \\ 0 & 1 & -\ell_2 \end{bmatrix} \vdots -\begin{bmatrix} \frac{a}{\ell_3} & 0 & 1-\frac{a}{\ell_3} & 0 & 0 & 0 \\ 0 & \frac{a}{\ell_3} & 0 & 1-\frac{a}{\ell_3} & 0 & 0 \\ 0 & 0 & \frac{b}{\ell_4} & 0 & 1-\frac{b}{\ell_4} & 0 \\ 0 & 0 & 0 & \frac{b}{\ell_4} & 0 & 1-\frac{b}{\ell_4} \end{bmatrix}\right] \begin{Bmatrix} \begin{Bmatrix} DVH_1 \\ DVH_2 \\ DVH_3 \end{Bmatrix} \\ \begin{Bmatrix} \overline{ZH}_1 \\ \overline{ZH}_2 \\ \overline{ZH}_3 \\ \overline{ZH}_4 \\ \overline{ZH}_5 \\ \overline{ZH}_6 \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{C8}$$

Note that the number of individual equations represented by equation (C7) is NI where, as previously defined, NI is the number of variables in the vector $\{c\}$. The use of only elementary row operations reduces equation (C7) to,

$$\begin{bmatrix} [T1] & \vdots & -[T2] \\ [0] & \vdots & -[T3] \end{bmatrix} \begin{Bmatrix} \{DVH\} \\ \{\overline{ZH}\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{C9}$$

where $[T1]$ is square and nonsingular and the rows of both $[T2]$ and $[T3]$ are linearly independent. For the example, this is accomplished by interchanging rows 3 and 4 and then subtracting row 1 from row 4. These elementary row operations yield

$$\left[\begin{matrix} \begin{bmatrix} 1 & 0 & -\ell_1 \\ 0 & 1 & \ell_2 \\ 0 & 1 & -\ell_2 \end{bmatrix} & \vdots & -\begin{bmatrix} \frac{a}{\ell_3} & 0 & 1-\frac{a}{\ell_3} & 0 & 0 & 0 \\ 0 & \frac{a}{\ell_3} & 0 & 1-\frac{a}{\ell_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{\ell_4} & 0 & 1-\frac{b}{\ell_4} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \vdots & -\begin{bmatrix} -\frac{a}{\ell_3} & 0 & \frac{b}{\ell_4}+\frac{a}{\ell_3}-1 & 0 & 1-\frac{b}{\ell_4} & 0 \end{bmatrix} \end{matrix}\right] \begin{Bmatrix} \begin{Bmatrix} DVH_1 \\ DVH_2 \\ DVH_3 \end{Bmatrix} \\ \begin{Bmatrix} \overline{ZH}_1 \\ \overline{ZH}_2 \\ \overline{ZH}_3 \\ \overline{ZH}_4 \\ \overline{ZH}_5 \\ \overline{ZH}_6 \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \\ \{0\} \end{Bmatrix} \tag{C10}$$

which is of the form of equation (C9). Such a reduction can be carried out systematically by one of the various forms of Gaussian elimination using only elementary row operations. Formally any such Gaussian elimination procedure applied to equation (C7) can be expressed as the following two equations:

$$[GER][THR] = \begin{bmatrix} [T1] \\ [0] \end{bmatrix} \tag{C11a}$$

$$[GER][THA] = \begin{bmatrix} [T2] \\ [T3] \end{bmatrix} \tag{C11b}$$

where $[GER]$ is a square nonsingular matrix reflecting the elementary row operations. For the example,

$$[GER] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{C12}$$

In cases where coupling the rotor and airframe equations implies constraints on the airframe equations, a logical division of responsibilities would appear to be the following:

1. The rotor analyst provides $[THR]$ and $[GER]$

2. The airframe analyst provides $[THA]$

with both working to a commonly accepted definition of the connection points and the connection variables $\{c\}$.

Equation (C9) is now written as two equations:

$$\{DVH\} = [\overline{TH}]\{\overline{ZH}\} \tag{C13a}$$

$$[\overline{TC}]\{\overline{ZH}\} = \{0\} \tag{C13b}$$

where

$$[\overline{TH}] = [T1]^{-1}[T2] \tag{C14a}$$

$$[\overline{TC}] = [T3] \tag{C14b}$$

The numbers of individual equations represented by equations (C13a) and (C13b) are NVH and (NI - NVH), respectively, where, as previously defined, NVH is the number of rotor variables which explicitly appear in the relations connecting the rotor to the airframe. The rows of both $[\overline{TH}]$ and $[\overline{TC}]$ are linearly independent.

In the special case where the number of variables in $\{DVH\}$ equals the number of variables in $\{c\}$, the matrix $[THR]$ appearing in equation (C7) is square and nonsingular. In this case, no airframe constraints as represented by equation (C13b) are defined, and the equation defining the matrix $[\overline{TH}]$ which appears in equation (C13a) follows directly from the basic relations as

$$[\overline{TH}] = [THR]^{-1}[THA] \tag{C15}$$

As has been shown, equations (C13) are equivalent to the linearized basic relations given by equations (C2). Equation (C13a) expresses the rotor interface variables $\{DVH\}$ explicitly in terms of the airframe interface variables $\{\overline{ZH}\}$. Equation (C13b) involves only the airframe interface variables and can be recognized as constraints on the airframe finite-element model inherent in the formulation of the rotor model. Equations (C13) are adopted in this paper as the general form of the coupling equations.

Approximate Formulation of the Coupling Equations

As stated, equations (C13) follow exactly from the basic relations given by equations (C2). As a practical matter, engineers sometimes formulate coupling equations which are more or less inconsistent with the basic relations. Such formulations may ignore the basic relations altogether as when fictitious massless structural elements are incorporated to establish coupling, or the basic relations may be taken into account in an approximate way. Two such approximate procedures are noted here.

In the first procedure, certain of the constraints represented by equation (C13b) are ignored. This is permissible if it can be assessed that the airframe is constructed so that it behaves approximately as specified in some or all of the constraint equations. For the example of figure C2, the constraint equation (C13b) becomes the single equation,

$$\begin{bmatrix} -\dfrac{a}{\ell_3} & 0 & \dfrac{b}{\ell_4} + \dfrac{a}{\ell_3} - 1 & 0 & 1 - \dfrac{b}{\ell_4} & 0 \end{bmatrix} \begin{Bmatrix} \overline{ZH}_1 \\ \overline{ZH}_2 \\ \overline{ZH}_3 \\ \overline{ZH}_4 \\ \overline{ZH}_5 \\ \overline{ZH}_6 \end{Bmatrix} = 0 \tag{C16}$$

Assume for simplicity that $\ell_3 = \ell_4 = \ell$ and that $a = b = \ell/2$. The equation then reduces to

$$\overline{ZH}_1 = \overline{ZH}_5 \tag{C17}$$

That is, the horizontal displacements at the two finite-element nodes indicated in figure C2 are equal. Since these nodes are spanned by axial members, it might very well be judged that the pertinent stiffnesses are large, so that equation (C17) is approximately satisfied. This procedure may often be justified. However, it is judicious to derive equation (C13b) and evaluate separately the implications of retaining or discarding each one of the individual constraint equations.

In the second procedure equation (C7) is approximately satisfied on a least squares basis. Explicit equations of constraint corresponding to equation (C13b) do not emerge. The coupling relation corresponding to equation (C13a) is obtained by minimizing the quadratic form,

$$\begin{Bmatrix} \{DVH\} \\ \{\overline{ZH}\} \end{Bmatrix}^T \Big[[THR] \;\vdots\; -[THA] \Big]^T \Big[[THR] \;\vdots\; -[THA] \Big] \begin{Bmatrix} \{DVH\} \\ \{\overline{ZH}\} \end{Bmatrix}$$

with respect to the rotor variables {DVH} which yields

$$\{DVH\} = \Big[[THR]^T [THR] \Big]^{-1} [THR]^T [THA] \{\overline{ZH}\} \tag{C18}$$

Equation (C18) is in the form of equation (C13a) where

$$[\overline{TH}] = \Big[[THR]^T [THR] \Big]^{-1} [THR]^T [THA] \tag{C19}$$

This least squares procedure may be useful in preliminary design where the details of the mechanical connection between the rotor and the airframe are not yet defined.

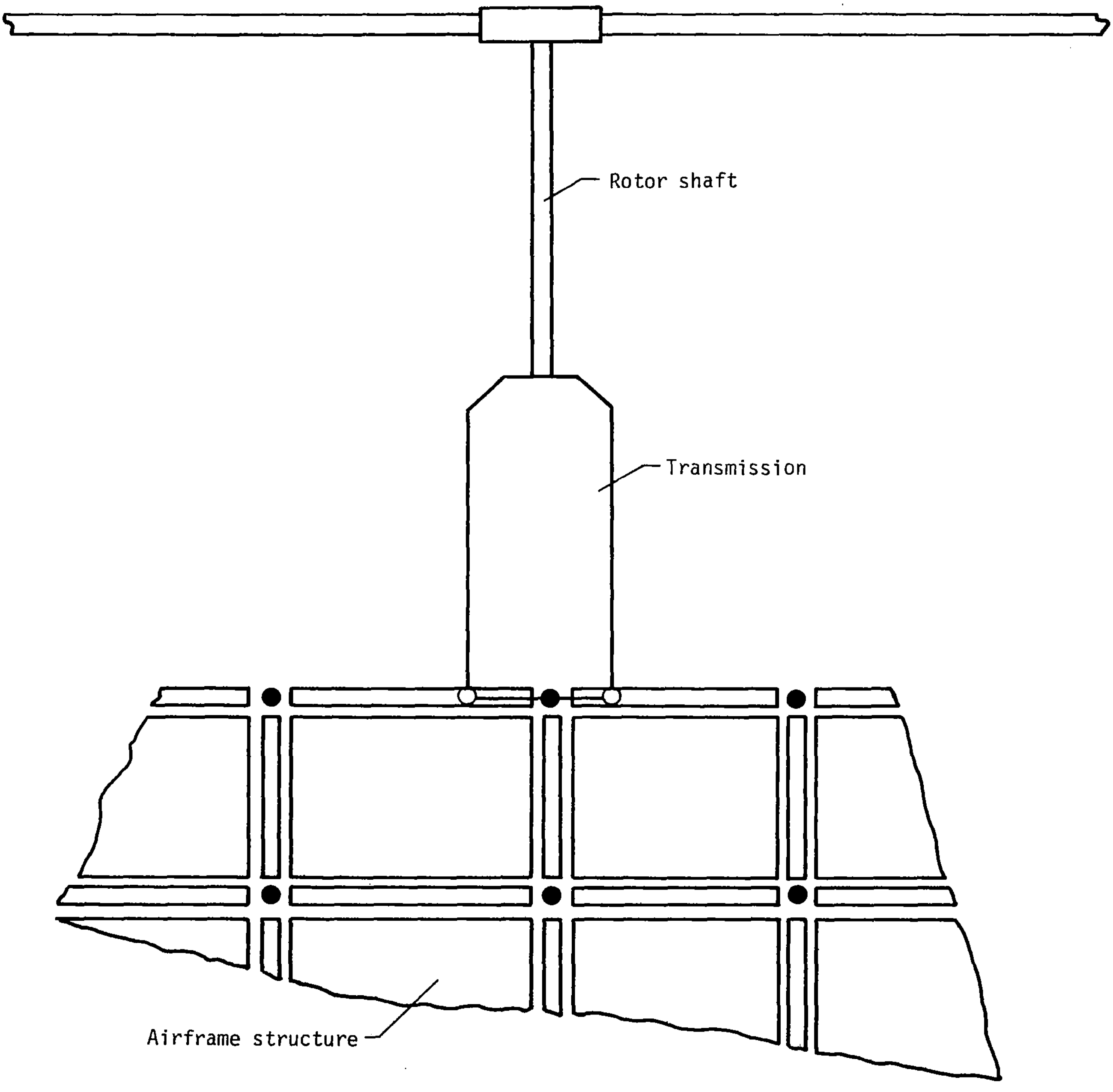


Figure C1.- Illustration of coupled rotor and airframe.

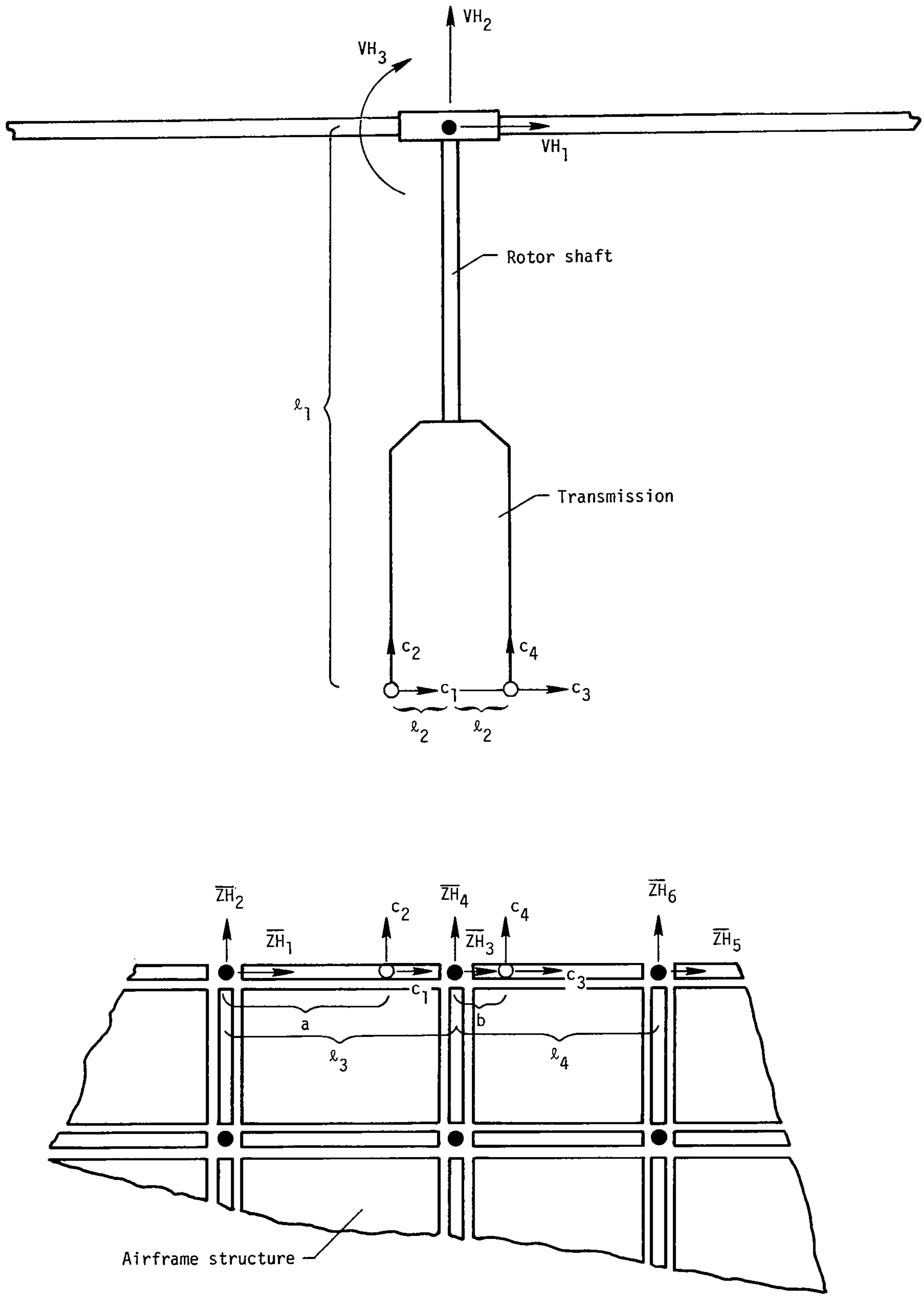


Figure C2.- Configuration used to illustrate specification of basic relations and coupling equations.

