NASA TECHNICAL MEMORANDUM NASA TM-77770

WAVE GUIDE BRANCHINGS (n-PORT) TYPES OF REPRESENTATION OF LINEAR, TIME-INDEPENDENT AND SOURCELESS n-PORTS

Hans Brand

Translation of "3. Wellenleiterverzweigungen (n-Tore): 3.4. Darstellungsarten Linearer, Zeitunabhaengiger und Quellenloser n-Tore", from the book: "Schaltungslehre Linearer Mikrowellennetze", (Circuit Studies of Linear Microwave Networks), Stuttgart, S. Hirzel Verlag, 1970, pp. 118-137.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 JANUARY 1982

(NASA-TM-77770) WAVE GUIDE BRANCHINGS (n-PORT) TYPES OF REPRESENTATION OF LINEAR, TIME-INDEPENDENT AND SOURCELESS n-PORTS (National Aeronautics and Space Administration) 26 p HC A03/MF A01 CSCL 20N G3/32

N85-10233

Unclas
24247

STANDARD TITLE PAGE

1. Report No. NASA TM-77770	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle WAVE GUIDE BRANCHINGS (n-PORT) TYPES OF REPRESENTATION OF LINEAR, TIME-INDEPENDENT AND SOURCELESS n-PORTS		5. Report Date ~~January 1982~~ 6. Performing Organization Code
7. Author(s) HANS BRAND		8. Performing Organization Report No. 10. Work Unit No.
9. Performing Organization Name and Address SCITRAN 1482 East Valley Rd., Santa Barbara, CA 93108		11. Contract or Grant No. ~~NASW-3542~~ 13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20456		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "3. Wellenleiterverzweigungen (n-Tore): 3.4. Darstellungsarten Linearer, Zeitunabhaengiger und Quellenloser n-Tore", from the book: "Schaltungslehre Linearer Mikrowellennetze", (Circuit Studies of Linear Microwave Networks), Stuttgart, S. Hirzel Verlag, 1970, pp. 118-137.

16. Abstract

The wave representation and scattering matrix of n-ports is discussed.

ORIGINAL PAGE IS OF POOR QUALITY

17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified-Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 26	22. Price

NASA-HQ

3.42 Wave representation, scattering matrix

If the n-port is described by a relationship between input and output waves, then we talk about a wave representation. We have already repeatedly pointed out that this representation is particularly suited to the treatment of microwave switching. The scalar variables of state of wave representation are the input and output voltages U_k^+, U_k^-, respectively the input and output currents I_k^+, I_k^- or the input and output variables a_k, b_k defined in section 2.52. ** These reduced variables of state a_k or, respectively, b_k--the half-squared quantity of which represent the actual output transmitted by the input or, respectively, output wave (cf. equation 2.5/22)--are particularly descriptive, physically, precisely because of this direct relationship to the actual output transmitted as already mentioned in section 2.52. In addition, they are more suitable than current and voltage "waves" $I_k^\pm$ and $U_k^\pm$ in many instances of microwave measurement technology. Hence, in what follows we shall base the description and calculations of microwave n-ports and networks on these wave variables. */118

Now we must establish a suitable pattern to inter-relate the 2n wave variables a_k, b_k. From it will depend the physical meaning of the n^2 coefficients, which will then represent the n-port in the scalar relations. To begin with, we shall investigate a pattern in which the output waves at each port are described explicitly as a function of all input waves. In that case, the n linear equations are

$$
\begin{aligned}
b_1 &= s_{11} a_1 + s_{12} a_2 \ldots + s_{1k} a_k \ldots + s_{1n} a_n, \\
b_2 &= s_{21} a_1 + s_{22} a_2 \ldots + s_{2k} a_k \ldots + s_{2n} a_n, \\
&\;\vdots \\
b_k &= s_{k1} a_1 + s_{k2} a_2 \ldots + s_{kk} a_k \ldots + s_{kn} a_n, \\
&\;\vdots \\
b_n &= s_{n1} a_1 + s_{n2} a_2 \ldots + s_{nk} a_k \ldots + s_{nn} a_n
\end{aligned}
\qquad (3.4/3a)
$$

* Numbers in the margin indicate pagination in the foreign text.

** Translator's note. The reader is referred to the original foreign text.

or, in matrix notation,

$$\begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \cdots & s_{1k} \cdots & s_{1n} \\ s_{21} & s_{22} & s_{2k} & s_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ s_{k1} & s_{k2} \cdots & s_{kk} & s_{kn} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ s_{n1} & s_{n2} \cdots & s_{nk} \cdots & s_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix}. \qquad (3.4/3b)$$

The n-row square matrix of the coefficients s_{kl}

$$S = \begin{pmatrix} s_{11} & s_{12} \cdots & s_{1k} \cdots & s_{1n} \\ s_{21} & s_{22} & s_{2k} & s_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ s_{k1} & s_{k2} \cdots & s_{kk} & s_{kn} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ s_{n1} & s_{n2} \cdots & s_{nk} \cdots & s_{nn} \end{pmatrix} \qquad (3.4/4)$$

is designated "scattering matrix" or, sometimes also distribution matrix. Correspondingly, the coefficients s_{kl} are called coefficients of the scattering matrix or, in short, scattering coefficients. The column matrices

