NASA TECHNICAL MEMORANDUM NASA TM-77052

(NASA-TM-77052) DYNAMIC ANALYSIS OF EVOLUTIVE CONSERVATIVE SYSTEMS. DISCUSSION OF EIGENMODE CROSSINGS (National Aeronautics and Space Administration) 47 p HC A03/MF A01 CSCL 22B G3/15

N84-34462

Unclas 24064

DYNAMIC ANALYSIS OF EVOLUTIVE CONSERVATIVE SYSTEMS. DISCUSSION OF "EIGENMODE CROSSINGS"

H. J.-P. Morand

Translation of "Analyse Dynamique de Systemes Conservatifs Evolutifs. Discussion des "Croisements de Modes", Association Technique Maritime et Aeronautique, Bulletin, no. 76, 1976, pp. 329-350; Discussion, pp. 351, 352 and report ONERA TP 1976-54, pp 1-22

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON D.C. 20546 SEPTEMBER 1984

DYNAMIC ANALYSIS OF EVOLUTIVE CONSERVATIVE SYSTEMS. DISCUSSION OF "EIGENMODE CROSSINGS"

H., J.-P. Morand

ANNOTATION

Following an analysis of the close link that exists between the symmetries of a dynamic system and the multiplicity of its natural vibration frequencies, we show , on variational grounds, that for a system with a fixed symmetry, the natural frequencies associated with natural modes of a given type of symmetry do not cross each other during the evolution of this system.

The theoretical method has been put to use numerically and applied to the analysis of the evolution of the axisymmetric hydroelastic modes of the "Ariane" launcher during first-stage combustion.

DYNAMIC ANALYSIS OF EVOLUTIVE CONSERVATIVE SYSTEMS*. DISCUSSION OF "EIGENMODE CROSSINGS"

H., J.-P. Morand**

INTRODUCTION

To set the stage, let us consider the evolution of the longitudinal modes of vibration of a liquid propellant launcher during combustion of one stage. In practice, it is convenient to characterize the configuration of such an evolutive dynamic system by means of the draining rate of the propellants in the functioning stage. We might think that the change, a priori considerable, of the natural frequencies resulting from changes in the mass of the vehicle /2***

* Research financed by the Centre National d'Etudes Spatials (National Center for Space Research)
** Engineer at the Office National d'Etudes et de Recherches Aerospatiales
*** Numbers in margin indicate foreign pagination

is necessarily accompanied by frequencies crossing each other (Figure 1). However, when the experimenters track the natural frequencies of an evolutive system, they observe, in fact, events of the type depicted in Figure 2 and that we have agreed to call "modal interaction".

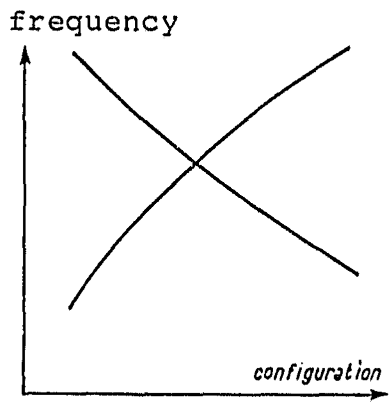


Fig. 1 - Frequency Crossing.

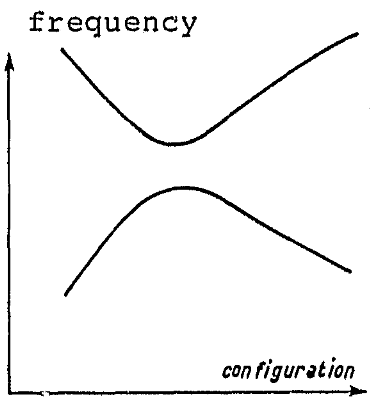


Fig. 2 - "Modal interaction".

One of the goals of this study is to show -as if there was a need!- that experimental reality matches theoretical reality and that, with the exception of relatively academic cases that we will discuss, the eigenmode crossings must be considered as "accidental" with respect to modal interaction events.

If we were only discussing the physical reality of eigenmode crossings, our intent could appear to be somewhat academic and provide only a principle for plotting curves describing the evolution of frequencies. However, it happens that during modal interaction, the ability to excite a mode of vibration with a given type of excitation can undergo "catastrophic" variations: And this excitability measured by the effective mass μ_e (defined below) proves to be playing an essential function in predicting pogo instabilities of launchers using liquid propellant for which we now recall the hydraulic and mechanical origins (ref. 1).

In flight, the thrust provided by the engines can fluctuate slightly on either side of an average value. These fluctuations induce vibrations in the launcher structure, thus subjecting it to accelerations around natural resonant frequencies. These accelerations of the structure are in turn coupled to the system supplying fuels which oscillate at the resonant frequencies of the tanks and the supply lines /2
to the engines. These last oscillations generate pressure peaks in the propellants. These pressure peaks act on the engines by causing the propellant flows to vary, thus causing thrust fluctuations at the frequency of the pressure fluctuations. If the natural frequencies of the structure and of the propulsion systems coincide, the pressure fluctuation resulting from the thrust fluctuation increases. It can even diverge if the thrust fluctuation generates a pressure fluctuation greater than the one that created it: it is the pogo instability.

This basic analysis clearly shows why the investigation of the pogo stability loop requires, in particular, that the frequencies and the natural forms of hydroelastic vibrations of the launcher be known. The parameter measuring the sensitivity of a vibration mode of the tank system to an excitation at the anchor point M for the engine structures is the "effective mass" μ_g defined by:

$$\mu_e = \frac{\mu_g}{u^2(M)}, \qquad (1)$$

with μ_g designating the generalized mass of the vibration mode for an arbitrary normalization and $u^2(M)$ being the square of the modal amplitude of the displacement of the anchor point M.

Now, we are going to analyze the interactions of modes, as well as resulting effective mass variations, in the general case of an evolutive conservative dynamic system. By conservative dynamic system, we essentially mean a system with harmonic variations described by a variational principle involving a potential energy and a kinetic energy that can be expressed as quadratic forms of acceptable displacements.

N.B. The qualifier "conservative" implies that we neglect any dissipating effect in the vibration study and that, in the case of a system varying as a function of time, the mechanical characteristics of the system (stiffness, mass) can be considered as constant over several periods of the vibrations under study. In concrete terms, in the case of the Ariane launcher, the combustion time for the first stage is 140 seconds whereas the period of the vibrations studied is less than one tenth of a second; it is therefore proper in such a situation to compute the vibrations at a given time by freezing the draining rate τ which describes the corresponding instantaneous configuration.

During the first part of this study we will try to define, as generally as possible, the forms and natural frequencies of such a dynamic system.

We will consider the close link that exists between the multiplicity of the natural frequencies and the symmetries of a dynamic system. This indispensable tendency to decrease will enable us, on the one hand, to show theoretical cases of natural frequency crossings and, on the other, to state in a constructive manner the problem of the evolution of the vibration modes of a system.

We will then show how a so-called "two-mode" approach to the solution of the variational problem governing the natural forms and frequencies makes it possible to prevent the expected frequency crossings in the one-mode approach.

We will also give a geometric interpretation of the interaction of the two modes and of the remarkable nature of the effective mass that can result from it. /3

During the second part of this report, we will illustrate the general analysis outlined above by applying it to the discussion of the evolution of the hydroelastic modes of the Ariane launcher during first-stage combustion.

We will present some numerical results involving an analytical simplification of the evolution problem of the vibration modes of a liquid launcher valid for the investigation of the configurations close to a configuration for which we know the modes of vibration (in the present case these natural modes have been computed using the method of finite elements).