APPENDIX D

CALCULATION OF AIRFRAME TRIM SOLUTION USING AIRFRAME HARMONIC FORCED RESPONSES

The differential equations defining the nonzero portions $(\{ZSO\},\{FHO\})$ of the airframe trim displacement vector and the trim force vector are given by equations (29) and (30). Writing equation (21) in harmonic balance form while taking into account equations (29), (30), (26), and (28) leads to equations of the following form to be solved for the harmonic components $(N \neq 0)$ of the trim solution:

$$\left[\begin{array}{c|c} \begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11] & [MA12] \\ [MA21] & [MA22] \end{bmatrix} & - N\Omega \begin{bmatrix} [CA11] & [CA12] \\ [CA21] & [CA22] \end{bmatrix} \\ \hline + N\Omega \begin{bmatrix} [CA11] & [CA12] \\ [CA21] & [CA22] \end{bmatrix} & \begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11] & [MA12] \\ [MA21] & [MA22] \end{bmatrix} \end{array}\right] \left\{\begin{array}{c} \begin{Bmatrix} \{0\} \\ \{ZSONS\} \end{Bmatrix} \\ \hline \begin{Bmatrix} \{0\} \\ \{ZSONC\} \end{Bmatrix} \end{array}\right\} = \left\{\begin{array}{c} \begin{Bmatrix} \{FHONS\} - \{LHNS\} \\ - \{LSNS\} \end{Bmatrix} \\ \hline \begin{Bmatrix} \{FHONC\} - \{LHNC\} \\ - \{LSNC\} \end{Bmatrix} \end{array}\right\} \qquad \text{(D1)}$$

For the purposes of this appendix, the solution for the displacement vector is now written in the form,

$$\left\{\begin{array}{c} \{0\} \\ \{ZSONS\} \\ \hline \{0\} \\ \{ZSONC\} \end{array}\right\} = \left\{\begin{array}{c} \{ZHFNS\} \\ \{ZSFNS\} \\ \hline \{ZHFNC\} \\ \{ZSFNC\} \end{array}\right\} + \left[\begin{array}{cc} [UZHIN] & [UZHON] \\ [UZSIN] & [UZSON] \\ \hline -[UZHON] & [UZHIN] \\ -[UZSON] & [UZSIN] \end{array}\right] \begin{Bmatrix} \{WS\} \\ \{WC\} \end{Bmatrix} \qquad \text{(D2)}$$

where the vectors $\{WS\}$ and $\{WC\}$ are as yet undetermined and the first vector on the right side is obtained from solution of

$$\left[\begin{array}{c|c} \begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11] & [MA12] \\ [MA21] & [MA22] \end{bmatrix} & - N\Omega \begin{bmatrix} [CA11] & [CA12] \\ [CA21] & [CA22] \end{bmatrix} \\ \hline + N\Omega \begin{bmatrix} [CA11] & [CA12] \\ [CA21] & [CA22] \end{bmatrix} & \begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11] & [MA12] \\ [MA21] & [MA22] \end{bmatrix} \end{array}\right] \left\{\begin{array}{c} \{ZHFNS\} \\ \{ZSFNS\} \\ \hline \{ZHFNC\} \\ \{ZSFNC\} \end{array}\right\} = - \left\{\begin{array}{c} \{LHNS\} \\ \{LSNS\} \\ \hline \{LHNC\} \\ \{LSNC\} \end{array}\right\} \qquad \text{(D3)}$$

Note that the harmonic forced responses defined in conjunction with equation (60) and sketch C appear in equation (D2). The vectors {LHNS}, {LSNS}, {LHNC}, and {LSNC} appearing on the right of equation (D3) are associated with any oscillatory forces impinging on the airframe from external sources, and these vectors are assumed to be known. The vectors {WS} and {WC} are chosen to satisfy

$$\begin{Bmatrix} \{ZHFNS\} \\ \{ZHFNC\} \end{Bmatrix} + \begin{bmatrix} [UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN] \end{bmatrix} \begin{Bmatrix} \{WS\} \\ \{WC\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{D4}$$

which yields

$$\begin{Bmatrix} \{WS\} \\ \{WC\} \end{Bmatrix} = - \begin{bmatrix} [UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN] \end{bmatrix}^{-1} \begin{Bmatrix} \{ZHFNS\} \\ \{ZHFNC\} \end{Bmatrix} \tag{D5}$$

Substituting equation (D5) into equation (D2) then gives the nonzero portions of the airframe trim displacement vector as

$$\begin{Bmatrix} \{ZSONS\} \\ \{ZSONC\} \end{Bmatrix} = \begin{Bmatrix} \{ZSFNS\} \\ \{ZSFNC\} \end{Bmatrix} - \begin{bmatrix} [UZSIN] & [UZSON] \\ -[UZSON] & [UZSIN] \end{bmatrix} \begin{bmatrix} [UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN] \end{bmatrix}^{-1} \begin{Bmatrix} \{ZHFNS\} \\ \{ZHFNC\} \end{Bmatrix} \tag{D6}$$

The corresponding airframe trim force vector is given by

$$\begin{Bmatrix} \{FHONS\} \\ \{0\} \\ --- \\ \{FHONC\} \\ \{0\} \end{Bmatrix} = \begin{bmatrix} [FUIN] & [FUON] \\ [0] & [0] \\ ---- & ---- \\ -[FUON] & [FUIN] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{WS\} \\ \{WC\} \end{Bmatrix} \tag{D7}$$

which yields

$$\begin{Bmatrix} \{FHONS\} \\ \{FHONC\} \end{Bmatrix} = - \begin{bmatrix} [FUIN] & [FUON] \\ -[FUON] & [FUIN] \end{bmatrix} \begin{bmatrix} [UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN] \end{bmatrix}^{-1} \begin{Bmatrix} \{ZHFNS\} \\ \{ZHFNC\} \end{Bmatrix} \tag{D8}$$

APPENDIX D

As pointed out in the main text, it is not necessary to account for the static external forces acting on the airframe. However, if desired, static external forces may be designated to act on the airframe and can be accounted for by allowing zeroth harmonic components in the trim solution. The needed zeroth harmonic equations corresponding to equations (D1), (D3), (D6), and (D8) are

$$\begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} \begin{Bmatrix} \{0\} \\ \{ZSOOC\} \end{Bmatrix} = \begin{Bmatrix} \{FHOOC\} - \{LHOC\} \\ -\{LSOC\} \end{Bmatrix} \tag{D9a}$$

$$\begin{bmatrix} [KA11] & [KA12] \\ [KA21] & [KA22] \end{bmatrix} \begin{Bmatrix} \{ZHFOC\} \\ \{ZSFOC\} \end{Bmatrix} = -\begin{Bmatrix} \{LHOC\} \\ \{LSOC\} \end{Bmatrix} \tag{D9b}$$

$$\{ZSOOC\} = \{ZSFOC\} - [UZSIO][UZHIO]^{-1}\{ZHFOC\} \tag{D9c}$$

$$\{FHOOC\} = -[FUIO][UZHIO]^{-1}\{ZHFOC\} \tag{D9d}$$

The vector {FHOOC} appears in the harmonic balance equations (see eqs. (49) and (65)). However, note that if the procedures leading to equation (55) are used, the equations containing {FHOOC} are deleted; therefore {FHOOC} has no effect on calculation of the vibratory responses. The only effect of static forces on the airframe is to superimpose a static displacement on the vibratory responses (see eq. (33)).

APPENDIX E

RECOMPUTATION OF AIRFRAME HARMONIC FORCED RESPONSES AND AIRFRAME TRIM SOLUTION BY MATRIX PARTITIONING

In design studies, the need often arises to recompute the airframe contributions to the harmonic balance equations to study the effects of varying structural members and masses or varying the impedances of vibration control devices. This appendix formulates a computational procedure based on the use of simple matrix partitioning to recompute both the harmonic forced responses of the airframe and the airframe trim solution. When relatively few masses and structural members are candidates for variation considerable savings in computing effort can be realized from this procedure. Reference 23 is a recent report on the same subject.

For purposes of this appendix, the airframe forced response equations for the Nth harmonic ($N \neq 0$) given in equation (60) are written in the more general partitioned form,

$$\left[\begin{array}{ccc|ccc} [A11] & [A12] & [A13] & -[B11] & -[B12] & -[B13] \\ [A21] & [A22] & [A23] & -[B21] & -[B22] & -[B23] \\ [A31] & [A32] & [A33] & -[B31] & -[B32] & -[B33] \\ \hline [B11] & [B12] & [B13] & [A11] & [A12] & [A13] \\ [B21] & [B22] & [B23] & [A21] & [A22] & [A23] \\ [B31] & [B32] & [B33] & [A31] & [A32] & [A33] \end{array}\right] \left[\begin{array}{c} [X1] \\ [X2] \\ [X3] \\ \hline [Y1] \\ [Y2] \\ [Y3] \end{array}\right] = \left[\begin{array}{c} [F1] \\ [F2] \\ [F3] \\ \hline [G1] \\ [G2] \\ [G3] \end{array}\right] \qquad \text{(E1)}$$

The partitioning in equation (E1) reflects categorization of the airframe variables as follows:

1. The first and fourth block rows correspond respectively to the sine and cosine components of the interface variables {DZH}. The number of equations in each of these block rows is NZH.

2. The second and fifth block rows correspond respectively to the sine and cosine components of variables which define the deflections of the elements which are varied. The number of equations in each of these block rows is NZSV.

3. The third and sixth block rows correspond respectively to the sine and cosine components of the remaining variables. The number of equations in each of these block rows is NZSF.

It is recognized that to categorize variables in this manner and to cast equation (60) into the form of equation (E1), in general, requires renumbering of variables and reordering of rows and columns. For simplicity of discussion, no rearranging is assumed. When rearranging is necessary, the appropriate procedures should be apparent.

As an illustration of this categorization of airframe variables, consider the simplified finite-element model of a helicopter airframe given previously in figure B1. Assume that the interface with the rotor (rotor not shown) is the single node numbered 15 which corresponds to the rotor hub. Assume further that masses and structural members to be varied are distributed throughout the airframe as depicted in figure E1. Then the degrees of freedom associated with node point 15 would be in the first category, the degrees of freedom associated with node points 1, 11, 14, 18, 19, 20, and 30 to 38 would be in the second category, and the degrees of freedom associated with the remaining node points would be in the third category.

Equation (E1) can be reduced to the following two equations:

$$
\left[\begin{bmatrix} [A11] & [A12] & -[B11] & -[B12] \\ [A21] & [A22] & -[B21] & -[B22] \\ [B11] & [B12] & [A11] & [A12] \\ [B21] & [B22] & [A21] & [A22] \end{bmatrix} - \begin{bmatrix} [A13] & -[B13] \\ [A23] & -[B23] \\ [B13] & [A13] \\ [B23] & [A23] \end{bmatrix} \begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [A31] & [A32] & -[B31] & -[B32] \\ [B31] & [B32] & [A31] & [A32] \end{bmatrix}\right] \begin{bmatrix} [X1] \\ [X2] \\ [Y1] \\ [Y2] \end{bmatrix}
$$

$$
= \begin{bmatrix} [F1] \\ [F2] \\ [G1] \\ [G2] \end{bmatrix} - \begin{bmatrix} [A13] & -[B13] \\ [A23] & -[B23] \\ [B13] & [A13] \\ [B23] & [A23] \end{bmatrix} \begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [F3] \\ [G3] \end{bmatrix} \qquad \text{(E2a)}
$$

$$
\begin{bmatrix} [X3] \\ [Y3] \end{bmatrix} = - \begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [A31] & [A32] & -[B31] & -[B32] \\ [B31] & [B32] & [A31] & [A32] \end{bmatrix} \begin{bmatrix} [X1] \\ [X2] \\ [Y1] \\ [Y2] \end{bmatrix} + \begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [F3] \\ [G3] \end{bmatrix} \qquad \text{(E2b)}
$$

Equation (E2a) is solved for [X1], [X2], [Y1], and [Y2], and the result is substituted into equation (E2b) to obtain [X3] and [Y3]. The coefficient matrix in equation (E2a) is relatively small (order equal to twice the total number of degrees of freedom in categories 1 and 2), but contains the inverse of a large matrix (order equal to twice the number of degrees of freedom in category 3). However, the matrix which must be inverted is not affected by varied masses and structural members, and thus the inverse need be computed only once. In this connection, it should be emphasized that while this matrix is formally indicated as an inverse matrix in equations (E2), in practice matrix decomposition techniques would be employed to compute matrix products involving an inverse as in these equations. These decomposition techniques require much less computational effort than does formal inversion. The effect of varied masses and structural members is contained solely in the submatrices [A22] and [B22], which appear only in the left term of the coefficient matrix in equation (E2a).

To compute the harmonic forced responses of the airframe using equations (E2), the following identifications are made:

$$\begin{bmatrix} [X1] \\ [X2] \\ [X3] \\ -- \\ [Y1] \\ [Y2] \\ [Y3] \end{bmatrix} = \begin{bmatrix} [UZHIN] & [UZHON] \\ [UZSIN2] & [UZSON2] \\ [UZSIN3] & [UZSON3] \\ ----- & ----- \\ -[UZHON] & [UZHIN] \\ -[UZSON2] & [UZSIN2] \\ -[UZSON3] & [UZSIN3] \end{bmatrix} \tag{E3a}$$

$$\begin{bmatrix} [F1] \\ [F2] \\ [F3] \\ -- \\ [G1] \\ [G2] \\ [G3] \end{bmatrix} = \begin{bmatrix} [FUIN] & [FUON] \\ [0] & [0] \\ [0] & [0] \\ ---- & ---- \\ -[FUON] & [FUIN] \\ [0] & [0] \\ [0] & [0] \end{bmatrix} \tag{E3b}$$

where

$$[UZSIN] = \begin{bmatrix} [UZSIN2] \\ [UZSIN3] \end{bmatrix} \tag{E4a}$$

$$[UZSON] = \begin{bmatrix} [UZSON2] \\ [UZSON3] \end{bmatrix} \tag{E4b}$$

As a basis for computing the solution for the airframe trim displacement vector, equation (D2) must be formed. Equation (E3a) provides the ingredients of the coefficient matrix of the second term on the right of equation (D2). To compute the first column on the right of equation (D2), the following identifications are made in equations (E2):

$$\begin{bmatrix} [X1] \\ [X2] \\ [X3] \\ -- \\ [Y1] \\ [Y2] \\ [Y3] \end{bmatrix} = \begin{Bmatrix} \{ZHFNS\} \\ \{ZSFNS2\} \\ \{ZSFNS3\} \\ ---- \\ \{ZHFNC\} \\ \{ZSFNC2\} \\ \{ZSFNC3\} \end{Bmatrix} \tag{E5a}$$

$$\begin{bmatrix} [F1] \\ [F2] \\ [F3] \\ -- \\ [G1] \\ [G2] \\ [G3] \end{bmatrix} = -\begin{Bmatrix} \{LHNS\} \\ \{LSNS2\} \\ \{LSNS3\} \\ --- \\ \{LHNC\} \\ \{LSNC2\} \\ \{LSNC3\} \end{Bmatrix} \tag{E5b}$$

where

$$\{ZSFNS\} = \begin{Bmatrix} \{ZSFNS2\} \\ \{ZSFNS3\} \end{Bmatrix} \tag{E6a}$$

$$\{ZSFNC\} = \begin{Bmatrix} \{ZSFNC2\} \\ \{ZSFNC3\} \end{Bmatrix} \tag{E6b}$$

$$\{LSNS\} = \begin{Bmatrix} \{LSNS2\} \\ \{LSNS3\} \end{Bmatrix} \tag{E6c}$$

$$\{LSNC\} = \begin{Bmatrix} \{LSNC2\} \\ \{LSNC3\} \end{Bmatrix} \tag{E6d}$$

The previous two computations can be combined by identifying

$$\begin{bmatrix} [X1] \\ [X2] \\ [X3] \\ -- \\ [Y1] \\ [Y2] \\ [Y3] \end{bmatrix} = \begin{bmatrix} [UZHIN] & [UZHON] & \{ZHFNS\} \\ [UZSIN2] & [UZSON2] & \{ZSFNS2\} \\ [UZSIN3] & [UZSON3] & \{ZSFNS3\} \\ \hline -[UZHON] & [UZHIN] & \{ZHFNC\} \\ -[UZSON2] & [UZSIN2] & \{ZSFNC2\} \\ -[UZSON3] & [UZSIN3] & \{ZSFNC3\} \end{bmatrix} \tag{E7a}$$

$$\begin{bmatrix} [F1] \\ [F2] \\ [F3] \\ -- \\ [G1] \\ [G2] \\ [G3] \end{bmatrix} = \begin{bmatrix} [FUIN] & [FUON] & -\{LHNS\} \\ [0] & [0] & -\{LSNS2\} \\ [0] & [0] & -\{LSNS3\} \\ \hline -[FUON] & [FUIN] & -\{LHNC\} \\ [0] & [0] & -\{LSNC2\} \\ [0] & [0] & -\{LSNC3\} \end{bmatrix} \tag{E7b}$$