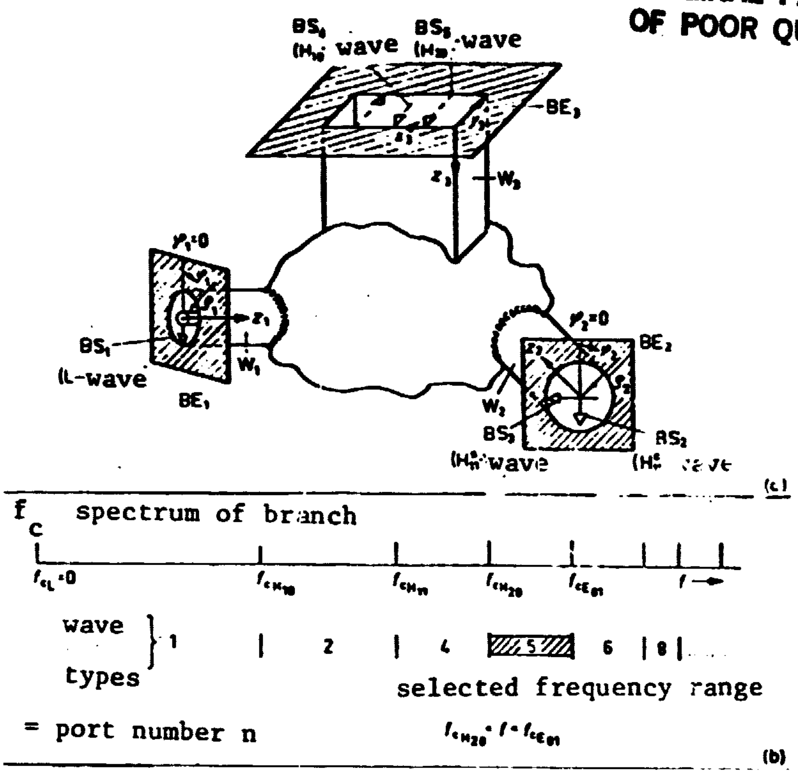
$$A = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \quad (3.4/5a) \quad \text{or} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_k \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} \quad (3.4/5b)$$

encompass the totality of the input wave variables a_k, respectively that of the output wave variables b_k in a fixed arrangement. Thereby they obtain the meaning of independent, n-dimensional variables of state. The n equations 3.4/3a,b can then be expressed, using those abbreviations, as

$$B = SA \qquad (3.4/3c)$$

(see also [B13], [B15] to [B17] and [7] to [11]).

What, then, is the physical meaning of the coefficients s_{kl} in the pattern established by equations 3.3/4, respectively 3.4/4. To answer this question we shall perform, in our imagination, a series of characteristic measurements, looking for an example at the branching sketched in Figure 3/7.

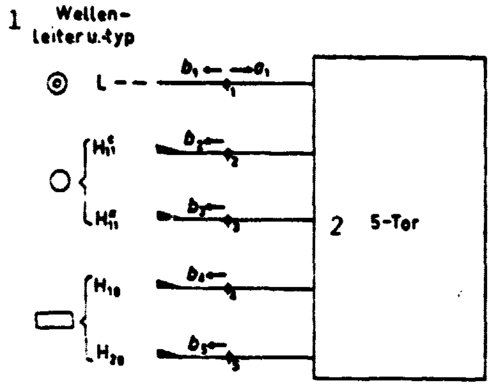


Port configuration (Wavetype numbering) for selected frequency range

Type no = port no.	name of type	reference plane	structure function
1	L	BE_1	$t_1 = t_L(\rho_1, \varphi_1)$
2	H^c_{11}	BE_2	$t_2 = t_{H^c_{11}}(\rho_2, \varphi_2)$
3	H^s_{11}	BE_2	$t_3 = t_{H^s_{11}}(\rho_2, \varphi_2)$
4	H_{10}	BE_3	$t_4 = t_{H_{10}}(x_3, y_3)$
5	H_{20}	BE_3	$t_5 = t_{H_{20}}(x_3, y_3)$

Figure 3/7 Example for a correspondence process for representation of a 5-port by non-structural related state variables;
a) definition of reference planes, coordinate systems and reference structures;
b) selection of a unique frequency range;
c) establishment of port order and structural functions.

The waveguides W_2 (circular waveguide) and W_3 (rectangular waveguide) are terminated by means of suitable ideal absorbers. A source is connected to the coaxial conductor W_1 that emits an L wave with a frequency within the definition range of the 5-port, towards the branching. This wiring can be symbolically represented by means of a replacement wiring diagram with unipolar connectors, as shown in Figure 3/8 below. Here, the function of the absorbers



(1) waveguide and wave type

(2) 5-port

Figure 3/8. For the determination of the scattering matrix coefficients, using the example of a 5-port, assigned according to the branching of Figure 3/7; excitation by means of a wave type at port 1

is represented by means of four separate, reflection-free 1-ports connected to ports 2, 3, 4 and 5. As an aid to memory, in front of each port are noted waveguides and wave types assigned each of them. Due to forced switching (absorber termination), at the reference planes BE_2 and BE_3--and hence, at the ports 2, 3, 4 and 5--there can only exist output waves represented by b_2, b_3, b_4 and b_5; this means that $a_2 = a_3 = a_4 = a_5 = 0$. At port 1, besides the exciting input wave a_1, in general we also have an output wave b_1.

According to equations 3.4/3, this special operating state can be described by the following equations of state:

$$\left.\begin{aligned} b_1 &= s_{11}\, a_1 \\ b_2 &= s_{21}\, a_1 \\ b_3 &= s_{31}\, a_1 \\ b_4 &= s_{41}\, a_1 \\ b_5 &= s_{51}\, a_1 \end{aligned}\right|_{a_2 = a_3 = a_4 = a_5 = 0.} \qquad (3.4/6)$$

Here, the coefficients s_{11} obviously indicates how much of the input wave a_1 is transformed into the output wave b_1, of the same wave type. Hence, in its physical meaning, it is a reflection factor. However, since it is only equal to the ratio b_1/a_1 when no input waves exist at all the remaining ports, it is appropriate to distinguish it also in its designation from factors that characterize the wave ratio b_1/a_1 under general operating conditions. Hence, we shall call s_{11} the self-reflection coefficient or, more briefly, the reflection coefficient of port 1.