1. - VARIATIONAL MODELING OF AN EVOLUTIVE DYNAMIC SYSTEM

1.1. Varational formulation of a problem of harmonic vibrations.

In order to introduce the general principles that enable us to describe a general conservative dynamic system, let us consider, as an example, the classical problem of the determination of the elastic vibration modes of a body (ref. 2) occupying a domain Ω (volume) and embedded over a portion Γ of its boundary $\partial\Omega$.

By definition, a natural mode of vibration is a stationary solution of the problem of small motions of the body about its static equilibrium position. If $\mathcal{U}(M,t)$ designates the displacement at time t of the particle of matter located at point M at equilibrium, we become interested in the solutions of the dynamic problem of the form:

$$\mathcal{U}(M, t) = u(M) \cos(\omega t + \varphi) \qquad (2)$$

with φ independent of M. Such a solution, when it exists, expresses a harmonic vibration, in phase, of every point of the elastic body.

For a body with finite dimensions we can show that such solutions exist for certain discrete values of ω: $\omega_0, \omega_1, \omega_2$, ... ω_n..., that constitute a series of numbers. ω_n is called a natural pulse and the associated field $u_n(M)$ is called a natural form.

In the particular problem considered, if we designate with u_α (α=1,2,3), the cartesian components of a certain natural form u , $\sigma_{\alpha\beta}$ the cartesian components of the Cauchy stress tensor considered as a function of the displacements by means of the behavior equation for the hyperelastic material, x_α the cartesian coordinates of a point in space, and n_α the components of the unit line perpendicular to $\partial\Omega$ and external to Ω , ω the natural pulse, the couple (u,ω) satisfies the equations:

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + \rho\omega^2 u_\alpha = 0 \quad \text{at every point of } \Omega \quad (3)$$

$$\sigma_{\alpha\beta}\, n_\beta = 0 \quad \text{over } \partial\Omega - \Gamma \quad (4)$$

$$u_\alpha = 0 \quad \text{over } \Gamma \quad (\alpha=1,2,3)) \quad (5)$$

(ρ = density per unit volume of the material)

It is convenient to write the preceding problem in an equivalent form by means of the virtual works principle. For this, we introduce the class C of the fields of regular vectors defined over Ω and equal to zero over Γ. The dynamic problem is now stated as follows: /4

Find ω and $u \in C$ so as to satisfy the equation

$$\int_\Omega \sigma_{\alpha\beta}(u)\,\epsilon_{\alpha\beta}(\delta u) - \omega^2 \int_\Omega \rho\, u_\alpha(\delta u)_\alpha = 0 \qquad (6)$$

regardless of the $\delta u \in C$

$$\left(\text{with} \quad \epsilon_{\alpha\beta}(u) = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$$

We say that C is the class of the "admissible forms" of the problem and that the condition at the limits (5) satisfied by the u elements of this class is essential.

As for the conditions at the limits (4), we observe that it no longer appears explicitely in equation (6): indeed, we can show that it results from satisfying the variational principle (6). This condition at the limits (4) which is included in the variational principle (6) is qualified as natural.

Finally, let us consider the ratio $[\omega^2]$

$$[\omega^2] = \frac{\frac{1}{2}\int_\Omega \sigma_{\alpha\beta}(u)\,\epsilon_{\alpha\beta}(u)}{\frac{1}{2}\int_\Omega \rho u^2} \quad ; \qquad (7)$$

considered as a function of $u \in \mathcal{C}; \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\,\epsilon_{\beta\alpha}$ and $\frac{1}{2}\int_\Omega \rho u^2$ constitute quadratic forms of $u \in C$ which have a direct physical meaning. Then, let us introduce into expression (7) for ω^2 a field of the type $u_n + \epsilon\delta u$, u_n being a solution of (6) for a "natural value $\lambda_n = \omega_n^2$ " and δu any class C field. We can then check, by noting that $\int \sigma_{\alpha\beta}(u)\,\epsilon_{\beta\alpha}(\delta u)$ and $\int \rho u\,\delta u$ are symmetrical bilinear forms, that the term in ε of the development of $[\omega^2]$ is zero. Stated otherwise, we note that the function (7) defined over $\mathcal{C}$ is stationary at points u_n of C and that the value taken by this function at such points is equal to ω_n^2. The reciprocal is direct; to give a general problem statement that we will use as a starting point for the analysis of the conservative dynamic systems, we introduce two symmetrical bilinear forms $K(u,v)$&$M(u,v)$ defined for a class of acceptable forms C verifying certain conditions at the essential limits. We will call the natural form u_n of the dynamic system described by K and M any stationary point of the function $[\omega^2]$ defined below:

$$\begin{cases} [\omega^2] = \dfrac{K(u,u)}{M(u,u)} & u \in \mathcal{C} \\ \delta[\omega^2] = 0 & \end{cases} \qquad (8)$$

(formal way of writing the condition for the standing state)

The value of $[\omega^2]$ at such points of C is the square of the associated natural pulse ω. (For the general study of the dynamic systems, we will use the standard printing character u to designate an acceptable form of C). /5

1.2. Classification of the types of an evolution problem.

We will now consider a conservative system depending continuously on a real parameter τ and we propose to analyze how the natural forms u_n and the natural pulses ω_n evolve as a function of τ.

We are led to distinguish between two main categories of mathematical problems.

Category A: "evolution of operators".

By definition, it includes cases where the C class of the admissible forms is independent of τ and where operators K and M are dependent on τ.

Let us note that stating that C is constant means in practice that the domains (volume, surface, ...)over which the unknown fields of the problem generally noted as "u" are defined are constant, and also that the conditions at the essential limits satisfied by u are constant.

As a typical example of a problem of this category, we can mention the problem of the determination of the vibration modes of a homogeneous body occupying a volume Ω and embedded over a fixed portion Γ of its boundary as a function of the density of this body, the elastic characteristics (behavior equation) being assumed as constant. Moreover, in this case, only the operator $M(u,v)=\int_\Omega \rho u v$ depends (linearly) on $\tau = \rho$.

Category B: "evolution of the class C".

We group in this category the problems of the perturbation of conditions at the essential limits (the domains remaining constant) and the problems of the perturbation of the boundaries of domains for which the domains depend continuously on a parameter τ.

In the rest of this report, we will only consider problems of the first category. Yet, the example of liquid-propellant launchers that we propose to treat seems to typically belong to the second category! Nevertheless, we will show that through a judicious choice of the unknown fields, we can go back to a problem of the evolution of operators.

The first question that we wish to answer is the following: when τ varies, can two distinct natural frequencies for $\tau = 0$ that remain continuous as a function of τ merge for a particular value of τ ? In other words, can there be a frequency crossing in the evolution curves for the natural frequencies of the dynamic system as a function of τ ?

We observe that such a crossing (Figure 1) must be accompanied by a degeneracy (or multiplicity) of a natural frequency.

Therefore, we are naturally led to analyze under which conditions such a degeneracy is predictable.

1.3. Degeneracy and symmetry

We can say that as a general rule the degeneracy of the natural frequencies of a system is closely linked to the symmetries of this system. To define this link, we will now recall a few results from the "group theory" (ref. 3). /6

Let G be the group of the symmetries that keeps a dynamic system constant. In concrete terms, in the case of a bounded system, the possible symmetries are by necessity one of the types: rotations about an axis, planes of symmetry or center of symmetry, with the exception of translations for obvious reasons.