For the zeroth harmonic, the equations corresponding to equations (E1), (E2), and (E7) are

$$\begin{bmatrix} [A11] & [A12] & [A13] \\ [A21] & [A22] & [A23] \\ [A31] & [A32] & [A33] \end{bmatrix} \begin{bmatrix} [Y1] \\ [Y2] \\ [Y3] \end{bmatrix} = \begin{bmatrix} [G1] \\ [G2] \\ [G3] \end{bmatrix} \tag{E8}$$

$$\left[\begin{bmatrix}[A11] & [A12]\\ [A21] & [A22]\end{bmatrix} - \begin{bmatrix}[A13]\\ [A23]\end{bmatrix}[A33]^{-1}\begin{bmatrix}[A31] & [A32]\end{bmatrix}\right]\begin{bmatrix}[Y1]\\ [Y2]\end{bmatrix} = \begin{bmatrix}[G1]\\ [G2]\end{bmatrix} - \begin{bmatrix}[A13]\\ [A23]\end{bmatrix}[A33]^{-1}[G3] \tag{E9a}$$

$$[Y3] = -[A33]^{-1}\begin{bmatrix}[A31] & [A32]\end{bmatrix}\begin{bmatrix}[Y1]\\ [Y2]\end{bmatrix} + [A33]^{-1}[G3] \tag{E9b}$$

$$\begin{bmatrix}[Y1]\\ [Y2]\\ [Y3]\end{bmatrix} = \begin{bmatrix}[UZHIO] & \{ZHFOC\}\\ [UZSIO2] & \{ZSFOC2\}\\ [UZSIO3] & \{ZSFOC3\}\end{bmatrix} \tag{E10a}$$

$$\begin{bmatrix}[G1]\\ [G2]\\ [G3]\end{bmatrix} = \begin{bmatrix}[FUIO] & -\{LHOC\}\\ [0] & -\{LSOC2\}\\ [0] & -\{LSOC3\}\end{bmatrix} \tag{E10b}$$

where

$$[UZSIO] = \begin{bmatrix}[UZSIO2]\\ [UZSIO3]\end{bmatrix} \tag{E11a}$$

$$\{ZSFOC\} = \begin{Bmatrix}\{ZSFOC2\}\\ \{ZSFOC3\}\end{Bmatrix} \tag{E11b}$$

$$\{LSOC\} = \begin{Bmatrix}\{LSOC2\}\\ \{LSOC3\}\end{Bmatrix} \tag{E11c}$$

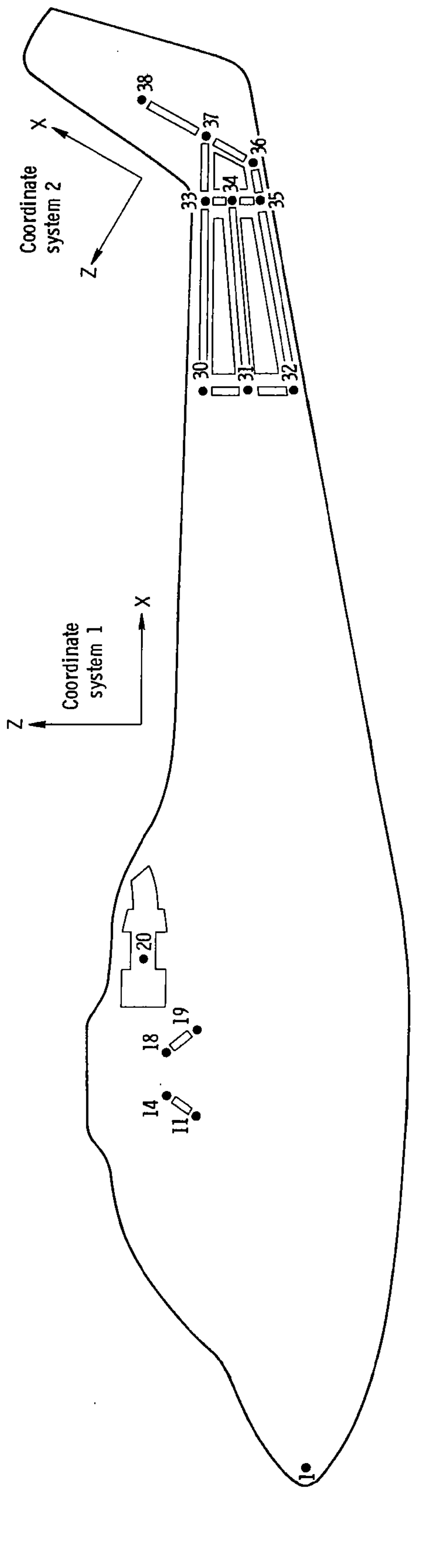


Figure E1.- Masses and structural members of airframe finite-element model which are to be varied.

APPENDIX F

AIRFRAME FORCES WHICH ARE PRESENTED IN TERMS OF IMPEDANCES

The need may arise to introduce forces acting on the airframe which are specified in terms of impedances. For example, the traditional method of representing frequency-independent structural damping leads to forces of this type. Also giving rise to such forces are impedance representations of vibration control devices, effects of drive system rotation, and effects of engine rotation. Forces represented by impedances are introduced directly into the harmonic balance equations given in equation (49). Let these additional forces be designated by the vector {FDZ}. The Fourier series expansion of this vector, in partitioned form, may be written as

$$\{FDZ\} = \begin{Bmatrix} \{FDZH\} \\ \{FDZS\} \end{Bmatrix} = \begin{Bmatrix} \{FDZH0C\} \\ \{FDZS0C\} \end{Bmatrix} + \begin{Bmatrix} \{FDZH1S\} \\ \{FDZS1S\} \end{Bmatrix} \sin \Omega t + \begin{Bmatrix} \{FDZH1C\} \\ \{FDZS1C\} \end{Bmatrix} \cos \Omega t$$

$$+ \begin{Bmatrix} \{FDZH2S\} \\ \{FDZS2S\} \end{Bmatrix} \sin 2\Omega t + \begin{Bmatrix} \{FDZH2C\} \\ \{FDZS2C\} \end{Bmatrix} \cos 2\Omega t + \ldots \tag{F1}$$

where

$$\begin{Bmatrix} \begin{Bmatrix} \{FDZH0C\} \\ \{FDZS0C\} \end{Bmatrix} \\ \begin{Bmatrix} \{FDZH1S\} \\ \{FDZS1S\} \end{Bmatrix} \\ \begin{Bmatrix} \{FDZH1C\} \\ \{FDZS1C\} \end{Bmatrix} \\ \begin{Bmatrix} \{FDZH2S\} \\ \{FDZS2S\} \end{Bmatrix} \\ \begin{Bmatrix} \{FDZH2C\} \\ \{FDZS2C\} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} \{FDZHNS\} \\ \{FDZSNS\} \end{Bmatrix} \\ \begin{Bmatrix} \{FDZHNC\} \\ \{FDZSNC\} \end{Bmatrix} \\ \vdots \end{Bmatrix} = \begin{bmatrix} [HI0] & & & & & & & & \\ & [HI1] & [HO1] & & & & & & \\ & -[HO1] & [HI1] & & & & & & \\ & & & [HI2] & [HO2] & & & & \\ & & & -[HO2] & [HI2] & & & & \\ & & & & & \ddots & & & \\ & & & & & & [HIN] & [HON] & \\ & & & & & & -[HON] & [HIN] & \\ & & & & & & & & \ddots \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \{DZH0C\} \\ \{DZS0C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH1S\} \\ \{DZS1S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH1C\} \\ \{DZS1C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH2S\} \\ \{DZS2S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH2C\} \\ \{DZS2C\} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} \{DZHNS\} \\ \{DZSNS\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZHNC\} \\ \{DZSNC\} \end{Bmatrix} \\ \vdots \end{Bmatrix} \tag{F2}$$

and where [HIO] is the constant (zeroth harmonic) impedance and [HIN] and [HON] are the in-phase and out-of-phase Nth harmonic ($N \neq 0$) impedances. Equation (F2) is in the nature of a collection of impedance relationships expressing force amplitudes as linear functions of displacement amplitudes. The typical (Nth harmonic) impedance equation in this collection is given by

$$\begin{Bmatrix} \begin{Bmatrix} \{FDZHNS\} \\ \{FDZSNS\} \end{Bmatrix} \\ \\ \begin{Bmatrix} \{FDZHNC\} \\ \{FDZSNC\} \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} HIN \end{bmatrix} & \begin{bmatrix} HON \end{bmatrix} \\ \\ -\begin{bmatrix} HON \end{bmatrix} & \begin{bmatrix} HIN \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \{DZHNS\} \\ \{DZSNS\} \end{Bmatrix} \\ \\ \begin{Bmatrix} \{DZHNC\} \\ \{DZSNC\} \end{Bmatrix} \end{Bmatrix} \qquad \text{(F3)}$$

The coefficient matrix in equation (F3) is known as an impedance matrix and is determined by the linearized characteristics (mechanical, electrical, or otherwise) of a particular device. It is a square matrix, constant in time and generally a function of the rotational frequency Ω. The block diagonal form of equation (F2), which uncouples the harmonics, is appropriate for devices represented by linear equations.

The effect of the resulting additional airframe forces on equation (49) is reflected in the following modifications to equations (50):

$$\begin{bmatrix} \blacksquare \end{bmatrix} = \begin{bmatrix} \begin{matrix} [KA11][KA12] \\ [KA21][KA22] \end{matrix} \end{bmatrix} + \begin{bmatrix} [HIO] \end{bmatrix} \qquad \text{(F4a)}$$

$$\begin{bmatrix} \begin{bmatrix} \blacksquare \end{bmatrix} & \begin{bmatrix} \blacksquare \end{bmatrix} \\ \begin{bmatrix} \blacksquare \end{bmatrix} & \begin{bmatrix} \blacksquare \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} [KA11][KA12] \\ [KA21][KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11][MA12] \\ [MA21][MA22] \end{bmatrix} \end{bmatrix} & \begin{bmatrix} -N\Omega \begin{bmatrix} [CA11][CA12] \\ [CA21][CA22] \end{bmatrix} \end{bmatrix} \\ \begin{bmatrix} +N\Omega \begin{bmatrix} [CA11][CA12] \\ [CA21][CA22] \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} [KA11][KA12] \\ [KA21][KA22] \end{bmatrix} - (N\Omega)^2 \begin{bmatrix} [MA11][MA12] \\ [MA21][MA22] \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} \begin{bmatrix} [HIN] \end{bmatrix} & \begin{bmatrix} [HON] \end{bmatrix} \\ \begin{bmatrix} -[HON] \end{bmatrix} & \begin{bmatrix} [HIN] \end{bmatrix} \end{bmatrix} \qquad \text{(F4b)}$$

APPENDIX F

For the special case where the additional forces represent frequency-independent structural damping, the matrix [HIN] is null and the matrix [HON] is independent of the rotational frequency Ω.

APPENDIX G

ALGORITHM FOR COMPUTING THE PARAMETERS AND INDICES IN EQUATION (52)

The formula given in equation (52) for the matrix $[DX_1X_2X_3X_4]$ contains the nine integer parameters - UOK, ULK, UHK, UOC, ULC, UHC, UOM, ULM, and UHM - and the four indices - P, Q, L, and H. The algorithm for computing these items is given in this appendix. Some of the conditions for computing the parameters are not mutually exclusive. In such cases, the last condition always takes precedence.

Computation of indices

$$L = |X_1 - X_3|$$

$$H = X_1 + X_3$$

$$P = C \qquad Q = S \qquad (\text{If } X_2 = X_4)$$

$$P = S \qquad Q = C \qquad (\text{If } X_2 \neq X_4)$$

Computation of parameters for stiffness terms

$$UHK = \begin{cases} -1 & (\text{If } X_2 = S \text{ and } X_4 = S) \\ 1 & (\text{If } X_2 = C \text{ and } X_4 = C) \\ 1 & (\text{If } X_2 \neq X_4) \\ 0 & (\text{If } X_1 = 0 \text{ and } X_3 = 0) \end{cases}$$

$$ULK = \begin{cases} 1 & (\text{If } X_2 = X_4) \\ -1 & (\text{If } X_2 = S \text{ and } X_4 = C \text{ and } X_3 > X_1) \\ 1 & (\text{If } X_2 = C \text{ and } X_4 = S \text{ and } X_3 > X_1) \\ 1 & (\text{If } X_2 = S \text{ and } X_4 = C \text{ and } X_3 < X_1) \\ -1 & (\text{If } X_2 = C \text{ and } X_4 = S \text{ and } X_3 < X_1) \\ 0 & (\text{If } X_1 = X_3) \\ 0 & (\text{If } X_1 = 0) \end{cases}$$

$$UOK = \begin{cases} 1 & (\text{If } X_1 = X_3 \text{ and } X_2 = X_4) \\ 0 & (\text{If } X_1 \neq X_3 \text{ or } X_2 \neq X_4) \end{cases}$$

Computation of parameters for damping terms

$$SN = \begin{cases} 1 & (\text{If } X_4 = S) \\ -1 & (\text{If } X_4 = C) \end{cases}$$

$$UHC = \begin{cases} 1 & (\text{If } X_2 \neq X_4) \\ SN & (\text{If } X_2 = X_4) \\ 0 & (\text{If } X_3 = 0) \end{cases}$$

$$ULC = \begin{cases} SN & (\text{If } X_2 \neq X_4) \\ -1 & (\text{If } X_2 = X_4 \text{ and } X_3 > X_1) \\ 1 & (\text{If } X_2 = X_4 \text{ and } X_3 < X_1) \\ 0 & (\text{If } X_1 = X_3) \\ 0 & (\text{If } X_1 = 0) \\ 0 & (\text{If } X_3 = 0) \end{cases}$$

$$UOC = \begin{cases} SN & (\text{If } X_1 = X_3 \text{ and } X_2 \neq X_4) \\ 0 & (\text{If } X_1 \neq X_3 \text{ or } X_2 = X_4) \\ 0 & (\text{If } X_1 = 0) \\ 0 & (\text{If } X_3 = 0) \end{cases}$$

Computation of parameters for mass terms

$$\left.\begin{aligned} UHM &= UHK \\ ULM &= ULK \\ UOM &= UOK \end{aligned}\right\} \quad (\text{If } X_3 \neq 0)$$

$$UHM = 0 \qquad ULM = 0 \qquad UOM = 0 \qquad (\text{If } X_3 = 0)$$

APPENDIX H

REPRESENTATION OF THE ROTOR SYSTEM BY IMPEDANCES

In this paper, the rotor contributions to the harmonic balance equations (see sketch B and eqs. (51) and (52)) have been derived analytically on the basis of linearization of the rotor equations and harmonic expansion of the unknown rotor displacements (see eqs. (1) and (49)). It has been suggested in the literature (see, for example, ref. 8) that it would be feasible to formulate the rotor contributions in the form of impedances computed directly by numerically integrating the fundamental, generally nonlinear, equations of motion of the rotor system. This appendix explains this approach.