Under the same special operating conditions, the coefficient s_{21} in equations 3.4/6, for $a_2 = a_3 = a_4 = a_5 = 0$ indicates how much of the input wave a_1 (here of the L wave type) is transmitted to the output wave b_2 of a different type (here of the H^c_{11} wave). Correspondingly, we call s_{21} the self-transmission coefficient or, in short, the transmission coefficient between ports 1 and 2. The significance and designation of the coefficients s_{31}, respectively s_{41} and respectively s_{51} becomes obvious when we consider the transmission from port 1 to ports 3, 4 and 5, respectively.

We may generalize the interpretation and call
s_{kk} the (self) reflection coefficient of port k
and
s_{kl} the (self) transmission coefficients between ports l and k.

The customary sequence of the two scattering coefficient subindices is more readily remembered, based on the equation $b_k = s_{kl}a_l$ of equations 3.4/6, according to the following rule: the first subindex is equal to the index of the output wave (response), while the second subindex equals that of the input (excitation) wave. The n^2 coefficients s_{kl} are the quantities that characterize the n-port in scattering matrix representation. Since they relate variables of state of the same dimensions in vector form, the coefficients themselves are dimensionless and in general complex and frequency dependent.

It is often necessary to represent an n-port not by means of the n^2 individual scattering coefficients, but by appropriately formed groups of them. Let us become familiar with this method using the branching sketched in Figure 3/7 as an example. The three waveguides provide a meaningful sub-pattern. We shall begin by setting up the equations of state for the 5-port in the notation corresponding to equation 3.4/3b; to this end we shall combine the wave variables of the same waveguide into groups corresponding to the scattering coefficients:

$$\begin{pmatrix} b_1 \\ \hdashline b_2 \\ b_3 \\ \hdashline b_4 \\ b_5 \end{pmatrix} = \left(\begin{array}{c:cc:cc} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ \hdashline s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} \\ \hdashline s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} \end{array}\right) \begin{pmatrix} a_1 \\ \hdashline a_2 \\ a_3 \\ \hdashline a_4 \\ a_5 \end{pmatrix}. \qquad (3.4/7a)$$

ORIGINAL PAGE IS OF POOR QUALITY

Using the following abbreviations for the sub-columns.

$$A_1 = a_1, \quad A_2 = \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_4 \\ a_5 \end{pmatrix} \qquad (3.4/8a)$$

and

$$B_1 = b_1, \quad B_2 = \begin{pmatrix} b_2 \\ b_3 \end{pmatrix}, \quad B_3 = \begin{pmatrix} b_4 \\ b_5 \end{pmatrix} \qquad (3.4/8b)$$

as well as for the sub-matrices

$$\begin{aligned} S_{11} &= s_{11}, & S_{12} &= (s_{12} \ \ s_{13}), & S_{13} &= (s_{14} \ \ s_{15}). \\ S_{21} &= \begin{pmatrix} s_{21} \\ s_{31} \end{pmatrix}, & S_{22} &= \begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix}, & S_{23} &= \begin{pmatrix} s_{24} & s_{25} \\ s_{34} & s_{35} \end{pmatrix}, \\ S_{31} &= \begin{pmatrix} s_{41} \\ s_{51} \end{pmatrix}, & S_{32} &= \begin{pmatrix} s_{42} & s_{43} \\ s_{52} & s_{53} \end{pmatrix}, & S_{33} &= \begin{pmatrix} s_{44} & s_{45} \\ s_{54} & s_{55} \end{pmatrix} \end{aligned} \qquad (3.4/8c)$$

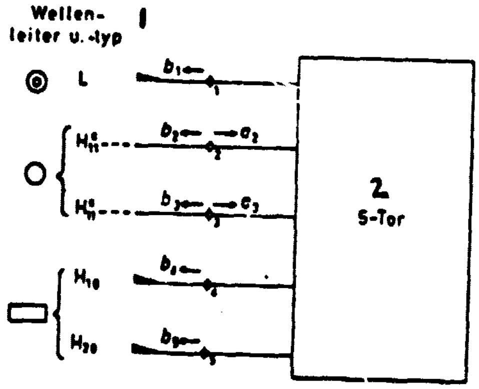
we then obtain, from equation 3.4/7a

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}. \qquad (3.4/7b)$$

In the first place, by this grouping into subcolumns and submatrices we can considerably reduce the writing effort in comparison to the more detailed notation of equation 3.4/7a. The deeper significance of this technique, however, will only become clear when we think of the task to be dealt with in Chapter 4, i.e., branchings with several wave types on the waveguide (multitype waveguides) connected to other, suitable branchings. In the example according to Figure 3/7 considered here, in any wiring of the circular waveguide, the two H_{11} wave types must always be treated as a "package", in wiring. This package "wave type group" is here represented by the two sub-columns A_2 and B_2.

In general, the submatrices relating the sub-columns are rectangular; in this example, both S_{12} and S_{13} are single-row and two-column matrices, and S_{21} and S_{31} are two-row and single-column. In special cases the submatrices may be square, as S_{22}, S_{23}, S_{32} and S_{33} in our example, or even single element matrices, i.e., scalar coefficients such as S_{11} in our example.

The physical meaning of the submatrices shall be discussed, once more, performing an imaginary experiment. Corresponding to Figure 3/9 below, this time we shall connect the coaxial conductor



(1) waveguide and wave type

(2) 5-port

Figure 3/9. To determine the scattering matrix coefficient of a 5-port arranged in conformance to the branching in Figure 3/7; excitation by a wave type group at ports 2 and 3

(i.e., port 1) and the rectangular waveguide (i.e., ports 4 and 5) to suitable terminations, such as ideal absorbers, and feed the branching via the circular waveguide. Here we shall have to assume, in general, that both a H^c_{11} wave (a_2) and an H^s_{11} wave (a_3) approach the branching. In general, there will be output waves at all ports. This special operating condition can be described--now expressed in terms of subcolumns and submatrices--by means of the following equations of state:

$$\left.\begin{aligned} B_1 &= S_{12} A_2 \\ B_2 &= S_{22} A_2 \\ B_3 &= S_{32} A_2 \end{aligned}\right| \; A_1 = 0, A_3 = 0. \qquad (3.4/9)$$

The submatrix S_{22} here describes the interaction of the two H_{11} wave types within the same waveguide. Physically, it represents the reflection of a wave type group when the input waves of all other wave type groups disappear. Referring to the designation self-reflection coefficient, we shall thus call this submatrix self-reflection matrix or, in short, reflection matrix of the wave type group 2.