Let us then designate as E_ω the entire set of natural forms corresponding to a natural pulse ω. Let us choose within E a base $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ of natural forms so that any natural form $\phi \in E_\omega$ can be represented in a unique manner through a linear combination of these n forms (n is by definition the multiplicity of ω).

Under these conditions, to state that the system studied is fixed with a $\sigma \in G$ symmetry means that if φ is a natural form, the $\sigma\phi$ form obtained "by forcing the symmetry operation σ on the natural form ϕ " is also a natural form for the same frequency and, consequently, belongs to E_ω. For any $\phi \in E_m$. We can therefore write $\sigma\phi$ in the form of a linear combination of $\phi_1, \phi_2, \phi_3, \ldots \phi_n$.

We thus verify that for any element of symmetry $\sigma \in G$ there is a certain linear substitution denoted as $T\sigma$ within E_ω .

Such an identity $\sigma : T \to T\sigma$ which associates any σ element of the G group, a linear substitution in the vector space E_ω is called a linear representation of the G group within E_ω.

We can then attempt to decompose E_ω into a sum of separate spaces $E_\omega^{(1)} + E_\omega^{(2)} + \ldots + E_\omega^{(p)}$ which are independently constant within G.

We show that there is such a unique ultimate decomposition and we say that it provides the reduction of the representation T inside E_ω.

The identity $\sigma : T \to T\sigma$ induced in each "irreducible sub-space" $E_\omega^{(j)}$ is called an irreducible representation.

The dimension of the $E_\omega^{(j)}$ space is called the dimension of the corresponding irreducible representation.

The theory of the linear representation of the groups has for framework the vector spaces on the body of the $\mathbf{C}$ complexes, and to an irreducible representation of dimension s by $\mathbf{C}$ there can correspond some physically irreducible representations of dimension s or $2s$ in the vector spaces on the body of the real numbers like the E_ω space considered above.

This remark is significant when this theory is to be applied to the classification of the natural modes of a conservative system and to the anticipation of the multiplicity of the natural frequencies.

In the discussion that will follow we will use the following two results:

The natural modes of a conservative dynamic system depicting a group with G symmetry are classified by type depending on the irreducible representations T_j in G.

- The n degeneracy of a natural space E_ω of type T_j is simply connected to the dimension s_j of the irrreducible representation considered ($n = s_j$ or $n = 2s_j$ depending on the cases).

In order to illustrate the purely geometrical meaning of the normal degeneracies of the natural frequencies, let us mention two theorems valid for the finite groups.

- Burnside's Theorem: the sum of the squares of the dimensions of all the distinct irreducible representations of a finite group is equal to the number of elements of that group.

- The number of irreducible representations of a finite group is equal to the number of classes of that group (let us recall that we call a class from a G group a sub-set of G composed /7
of elements that are "conjugate" elements of a given element σ, that is to say generated by $x^{-1}\sigma x$ x covering G; the

set of classes constitutes a partition of the group).

The preceding discussion enables us to consider some cases of evolution of the dynamic systems necessarily leading to theoretical mode crossings: that is the case of systems such that for a certain value of τ, a sudden change of symmetry takes place.

For example, let us consider the modes of vibration of a rectangular plate with sides a and b as a function of $\frac{b}{a} = \tau$ (a constant). We observe that for $\tau = 1$ (square plate) the symmetry group of the structure acquires a fourth-order axis of rotation perpendicular to the plate at its center. But it happens that the symmetry group of the square plate has real irreducible two-dimensional representations whereas the symmetry group of the rectangular plate allows only real irreducible one-dimensional representations. Consequently, we expect, on the basis of these purely geometrical considerations, that for $\tau = 1$, certain vibration frequencies of the rectangular plate merge in pairs.

- We eliminate in the rest of our report such a case of theoretical crossings by considering only evolutive dynamic systems that have a symmetry group independent of τ.

In accordance with a remark made above, we can classify the various natural modes of a dynamic system according to the various irreducible representations of the symmetry group.

- We can demonstrate that the determination of the natural modes of distinct types T_j can take place by means of distinct variational formulas of type (8); in addition, we can arrange it so that each natural frequency with a given natural multiplicity (which only depends on the irreducible representation T_j) appears only once among the solutions of this variational formula.

We will not consider here the considerable practical consequences of these last statements with regard to the computation of the vibration (or buckling) modes of structures displaying symmetries (repetitive, for example); we will limit ourselves to draw the following conclusion that is useful for the purpose of this report:

Given an evolutive dynamic system with a constant geometry G; then, the natural modes and frequencies of distinct types T_j being governed by distinct mathematical problems, nothing prevents the natural frequencies of type T_j to "cross" natural frequencies of type $T_k \neq T_j$.

1.4. Definition of the evolution problem

In order to set aside from the discussion the two cases of theoretical crossings analyzed above, when we again talk of the evolution of natural frequencies within this report, we will mandatorily imply that we are considering the evolution of the natural frequencies and forms of a given T_j type for a dynamic system of a fixed symmetry group G.

Let us consider a diagram showing the evolution of natural frequencies as a function of a parameter τ as depicted in Figure 3.

We propose to analyze the evolution of the natural modes and frequencies in the neighborhood of a value of τ that we choose as equal to zero, thanks to a judicious translation of the parameter τ. /8

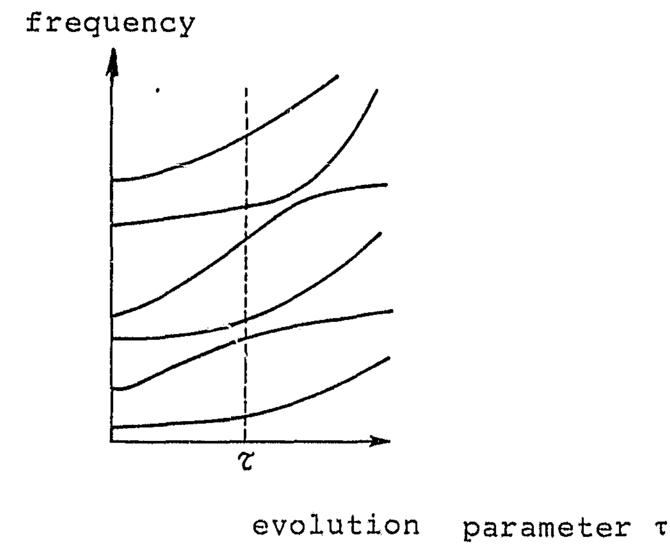


Figure 3. - Evolution of natural frequencies

In accordance with a remark in the preceding paragraph, we will assume, without the restriction of a generality, that formula (8) has been selected so that the natural frequencies are not degenerated (except perhaps at frequency crossings, if they exist !?).

In order to simplify notations, we will also assume that only the operator M is dependent of τ and we will even limit ourselves to a linear dependency (these two simplifications are optional and the reader will be able to adapt the following analysis to the general case). We will also use the notation λ_i for the natural value that is the square of a natural pulse ω_i.

For $\tau=0$ the natural values $\lambda_i^{(0)}$ are governed, as well as the corresponding varational forms $u_i^{(0)}$,by the variational principle:

$$0=\delta[\lambda]=\delta\frac{K(u,u)}{M(u,u)};\qquad u\in\mathcal{C} \tag{9}$$

For $\tau\neq 0$ the natural values λ_i^* are governed, as well as the new forms obtained u_i^*, by the variational principle

$$0=\delta[\lambda^*]=\delta\frac{K(u,u)}{M(u,u)+\tau V(u,u)};\qquad u\in\mathcal{C} \tag{10}$$

τV is the finite perturbation constituted by the increase of the quadratic form $M(u,u)$..