Imagine the rotor system trimmed in a steady flight condition. Assume that the rotor is idealized and represented by fundamental nonlinear equations as discussed in appendix A. Then the rotor variables involved in the rotor-airframe interface are denoted by {VH}, and the corresponding generalized forces required to maintain the trim condition are denoted by {QH}. If the rotor displacements are assumed to vary only slightly from a trim solution as discussed in appendix A, then the variables {VH} may be represented by the small motion variables {DVH} which have the following Fourier series form:

$$\{DVH\} = \{DVH0C\} + \{DVH1S\} \sin \Omega t + \{DVH1C\} \cos \Omega t + \ldots \tag{H1}$$

The total forces represented by {QH} and the trim solution forces represented by {QH0} may be expressed in Fourier series forms as

$$\{QH\} = \{QH0C\} + \{QH1S\} \sin \Omega t + \{QH1C\} \cos \Omega t + \ldots \tag{H2}$$

$$\{QH0\} = \{QH00C\} + \{QH01S\} \sin \Omega t + \{QH01C\} \cos \Omega t + \ldots \tag{H3}$$

Under the assumption of small displacements, the coefficients {QH0C}, {QH1S}, and so forth in equation (H2) may be expressed by a linear equation of the form,

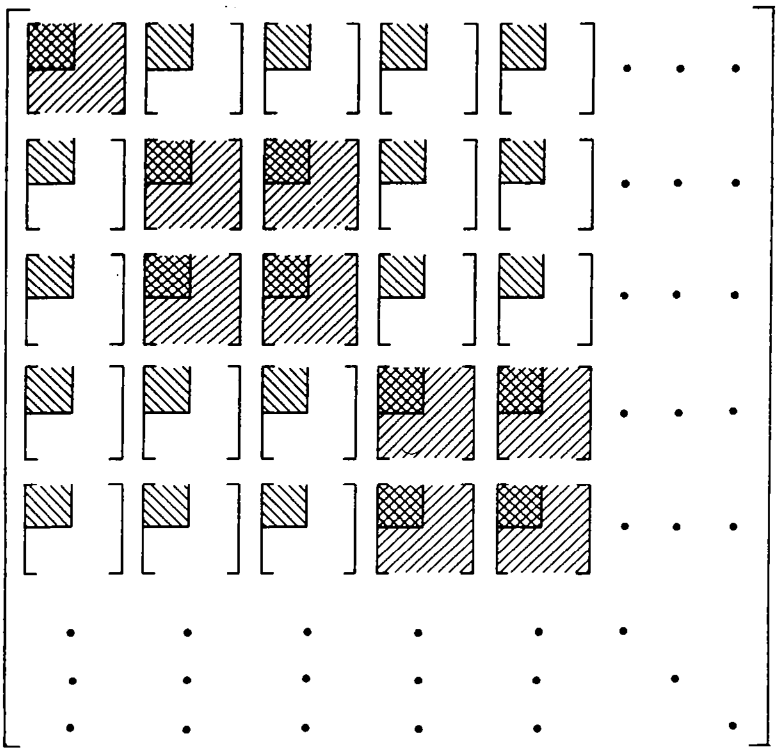
$$\begin{Bmatrix} \{QH0C\} \\ \{QH1S\} \\ \{QH1C\} \\ \{QH2S\} \\ \{QH2C\} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \{QH00C\} \\ \{QH01S\} \\ \{QH01C\} \\ \{QH02S\} \\ \{QH02C\} \\ \vdots \end{Bmatrix} + \begin{bmatrix} [KU0C0C][KU0C1S][KU0C1C][KU0C2S][KU0C2C] & \ldots \\ [KU1S0C][KU1S1S][KU1S1C][KU1S2S][KU1S2C] & \ldots \\ [KU1C0C][KU1C1S][KU1C1C][KU1C2S][KU1C2C] & \ldots \\ [KU2S0C][KU2S1S][KU2S1C][KU2S2S][KU2S2C] & \ldots \\ [KU2C0C][KU2C1S][KU2C1C][KU2C2S][KU2C2C] & \ldots \\ \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \{DVH0C\} \\ \{DVH1S\} \\ \{DVH1C\} \\ \{DVH2S\} \\ \{DVH2C\} \\ \vdots \end{Bmatrix} \tag{H4}$$

Equation (H4) expresses the force-displacement relationship for the rotor system in terms of the interface degrees of freedom. The vector of trim solution forces given by the first term on the right side of equation (H4) and the submatrices $[KUX_1X_2X_3X_4]$ which comprise the coefficient matrix in equation (H4) can be computed as described in reference 8 by numerically integrating the nonlinear equations of motion of the rotor system. It should be recognized that these quantities must in principle be derived anew whenever the flight condition changes because the validity of equation (H4) is predicated on linear behavior of the rotor system in the neighborhood of a trim solution.

Equation (H4) has been set forth to determine the generalized forces {QH} representing external forces required to maintain specified generalized displacements {DVH}. Alternatively, the formulation may be viewed as equations, in harmonic balance form, which determine the displacements {DVH} corresponding to specified generalized forces {QH}. The harmonic components of the specified generalized forces appear in the left column. In the special case for which the left column is nulled, as in the following equation, the equation represents a free-body rotor system with no forces at the hub:

$$\begin{bmatrix} [KU0C0C] & [KU0C1S] & [KU0C1C] & [KU0C2S] & [KU0C2C] & \cdots \\ [KU1S0C] & [KU1S1S] & [KU1S1C] & [KU1S2S] & [KU1S2C] & \cdots \\ [KU1C0C] & [KU1C1S] & [KU1C1C] & [KU1C2S] & [KU1C2C] & \cdots \\ [KU2S0C] & [KU2S1S] & [KU2S1C] & [KU2S2S] & [KU2S2C] & \cdots \\ [KU2C0C] & [KU2C1S] & [KU2C1C] & [KU2C2S] & [KU2C2C] & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \{DVH0C\} \\ \{DVH1S\} \\ \{DVH1C\} \\ \{DVH2S\} \\ \{DVH2C\} \\ \vdots \end{Bmatrix} = - \begin{Bmatrix} \{QH0C\} \\ \{QH0S\} \\ \{QH0C\} \\ \{QH0S\} \\ \{QH0C\} \\ \vdots \end{Bmatrix} \quad \text{(H5)}$$

Equation (H5) can be specified at the outset of computations in lieu of equations (2) to represent the rotor system. In this event, the harmonic balance equations (eq. (49)) have the form



$$\begin{Bmatrix} \{DZH0C\} \\ \{DZS0C\} \\ \{DZH1S\} \\ \{DZS1S\} \\ \{DZH1C\} \\ \{DZS1C\} \\ \{DZH2S\} \\ \{DZS2S\} \\ \{DZH2C\} \\ \{DZS2C\} \\ \vdots \end{Bmatrix} = - \begin{Bmatrix} [TH]^T\{QH00C\} + \{FH00C\} \\ \{0\} \\ [TH]^T\{QH01S\} + \{FH01S\} \\ \{0\} \\ [TH]^T\{QH01C\} + \{FH01C\} \\ \{0\} \\ [TH]^T\{QH02S\} + \{FH02S\} \\ \{0\} \\ [TH]^T\{QH02C\} + \{FH02C\} \\ \{0\} \\ \vdots \end{Bmatrix} \qquad (H6)$$

Equation (51) for computation of the rotor contributions to the coefficient matrix is replaced by

$$\boxed{X_1X_2X_3X_4} = [TH]^T[KUX_1X_2X_3X_4][TH] \qquad (H7)$$

Note that in this approach the variables {DVB} do not appear.

Further, when the rotor system is represented by impedances in this manner, the transformation equation (63) takes the form

$$\begin{Bmatrix} \begin{Bmatrix} \{DZH0C\} \\ \{DZS0C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH1S\} \\ \{DZS1S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH1C\} \\ \{DZS1C\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH2S\} \\ \{DZS2S\} \end{Bmatrix} \\ \begin{Bmatrix} \{DZH2C\} \\ \{DZS2C\} \end{Bmatrix} \\ \vdots \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} [UZHI0] \\ [UZSI0] \end{bmatrix} & & & \\ & \begin{bmatrix} [UZHI1] & [UZHO1] \\ [UZSI1] & [UZSO1] \\ -[UZHO1] & [UZHI1] \\ -[UZSO1] & [UZSI1] \end{bmatrix} & & \\ & & \begin{bmatrix} [UZHI2] & [UZHO2] \\ [UZSI2] & [UZSO2] \\ -[UZHO2] & [UZHI2] \\ -[UZSO2] & [UZSI2] \end{bmatrix} & \\ & & & \ddots \end{bmatrix} \begin{Bmatrix} \{QZH0C\} \\ \{QZH1S\} \\ \{QZH1C\} \\ \{QZH2S\} \\ \{QZH2C\} \\ \vdots \end{Bmatrix} \tag{H8}$$

and the structure of the reduced harmonic balance equation (65) becomes

$$\begin{bmatrix} [\;\;] & [\;\;] & [\;\;] & [\;\;] & [\;\;] & \cdots \\ [\;\;] & [\;\;] & [\;\;] & [\;\;] & [\;\;] & \cdots \\ [\;\;] & [\;\;] & [\;\;] & [\;\;] & [\;\;] & \cdots \\ [\;\;] & [\;\;] & [\;\;] & [\;\;] & [\;\;] & \cdots \\ [\;\;] & [\;\;] & [\;\;] & [\;\;] & [\;\;] & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \{QZH0C\} \\ \{QZH1S\} \\ \{QZH1C\} \\ \{QZH2S\} \\ \{QZH2C\} \\ \vdots \end{Bmatrix} = - \begin{Bmatrix} [TH]^T\{QHO0C\} + \{FHO0C\} \\ [TH]^T\{QHO1S\} + \{FHO1S\} \\ [TH]^T\{QHO1C\} + \{FHO1C\} \\ [TH]^T\{QHO2S\} + \{FHO2S\} \\ [TH]^T\{QHO2C\} + \{FHO2C\} \\ \vdots \end{Bmatrix} \tag{H9}$$

The submatrices of the coefficient matrix in equation (H9) are denoted by $[GX_1X_2X_3X_4]$ and are given by

$$[GX_1X_2X_3X_4] = [TH]^T[KUX_1X_2X_3X_4][TH][UZHIX_3]$$

$$- \alpha[TH]^T[KUX_1X_2X_3P][TH][UZHOX_3]$$

$$+ \beta[FUIX_3] + \gamma[FUOX_3] \tag{H10}$$

which is a specialization of equation (67d). As discussed in connection with equation (67d), the parameters α, β, and γ and the index P appearing in equation (H10) are computed by rules given in appendix J.

The total airframe displacements, and the corresponding velocities and accelerations, are recovered with equations (71). When impedances are used to represent the rotor as discussed in this appendix, the harmonic solutions for the airframe displacements {DZH} and {DZS} are obtained by substituting the solution of equation (H9) into equation (H8). These results, and the airframe trim solution, are then substituted into equations (71) for the final recovery of the displacements, velocities, and accelerations. If needed, the resultant forces acting on the airframe at the interface with the rotor are obtained by substituting the solution of equation (H9) into equation (72).

APPENDIX I

CALCULATION OF HARMONIC FORCED RESPONSES FOR UNIT IMPOSED HARMONIC DISPLACEMENTS

Finite-element computer codes for computing harmonic forced responses are generally predicated on imposed harmonic forces rather than on imposed harmonic displacements. However, if needed, harmonic forced responses associated with unit imposed displacements are readily obtained from the harmonic forced responses calculated for imposed forces. Using the harmonic forced responses resulting from imposed forces for the Nth harmonic ($N \neq 0$), one can define

$$\begin{bmatrix} [f] & [g] \\ -[g] & [f] \end{bmatrix} = \begin{bmatrix} [UZHIN] & [UZHON] \\ -[UZHON] & [UZHIN] \end{bmatrix}^{-1} \tag{I1}$$

Postmultiplying equation (60) by the left matrix in equation (I1) results in response and force matrices of the forms shown in figure I1. The harmonic forced responses corresponding to unit imposed displacements are given in figure I1(a) and the corresponding set of forces required to maintain the unit displacements are given in figure I1(b).

The corresponding matrices for the zeroth harmonic are obtained by defining

$$[f] = [UZHIO]^{-1} \tag{I2}$$

and postmultiplying equation (61) by $[f]$. This yields the responses and forces shown in figure I2.

$$\left[\begin{array}{cc} [I] & [0] \\ [UZSIN][f] - [UZSON][g] & [UZSIN][g] + [UZSON][f] \\ \hline [0] & [I] \\ -[UZSON][f] - [UZSIN][g] & -[UZSON][g] + [UZSIN][f] \end{array}\right]$$

(a) Responses.

$$\left[\begin{array}{cc} [FUIN][f] - [FUON][g] & [FUIN][g] + [FUON][f] \\ [0] & [0] \\ \hline -[FUON][f] - [FUIN][g] & -[FUON][g] + [FUIN][f] \\ [0] & [0] \end{array}\right]$$

(b) Forces.

Figure I1.- Harmonic forced responses corresponding to unit imposed displacements calculated from responses corresponding to imposed forces.

$$\left[\begin{array}{c} [I] \\ [UZSIO][f] \end{array}\right] \qquad \left[\begin{array}{c} [FUIO][f] \\ [0] \end{array}\right]$$

(a) Responses. (b) Forces.

Figure I2.- Zeroth harmonic responses corresponding to unit imposed displacements calculated from responses corresponding to imposed forces.

APPENDIX J

ALGORITHM FOR COMPUTING THE PARAMETERS AND INDEX IN EQUATIONS (67)

The formulas given in equations (67) for the submatrices into which matrix $[EX_1X_2X_3X_4]$ is partitioned contain the three integer parameters α, β, and γ and the index P. The algorithm for computing these items is as follows:

$$\alpha = 0 \qquad (\text{If } X_3 = 0)$$

$$\alpha = 1 \qquad P = C \qquad (\text{If } X_4 = S)$$

$$\alpha = -1 \qquad P = S \qquad (\text{If } X_3 \neq 0 \text{ and } X_4 = C)$$

$$\beta = 1 \qquad \gamma = 0 \qquad (\text{If } X_1 = X_3 \text{ and } X_2 = X_4)$$

$$\beta = 0 \qquad \gamma = 1 \qquad (\text{If } X_1 = X_3 \text{ and } X_2 = S \text{ and } X_4 = C)$$

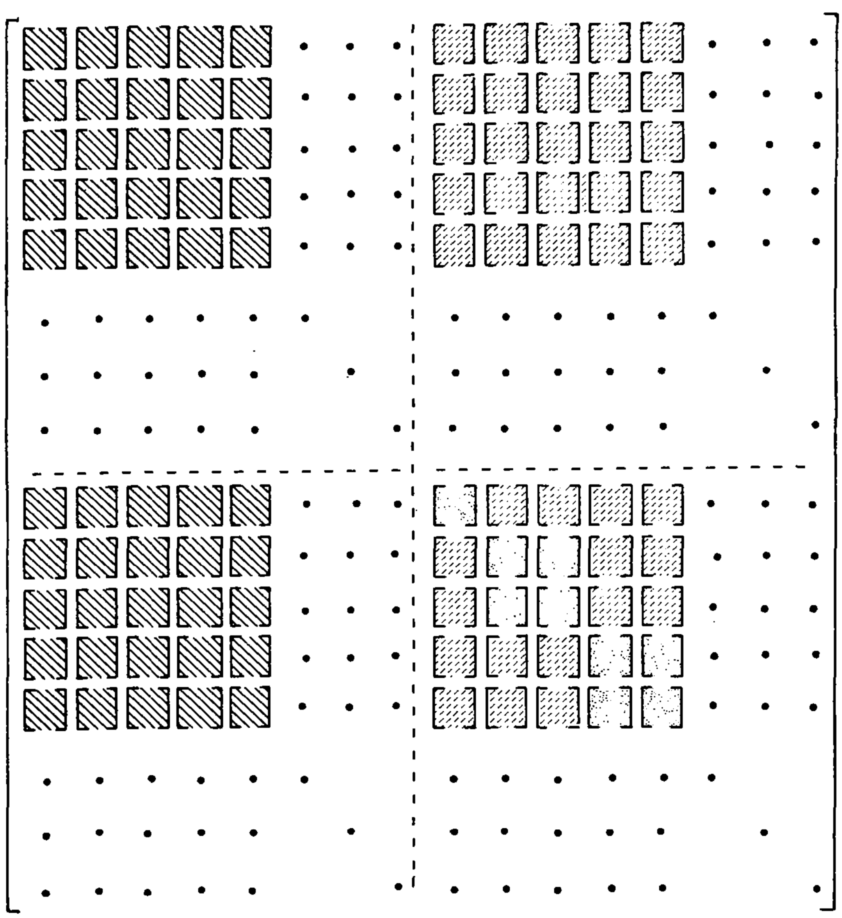
$$\beta = 0 \qquad \gamma = -1 \qquad (\text{If } X_1 = X_3 \text{ and } X_2 = C \text{ and } X_4 = S)$$

$$\beta = 0 \qquad \gamma = 0 \qquad (\text{If } X_1 \neq X_3)$$

APPENDIX K

SOLUTION OF REDUCED HARMONIC BALANCE EQUATIONS BY MATRIX PARTITIONING

As pointed out in the main text and in appendix E, the need often arises in design studies to recompute the airframe contributions to the harmonic balance equations in order to assess the effects of varying structural members and masses or varying impedances of vibration control devices. This means that the reduced harmonic balance equations given in equation (65) must be solved each time the airframe structure is changed. This appendix formulates a computational procedure based on the use of simple matrix partitioning to solve the reduced harmonic balance equations. In recomputation, considerable savings in computing effort can be realized from this procedure when the rotor representation involves many degrees of freedom relative to the number of degrees of freedom required to describe the interface with the airframe. To this end, equation (65) is first rearranged as follows:



$$\left\{\begin{array}{c} \{DVB0C\} \\ \{DVB1S\} \\ \{DVB1C\} \\ \{DVB2S\} \\ \{DVB2C\} \\ \cdot \\ \cdot \\ \cdot \\ \hdashline \{QZH0C\} \\ \{QZH1S\} \\ \{QZH1C\} \\ \{QZH2S\} \\ \{QZH2C\} \\ \cdot \\ \cdot \\ \cdot \end{array}\right\} = -\left\{\begin{array}{c} \{QB00C\} \\ \{QB01S\} \\ \{QB01C\} \\ \{QB02S\} \\ \{QB02C\} \\ \cdot \\ \cdot \\ \cdot \\ \hdashline [TH]^T\{QH00C\} + \{FH00C\} \\ [TH]^T\{QH01S\} + \{FH01S\} \\ [TH]^T\{QH01C\} + \{FH01C\} \\ [TH]^T\{QH02S\} + \{FH02S\} \\ [TH]^T\{QH02C\} + \{FH02C\} \\ \cdot \\ \cdot \\ \cdot \end{array}\right\} \qquad \text{(K1)}$$