Submatrix S_{32} describes the transmission from the waveguide W_2 to a different waveguide W_3, i.e., the coupling of a wave type group (H_{10} and H_{20}), again under the same special operating conditions, that the input waves disappear in waveguides W_1 and W_3. Hence, we shall call this submatrix S_{32} self-transmission matrix or, in short, transmission matrix[6] between the port (or respectively, wave type) group at W_2 and the port (or respectively, wave type) group at W_3. Correspondingly, we shall call S_{12} the transmission matrix between the port groups at W_2 and the port group at W_1, which here consists of a single L-type.

We can now once again generalize the interpretation, and call

S_{KK} the (self) reflection matrix of port group K

and

S_{KL} the (self) transmission matrix between port group L and port group K.

---

[6] Let us point out that the submatrix S_{KL} of the scattering matrix called transmission matrix here, is not identical to the wave chain matrix dealt with in section 3.44 which is often also called transmission matrix.

There is always a reflection matrix S_{KK} in the main diagonal of complete scattering matrix, designated as a supermatrix, in this case, since its elements are matrices themselves. In addition, a reflection matrix is always square.

To conclude, we shall consider some transformations of the equation of state 3.4/3. If we resolve it for the column of the input waves A, we obtain

$$\boldsymbol{A} = \boldsymbol{S}^{-1} \boldsymbol{B} \qquad (3.4/10)$$

with the inverse scattering matrix S^{-1}. While a scattering matrix can be indicated, in principle, for every technically feasible, linear, time-independent and sourceless n-port, the inverse scattering matrix exists only in exceptional cases. Hence, it is not generally suitable for the representation of n-ports. In the calculation of n-port networks on the basis of scattering matrix representation, however, the inverse is occasionally required. In each individual case it must then be tested whether S^{-1} exists.

If we transpose the matrix equation 3.4/3c, we obtain

$$\boldsymbol{B}^{T} = \boldsymbol{A}^{T} \boldsymbol{S}^{T} \qquad (3.4/11)$$

with the row matrices A^T and B^T and the transposed scattering matrix S^T (see, for instance [B3]). In comparison to equation 3.4/3, equation 3.4/11 contains no new physical information. It is, however, necessary in several calculations, such as in the formation of scalar products for the determination of the energy balance of n-ports. In this context a representation by submatrices is often useful. In terms of submatrices, the transposed scattering matrix--using the example of equation 3.4/7b--can be written

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}^{T} = \begin{pmatrix} S_{11}^{T} & S_{21}^{T} & S_{31}^{T} \\ S_{12}^{T} & S_{22}^{T} & S_{32}^{T} \\ S_{13}^{T} & S_{23}^{T} & S_{33}^{T} \end{pmatrix}. \qquad (3.4/12)$$

Below we shall consider the scattering matrix representation always as the basic form for the treatment of microwave n-ports and we shall refer all other forms of representation to it.

3.43 Voltage/current representation

When an n-port is represented by means of relationships between voltages and currents, we talk of a voltage/current representation. As is known, this kind of representation is particularly suited to the treatment of switchings of concentrated components. It is of use to microwave technology when the internal structure of a switching--and hence, the behavior of the corresponding n-port--can be predicted on the basis of a replacement wiring diagram of concentrated components. This is often the case especially for branchings of double conductors, used in the L-wave area and for which the dimensions of the "internal" components is still small in comparison to the operating wavelength.

Here we are considering only descriptions by means of reduced voltages u_k and reduced currents i_k, codimensional with the wave variables a_k, b_k and which, according to equation 2.5/12

$$u_k = a_k + b_k,$$
$$i_k = a_k - b_k$$

can be expressed directly by means of them.

Several patterns are known to relate voltages and currents. In the so called impedance form

$$\begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \cdot & \cdot & \\ \cdot & & \cdot \\ z_{n1} & & z_{nn} \end{pmatrix} \begin{pmatrix} i_1 \\ \cdot \\ \cdot \\ i_n \end{pmatrix} \qquad (3.4/13a)$$

or in abbreviated form,

$$u = z\,i \qquad (3.4/13b)$$

the port voltages u_k are given as explicit functions of the port currents i_k. Here,

$$u = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \quad (3.4/14a) \qquad \text{and} \qquad i = \begin{pmatrix} i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_n \end{pmatrix} \quad (3.4/(14b)$$

are the column matrices of the port voltages or port currents, respectively, and

$$z = \begin{pmatrix} z_{11} \cdots z_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \quad \cdot \\ \cdot \quad \quad \quad \cdot \\ z_{n1} \cdots z_{nn} \end{pmatrix} \qquad (3.4/15)$$

is the impedance matrix in normalized notation. The coefficients z_{kl} of this impedance matrix are dimensionless and in general, complex and frequency dependent.

If we solve the equation of state 3.4/13 for the port currents, we obtain the so-called admittance form

$$\begin{pmatrix} i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_n \end{pmatrix} = \begin{pmatrix} y_{11} \cdots y_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \quad \cdot \\ \cdot \quad \quad \quad \cdot \\ y_{n1} \cdot\cdot\ y_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \qquad (3.4/16a)$$

or, in abbreviated form,

$$i = y\,u. \qquad (3.4/16b)$$

The coefficient pattern

$$y = \begin{pmatrix} y_{11} \cdots y_{1n} \\ \cdot \quad \cdot \\ \cdot \quad \quad \cdot \\ \cdot \quad \quad \quad \cdot \\ y_{n1} \quad\ y_{nn} \end{pmatrix} \qquad (3.4/17)$$

is the admittance matrix in normalized notation. Its coefficients y_{kl} are also dimensionless and in general complex and frequency dependent.