We assume that for $\tau=0$, problem (9) is resolved. It is convenient for the considerations that will follow to assume that the $u_i^{(0)}$ series is normalized so that $M(u_i^{(0)},u_i^{(0)}=1$; indeed, if $v_i^{(0)}$ is a non-normalized series, it is sufficient to define $u_i^{(0)}=v_i^{(0)}/(M(v_i^{(0)},v_i^{(0)}))^{1/2}$. In consideration of this convention, the orthogonality relations through the solutions of (9) are written

$$\begin{cases} M(u_i^{(0)}, u_j^{(0)}) = \delta_{ij} \\ K(u_i^{(0)}, u_j^{(0)}) = \lambda_i^{(0)} \delta_{ij} \end{cases} \qquad (\delta_{ij} \text{ Knonecker's symbol}) \qquad (11)$$

1.5. Variational resolution of the perturbed problem

We wish to resolve problem (10). We then assume that the $u_i^{(0)}$ series makes it possible to represent with as much accuracy as we want an admissible form $u \in \mathcal{C}$. /9

Therefore we can look for the solution of problem (10) in the form of a linear combination

$$u^* = \sum_{i=1}^{N} \alpha_i u_i^{(0)}, \qquad (12)$$

the α_i values being determined by the condition of stationarity of $[\lambda^*]$. It is proper to verify that the "best values of α_i" for a given N satisfy the equation at the following natural values where λ^* is the corresponding natural value.

$$\lambda_i^{(0)} \delta_{ij} \alpha_j - \lambda^* [\delta_{ij} + \tau V(u_i^{(0)}, u_j^{(0)})] \alpha_j = 0 \qquad (13)$$

(i and j varying from 1 to N).

The preceding method of resolution makes it possible, through the introduction of an increasing number N of natural forms of the $u_i^{(0)}$ series, to determine with an arbitrary accuracy the solution of the perturbed dynamic problem.

Also, let us note that [λ*] being stationary at the u_i^* points of **C**, any error of order ε in the natural form u_i^* is not accompanied by an error in ε^2 in the natural value λ_i^*. Therefore, it is reasonable to expect accurate estimates of the natural frequencies, even with relatively rough approximations of the natural forms (ref. 4).

1.6. One-mode approximation

The simplest approximation that we could make to solve (10) consists of stating that $u_i^{(0)}$ is, as a first approximation, a natural form of the perturbed system (10). The corresponding natural value is then obtained by injecting the "test function" $u_i^{(0)}$ into the expression for the Rayleigh quotient [λ*]. This estimate will be annotated $\lambda_i^{(1)}$; taking (11) into account, we therefore have:

$$\lambda_i^* \sim \lambda_i^{(1)} = \frac{\lambda_i^{(0)}}{1 + \tau V(u_i^{(0)}, u_i^{(0)})} \qquad (14)$$

By repeating this computation for each natural form $u_i^{(0)}$, we obtain for each value of τ what we will call the estimate of the evaluation of the natural values in the one-mode approximation. In Figure 4, we have depicted a typical variation of these estimates; let us assume that the examination of the approximate evolution curves thus obtained leads to a frequency crossing for $\tau = \tau_0$ (Figure 4).

In the neighborhood of such a value, it is legitimate to question the validity of the one-mode approximation. We are then led to improve our method for estimating the solutions of equation (10) as follows:

1.7. Two-mode approximation

We wish to examine "under a magnifying glass" the area of the diagram (Figure 4) enclosed by the dotted line. The simplest refinement of the preceding model consists of looking for the solution of equation (10) in the form $\alpha_i u_i^{(0)} + \alpha_j u_j^{(0)}$, i and j being the modes involved by this crossing possibility. /10

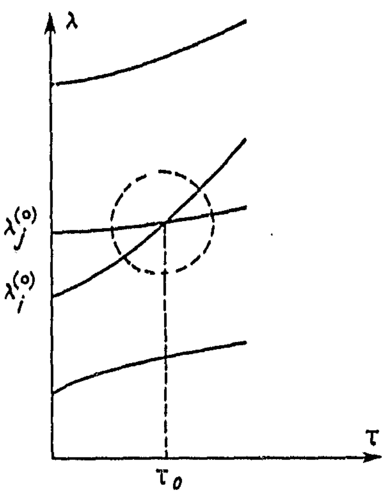


Figure 4.-Evolution of the natural values in the "one-mode" approximation.

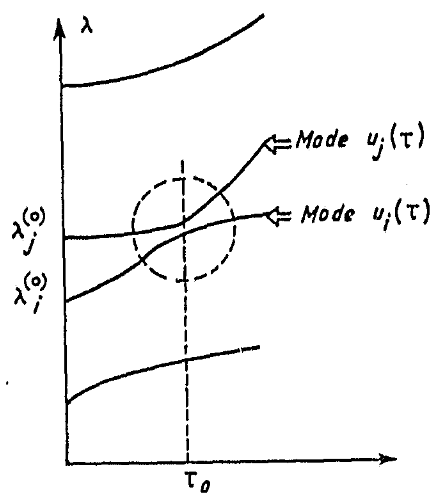


Figure 5.-Evolution of the natural values in the "two-mode" approximation.

The best α_i and α_j verify equation (13) which is explicitly written:

$$\begin{bmatrix} \lambda_i^{(0)} & 0 \\ 0 & \lambda_j^{(0)} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \alpha_j \end{bmatrix} = \lambda \begin{bmatrix} 1+\tau V_{ii} & \tau V_{ij} \\ \tau V_{ij} & 1+\tau V_{jj} \end{bmatrix} \begin{bmatrix} \alpha_i \\ \alpha_j \end{bmatrix} \qquad (15$$

(we have used the abbreviated notation V_{ij} for $V(u_i^{(0)},u_j^{(0)})$).

We note immediately that the preceding couple of equations is formally identical to the equations governing the vibrations of a system composed of two coupled harmonic oscillators. The natural frequencies of the two oscillators in question are precisely the frequencies obtained in the one-mode approximation according to (14).

The detailed discussion of the problem at the natural values being elementary, we will only indicate the resulting consequences within the scope of our analysis; they are easily understood thanks to the analogy that have just mentioned.

- As long as the natural frequencies of the oscillators are sufficiently removed, the effect of the τV_{ij} coupling on the natural modes and frequencies is negligible: in our problem, that means that outside the area bounded by the dotted line, the estimates of the frequencies in the one-mode approach are very close to the exact values.

On the other hand, the τV_{ij} interaction plays a crucial role "when the frequencies of the two harmonic oscillators are very close": in fact, the interaction effect is to reveal a "forbidden frequency range" (or "gap" in English-language terminology) where the one-mode approach allowed for a prediction of a frequency crossing!

We should now check that the simultaneously consideration of an increasing number of $u_i^{(0)}$ modes to solve (10) does not modify the qualitative conclusions depicted in Figure 5. There also remains the analysis of the interaction /11 situations with more than one mode but this is outside the scope of the work that we have set for ourselves here.

The entire prêceding discussion bears on the fact that when $\tau V(u_i^{(0)} u_j^{(0)}$ reaches zero, it cannot do so accidentally. If we had not taken care to consider only modes of a T_j type of a system of symmetry G , it would have been otherwise. Indeed, V is by hypothesis an operator that is constant within G (since it respects the fixed symmetries of the system considered); but there is a theorem in theoretical physics under the name of "selection rule" which is stated as follows:

- Given two natural forms $u^{[a]}$ and $u^{[b]}$ belonging to two natural sub-spaces that transform into two different irreducible representations T_a and T_b of the G group, and given a bilinear operator that is constant within G ; under these conditions $V(u^{[a]},u^{[b]})$ is equal to zero.