APPENDIX K

Let equation (K1) be represented by the equation

$$\left[\begin{array}{c:c} [A] & [B] \\ \hdashline [C] & [D] \end{array}\right] \left\{\begin{array}{c} \{X1\} \\ -- \\ \{X2\} \end{array}\right\} = \left\{\begin{array}{c} \{P1\} \\ -- \\ \{P2\} \end{array}\right\} \tag{K2}$$

Equation (K2) can be reduced to the following two equations:

$$\Big[[D] - [C][A]^{-1}[B]\Big]\{X2\} = \{P2\} - [C][A]^{-1}\{P1\} \tag{K3a}$$

$$\{X1\} = [A]^{-1}\Big\{\{P1\} - [B]\{X2\}\Big\} \tag{K3b}$$

Equation (K3a) is solved for $\{X2\}$ and the result substituted into equation (K3b) to obtain $\{X1\}$. The coefficient matrix in equation (K3a) is relatively small (maximum order equal to (NT)(NZH) by (NT)(NZH)) but contains the inverse of a relatively large matrix (maximum order equal to (NT)(NVB) by (NT)(NVB)). However, the matrix which must be inverted does not contain the effects of airframe changes and thus the inverse need be computed only once. As noted in appendix E, matrix decomposition techniques would be used rather than formal inversion. The effects of airframe changes are contained solely in the submatrices $[B]$ and $[D]$. The computations needed to evaluate the matrix products involving $[B]$ can be reduced in situations requiring recomputation by writing $[B]$ in the form

$$[B] = [B1][E1] + [B2][E2] \tag{K4}$$

where

$$[B1][E1] = \begin{bmatrix} [D0C0C12][D0C1S12][D0C1C12][D0C2S12][D0C2C12] \; . \; . \; . \\ [D1S0C12][D1S1S12][D1S1C12][D1S2S12][D1S2C12] \; . \; . \; . \\ [D1C0C12][D1C1S12][D1C1C12][D1C2S12][D1C2C12] \; . \; . \; . \\ [D2S0C12][D2S1S12][D2S1C12][D2S2S12][D2S2C12] \; . \; . \; . \\ [D2C0C12][D2C1S12][D2C1C12][D2C2S12][D2C2C12] \; . \; . \; . \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \end{bmatrix} \begin{bmatrix} [TH][UZHI0] & & & & & \\ & [TH][UZHI1] & & & & \\ & & [TH][UZHI1] & & & \\ & & & [TH][UZHI2] & & \\ & & & & [TH][UZHI2] & \\ & & & & & \ddots \end{bmatrix} \tag{K5a}$$

and

$$
[\mathrm{B2}][\mathrm{E2}] = \begin{bmatrix}
[\mathrm{D0C0C12}][\mathrm{D0C1C12}][\mathrm{D0C1S12}][\mathrm{D0C2C12}][\mathrm{D0C2S12}] \ . \ . \ . \\
[\mathrm{D1S0C12}][\mathrm{D1S1C12}][\mathrm{D1S1S12}][\mathrm{D1S2C12}][\mathrm{D1S2S12}] \ . \ . \ . \\
[\mathrm{D1C0C12}][\mathrm{D1C1C12}][\mathrm{D1C1S12}][\mathrm{D1C2C12}][\mathrm{D1C2S12}] \ . \ . \ . \\
[\mathrm{D2S0C12}][\mathrm{D2S1C12}][\mathrm{D2S1S12}][\mathrm{D2S2C12}][\mathrm{D2S2S12}] \ . \ . \ . \\
[\mathrm{D2C0C12}][\mathrm{D2C1C12}][\mathrm{D2C1S12}][\mathrm{D2C2C12}][\mathrm{D2C2S12}] \ . \ . \ . \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \ddots
\end{bmatrix}
\begin{bmatrix}
[\mathrm{TH}][0] & & & & & \\
& -[\mathrm{TH}][\mathrm{UZHO1}] & & & & \\
& & [\mathrm{TH}][\mathrm{UZHO1}] & & & \\
& & & -[\mathrm{TH}][\mathrm{UZHO2}] & & \\
& & & & [\mathrm{TH}][\mathrm{UZHO2}] & \\
& & & & & \ddots
\end{bmatrix}
$$

(K5b)

Substituting equation (K4) into equations (K3) results in

$$
\Big[[\mathrm{D}] - \big[[\mathrm{C}][\mathrm{A}]^{-1}[\mathrm{B1}][\mathrm{E1}] + [\mathrm{C}][\mathrm{A}]^{-1}[\mathrm{B2}][\mathrm{E2}]\big]\Big]\{\mathrm{X2}\} = \{\mathrm{P2}\} - [\mathrm{C}][\mathrm{A}]^{-1}\{\mathrm{P1}\} \qquad \text{(K6a)}
$$

$$
\{\mathrm{X1}\} = [\mathrm{A}]^{-1}\{\mathrm{P1}\} - \big[[\mathrm{A}]^{-1}[\mathrm{B1}][\mathrm{E1}] + [\mathrm{A}]^{-1}[\mathrm{B2}][\mathrm{E2}]\big]\{\mathrm{X2}\} \qquad \text{(K6b)}
$$

In this form of equations (K3), the effects of structural changes in the airframe are contained solely in [D], [E1], and [E2].

APPENDIX L

COMMENTS ON ORDER AND BANDEDNESS OF REDUCED HARMONIC BALANCE EQUATIONS

Original Equations

The indefinite form of the original reduced harmonic balance equations is given in equation (65). As has been previously discussed, to effect a harmonic balance solution in practice, the series in equation (48) must be truncated. The assumption of a definite (i.e., finite) form for the series given by equation (48) results in a definite form of equation (65); that is, the coefficient matrix becomes a matrix of finite order. Truncations in the series represented by equations (2a) to (2c) do not affect the order of the coefficient matrix of equation (65), although truncations in these series can affect the accuracy of the computations of the elements of the coefficient matrix. On the other hand, truncations in the series given in equations (2a) to (2c) do affect the bandwidth of the coefficient matrix, a characteristic which can be an important consideration relating to the computational effort required to solve equation (65). Explicit formulas can be given for maximum values of order and bandwidth of the coefficient matrix. Let NT denote the number of terms retained in the series of equation (48). (For the example given by eq. (57), NT = 9.) Then the maximum order of the coefficient matrix in equation (65) is NT(NVB + NZH) by NT(NVB + NZH). Let MH denote the maximum harmonic number appearing in the series given by equations (2a) to (2c). Then the resulting coefficient matrix has the banded form indicated in figure L1 with maximum bandwidth given by $2[1 + 2(MH)](NVB + NZH)$.

Equations Rearranged for Solution by Matrix Partitioning

Consistent with previous discussion related to maximum order and bandwidth of definite forms of equation (65), the maximum order of the coefficient matrix in equation (K2) is NT(NVB + NZH) by NT(NVB + NZH). However, because of the row and column interchanges necessary to transform equation (65) to the form given in equation (K1), the coefficient matrix of equation (K2) has the banded form depicted in figure L2. The maximum bandwidths of each of the four submatrices of the coefficient matrix in equation (K2) are indicated.

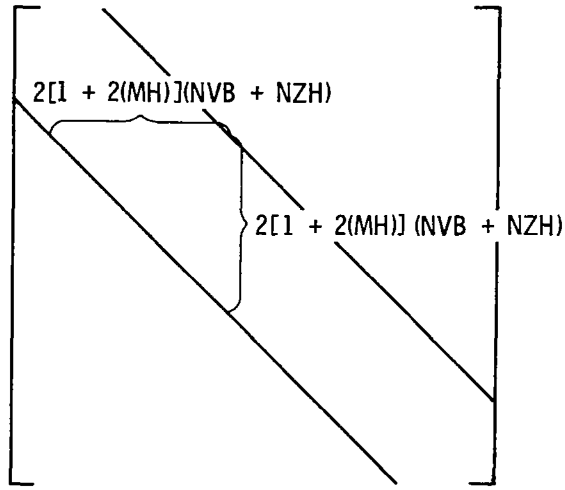


Figure L1.- Banded form of reduced harmonic balance equations.

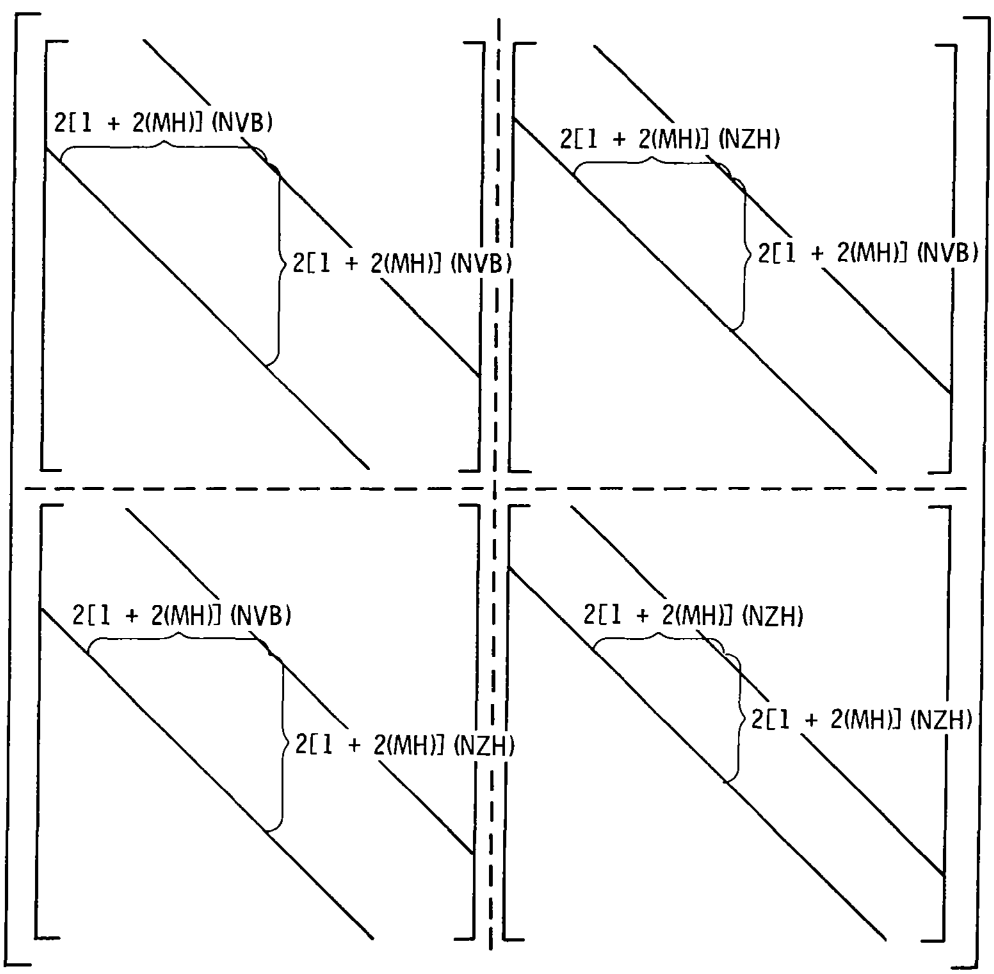


Figure L2.- Banded form of reduced harmonic balance equations rearranged for solution by matrix partitioning.

APPENDIX M

ADDITIONAL COMPUTING SEQUENCES

Two additional computing sequences for calculating airframe vibrations are described here to supplement the basic computing sequence given in the main text. The first sequence includes steps for reanalysis by the matrix partitioning method of appendix E. The second is a sequence which is appropriate when the rotor is represented by impedances as described in appendix H. For each sequence, a block diagram is presented in which the tasks to be performed are arranged in three columns to distinguish work done by rotor analysts, work done by airframe analysts, and joint work.

Computing Sequence With Reanalysis

Figure M1 shows a block diagram indicating the sequence of tasks for calculating airframe vibrations including reanalysis when members are varied. Supplementary notes follow, keyed by number to the individual blocks:

1. Specify the linearized rotor model: The specification is made by providing the coefficient matrices in the Fourier series expansions of [MR], [CR], [KR], and {QRO} represented by equations (2).

2. Identify rotor-airframe connection points: The rotor and airframe analysts agree on an arrangement of discrete points at which to designate connections. The displacements and rotations at these points needed to express connectivity are identified. This defines the vector {c} discussed in conjunction with equations (C2).

3. Partition the rotor equations to isolate interface variables: Among the rotor variables {DVR} appearing in the linearized rotor equations, a subset {DVH} is identified as the variables which explicitly appear in expressions characterizing connections of the rotor model to the airframe model (eq. (C2a)). The matrices specifying the linear rotor model are partitioned accordingly. See discussion of equation (3) and following.

4. Form rotor contributions to the coupling equations: The matrices [THR] and [GER] are formed. See equations (C2a) and (C11a) and related discussion.

5. Identify harmonics in assumed solution: The steady-state solution given by equation (48) is specialized to a definite form agreed on by the rotor and airframe analysts. See equation (57) for an example of a definite form.

6. Specify loads impinging directly on airframe: The distribution of external oscillating loads impinging directly on the airframe must be described to enable the airframe analyst to define the vector $\{\overline{L}\}$ appearing in equation (5). Static components may be included, but it is not necessary to include them. Loads transmitted from the rotor to the airframe through the rotor-airframe connections are not included in this specification.

7. Form original finite-element model: A finite-element code is employed to generate the matrices $[\overline{MA}]$, $[\overline{CA}]$, and $[\overline{KA}]$, appearing in equation (5). Also generated are the load coefficients $\{\overline{L0C}\}$, $\{\overline{L1S}\}$, $\{\overline{L1C}\}$, ..., appearing in equation (6).

8. Partition the airframe equations to isolate the interface variables: Among the airframe variables $\{\bar{Z}\}$, a subset $\{\overline{ZH}\}$ is identified to be the variables which explicitly appear in the expressions characterizing connections of the airframe model to the rotor model (eq. (C2b)). The matrices characterizing the airframe model are partitioned accordingly. See discussion of equation (7) and following.

9. Form airframe contribution to the coupling equations: The matrix [THA] is formed. See equation (C2b) and related discussion.

10. Form the coupling equations: Compute [T1] from equation (C11a). Compute [T2] and [T3] from equation (C11b). Compute $[\overline{TH}]$ from equation (C14a). Compute $[\overline{TC}]$ from equation (C14b). This establishes the coupling equations (9a) and (9b).

11. Modify the coupling equations: Identify the submatrices $[\overline{TCD}]$ and $[\overline{TCI}]$ associated with the matrix $[\overline{TC}]$. See equation (10) and related discussion. Identify the submatrices $[\overline{THD}]$ and $[\overline{THI}]$ associated with the matrix $[\overline{TH}]$. See equation (12) and related discussion. Compute the matrix [TH] from equation (14). The coupling equations are thus reduced to a single equation, equation (13).

12. Modify the airframe equations: The matrix $[\overline{TI}]$ in equation (15) is formed from equation (11), as discussed in the text. Operating with $[\overline{TI}]$ on the partitioned form of the original finite-element equations, as shown by equation (16), establishes the modified form of the airframe equations (eq. (21)). Note that this task has been to some extent routinized in the NASTRAN code through the feature called multipoint constraints.

13. Partition airframe equations to isolate members to be varied: Identify the airframe members which are candidates for variation. Partition equation (60) to isolate these members in accordance with the discussion associated with equation (E1).