From a comparison of the two equations, 3.4/13 and 3.4/16, it follows that the admittance matrix is the inverse of the impedance matrix and that hence we have

$$y z = z y = E \qquad (3.4/18)$$

with the n-row unit matrix E.

The physical meaning of the coefficients of the impedance matrix or admittance matrix, respectively, results from open-circuit load or short-circuit measurements, respectively. If, for instance, we supply the n-port at the 1-port ($i_1 \neq 0$) and leave all other ports in open circuit load mode ($i_k = 0$ for $k = 1, \ldots, n$, $k \neq 1$), then we obtain voltages that are proportional to i_1:

$$u_k = z_{k1} i_1.$$

for k = 1, voltages and currents at the same port are related to each other; hence the quantity z_{11} is a directly measurable impedance that we shall call open circuit load input impedance of port 1. For $k \neq 1$, the coefficients have the meaning of a "current-to-voltage" transmission factor, also called core impedance in the four-pole theory [B11]. The coefficients of the admittance matrix are obtained in a corresponding manner, by supplying port 1 ($u_1 \neq 0$) and short-circuiting all other ports ($u_k = 0$ for $k = 1, \ldots, n$, $k \neq 1$). Thus n currents are obtained, proportional to u_1:

$$i_k = y_{k1} u_1.$$

The quantity y_{11} is here called short-circuit input admittance of port 1. For $k \neq 1$, the coefficients have the meaning of a "voltage-to-current" transmission factor, called core admittance in the four-pole theory [B11].

The column matrices A, B of the scattering form and the column matrices u, i of the impedance or admittance form, respectively, have in common that in each case they contain only variables of state of the same kind, that are arranged in the same sequence in all four cases (port numbering). For this reason, it is easy to convert into each other state columns and the matrices relating them. Instead of the n times four equations 2.5/12a, b and 2.5/13a, b we then obtain the matrix equations

$$u = A + B, \quad (3.4/19a)$$
$$i = A - B \quad (3.4/19b)$$

respectively

$$A = \tfrac{1}{2}(u + i), \quad (3.4/20a)$$
$$B = \tfrac{1}{2}(u - i). \quad (3.4/20b)$$

in the scatter form

$$B = S A$$

we now replace the columns of the wave variables with equations 3.4/20a, b by means of the columns u and i and arrange the equations in terms of these columns:

$$u - i = S u + S i,$$
$$u - S u = i + S i.$$

By means of the unit matrix, u and i can be factored out

$$(E - S)\, u = (E + S)\, i$$

and assuming that (E - S) is not singular, the equations can then be solved for u(i):

$$u = (E - S)^{-1} (E + S)\, i.$$

As can be seen from a comparison with equation 3.4/13b, the resulting matrix is the impedance matrix, z. By means of a corresponding solving for i(u)--or directly, by inversion of z--we obtain the admittance matrix y as a function of the scattering matrix. The result is

$$z = (E - S)^{-1} (E + S) = (E + S)(E - S)^{-1} \quad (3.4/21a)$$

and

$$y = (E + S)^{-1} (E - S) = (E - S)(E + S)^{-1}. \quad (3.4/21b)$$

If one starts with the impedance or, respectively, admittance equation 3.4/13 or, respectively, 3.4/16, and solve for B(A), then the scattering matrix S is obtained, expressed in terms of the impedance and, respectively, admittance matrix:

$$\text{and} \quad \begin{aligned} S &= (z + E)^{-1}(z - E) = (z - E)(z + E)^{-1} & (3.4/21c) \\ -S &= (y + E)^{-1}(y - E) = (y - E)(y + E)^{-1}. & (3.4/21d) \end{aligned}$$

A precondition for the validity of the transformation 3.4/21 is the existence of the matrix inversions. We should furthermore point out that matrix products of the above form are always interchangeable. The equations 3.4/21 can formally be considered as n-dimensional generalizations of the equations 3.2/9, already known from the 1-port, between the reflection factor and the normalized impedance or, respectively, admittance. The scalar operator variables r, z or, respectively, y here correspond to the matrix-operator variables S, z or, respectively, y.

Besides the impedance, respectively admittance form, there are other forms of relating in the voltage/current representation, among which the series-parallel form is of some significance. For a two-gate ([B11], [B12]), it is

$$\begin{pmatrix} u_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} i_1 \\ u_2 \end{pmatrix} \qquad (3.4/22)$$

and it is preferred primarily for the description of the small-signal behavior of transistors. The connecting matrix h is here called hybrid matrix. The experimental determination and hence also the physical meaning of the coefficients h_{kl} is again derived from the equating to zero of the state variables i_1 or, respectively, u_2, i.e., by means of short-circuit measurements at port 2, respectively open-circuit load measurements at port 1.

$h_{11} = \left.\frac{u_1}{i_1}\right|_{u_2=0}$ — short-circuit input impedance of port 1

$h_{21} = \left.\frac{i_2}{i_1}\right|_{u_2=0}$ — short-circuit current ratio of port-1 to port-2

$h_{12} = \left.\frac{u_1}{u_2}\right|_{i_1=0}$ — open-circuit load voltage ratio of port-2 to port-1

$h_{22} = \left.\frac{i_2}{u_2}\right|_{i_1=0}$ — open-circuit load input admittance of port-2

In the microwave area, short-circuit and open-circuit load measurements are accompanied by great difficulties, on the one hand, because of the realization of these operating conditions and on the other, because of the tendency to oscillation (instability) of active systems *under these operating conditions. These difficulties do not occur in measuring circuits as in Figures 3/8 or, respectively, 3/9--i.e., when the ports that are not supplied are terminated with ideal absorbers. For this reason, it is appropriate to also characterize transistor amplifiers in the microwave range by means of scattering coefficients. The frequently needed recalculations between the coefficients of the voltage-current representation, z_{kl}, y_{kl}, h_{kl} and the scattering coefficients s_{kl}, are shown in Table 5 for a 2-port.