It is from this property that we have the uncoupling of the mathematical problems caused by the search for the modes of distinct types that we have previoulsy mentioned. Let us return to the physical interpretation of Figure 5. If we "follow" the natural form $u_j(\tau)$, we note that outside the area defined by the dotted line, this form differs very little from $u_j(0)$ as long as $\tau < \tau_0$ and that it is very close to $u_i(0)$ for $\tau > \tau_0$. On the other hand, for $\tau \approx \tau_0$ it has the hybrid characteristics of $u_i(0)$ and $u_j(0)$.

The reversal of the physical characteristics of the vibration modes takes place naturally for the solution corresponding to the lower branch ($u_i(\tau)$).

This reversal of physical characteristics of the vibration modes of a structure within an interval of variation of the parameter τ which defines its configuration is a characteristic of a two-mode interaction.

1.8. Geometrical interpretation

We can give the following geometrical interpretation of an interaction event which will be useful in discussing the evolution of the effective mass μ_e. When using the approach consisting of having the dependency in τ of the dynamic system rely entirely on $M(u,u)$, it is convenient to consider that the $u_i(\tau)$ series constitutes a base of **C** that is orthogonal to the direction of the scalar product $K(u,v)$ (on the condition of removing from the discussion the rigid body modes that have a zero "length" for the "standard" associated with this scalar product).

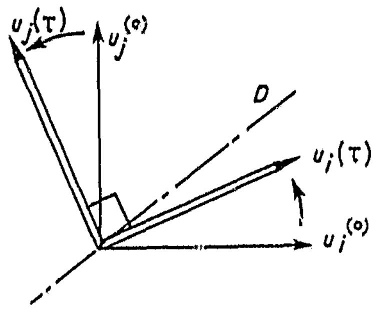
We can then condense the description of the evolution of the two natural mode solutions of (15) by saying that during the interaction, the directions of $u_i(\tau) u_j(\tau)$ perform a simultaneous rotation of about 90 degrees while remaining perpendicular to each other and while remaining located within the plane defined by $u_i(0)$ and $u_j(0)$.

1.9. Evolution of the effective mass during a modal interaction.

We wish to show that during a two-mode interaction, the effective mass μ_e can become infinite for one of the modes. Let us return to the simple case of a structure excited at a point M along a given direction Δ, which was useful to us in introducing the concept of an effective mass (1). Let us then consider within $\mathbf{C}$ all of the admissible forms $\boldsymbol{u}$ such that the displacement $\boldsymbol{u}(M)$ along the direction Δ of the excitation force is equal to zero. All of these represent a hyperplane of $\mathbf{C}$. If we represent the plotted line $\mathcal{D}$ of this hyperplane in the plane generated by $u_i(0)$ and $u_j(0)$ (Figure 6) we note that during the rotation as a unit of the couple of the directions of $u_i(\tau)$ $u_j(\tau)$, the end of one of the two representative vectors crosses the line $\mathcal{D}$ for a given value of τ. Consequently, for this value of τ the effective mass μ_e is infinite! /12

This same geometrical interpretation makes it possible to

Figure 6 . Geometrical representation of the evolution of the natural forms.



interpret the effective mass minimum of the other mode sometimes observed in the neighborhood of a modal intersection. In practice, these properties are not observed with such clarity: that is because we are seldom in the presence of "pure" two-mode interactions and because the interactions with the other modes can alter this simple scheme. We will present in II a concrete example of such "catastrophic" variations of the effective mass resulting from the preceding interpretation.

2. - APPLICATION TO LIQUID-PROPELLANT LAUNCHERS

As we have indicated in the introduction, a liquid-propellant launcher is a typical example of an evolutive system (and it is within this frame of reference that we have approached the problem); let us recall that in the case of the Ariane launcher the mass of the propellants at first-stage ignition represents about 90% of the total mass of the vehicle!

Moreover, we can assume with good reliability that such a dynamic system is symmetrical about its axis, that is to say, in a precise way, that it has as G symmetry group, the rotations of any angle about the axis of the launcher and all of the planes of symmetry passing through this axis. For a system displaying such symmetries, we know that the modes are classified by categories depending on the value of a circumferential whole number n which constitutes a numbering of the different irreducible representations (real) of G. In accordance with the general conclusions presented in I, nothing prevents the natural frequencies of modes with different n's to cross each other.

In this case, we are mainly interested in the evolution of the vibration modes called longitudinal or axisymmetrical ($n=0$). The corresponding irreducible reproduction of G being one-dimensional, these modes have a natural degeneracy of 1. The practical interest of these modes in the discussion of the pogo instabilities of the launcher lies in the fact that they are the only ones that have a pressure fluctuation along the axis of the vehicle, in the immediate neighborhood of which the fuel lines of the propulsion systems are actually located. /13

2.1. Hydroelastic vibrations: generalities

From a mechanical standpoint a liquid-propellant launcher can be represented by an elastic structure containing a liquid that we will assume to be incompressible and perfect.

In order not to burden this report we will not consider here the effects of the stresses induced by the internal pressurization (reference 5) in particular, as well as the accoustical effects associated with the compressibility of the liquid and of the gas used for pressurization (reference 6). In addition, we will neglect the effects due to gravity g and to the launcher acceleration γ .

This last approximation is fundamental to the analysis that will follow and therefore requires several clarifications. Given a hydroelastic system in an apparent gravity field $g+\gamma$ (we neglect coriolis forces); it is convenient to introduce two types of vibration modes defined as follows:

- The hydroelastic modes computed by neglecting the potential energy terms associated with $g+\gamma$.

- The sloshing modes of the liquids computed by assuming the structure to be rigid and immobile.

These two series of modes that we can consider as near-solutions to the entire problem must be considered as constituting a base for the admissible forms $\boldsymbol{G}$ of the real problem or, if we wish, as a system of "generalized coordinates" for the solution of this problem.

The coupling between these near-solutions coming from satisfying the variational principle of the exact problem then leads to the discussion of a problem with the natural values of type (13). With a one-mode approach, we will say that the effect at the lowest order of this coupling leads to a renormalization of the natural frequencies of each mode; with a multi-mode approach, we will speak of an interaction between hydroelastic modes and sloshing modes. An exact pogo mode can then be described as a hydroelastic mode "surrounded" by sloshing modes. This representation of the vibration modes of complex structures based on the description of the resonant excitations in quantum physics can then be applied to the computation of the effects of apparent gravity $g+\gamma$ on the pogo modes of a liquid-propellant launcher in flight or of modes undergoing vibration testing.

2.2. Hydroelastic vibrations: variational formula.

We will now show a heuristic derivation of the variational principle governing the hydroelastic modes in "weightlessness". With regard to the rigorous derivation of the equations for this problem and their variational treatment, the reader will consult reference [8].

To be able to use the method developed in Part I, we need a type (8) variational principle defined over a class of acceptable forms that does not depend on the fuel level of the elastic structure.

Figure 7 graphically depicts an elastic medium occupying a domain Ω in contact with a liquid with a density of ρ_F, occupying a domain Ω_F, along a surface Σ. Γ designates the exposed (or free)surface of the liquid (we will again use the bold letter $\boldsymbol{u}$ to designate a vector field).