14. Carry out matrix operations which are unaffected by member changes: For the Nth harmonic ($N \neq 0$) use equation (E7b) to compute the right side of equation (E2a) and compute the term

$$\begin{bmatrix} [A13] & -[B13] \\ [A23] & -[B23] \\ [B13] & [A13] \\ [B23] & [A23] \end{bmatrix} \begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [A31] & [A32] & -[B31] & -[B32] \\ [B31] & [B32] & [A31] & [A32] \end{bmatrix}$$

which appears in the coefficient matrix of equation (E2a). Note that intermediate steps in this process yield the terms

$$\begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [A31] & [A32] & -[B31] & -[B32] \\ [B31] & [B32] & [A31] & [A32] \end{bmatrix}$$

and

$$\begin{bmatrix} [A33] & -[B33] \\ [B33] & [A33] \end{bmatrix}^{-1} \begin{bmatrix} [F3] \\ [G3] \end{bmatrix}$$

which appear in equation (E2b). For the zeroth harmonic, use equation (E10b) to compute the right side of equation (E9a) and compute the term

$$\begin{bmatrix} [A13] \\ [A23] \end{bmatrix} [A33]^{-1} \begin{bmatrix} [A31] & [A32] \end{bmatrix}$$

which appears in the coefficient matrix of equation (E9a). Note that intermediate steps in this process yield the terms

$$[A33]^{-1} \begin{bmatrix} [A31] & [A32] \end{bmatrix}$$

and

$$[A33]^{-1} [G3]$$

which appear in equation (E9b).

15. Solve for [X1], [X2], [X3], [Y1], [Y2], [Y3]: For the nonzero harmonics, equation (E2a) is solved for [X1], [X2], [Y1], and [Y2] and the result substituted into equation (E2b) to obtain [X3] and [Y3]. For the zeroth harmonic, equation (E9a) is solved for [Y1] and [Y2] and the result substituted into equation (E9b) to obtain [Y3].

16. Assemble harmonic forced responses: For the nonzero harmonics, extract the matrix

$$\begin{bmatrix} [UZHIN] & [UZHON] \\ [UZSIN2] & [UZSON2] \\ [UZSIN3] & [UZSON3] \\ \hline -[UZHON] & [UZHIN] \\ -[UZSON2] & [UZSIN2] \\ -[UZSON3] & [UZSIN3] \end{bmatrix}$$

from the matrix

$$\begin{bmatrix} [X1] \\ [X2] \\ [X3] \\ --- \\ [Y1] \\ [Y2] \\ [Y3] \end{bmatrix}$$

as indicated by equation (E7a) (the latter matrix was computed in task 15). Appropriate rearrangement of the rows of the extracted matrix (reversing the reordering of rows and columns employed in arriving at equation (E1)) directly yields the matrix of harmonic forced responses shown in sketch C. For the zeroth harmonic, extract the matrix

$$\begin{bmatrix} [UZHI0] \\ [UZSI02] \\ [UZSI03] \end{bmatrix}$$

from the matrix

$$\begin{bmatrix} [Y1] \\ [Y2] \\ [Y3] \end{bmatrix}$$

as indicated in equation (E10a) (the latter matrix was computed in task 15). Appropriate rearrangement of the rows of the extracted matrix yields the matrix of responses shown in sketch D.

17. Complete airframe trim solution: The nonzero portions of the airframe trim displacement vector and the trim force vector are given by equations (D6) and (D8). The displacements

$$\begin{Bmatrix} \{ZHFNS\} \\ \{ZHFNC\} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \{ZSFNS\} \\ \{ZSFNC\} \end{Bmatrix}$$

which appear in these equations are computed by extracting the third column of equation (E7a) (obtained in task 15) and rearranging rows. This result and the results of task 16 are used to compute the trim displacements and forces given by equations (D6) and (D8). The corresponding results for the zeroth harmonic are computed from equations (D9c) and (D9d).

18. Form reduced harmonic balance equations: The matrices $[DX_1X_2X_3X_4]$ are computed using equation (52) and the algorithm of appendix G. Equations (67) and the algorithm of appendix J are used to compute the coefficient matrix of equation (65). In this process, the only rows and columns generated are those corresponding to retained harmonics (see task 5).

19. Solve reduced harmonic balance equations: Equation (65) is solved.

20. Recover airframe responses: Equations (63) and (71) are used. If interface forces are desired, use equations (72) and (73).

21. Vary members: Changes are made to any of the members which were previously identified as candidates for variation (see task 13). Go back to task 15.

Computing Sequence When Rotor Is Represented by Impedances

Figure M2 shows a block diagram indicating the sequence of tasks for calculating airframe vibrations when the rotor system is represented by impedances, as discussed in appendix H. Supplementary notes follow, keyed by number to the individual blocks:

1. Identify harmonics in assumed solution: The steady-state solution given by equation (48) is specialized to a definite form agreed on by the rotor and airframe analysts. See equation (57) for an example of a definite form.

2. Identify rotor-airframe connection points: The rotor and airframe analysts agree on an arrangement of discrete points at which to designate connections. The displacements and rotations at these points needed to express connectivity are identified. This defines the vector $\{c\}$ discussed in conjunction with equations (C2).

3. Identify interface variables in nonlinear rotor equations: Among the rotor variables $\{VR\}$ appearing in the nonlinear rotor equations, a subset $\{VH\}$ is identified as the variables which explicitly appear in the expressions characterizing connections of the rotor model to the airframe model (eq. (C1a)).

4. Specify the linearized rotor model: The specification is made by providing the impedance matrix (the coefficient matrix) in equation (H5) and the vector of forces appearing on the right side of that equation.

5. Form rotor contributions to the coupling equations: The matrices $[THR]$ and $[GER]$ are formed. See equations (C2a) and (C11a) and related discussion.

6. Specify loads impinging directly on airframe: The distribution of external oscillating loads impinging directly on the airframe must be described to enable the airframe analyst to define the vector $\{\bar{L}\}$ appearing in equation (5). Static components may be included, but it is not necessary to include them. Loads transmitted from the rotor to the airframe through the rotor-airframe connections are not included in this specification.

7. Form original finite-element model: A finite-element code is employed to generate the matrices $[\overline{MA}]$, $[\overline{CA}]$, and $[\overline{KA}]$, appearing in equation (5). Also generated are the load coefficients $\{\overline{L0C}\}$, $\{\overline{L1S}\}$, $\{\overline{L1C}\}$, ..., appearing in equation (6).

8. Partition the airframe equations to isolate the interface variables: Among the airframe variables $\{\overline{Z}\}$, a subset $\{\overline{ZH}\}$ is identified to be the variables which explicitly appear in the expressions characterizing connections of the airframe model to the rotor model (eq. (C2b)). The matrices characterizing the airframe model are partitioned accordingly. See discussion of equation (7) and following.

9. Form airframe contribution to the coupling equations: The matrix [THA] is formed. See equation (C2b) and related discussion.

10. Form the coupling equations: Compute [T1] from equation (C11a). Compute [T2] and [T3] from equation (C11b). Compute $[\overline{TH}]$ from equation (C14a). Compute $[\overline{TC}]$ from equation (C14b). This establishes the coupling equations (9a) and (9b).

11. Modify the coupling equations: Identify the submatrices $[\overline{TCD}]$ and $[\overline{TCI}]$ associated with the matrix $[\overline{TC}]$. See equation (10) and related discussion. Identify the submatrices $[\overline{THD}]$ and $[\overline{THI}]$ associated with the matrix $[\overline{TH}]$. See equation (12) and related discussion. Compute the matrix [TH] from equation (14). The coupling equations are thus reduced to a single equation, equation (13).

12. Modify the airframe equations: The matrix $[\overline{TI}]$ in equation (15) is formed from equation (11), as discussed in the text. Operating with $[\overline{TI}]$ on the partitioned form of the original finite-element equations, as shown by equation (16), establishes the modified form of the airframe equations (eq. (21)). Note that this task has been to some extent routinized in the NASTRAN code through the feature called multipoint constraints.

13. Compute harmonic forced responses: Equation (60) is solved for each nonzero harmonic of interest (see task 1) and the results are assembled as shown in sketch C. For any zeroth harmonic responses, equation (61) is solved and the results assembled as shown in sketch D.

14. Compute airframe trim solution: Equation (D3) is solved for {ZHFNS}, {ZSFNS}, {ZHFNC}, and {ZSFNC}. The vectors representing trim displacements and forces are computed from equations (D6) and (D8). For the zeroth harmonic, equations (D9b) to (D9d) are used.

15. Form reduced harmonic balance equations: Equation (H10) and the algorithm of appendix J are used to compute the coefficient matrix in equation (H9). In this process, the only rows and columns generated are those corresponding to retained harmonics (see task 1).

16. Solve reduced harmonic balance equations: Equation (H9) is solved.

17. Recover airframe responses: Equations (H8) and (71) are used. If interface forces are desired, use equations (72) and (73).

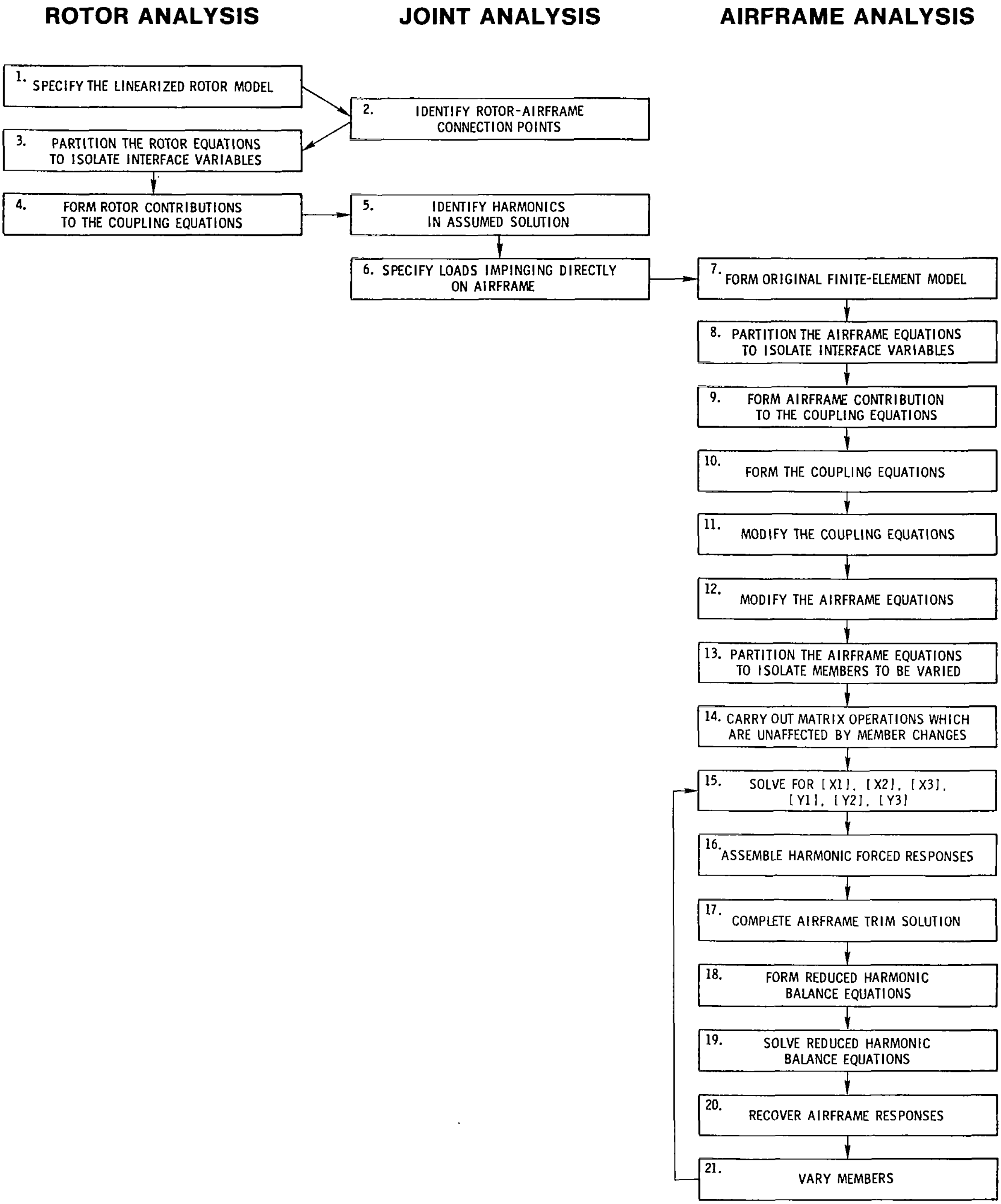


Figure M1.- Block diagram indicating the sequence of tasks for calculating airframe vibrations including reanalysis when members are varied.

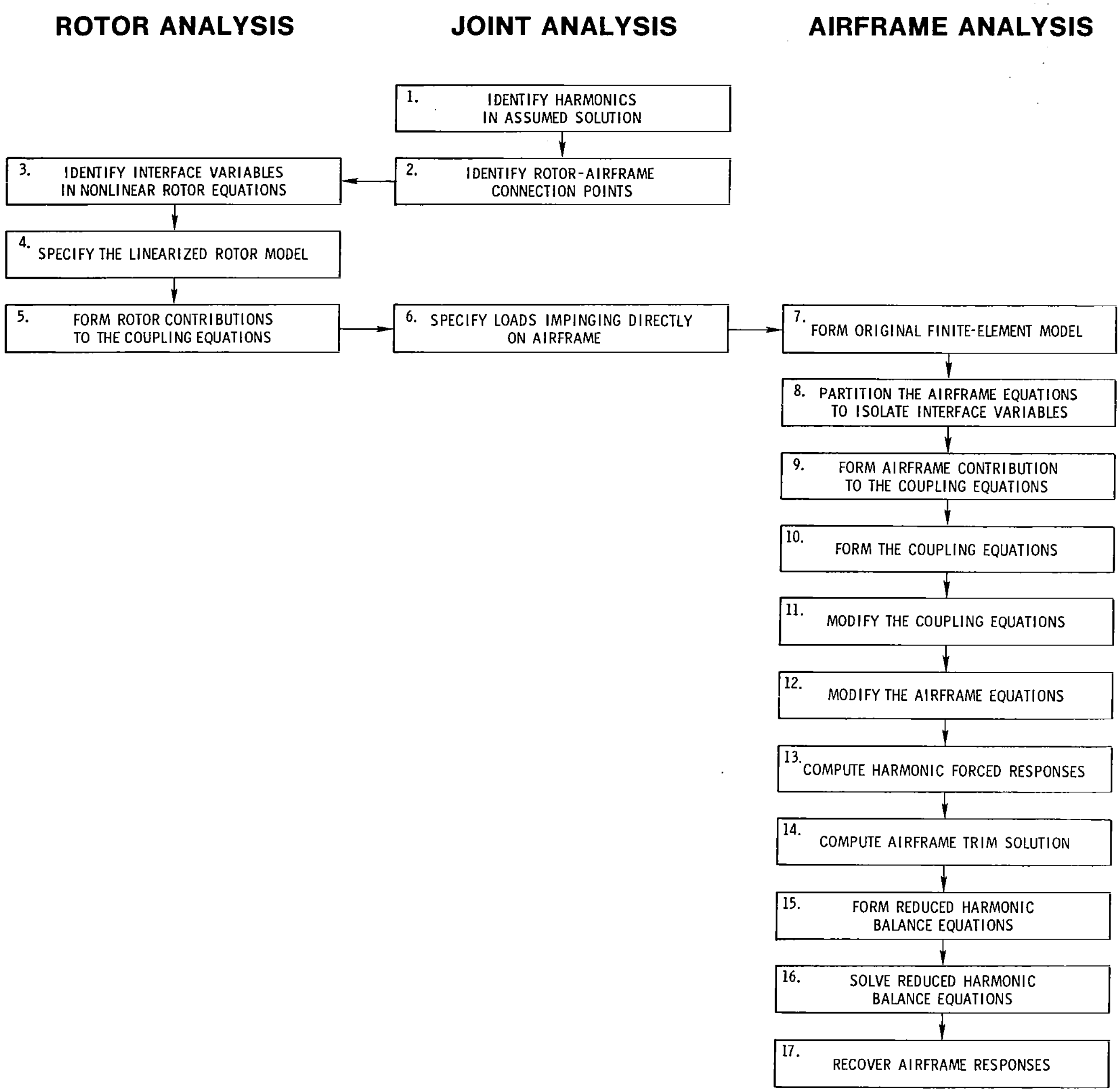


Figure M2.- Block diagram indicating the sequence of tasks for calculating airframe vibrations when rotor is represented by impedances.

REFERENCES

1. The NASTRAN® User's Manual (Level 17.5). NASA SP-222(05), 1978.

2. Johnston, Robert A.; and Cassarino, S. J.: Aeroelastic Rotor Stability Analysis. USAAMRDL-TR-75-40, U.S. Army, Jan. 1976. (Available from DTIC as AD A020 871.)