TABLE 5. Relationships between coefficients s_{kl} and z_{kl}, y_{kl}, h_{kl} of a 2-port

$z_{11} = \frac{(1+s_{11})(1-s_{22})+s_{12}s_{21}}{(1-s_{11})(1-s_{22})-s_{12}s_{21}}$ $z_{12} = \frac{2 s_{12}}{(1-s_{11})(1-s_{22})-s_{12}s_{21}}$ $z_{21} = \frac{2 s_{21}}{(1-s_{11})(1-s_{22})-s_{12}s_{21}}$ $z_{22} = \frac{(1-s_{11})(1+s_{22})+s_{12}s_{21}}{(1-s_{11})(1-s_{22})-s_{12}s_{21}}$	$y_{11} = \frac{(1-s_{11})(1+s_{22})+s_{12}s_{21}}{(1+s_{11})(1+s_{22})-s_{12}s_{21}}$ $y_{12} = \frac{-2 s_{12}}{(1+s_{11})(1+s_{22})-s_{12}s_{21}}$ $y_{21} = \frac{-2 s_{21}}{(1+s_{11})(1+s_{22})-s_{12}s_{21}}$ $y_{22} = \frac{(1+s_{11})(1-s_{22})+s_{12}s_{21}}{(1+s_{11})(1+s_{22})-s_{12}s_{21}}$	$h_{11} = \frac{(1+s_{11})(1+s_{22})-s_{12}s_{21}}{(1-s_{11})(1+s_{22})+s_{12}s_{21}}$ $h_{12} = \frac{2 s_{12}}{(1-s_{11})(1+s_{22})+s_{12}s_{21}}$ $h_{21} = \frac{-2 s_{21}}{(1-s_{11})(1+s_{22})+s_{12}s_{21}}$ $h_{22} = \frac{(1-s_{11})(1-s_{22})-s_{12}s_{21}}{(1-s_{11})(1+s_{22})+s_{12}s_{21}}$
$s_{11} = \frac{(z_{11}-1)(z_{22}+1)-z_{12}z_{21}}{(z_{11}+1)(z_{22}+1)-z_{12}z_{21}}$ $s_{12} = \frac{2 z_{12}}{(z_{11}+1)(z_{22}+1)-z_{12}z_{21}}$ $s_{21} = \frac{2 z_{21}}{(z_{11}+1)(z_{22}+1)-z_{12}z_{21}}$ $s_{22} = \frac{(z_{11}+1)(z_{22}-1)-z_{12}z_{21}}{(z_{11}+1)(z_{22}+1)-z_{12}z_{21}}$	$s_{11} = \frac{(1-y_{11})(1+y_{22})-y_{12}y_{21}}{(1+y_{11})(1+y_{22})-y_{12}y_{21}}$ $s_{12} = \frac{-2 y_{12}}{(1+y_{11})(1+y_{22})-y_{12}y_{21}}$ $s_{21} = \frac{-2 y_{21}}{(1+y_{11})(1+y_{22})-y_{12}y_{21}}$ $s_{22} = \frac{(1+y_{11})(1-y_{22})-y_{12}y_{21}}{(1+y_{11})(1+y_{22})-y_{12}y_{21}}$	$s_{11} = \frac{(h_{11}-1)(h_{22}+1)-h_{12}h_{21}}{(h_{11}+1)(h_{22}+1)-h_{12}h_{21}}$ $s_{12} = \frac{2 h_{12}}{(h_{11}+1)(h_{22}+1)-h_{12}h_{21}}$ $s_{21} = \frac{-2 h_{21}}{(h_{11}+1)(h_{22}+1)-h_{12}h_{21}}$ $s_{22} = \frac{(1+h_{11})(1-h_{22})+h_{12}h_{21}}{(h_{11}+1)(h_{22}+1)-h_{12}h_{21}}$
$s_{kl} \longleftrightarrow z_{kl}$	$s_{kl} \longleftrightarrow y_{kl}$	$s_{kl} \longleftrightarrow h_{kl}$

---

* For a definition of the concepts active, passive, etc. see 3.6.

3.44 Chain forms

In the n-port representation of the scattering, impedance and admittance forms, we combined state variables of the same kind in columns and inter-related them in appropriate manners. It is possible, however, to combine variables of state of different kinds in other meaningful ordering systems, if they refer to the same port; for instance, into columns that can then be connected by n-port matrices. The chain forms are constructed according to this pattern and we shall, initially, discuss them for a 2-port. In the representation, we shall use reduced variables of state throughout.

Using these reduced variables of state, the best known of these chain forms becomes, in the voltage/current representation

$$\begin{pmatrix} u_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} u_2 \\ -i_2 \end{pmatrix} \qquad (3.4/23a)$$

with the normalized chain matrix

$$k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}. \qquad (3.4/23b)$$

Their coefficients are dimensionless. Since usually they are defined between variables of state in asymmetrical or chain representation (u_E, i_E and u_A, i_A) according to Figure 3/1, but we want to use the symmetrical representation throughout, we have replaced i_A in equation 3.4/23a with $-i_2$. The transmission equation 3.1/5 is a special example of the chain form 3.4/23a.