Our objective now is to modify the Rayleigh principle (7) governing the vibrations of an elastic body so as to take into account the effect of the liquid on the wall Σ of the elastic structure. To the denominator $\frac{1}{2}\int_{\Omega} \boldsymbol{u}^2$, there must /14 therefore be added an additional term $\frac{1}{2}\int_{\Omega_F} \rho_F\, \boldsymbol{u}_F^2$ associated with the displacement $\boldsymbol{u}_F$ of the particles of the liquid.

But, in the approximation of an incompressible perfect fluid, u_F derives from a displacement potential φ(reference 8)

$$u_F = \operatorname{grad} \varphi \qquad (15)$$

which has a modal amplitude simply related to the non-stationary pressure p according to (reference 8):

$$p = \rho_F\, \omega^2 \varphi \quad \text{(linearization of the Bernouilli Theorem)} \qquad (16)$$

for g=0

In addition, φ verifies the following equations:

$$\begin{cases} \Delta\varphi = 0 & \\ \dfrac{\partial\varphi}{\partial n} = u\,.\,n & \text{over } \Sigma \\ \varphi = 0 & \text{over } \Gamma \end{cases} \qquad (17)$$

expressing, respectively, the incompressibility of the liquid, the condition of the contact between the wall and liquid over Σ, and the zero value of the fluctuation of the pressure over the exposed surface Γ.

We then note that the preceding equations (17) make it possible to formally "express" φ "as a function" of the displacements of the structure along the contact surface Σ.

Consequently, the following equality,

$$M_A(u\,,u) = \int_{\Omega_F} \rho_F\, u_F^2 = \int_{\Omega_F} \rho_F\, (\mathrm{grad}\,\varphi)^2 \qquad u \in \mathcal{C} \tag{18}$$

defines a quadratic form of the displacements of the elastic structure. The notation M_A has been selected to underline the connection between the quadratic form formally defined by (18) and the concept of added mass (reference 9).

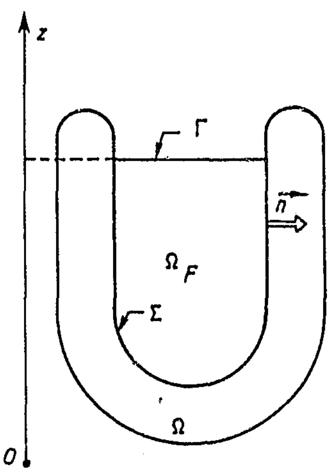


Fig. 7.-Graphical representation of a hydroelastic system.

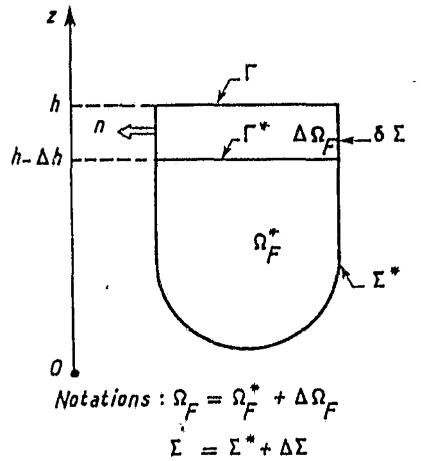


Fig. 8 .-Representation of the liquid domains for two fuel levels.

It is then possible to demonstrate that the stationarity principle

$$0 = \delta[\omega^2] = \delta \frac{\frac{1}{2}\int_\Omega \sigma_{ij}(u)\,\epsilon_{ij}(u)}{\frac{1}{2}M_A(u\,,u) + \frac{1}{2}\int_\Omega \rho u^2} \;; \qquad u \in \mathcal{C} \qquad (19)$$

where $[\omega^2]$ is a function of only the structure displacements, totally defines the natural forms u_i and the natural pulses ω_i associated with the hydroelastic vibration modes

2.3. Evolution of the hydroelastic modes

We now consider a fuel level configuration of an elastic structure temporarily defined by the vertical dimension of the horizontal exposed surface of the liquid. In the new configuration the liquid occupies a domain Ω^*_F, has an exposed surface Γ^* and wets the structure over a surface Σ^*. We designate by means of $\Delta\Omega_F$ (Figure 8) the variation of the fluid domain bounded by the exposed surfaces Γ and Γ^* and the contact surface $\Delta\Sigma$.

Let $M^*_A(u\,,u)$ be the quadratic form of the "added masses" for this new configuration. It is convenient to formally introduce the increase of the quadratic form M_A between the configurations defined by Γ and Γ^* according to:

$$\Delta M_A(u\,,u) = M^*_A(u\,,u) - M_A(u\,,u) \;;\; u \in \mathcal{C}. \qquad (20)$$

The variational problem governing the hydroelastic vibrations u^*, ω^* of this new problem can then be written:

$$0 = \delta[\omega^{*2}] = \delta \frac{\frac{1}{2}\int_{\Omega} \sigma_{ij}(u)\, \epsilon_{ij}(u)}{\frac{1}{2} M_A(u, u) + \frac{1}{2} \Delta M_A(u, u) + \frac{1}{2}\int_{\Omega} \rho u^2}, \quad u \in \mathcal{C}. \qquad (21)$$

By looking for its u^* solutions in the form of a linear combination $\Sigma\alpha_j u_j$ and by noting the orthogonality relations (see equation (11)) verified by u_j, the "best" α_j and the associated natural frequencies verify the following equation analogous to (13):

$$\omega_i^2\, \delta_{ij}\, \alpha_j - \omega^2\, [\delta_{ij} + \Delta M_A(u_i, u_j)]\, \alpha_j = 0 \qquad (22)$$

By using discretization and reduction techniques associated with the method of finite elements (see [8] and [6]) we can compute $\Delta M_A\ (u_i\ , u_j)$ directly and properly reach the numerical solution of the preceding natural value problem over a truncated base of natural forms $u_0\ , u_1, \ldots u_N$. This method is being developed at ONERA for the computation of the hydroelastic modes of liquid-propellant launchers. It should make it possible to compute "by finite elements" (very accurate but rather burdensome method) only a limited number of launcher configurations and to have access to the dynamic characteristics for intermediate configurations through this simple variational method and we could also show that it is related to the methods called "of dynamic sub-structuring". /16

Within the scope of this report, we will again simplify problem (19) and establish an analytical expression of ΔM_A, valid for Γ^* configurations very close to Γ.

We start from an exact expression of $\Delta M_A(u,u)$ obtained by using definition (18) of M_A and the properties of φ and φ^* expressed by equations (17):

$$\Delta M_A(u,u) = \int_{\Omega_F^*} \rho_F(\text{grad}\,\varphi^*)^2 - \int_{\Omega_F} \rho_F(\text{grad}\,\varphi)^2 = \int_{\Gamma^*} \rho_F \varphi \frac{\partial \varphi^*}{\partial Z} - \int_{\Delta\Sigma} \rho_F \varphi u.n \qquad (23)$$

φ^* is of course the solution of equations (17) for the same value of u $\mathbf{C}$, but for the Ω_F^* domain bounded by Γ^* and Σ^*).

We now propose not to keep the Δh terms of equation (23) and to neglect the higher-order terms.

Discussion of the term $\int_{\Gamma^*} \varphi \frac{\partial \varphi^*}{\partial Z}\Big|$; in the neighborhood of Γ, taking into account the zero value condition of φ over Γ (17), we can, within this order of approximation, replace φ with $\Delta h \frac{\partial \varphi}{\partial Z}\Big|_\Gamma$ and neglect the difference between $\frac{\partial \varphi}{\partial Z}\Big|_\Gamma$ and $\frac{\partial \varphi^*}{\partial Z}\Big|_{\Gamma^*}$.