3. Anderson, W. D.; Conner, F.; Kretsinger, P.; and Reaser, J. S.: REXOR Rotorcraft Simulation. Volume I - Engineering Documentation. USAAMRDL-TR-76-28A, U.S. Army, July 1976. (Available from DTIC as AD A028 314.)

4. McLarty, Tyce T.: Rotorcraft Flight Simulation With Coupled Rotor Aeroelastic Stability Analysis. Volume I - Engineer's Manual. USAAMRDL-TR-76-41A, U.S. Army, May 1977. (Available from DTIC as AD A042 462.)

5. Warmbrodt, William; and Friedmann, Peretz: Formulation of Coupled Rotor/Fuselage Equations of Motion. Vertica, vol. 3, no. 3/4, 1979, pp. 245-271.

6. Johnson, Wayne: A Comprehensive Analytical Model of Rotorcraft Aerodynamics and Dynamics. Part 1: Analysis Development. NASA TM-81182, AVRADCOM TR-80-A-5, 1980.

7. Gerstenberger, Walter; and Wood, Edward R.: Analysis of Helicopter Aeroelastic Characteristics in High-Speed Flight. AIAA J., vol. 1, no. 10, Oct. 1963, pp. 2366-2381.

8. Novak, M. E.: Rotating Elements in the Direct Stiffness Method of Dynamic Analysis With Extensions to Computer Graphics. Shock & Vib. Bull., Bull. 40, Pt. 4, U.S. Dep. Def., Oct. 1969, pp. 41-46.

9. Staley, J. A.; and Sciarra, J. J.: Coupled Rotor/Airframe Vibration Prediction Methods. Rotorcraft Dynamics, NASA SP-352, 1974, pp. 81-90.

10. Anderson, W. D.; and Gaidelis, J. A.: Vibration Inputs and Response of a Hingeless Rotor Compound Helicopter. Rep. LR 26523 (Contract No. DAAJ01-73-C-0286, Modification P00006), Lockheed-California Co., Rotary Wing Div., June 1974.

11. Kunz, Donald L.: Effects of Rotor-Body Coupling in a Linear Rotorcraft Vibration Model. Preprint No. 80-26, 36th Annual Forum, American Helicopter Soc., May 1980.

12. Cassarino, Sebastian J.; and Mouzakis, Timoleon: Bifilar Analysis User's Manual - Volume II. NASA CR-159228, 1980.

13. Kunz, Donald L.: Response Characteristics of a Linear Rotorcraft Vibration Model. A Collection of Technical Papers, Part 2 - AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics & Materials Conference and AIAA Dynamics Specialists Conference, Apr. 1981, pp. 454-462. (Available as AIAA-81-0616.)

14. Berman, Alex: A Generalized Coupling Technique for the Dynamic Analysis of Structural Systems. J. American Helicopter Soc., vol. 25, no. 3, July 1980, pp. 22-28.

15. Peters, David A.; and Ormiston, Robert A.: Flapping Response Characteristics of Hingeless Rotor Blades by a Generalized Harmonic Balance Method. NASA TN D-7856, 1975.

16. Yen, Jing G.; and McLarty, Tyce T.: Analysis of Rotor-Fuselage Coupling and Its Effect on Rotorcraft Stability and Response. Vertica, vol. 3, no. 3/4, 1979, pp. 205-219.

17. Eipe, Abraham: Effect of Some Structural Parameters on Elastic Rotor Loads by an Iterative Harmonic Balance. Ph. D. Thesis, Washington Univ., Sever Inst. of Technol., Dec. 1979.

18. Hohenemser, Kurt H.; and Yin, Sheng-Kuang: The Role of Impedance in Vibration Analysis of Rotorcraft. Vertica, vol. 3, no. 3/4, 1979, pp. 189-204.

19. Hsu, T-K; and Peters, D. A.: Coupled Rotor/Airframe Vibration Analysis by a Combined Harmonic-Balance, Impedance-Matching Method. Preprint No. 80-21, 36th Annual Forum, American Helicopter Soc., May 1980.

20. Friedmann, P.: Editorial. Vertica, vol. 3, no. 3/4, 1979, pp. 187-188.

21. Cronkhite, James D.; Wilson, Henry E.; and Berry, Victor L.: Correlation of AH-1G Helicopter Flight Vibration Data and Tailboom Static Test Data With NASTRAN Results. NASA CR-145120, [1977].

22. McLaughlin, A.: Finite Element Dynamic Analysis of Production Aircraft. Proceedings of 4th European Rotorcraft and Powered Lift Aircraft Forum, Volume 1, Costruzioni Aeronautiche Giovanni Agusta S.P.A., 1978, pp. 20.0 - 20.9.

23. Flannelly, W. G.; Fabunmi, J. A.; and Nagy, E. J.: Analytical Testing. NASA CR-3429, 1981.

SYMBOLS

Symbols used throughout the main text and appendices in this paper are included in this list. However, for the sake of brevity and simplicity, symbols used and defined in one section of the main text or in one appendix are not included. The orders (or maximum orders) of all matrices appearing in this paper are given in table I after the symbol list.

The symbols OC, NC, and NS attached to a matrix symbol denote the following:

OC	constant (zeroth harmonic) term in Fourier series expansion
NC	Nth harmonic ($N \neq 0$) cosine term in Fourier series expansion
NS	Nth harmonic ($N \neq 0$) sine term in Fourier series expansion

For example, {DVBOC}, {DVBNC}, and {DVBNS} are the constant term, Nth harmonic cosine term, and Nth harmonic sine term in the Fourier series expansion of {DVB} (see eq. (48)).

[A11],[A12],[A13], [A21],[A22],[A23], [A31],[A32],[A33]	submatrices appearing in equation (E1)
[B11],[B12],[B13], [B21],[B22],[B23], [B31],[B32],[B33]	submatrices appearing in equation (E1)
$[\bar{C}]$	matrix characterizing linear equations of constraint (see eq. (B8))
[CA]	viscous damping matrix for modified airframe finite-element model (see eq. (19))
[CA11],[CA12], [CA21],[CA22]	submatrices of [CA] (see eq. (21))
$[\overline{CA}]$	viscous damping matrix for original airframe finite-element model (see eq. (B3))
$[\overline{CA11}]$,$[\overline{CA12}]$, $[\overline{CA21}]$,$[\overline{CA22}]$	submatrices of $[\overline{CA}]$ (see eq. (B6))
[CR]	matrix appearing in linearized generalized force expression for rotor (see eq. (A19))

$[CR11],[CR12],$ $[CR21],[CR22]$ submatrices of $[CR]$ (see eq. (A25))

{c} vector containing displacements and rotations associated with points connecting rotor model to original airframe finite-element model

{DVB} vector of small deviations from trim state {VBO}

{DVH} vector of small deviations from trim state {VHO}

{DVR} vector of small deviations from trim state {VRO}

$[DX_1X_2X_3X_4]$ matrix defined by equation (52)

$[DX_1X_2X_3X_411]$, $[DX_1X_2X_3X_412]$, $[DX_1X_2X_3X_421]$, $[DX_1X_2X_3X_422]$ submatrices of $[DX_1X_2X_3X_4]$ (see eq. (51))

{DZ} vector of small deviations from trim state {ZO}

{DZH} vector of small deviations from trim state {ZHO}

{DZS} vector of small deviations from trim state {ZSO}

$[EX_1X_2X_3X_4]$ designation for typical submatrix appearing in coefficient matrix of equation (65)

{F} vector of generalized force expressions for modified airframe finite-element model (see eq. (19))

$\{\bar{F}\}$ vector of generalized force expressions for original airframe finite-element model (see eq. (B3))

[F1],[F2],[F3] submatrices appearing in equation (E1)

{FH} vector of generalized force expressions for modified airframe finite-element model associated with {ZH}

{FHO} trim solution value for {FH}

{FHTR} total interface forces required to suppress mean interface displacements (see eq. (56))

{FO} trim solution for {F}

{FS} vector of generalized force expressions for modified airframe finite-element model associated with {ZS}

{FSO} trim solution value for {FS}

[FUIN],[FUON] matrices of in-phase and out-of-phase forces associated with Nth harmonic ($N \neq 0$) forced response (see eq. (60))

[FUIO] matrix of in-phase forces associated with zeroth harmonic forced response (see eq. (61))

[$FX_1X_2X_3X_4$] designation for typical submatrix appearing in coefficient matrix in equation (72) and defined in equation (73)

{FZHOC} constant (zeroth harmonic) term in Fourier series expansion of the resultant force imposed on the airframe by the rotor

{FZHNC} Nth harmonic ($N \neq 0$) cosine term in Fourier series expansion of the resultant force imposed on the airframe by the rotor

{FZHNS} Nth harmonic ($N \neq 0$) sine term in Fourier series expansion of the resultant force imposed on the airframe by the rotor

[G1],[G2],[G3] submatrices appearing in equation (E1)

[GER] matrix reflecting resultant of all elementary row operations

H,L,P,Q indices appearing in equation (52)

[KA] stiffness matrix for modified airframe finite-element model (see eq. (19))

[KA11],[KA12], [KA21],[KA22] submatrices of [KA] (see eq. (21))

[$\overline{KA}$] stiffness matrix for original airframe finite-element model (see eq. (B3))

[$\overline{KA11}$],[$\overline{KA12}$], [$\overline{KA21}$],[$\overline{KA22}$] submatrices of [$\overline{KA}$] (see eq. (B6))

[KR] matrix appearing in linearized generalized force expression for rotor (see eq. (A19))

[KR11],[KR12], [KR21],[KR22] submatrices of [KR] (see eq. (A25))

{L} vector of loads directly applied to modified airframe finite-element model from sources which are external to both the airframe and the rotor mechanical system (see eq. (19))

{LH} vector of modified external loads associated with {ZH}

$\{LS\}$ vector of modified external loads associated with $\{ZS\}$

$\{\bar{L}\}$ vector of loads directly applied to original airframe finite-element model from sources which are external to both the airframe and the rotor mechanical system (see eq. (B3))

$\{\overline{LH}\}$ vector of original external loads associated with $\{\overline{ZH}\}$

$\{\overline{LS}\}$ vector of original external loads associated with $\{\overline{ZS}\}$

$[MA]$ mass matrix for modified airframe finite-element model (see eq. (19))

$\left.\begin{matrix}[MA11],[MA12]\\ [MA21],[MA22]\end{matrix}\right\}$ submatrices of $[MA]$ (see eq. (21))

$[\overline{MA}]$ mass matrix for original airframe finite-element model (see eq. (B3))

$\left.\begin{matrix}[\overline{MA11}],[\overline{MA12}]\\ [\overline{MA21}],[\overline{MA22}]\end{matrix}\right\}$ submatrices of $[\overline{MA}]$ (see eq. (B6))

$[MR]$ matrix appearing in linearized generalized force expression for rotor (see eq. (A19))

$\left.\begin{matrix}[MR11]\ [MR12]\\ [MR21]\ [MR22]\end{matrix}\right\}$ submatrices of $[MR]$ (see eq. (A25))

$[MR0C11]$ constant (zeroth harmonic) term in Fourier series expansion of $[MR11]$

$[MR1S11]$ first harmonic sine term in Fourier series expansion of $[MR11]$

$[MR1C11]$ first harmonic cosine term in Fourier series expansion of $[MR11]$

N designates general Nth harmonic ($N \neq 0$) term

NI number of elements in $\{c\}$

$\overline{NRC}$ number of rows in $[\bar{C}]$

NT number of terms retained in equation (48)

NV number of elements in $\{v\}$

NVB number of elements in $\{VB\}$

NVC number of elements in $\{VC\}$

NVH number of elements in $\{VH\}$

NVR	number of elements in {VR}
NZ	number of elements in {Z}
$\overline{NZ}$	number of elements in {$\bar{Z}$}
NZH	number of elements in {ZH}
$\overline{NZH}$	number of elements in {$\overline{ZH}$}
NZS	number of elements in {ZS}
$\overline{NZS}$	number of elements in {$\overline{ZS}$}
NZSF	number of variables in {ZS} corresponding to masses and structural members which are not to be varied
NZSV	number of variables in {ZS} corresponding to masses and structural members which are to be varied
{Q}	vector of all generalized force expressions for rotor system
{QB}	vector of rotor generalized force expressions associated with {VB}
{QBO}	trim solution value for {QB}
{QC}	vector of rotor generalized force expressions associated with {VC}
{QCO}	trim solution value for {QC}
{QH}	vector of rotor generalized force expressions associated with {VH}
{QHTR}	specified mean value of {QH}
{QR}	vector of rotor generalized force expressions associated with {VR}
{QRO}	trim solution value for {QR}
{QZHOC}	constant (zeroth harmonic) term in Fourier series expansion of new airframe variables (see eq. (62a))
{QZHNC}	Nth harmonic ($N \neq 0$) cosine term in Fourier series expansion of new airframe variables (see eq. (62b))
{QZHNS}	Nth harmonic ($N \neq 0$) sine term in Fourier series expansion of new airframe variables (see eq. (62b))
[$\overline{TC}$]	matrix characterizing linear equations of constraint on original airframe finite-element model inherent in formulation of rotor model (see eq. (C13b))
[$\overline{TCD}$],[$\overline{TCI}$]	submatrices of [$\overline{TC}$] consistent with the partitioning of {$\overline{ZH}$} into {$\overline{ZHD}$} and {$\overline{ZHI}$}

[TH] matrix characterizing the linear relation connecting {DVH} to $\{\overline{\text{ZHI}}\}$ in the modified coupling equations (see eq. (13))

$[\overline{\text{TH}}]$ matrix characterizing the linear relationship connecting {DVH} to $\{\overline{\text{ZH}}\}$ in the original coupling equations (see eq. (C13a))

[THA] matrix characterizing the linearized relation connecting {c} to $\{\overline{\text{ZH}}\}$

[THR] matrix characterizing the linearized relation connecting {c} to {DVH}

$[\overline{\text{THD}}]$,$[\overline{\text{THI}}]$ submatrices of $[\overline{\text{TH}}]$ consistent with partitioning $\{\overline{\text{ZH}}\}$ into $\{\overline{\text{ZHD}}\}$ and $\{\overline{\text{ZHI}}\}$

$[\overline{\text{TI}}]$ matrix defined by equation (15)

[T1],[T2],[T3] submatrices appearing in equation (C9)

t time

[UZHIN], [UZHON], [UZSIN], [UZSON] submatrices of matrix of harmonic forced responses for Nth harmonic ($N \neq 0$) (see eq. (60) and sketch C)

[UZHIO], [UZSIO] submatrices of matrix of harmonic forced responses for zeroth harmonic (see eq. (61) and sketch D)

UOK,ULK,UHK, UOC,ULC,UHC, UOM,ULM,UHM parameters appearing in equation (52) and computed by the algorithm in appendix G

{VB} vector of rotor generalized coordinates not contained in {VH}, {VC}, and VT

{VBO} trim solution value for {VB}

{VC} vector of rotor generalized coordinates which describe actions of the flight control system

{VCO} trim solution value for {VC}

{VH} vector of rotor generalized coordinates which explicitly appear in expressions characterizing mechanical connections of the rotor system to the airframe system

{VHO} trim solution value for {VH}

$\{VR\}$ vector representing $\{VB\}$ and $\{VH\}$ collectively

$\{VRO\}$ trim solution value for $\{VR\}$

VT rotor generalized coordinate representing shaft rotation angle

$\{v\}$ vector containing all generalized coordinates of rotor mechanical system

δW_a virtual work for airframe system

δW_r virtual work for rotor system

δW virtual work of airframe system and rotor system combined

$[X1],[X2],[X3]$ submatrices appearing in equation (E1)

X_I, Y_I, Z_I inertial axis system

$X_1X_2X_3X_4$ four-character indexing system: the first character is an integer indicating row harmonic, the second character is the letter S or C indicating sine or cosine, the third character is an integer indicating column harmonic, and the fourth character is the letter S or C indicating sine or cosine

$[Y1],[Y2],[Y3]$ submatrices appearing in equation (E1)

$\{Z\}$ vector of all airframe generalized coordinates for modified finite-element model (see eq. (19))

$\{\bar{Z}\}$ vector of all airframe generalized coordinates for original finite-element model (see eq. (B3))

$\{ZH\}$ vector of airframe generalized coordinates which explicitly appear in expressions characterizing mechanical connections of the modified airframe finite-element model to the rotor model

$\{\overline{ZH}\}$ vector of airframe generalized coordinates which explicitly appear in expressions characterizing mechanical connections of the original airframe finite-element model to the rotor model