In the chain form of the wave representation the input and output wave variables of port 2 are transformed into the output and input wave variables of port 1 according to the pattern

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \qquad (3.4/24a)$$

The matrix of the coefficients

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \qquad (3.4/24b)$$

we call wave chain matrix or, in short, cascade matrix. The designation cascade matrix and the symbols C for the matrix and c_{kl} for its coefficients were adopted for two reasons. On the one side, based on the application of this form of representation to the calculation of chain or cascade switchings of 2-ports; and on the other, to make a distinction with the voltage/current chain matrix[8]. The coefficients c_{kl} are dimensionless wave relation variables. It is in general not possible to determine them by measurement, terminating a port through a short-circuit, open-circuit load or absorber, and they neither have the physical meaning of a reflection nor that of a transmission coefficient. Only c_{22} becomes the ratio a_1/b_2 for $a_2 = 0$ and can be directly determined as the inverse transmission coefficient s_{21}^{-1}.

In the third possible chain form, the wave variables of one port and the voltage/current variables of the other port are interrelated ([12], [13]). Thus, we here have a mixed manner of representation

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} u_2 \\ i_2 \end{pmatrix} \qquad (3.4/25a)$$

with the mixed chain matrix

$$m = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \qquad (3.4/25b)$$

This mixed form is particularly useful when in a microwave network statements in a voltage/current representation (for instance, with concentrated components) must be translated into wave representation. The dimensionless coefficients m_{kl} of the mixed chain matrix can be determined as a relationship between the input, respectively output waves at port-1 and short-circuit current, respectively open-circuit-load voltage at port-2. The three chain forms indicated are three forms of chain differing in principle. For each of the three forms 3.4/23, 3.4/24 and 3.4/25, it is possible to find an inverse,

[8] In the literature the cascade matrix is often called transmission matrix. But since its coefficients in general do not have the physical meaning of transmission coefficients, we submit the nomenclature developed here.

provided the matrices k, C and m do not happen to be singular. Other forms are obtained exchanging variables of state inside the columns or, in the mixed form, exchanging port numbers. All of these forms, however, do not result in new relationships and hence, we shall discuss them no further. The three chain matrices k, C, m are easily transformed one into the other because of their internal relatedness. To this end, first we shall recalculate the state columns by means of the following system of orthogonal matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.4/26)$$

From the fundamental equations 2.5/12

$$u_k = a_k + b_k,$$
$$i_k = a_k - b_k$$

for the variables of state at the same port k, we then obtain the following column transformations, with the aid of the E, J, K, L matrices: $\begin{pmatrix} u_1 \\ i_1 \end{pmatrix} = (E - J)\begin{pmatrix} b_1 \\ a_1 \end{pmatrix}$ (3.4/27a), $\begin{pmatrix} u_2 \\ -i_2 \end{pmatrix} = (E - J)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, (3.4/27b)

$\begin{pmatrix} u_2 \\ i_2 \end{pmatrix} = (K + L)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ (3.4/27c) as well as $\begin{pmatrix} u_2 \\ i_2 \end{pmatrix} = K\begin{pmatrix} u_2 \\ -i_2 \end{pmatrix}$. (3.4/27d)

From the voltage/current chain form

$$\begin{pmatrix} u_1 \\ i_1 \end{pmatrix} = k\begin{pmatrix} u_2 \\ -i_2 \end{pmatrix}$$

we now obtain, with equations 3.4/27a and 3.4/27b

$$(E - J)\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = k(E - J)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

and by inversion, the wave chain form

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = (E - J)^{-1} k (E - J)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \tfrac{1}{2}(E + J) k (E - J)\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

with the cascade matrix

$$C = \tfrac{1}{2}(E + J) k (E - J) = \tfrac{1}{2}(k - kJ + Jk - JkJ). \quad (3.4/28a)$$

By solving equation 3.4/28a for k(C), we obtain

$$k = \tfrac{1}{2}(E - J) C (E + J) = \tfrac{1}{2}(C + CJ - JC - JCJ). \quad (3.4/28b)$$

Following a similar procedure, we can interconverter the m and k matrix and the m and C matrix:

$$m = \tfrac{1}{2}(E + J)\,k\,K = \tfrac{1}{2}(k\,K + J\,k\,K). \qquad (3.4/29a)$$
$$k = (E - J)\,m\,K = m\,K - J\,m\,K \qquad (3.4/29b)$$

and

$$m = \tfrac{1}{2}\,C\,(K + L) = \tfrac{1}{2}(C\,K + C\,L). \qquad (3.4/30a)$$
$$C = m\,(K + L) = m\,K + m\,L. \qquad (3.4/30b)$$

In all three cases the coefficients c_{kl}, k_{kl} and m_{kl} are obtained as simple linear combinations of the coefficients of the corresponding other matrix.

A similarly simple, inversion-free transformation between the chain matrix and the scattering matrix is not possible.

We shall first consider the transformation of the scattering matrix into the wave-chain matrix or cascade matrix. In the system of equations

$$b_1 = s_{11}\,a_1 + s_{12}\,a_2 \qquad (3.4/31a)$$
$$b_2 = s_{21}\,a_1 + s_{22}\,a_2 \qquad (3.4/31b)$$

we solve the second line (3.4/31b) for a_1:

$$a_1 = -\,s_{21}^{-1}\,s_{22}\,a_2 + s_{21}^{-1}\,b_2 \qquad (3.4/31c)$$

and replace equation 3.4/31c in equation 3.4/31a:

$$b_1 = (s_{12} - s_{11}\,s_{21}^{-1}\,s_{22})\,a_2 + s_{11}\,s_{21}^{-1}\,b_2. \qquad (3.4/31d)$$

The two equations 3.4/31d and 3.4/31c then form, arranged after the pattern in 3.4/24a, the wave-chain form. By comparison, we obtain the coefficients c_{kl} expressed in terms of the scattering coefficients

$$c_{11} = s_{12} - s_{11}\,s_{21}^{-1}\,s_{22}, \quad c_{12} = s_{11}\,s_{21}^{-1}, \qquad (3.4/31e)$$
$$c_{21} = -\,s_{21}^{-1}\,s_{22}, \quad c_{22} = s_{21}^{-1}.$$

We can see here that it is possible to build the cascade matrix for 2-ports only if coupling exists between port-1 and port-2, i.e., when the transmission coefficients $s_{21} \neq 0$. If, however, the cascade matrix does not exist, then--because of their "linear relatedness" with C--it also will not be possible to build the other two chain matrices, k and m.