Discussion of the term $\int_{\Delta\Sigma} \varphi\, u.n$: $\Delta\Sigma$ is "of the same order as" Δh ; the same for φ ; consequently this term brings only negligible contributions.

Finally, we will write for very small changes in level Δh:

$$\Delta M_A(u,u) \sim -\Delta h \int_\Gamma \rho_F \left(\frac{\partial \varphi}{\partial Z}\right)^2 \qquad (24)$$

To discuss the practical interest of this approximated expression, let us now consider the case of a liquid-propellant launcher in flight. We usually introduce the draining rate parameter τ to characterize the fueling configuration of the operating stage. This τ parameter is defined as follows: given M_0 the total mass of fuel at ignition of the operating stage considered, and M the mass of the fuels of that same stage for a certain flight configuration, then, by definition:

$$\frac{\tau}{100} = 1 - \frac{M}{M_0}. \tag{25}$$

If M_1 and M_2 respectively designate the initial masses of fuels 1 and 2 $(M_0 = M_1 + M_2)$, it will be useful, later, to introduce the respective mass ratios of fuels 1 and 2:

$$\chi_1 = \frac{M_1}{M_1 + M_2}, \quad \chi_2 = \frac{M_2}{M_1 + M_2}. \tag{26}$$

We can now transform the analytical expression (24) of ΔM_A to approach the study of the evolution of vibrations of liquid launchers. By using the preceding definitions, the matrix elements of ΔM_A that enter into the solution of (22) can then be written, by introducing the notation $\partial\varphi^{(i)}/\partial Z_1$ to designate the vertical displacement of the exposed surface of the tank for fuel 1 for the ith mode, and <> for the average surface value of a quantity: /17

$$\Delta M_A(u_I, u_J) = -\Delta\tau \frac{M_0}{100} \left\{ \chi_1 \left\langle \frac{\partial\varphi^{(I)}}{\partial Z_1} \frac{\partial\varphi^{(J)}}{\partial Z_1} \right\rangle + \chi_2 \left\langle \frac{\partial\varphi^{(I)}}{\partial Z_2} \frac{\partial\varphi^{(J)}}{\partial Z_2} \right\rangle \right\} \tag{27}$$

2.4 Tangent to the frequency evolution curves.

For an infinitesimal small $\Delta\tau = \varepsilon$ of the draining rate, the analytical expression (27) for ΔM_A is correct for order ε, and we can use it within the one-mode approximation, to compute the frequency variation for order ε, and hence, the derivative $\partial\omega / \partial\tau$. By calling F the frequency $\omega/2\pi$, and by designating with μ_g the generalized mass of the mode "being followed" (for an arbitrary normalization), we obtain:

$$\frac{\partial F}{\partial \tau} = \frac{F}{200} \frac{M_0}{\mu_g} (\chi_1 \langle \eta_1^2 \rangle + \chi_2 \langle \eta_2^2 \rangle) \tag{28}$$

where $\langle \eta_1^2 \rangle$ (resp $\langle \eta_2^2 \rangle$) designates the average surface quadratic value of the vertical displacement of the exposed surface of the tank (resp 2).

This last expression (28) can be subjected to a direct physical interpretation which simultaneously thrusts light on the approximation (24)(27). Indeed, we can verify that the terms inside parentheses give, for a variation in the draining rate, the amount of mass removed at the exposed surface of each reservoir where the modal amplitude of the displacements (vertical) is precisely η_1 and η_2 (resp). From this standpoint (28) is nothing but "the formula of the displaced frequencies) well known to experimenters. (we note the expected result $\partial F/\partial v \geq 0$ expressing the fact that during the emptying of the tanks, the natural vibration frequencies can only increase).

2.5. Hydroelastic modal interactions.

Expression (27) clearly shows why two hydroelastic modes of the same type, let us say longitudinal for the purpose of illustration, necessarily "interact" during the evolution. Indeed, the average values of the products of vertical displacements of the exposed surface of each tank do not generally add to zero when we consider any two modes.

2.6. Application to the flight of the Ariane launcher.

We have numerically applied the solution to equations (22) simplified by the analytical approximation (27) and we have applied it to the case of first-stage combustion (L 140) of the Ariane launcher.

Figure 9 represents the results of computations by finite elements (reference 10) of the asymmetrical hydroelastic modes of the whole launcher for four flight configurations of the first stage (points o). On this same figure, arrows have been placed to reflect the tangents to the frequency evolution curves computed by means of analytical expression (28) by using the displacements of the exposed surface and the generalized masses resulting from computations by finite elements (reference 10).

Finally, the numerical results obtained thanks to the method of mode interaction that we have just presented have been indicated by means of crosses. The continuous curves /18 represent the most reasonable frequency evolution curves that we could plot considering the additional information obtained by means of the modal interaction model.

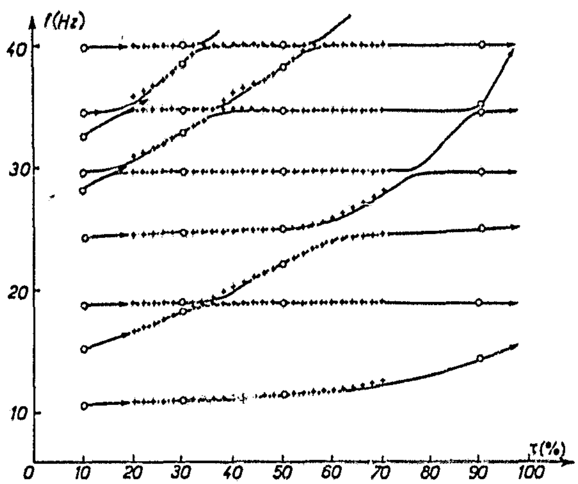


Figure 9.- Evolution diagram of the axisymmetrical vibration frequencies of the Ariane launcher (first-stage combustion).

o : computation by finite elements

+ : variational resolution of the approximate evolution problem.

Considerations to be noted:

- With respect to the bold move consisting of describing ΔM by means of its linear portion (27) for such an amplitude in the draining rate variation, the agreement between the estimates obtained from different reference configurations (30% and 50%) can be considered as very satisfactory, at least for the first hydroelastic modes.

- We observe a "strong" modal interaction between modes #3 and #4 between the 50% and 70% configurations.

- Diagram 9 includes modes of a practically constant frequency. This means, from (28), that in practice such modes are not accompanied by any displacement of the exposed surface inside the tanks in use. The examination of the natural forms then makes it possible to verify that these modes are "localized" to the structures of the upper stages.

- These "constant" modes interact correlatively very little with the others (very small $\Delta M_A(u_i, u_j)$): we are then in the presence of weak mode interactions giving rise to an extremely small frequency interval. Therefore, we can eliminate without risk such modes from the study of the evolution. This remark must make it possible to consider the computation of the pogo modes of a liquid-propellant launcher on the basis of pogo modes computed for a given configuration and selected as a function of the sufficiently rapid evolution criteria based on equation (28). This observation will be used profitably in the computations of finite evolution of the pogo modes by means of the method described above and which must lead to a "model with n significant pogo modes" of a liquid-propellant launcher. /19

2.7. Evolution of the effective masses.

Plate 10 depicts the evolution of the effective mass of the modes (defined for a normalization along the vertical displacement of the first-stage engine assembly) for an increasing number of "test forms" injected into the approximate variational principle. We note, and that is predictable, that while taking into consideration a relatively restricted number of modes makes possible a reasonable estimate of the natural frequencies, the precise estimate of the evolution of the effective masses requires the simultaneous taking into consideration of a large number of natural forms. /20

These diagrams make possible the observation of effective mass singularities and of minimum values of μ_ℓ for certain modes, in the neighborhood of a strong mode interaction.