$\{\overline{ZHD}\}$ variables in $\{\overline{ZH}\}$ which are rendered dependent

$\{\overline{ZHI}\}$ variables in $\{\overline{ZH}\}$ which remain independent

$\{ZHO\}$ trim solution value for $\{ZH\}$

$\{ZO\}$ trim solution for $\{Z\}$

$\{ZS\}$ vector of all airframe generalized coordinates of modified finite-element model not contained in $\{ZH\}$

$\{\overline{ZS}\}$ vector of all airframe generalized coordinates of original finite-element model not contained in $\{\overline{ZH}\}$

{ZSO}	trim solution value for {ZS}
{ZHFNC} {ZSFNC} {ZHFNS} {ZSFNS}	Nth harmonic ($N \neq 0$) cosine and sine responses of airframe due to Nth harmonic external loads acting on the airframe (see eq. (D3))
{ZHFOC} {ZSFOC}	static (zeroth harmonic) response of airframe due to static external loads acting on the airframe (see eq. (D9b))
α,β,γ	parameters appearing in equations (67c), (67d), and (H10) computed by the algorithm given in appendix J
$\delta(\ )$	virtual variation
[ ]	rectangular matrix
{ }	column matrix or vector
$[\ ]^T$	matrix transpose
$[\ ]^{-1}$	matrix inverse
[I]	unit matrix
[0]	null matrix
$(\ddot{\ })$	second time derivative, $\frac{d^2}{dt^2}$
$(\dot{\ })$	first time derivative, $\frac{d}{dt}$
$(\bar{\ })$	airframe term associated with original finite-element model

TABLE I.- ORDER OF MATRICES

Matrix	Order		Matrix	Order	
	Rows	Columns		Rows	Columns
[A][a]	(NVB) (NT)	(NVB) (NT)	[C][a]	(NZH) (NT)	(NVB) (NT)
[A11]	NZH	NZH	$[\overline{C}]$	$\overline{NRC}$	$\overline{NZ}$
[A12]	NZH	NZSV	[CA]	NZ	NZ
[A13]	NZH	NZSF	[CA11]	NZH	NZH
[A21]	NZSV	NZH	[CA12]	NZH	NZS
[A22]	NZSV	NZSV	[CA21]	NZS	NZH
[A23]	NZSV	NZSF	[CA22]	NZS	NZS
[A31]	NZSF	NZH	$[\overline{CA}]$	$\overline{NZ}$	$\overline{NZ}$
[A32]	NZSF	NZSV	$[\overline{CA11}]$	$\overline{NZH}$	$\overline{NZH}$
[A33]	NZSF	NZSF	$[\overline{CA12}]$	$\overline{NZH}$	$\overline{NZS}$
[B][a]	(NVB) (NT)	(NZH) (NT)	$[\overline{CA21}]$	$\overline{NZS}$	$\overline{NZH}$
[B1][a]	(NVB) (NT)	(NVH) (NT)	$[\overline{CA22}]$	$\overline{NZS}$	$\overline{NZS}$
[B2][a]	(NVB) (NT)	(NVH) (NT)	[CR]	NVR	NVR
[B11]	NZH	NZH	[CR11]	NVB	NVB
[B12]	NZH	NZSV	[CR12]	NVB	NVH
[B13]	NZH	NZSF	[CR21]	NVH	NVB
[B21]	NZSV	NZH	[CR22]	NVH	NVH
[B22]	NZSV	NZSV	[CROC]	NVR	NVR
[B23]	NZSV	NZSF	[CRNC]	NVR	NVR
[B31]	NZSF	NZH	[CRNS]	NVR	NVR
[B32]	NZSF	NZSV	{C}	NI	1
[B33]	NZSF	NZSF			

[a]Order indicated for matrix is maximum possible value.

TABLE I.- Continued

Matrix	Order		Matrix	Order	
	Rows	Columns		Rows	Columns
[D][a]	(NZH)(NT)	(NZH)(NT)	{DZSNC}	NZS	1
{DVB}	NVB	1	{DZSNS}	NZS	1
{DVBOC}	NVB	1			
{DVBNS}	NVB	1	[E1][a]	(NVH)(NT)	(NZH)(NT)
{DVBNC}	NVB	1	[E2][a]	(NVH)(NT)	(NZH)(NT)
{DVH}	NVH	1	$[EX_1X_2X_3X_4]$	NVB + NZH	NVB + NZH
{DVHOC}	NVH	1	$[EX_1X_2X_3X_411]$	NVB	NVB
{DVHNS}	NVH	1	$[EX_1X_2X_3X_412]$	NVB	NZH
{DVHNC}	NVH	1	$[EX_1X_2X_3X_421]$	NZH	NVB
{DVR}	NVR	1	$[EX_1X_2X_3X_422]$	NZH	NZH
$[DX_1X_2X_3X_4]$	NVR	NVR	{F}	NZ	1
$[DX_1X_2X_3X_411]$	NVB	NVB	$\{\bar{F}\}$	$\overline{NZ}$	1
$[DX_1X_2X_3X_412]$	NVB	NVH	[F1][a]	NZH	2(NZH) + 1
$[DX_1X_2X_3X_421]$	NVH	NVB	[F2][a]	NZSV	2(NZH) + 1
$[DX_1X_2X_3X_422]$	NVH	NVH	[F3][a]	NZSF	2(NZH) + 1
{DZ}	NZ	1	{FDZ}	NZ	1
{DZH}	NZH	1	{FDZH}	NZH	1
{DZHOC}	NZH	1	{FDZHOC}	NZH	1
{DZHNS}	NZH	1	{FDZHNC}	NZH	1
{DZHNC}	NZH	1	{FDZHNS}	NZH	1
{DZS}	NZS	1	{FDZS}	NZS	1
{DZSOC}	NZS	1	{FDZSOC}	NZS	1

[a]Order indicated for matrix is maximum possible value.

TABLE I.- Continued

Matrix	Order		Matrix	Order	
	Rows	Columns		Rows	Columns
{FDZSNC}	NZS	1	[G1][a]	NZH	2(NZH) + 1
{FDZSNS}	NZS	1	[G2][a]	NZSV	2(NZH) + 1
{FH}	NZH	1	[G3][a]	NZSF	2(NZH) + 1
{FHO}	NZH	1	[GER]	NI	NI
{FHOOC}	NZH	1	$[GX_1X_2X_3X_4]$	NZH	NZH
{FHONC}	NZH	1	[g]	NZH	NZH
{FHONS}	NZH	1	[HIO]	NZ	NZ
{FHTR}	NZH	1	[HIN]	NZ	NZ
{FO}	NZ	1	[HON]	NZ	NZ
{FS}	NZS	1			
{FSO}	NZS	1	[KA]	NZ	NZ
[FUIN]	NZH	NZH	[KA11]	NZH	NZH
[FUON]	NZH	NZH	[KA12]	NZH	NZS
[FUIO]	NZH	NZH	[KA21]	NZS	NZH
$[FX_1X_2X_3X_4]$	NZH	NZH	[KA22]	NZS	NZS
{FZHOC}	NZH	1	$[\overline{KA}]$	$\overline{NZ}$	$\overline{NZ}$
{FZHNS}	NZH	1	$[\overline{KA11}]$	$\overline{NZH}$	$\overline{NZH}$
{FZHNC}	NZH	1	$[\overline{KA12}]$	$\overline{NZH}$	$\overline{NZS}$
[f]	NZH	NZH	$[\overline{KA21}]$	$\overline{NZS}$	$\overline{NZH}$
$\{f_a\}$	NI	1	$[\overline{KA22}]$	$\overline{NZS}$	$\overline{NZS}$
$\{f_r\}$	NI	1	[KR]	NVR	NVR

[a]Order indicated for matrix is maximum possible value.

TABLE I.- Continued

Matrix	Order	
	Rows	Columns
[KR11]	NVB	NVB
[KR12]	NVB	NVH
[KR21]	NVH	NVB
[KR22]	NVH	NVH
[KROC]	NVR	NVR
[KRNC]	NVR	NVR
[KRNS]	NVR	NVR
$[KUX_1X_2X_3X_4]$	NVH	NVH
{L}	NZ	1
{LOC}	NZ	1
{LNC}	NZ	1
{LNS}	NZ	1
{LH}	NZH	1
{LHOC}	NZH	1
{LHNC}	NZH	1
{LHNS}	NZH	1
{LS}	NZS	1
{LSOC}	NZS	1
{LSOC2}	NZSV	1
{LSOC3}	NZSF	1
{LSNC}	NZS	1
{LSNC2}	NZSV	1
{LSNC3}	NZSF	1

Matrix	Order	
	Rows	Columns
{LSNS}	NZS	1
{LSNS2}	NZSV	1
{LSNS3}	NZSF	1
$\{\overline{L}\}$	$\overline{NZ}$	1
$\{\overline{LOC}\}$	$\overline{NZ}$	1
$\{\overline{LNS}\}$	$\overline{NZ}$	1
$\{\overline{LNC}\}$	$\overline{NZ}$	1
$\{\overline{LH}\}$	$\overline{NZH}$	1
$\{\overline{LHOC}\}$	$\overline{NZH}$	1
$\{\overline{LHNC}\}$	$\overline{NZH}$	1
$\{\overline{LHNS}\}$	$\overline{NZH}$	1
$\{\overline{LS}\}$	$\overline{NZS}$	1
$\{\overline{LSOC}\}$	$\overline{NZS}$	1
$\{\overline{LSNC}\}$	$\overline{NZS}$	1
$\{\overline{LSNS}\}$	$\overline{NZS}$	1
[MA]	NZ	NZ
[MA11]	NZH	NZH
[MA12]	NZH	NZS
[MA21]	NZS	NZH
[MA22]	NZS	NZS
$[\overline{MA}]$	$\overline{NZ}$	$\overline{NZ}$
$[\overline{MA11}]$	$\overline{NZH}$	$\overline{NZH}$
$[\overline{MA12}]$	$\overline{NZH}$	$\overline{NZS}$

TABLE I.- Continued

Matrix	Order		Matrix	Order	
	Rows	Columns		Rows	Columns
[$\overline{\text{MA21}}$]	$\overline{\text{NZS}}$	$\overline{\text{NZH}}$	{QBONS}	NVB	1
[$\overline{\text{MA22}}$]	$\overline{\text{NZS}}$	$\overline{\text{NZS}}$	{QC}	NVC	1
[MR]	NVR	NVR	{QCO}	NVC	1
[MROC]	NVR	NVR	{QH}	NVH	1
[MRNC]	NVR	NVR	{QHOC}	NVH	1
[MRNS]	NVR	NVR	{QHNC}	NVH	1
[MR11]	NVB	NVB	{QHNS}	NVH	1
[MR12]	NVB	NVH	{QHO}	NVH	1
[MR21]	NVH	NVB	{QHOOC}	NVH	1
[MR22]	NVH	NVH	{QHONC}	NVH	1
[MROC11]	NVB	NVB	{QHONS}	NVH	1
[MRNC11]	NVB	NVB	{QHTR}	NVH	1
[MRNS11]	NVB	NVB	{QO}	NV	1
{P1}[a]	(NVB) (NT)	1	{QR}	NVR	1
{P2}[a]	(NZH) (NT)	1	{QRO}	NVR	1
{Q}	NV	1	{QROOC}	NVR	1
{QB}	NVB	1	{QRONC}	NVR	1
{QBO}	NVB	1	{QRONS}	NVR	1
{QBOOC}	NVB	1	{QZHOC}	NZH	1
{QBONC}	NVB	1	{QZHNC}	NZH	1
			{QZHNS}	NZH	1

[a]Order indicated for matrix is maximum possible value.

TABLE I.- Continued

Matrix	Order	
	Rows	Columns
[$\overline{TC}$]	NI - NVH	$\overline{NZH}$
[$\overline{TCD}$]	NI - NVH	NI - NVH
[$\overline{TCI}$]	NI - NVH	$\overline{NZH}$ + NVH - NI
[TH]	NVH	NVH + $\overline{NZH}$ - NI
[THA]	NI	$\overline{NZH}$
[THR]	NI	NVH
[$\overline{TH}$]	NVH	$\overline{NZH}$
[$\overline{THD}$]	NVH	NI - NVH
[$\overline{THI}$]	NVH	$\overline{NZH}$ + NVH - NI
[$\overline{TI}$]	$\overline{NZH}$	$\overline{NZH}$ + NVH - NI
[T1]	NVH	NVH
[T2]	NVH	$\overline{NZH}$
[T3]	NI - NVH	$\overline{NZH}$
[UZHIN]	NZH	NZH
[UZHON]	NZH	NZH
[UZSIN]	NZS	NZH
[UZSON]	NZS	NZH
[UZHIO]	NZH	NZH
[UZSIO]	NZS	NZH
[UZSIN2]	NZSV	NZH
[UZSIN3]	NZSF	NZH
[UZSON2]	NZSV	NZH

Matrix	Order	
	Rows	Columns
[UZSON3]	NZSF	NZH
[UZSIO2]	NZSV	NZH
[UZSIO3]	NZSF	NZH
{VB}	NVB	1
{VBO}	NVB	1
{VC}	NVC	1
{VCO}	NVC	1
{VH}	NVH	1
{VHO}	NVH	1
{VO}	NV	1
{VR}	NVR	1
{VRO}	NVR	1
{v}	NV	1
{WC}	NZH	1
{WS}	NZH	1
[X1]	NZH	2(NZH) + 1
[X2]	NZSV	2(NZH) + 1
[X3]	NZSF	2(NZH) + 1
{X1}[a]	(NVB)(NT)	1
{X2}[a]	(NZH)(NT)	1

[a]Order indicated for matrix is maximum possible value.

TABLE I.- Concluded

Matrix	Order	
	Rows	Columns
[Y1][a]	NZH	2(NZH) + 1
[Y2][a]	NZSV	2(NZH) + 1
[Y3][a]	NZSF	2(NZH) + 1
{Z}	NZ	1
$\{\overline{Z}\}$	$\overline{NZ}$	1
{ZH}	NZH	1
$\{\overline{ZH}\}$	$\overline{NZH}$	1
$\{\overline{ZHD}\}$	NI - NVH	1
$\{\overline{ZHI}\}$	$\overline{NZH}$ + NVH - NI	1
{ZHFNS}	NZH	1
{ZHFNC}	NZH	1
{ZHFOC}	NZH	1
{ZHO}	NZH	1
{ZO}	NZ	1
{ZS}	NZS	1
$\{\overline{ZS}\}$	$\overline{NZS}$	1
{ZSFNS}	NZS	1
{ZSFNC}	NZS	1
{ZSFOC}	NZS	1
{ZSFNS2}	NZSV	1
{ZSFNS3}	NZSF	1
{ZSFNC2}	NZSV	1
{ZSFNC3}	NZSF	1
{ZSFOC2}	NZSV	1
{ZSFOC3}	NZSF	1
{ZSO}	NZS	1
{ZSOOC}	NZS	1
{ZSONC}	NZS	1
{ZSONS}	NZS	1

[a]Order indicated for matrix is maximum possible value.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA RP-1089		

4. Title and Subtitle	5. Report Date
A FORMULATION OF ROTOR-AIRFRAME COUPLING FOR DESIGN ANALYSIS OF VIBRATIONS OF HELICOPTER AIRFRAMES	June 1982
	6. Performing Organization Code 505-42-13-04
7. Author(s) Raymond G. Kvaternik and William C. Walton, Jr.	8. Performing Organization Report No. L-14243
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665	10. Work Unit No.
	11. Contract or Grant No.
	13. Type of Report and Period Covered Reference Publication
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The paper presents a comprehensive linear formulation of rotor-airframe coupling intended for vibration analysis in airframe structural design. The airframe is represented by a finite-element analysis model; the rotor is represented by a general set of linear differential equations with periodic coefficients; and the connections between the rotor and airframe are specified through general linear equations of constraint. Coupling equations are derived and then applied to the rotor and airframe equations to produce one set of linear differential equations governing vibrations of the combined rotor-airframe system. These equations are solved by the harmonic balance method for the system steady-state vibrations. A feature of the solution process is the representation of the airframe in terms of forced responses calculated at the rotor harmonics of interest. A method based on matrix partitioning is worked out for quick recalculations of vibrations in design studies when only relatively few airframe members are varied. All relations are presented in forms suitable for direct computer implementation.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Rotor-airframe coupling Helicopter vibrations Harmonic balance method Finite-element models Reanalysis	FEDD Distribution Subject Category 39

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	134	

Available: NASA's Industrial Applications Centers

NASA-Langley, 1982