N.B. The evolution of the first pogo mode (at non-zero frequency) is sensitive to the overall translation mode at a zero frequency (mode #0). The reader will be able to check that the interaction between this translation mode and the hydroelastic modes resulting from satisfying the variational principle (19) is directly associated with the principle of the conservation of momentum which, in the case of small motions of a free system, leads to the immobility of the center of gravity of this system.

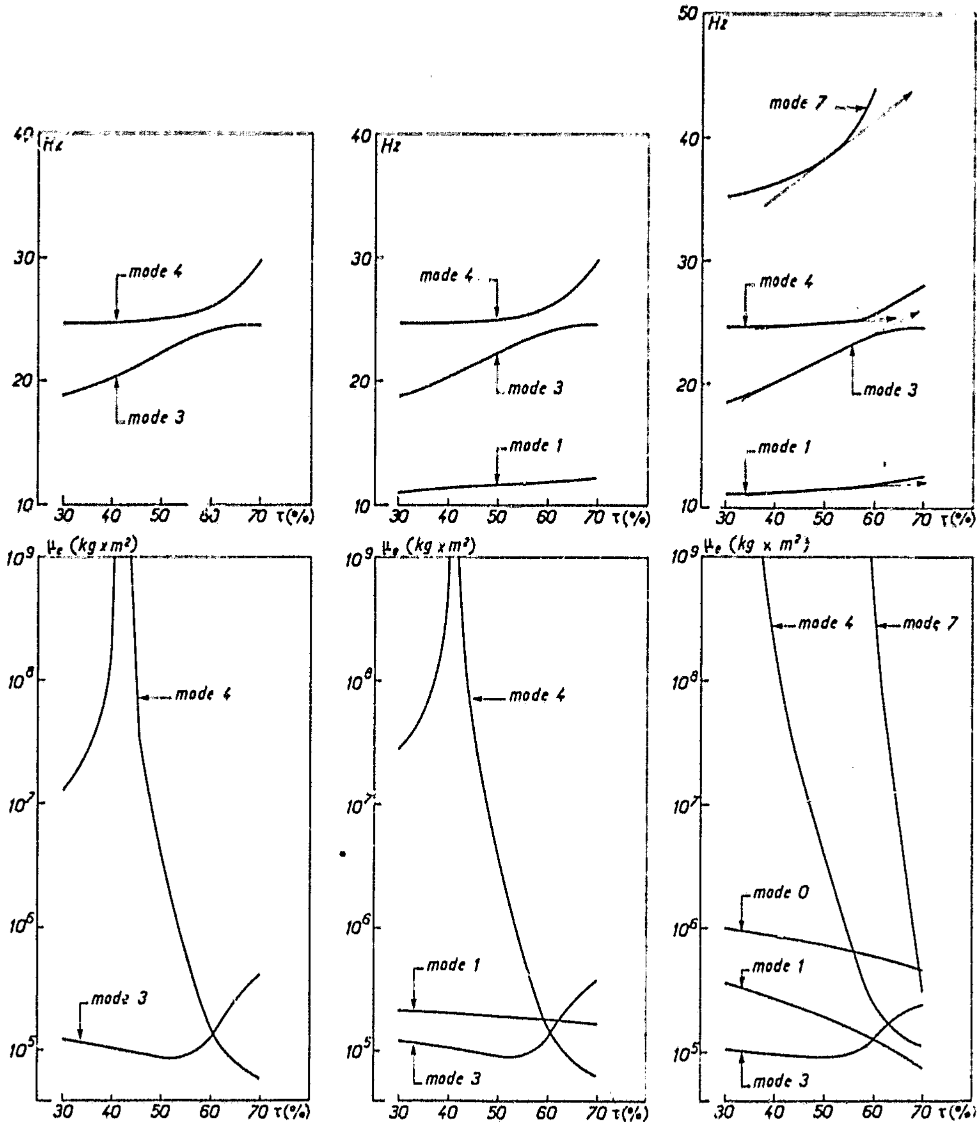


Figure 10. - Variational extrapolation of the vibratory characteristics of the Ariane launcher in the neighborhood of configurations computed by finite elements. Evolution of the frequencies F and of the effective masses μ_e as a function of the number of test natural forms.

CONCLUSION

During this study, we have used as theoretical cases of frequency crossings the case of the merging of natural frequencies of modes with different types of symmetry, and the case of the merging of natural frequencies of certain modes of a system for a particular configuration corresponding to a sudden increase of the symmetries.

Outside of these cases, frequency crossings can constitute only accidental events by comparison with the modal interactions revealed by the frequency evolution curves of the modes of a given type of symmetry, in the case of highly evolutive systems such as liquid-propellant launchers, missiles, etc...

This study seems to us to be equally as useful for the analysis of the modal interactions caused by a perturbation brought on by the measurement itself in the case, for example, of the experimental technique of shifted frequencies.

The variational method for treating evolutive conservative dynamic systems can be also applied to cases where the study of secondary effects is involved such as a change in the rigidity of a launcher structure resulting from tank pressurization (the τ parameter is then the internal pressure).

This method could then be extended to the case of non-conservative systems (by dropping variational reference (8)), a problem of this type arising particularly in the case of aeroelastic instabilities of aircraft (reference 11).

Finally, it is indispensable to mention that real structures do not have a perfect symmetry and that, consequently, the crossings of theoretical frequencies can be observed experimentally only as events of "weak" modal interaction which result, therefore, as a forbidden frequency band that is narrower with a smaller departure from perfect symmetries. These remarks must be taken into consideration to properly approach the problems of mode identification. /21

REFERENCES

1. Aviation Week & Space Technology, McGraw Hill Publication, February, 1969. Barrere, M., Bouttes J., Dordian J.J., Incidence of Combustion Phenomena on the Pogo Effect. La Recherche Aerospatiale, No. 1974-1.

2. Germain, P. Mecanique des milieux continus (Mechanics of Continuous Media). Masson, 1962.

3. Lyubarskiy, G.Ya. The Application of Group Theory in Physics. Pergamon Press, 1960.

4. Morse-Feshbach. Methods of Theoretical Physics, Part 2. McGraw Hill Book Company Inc.

5. Morand, J.H.P., Ohayon, R. Internal Pressure Effects on the Vibration of Partially Filled Elastic Tanks: World Congress on Finite Element Methods in Structural Mechanics. Bournemouth, Dorset, England, October 1975.

6. Morand, H.J.P., Ohayon, R. Investigation of Variational Formulations for the Electroelastic Vibration Problem. Submitted to the Second International Symposium on Finite Element Methods in Flow Problems. Rapallo, Italy, June 14-18, 1976.

7. Kittel, C. Quantum Theory of Solids. John Wiley & Sons, 1963.

8. Berger, H., Boujot J., Ohayon, R. On a Spectral Problem in Vibration Mechanics: Computation of Elastic Tanks Partially Filled with Liquids. Journal of Mathematical Analysis and Applications, Vol. 51, No. 2, August, 1975.

9. Zienkiewicz, O.C. The Finite Element Method in Engineering Science. McGraw Hill, 1971.

10. Berger H., Morand H.J.P., Ohayon R., Paquignon P. Vibrations axisymetriques du lanceur Ar iane (Axisymmetrical Vibrations of the Ariane Launcher). ONERA Document, unpublished, 1975.

11. Bisplingoff, R.L., Ashley, Halfman. Aeroelasticity. Addison Wesley Publishing Co. Inc., 1955, Cambridge.