If we once again solve the cascade form (3.4/31d, c) with the coefficients of equation 3.4/31e for the scattering form 3.4/31a, b, then we shall obtain the scattering coefficients expressed in terms of the cascade coefficients

$$s_{11} = c_{12}\,c_{22}^{-1}, \qquad s_{12} = c_{11} - c_{12}\,c_{22}^{-1}\,c_{21}, \qquad (3.4/31\text{f})$$
$$s_{21} = c_{22}^{-1}, \qquad s_{22} = -\,c_{22}^{-1}\,c_{21}.$$

With the help of equations 3.4/28 and 3.4/30, it then also becomes possible to express the coefficients k_{kl} and m_{kl} by means of the scattering coefficients and vice versa. These recalculations were combined in Table 6 below.

TABLE 6. Relations between the coefficients s_{kl} and c_{kl}, k_{kl}, m_{kl} of a 2-port

$c_{11} = \frac{-\det S}{s_{21}}$ $c_{12} = \frac{s_{11}}{s_{21}}$ $c_{21} = \frac{-s_{22}}{s_{21}}$ $c_{22} = \frac{1}{s_{21}}$	$k_{11} = \frac{s_{11} - s_{22} + 1 - \det S}{2 s_{21}}$ $k_{12} = \frac{s_{11} + s_{22} + 1 + \det S}{2 s_{21}}$ $k_{21} = \frac{-s_{11} - s_{22} + 1 + \det S}{2 s_{21}}$ $k_{22} = \frac{-s_{11} + s_{22} + 1 - \det S}{2 s_{21}}$	$m_{11} = \frac{s_{11} - \det S}{2 s_{21}}$ $m_{12} = \frac{-s_{11} - \det S}{2 s_{21}}$ $m_{21} = \frac{1 - s_{22}}{2 s_{21}}$ $m_{22} = \frac{-1 - s_{22}}{2 s_{21}}$
$s_{11} = \frac{c_{12}}{c_{22}}$ $s_{12} = \frac{\det C}{c_{22}}$ $s_{21} = \frac{1}{c_{22}}$ $s_{22} = \frac{-c_{21}}{c_{22}}$	$s_{11} = \frac{k_{11} + k_{12} - k_{21} - k_{22}}{k_{11} + k_{12} + k_{21} + k_{22}}$ $s_{12} = \frac{2 \det k}{k_{11} + k_{12} + k_{21} + k_{22}}$ $s_{21} = \frac{2}{k_{11} + k_{12} + k_{21} + k_{22}}$ $s_{22} = \frac{-k_{11} + k_{12} - k_{21} + k_{22}}{k_{11} + k_{12} + k_{21} + k_{22}}$	$s_{11} = \frac{m_{11} - m_{12}}{m_{21} - m_{22}}$ $s_{12} = \frac{-2 \det m}{m_{21} - m_{22}}$ $s_{21} = \frac{1}{m_{21} - m_{22}}$ $s_{22} = \frac{-m_{21} - m_{22}}{m_{21} - m_{22}}$
$s_{kl} \longleftrightarrow c_{kl}$	$s_{kl} \longleftrightarrow k_{kl}$	$s_{kl} \longleftrightarrow m_{kl}$

For many switching tasks it seems desirable and appropriate to be able to describe an n-port by means of chain forms also for port numbers n > 2. Below we shall investigate and discuss the permissibility of this extension and the always possible transformation into the scattering form only in the wave representation (cascade matrix).

To this end we first consider a branching to which can be assigned an n-port with an even number of ports n = 2m (in agreement with Figure 3/10a), and the 2m wave types or, respectively, ports can be meaningfully divided into two groups with the same number of m wave types, or ports, respectively. In correspondence to the procedure used in section 3.42 (cf. equation 3.4/7 and 3.4/8), we shall combine the input, or respectively, output wave variables of the same port groups into subcolumns:

$$A_1 = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}, \tag{3.4/32a}$$

$$A_2 = \begin{pmatrix} a_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} b_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}. \tag{3.4/32b}$$

thus obtaining the scattering form

$$B_1 = S_{11} A_1 + S_{12} A_2, \tag{3.4/33a}$$

$$B_2 = S_{21} A_1 + S_{22} A_2. \tag{3.4/33b}$$

Since in this case both port groups have the same number (m=n/2) of variables of state, here all the matrices S_{KL} will be square and of m rows. The equation of state 3.4/33b can thus be solved for instance for A_1 if only S_{21} is regular. We then obtain, in analogy to equation 3.4/31c,

$$A_1 = -S_{21}^{-1} S_{22} A_2 + S_{21}^{-1} B_2 \tag{3.4/33c}$$

and, replacing this equation in equation 3.4/33a and rearranging:

$$B_1 = (S_{12} - S_{11} S_{21}^{-1} S_{22}) A_2 + S_{11} S_{21}^{-1} B_2. \tag{3.4/33d}$$

We submit, now, that a wave-chain form

$$B_1 = C_{11} A_2 + C_{12} B_2, \tag{3.4/34a}$$

$$A_1 = C_{21} A_2 + C_{22} B_2 \tag{3.4/34b}$$

exists for the 2m-port to be built according to the same pattern as the wave-chain form of the 2-port (cf. equation 3.4/24a). The submatrices C_{KL} and the cascade matrix C are then obtained from

ORIGINAL PAGE IS OF POOR QUALITY

the submatrices S_{KL} by comparison of equations 3.4/33c, d and 3.4/34a, b, as

$$C_{11} = S_{12} - S_{11} S_{21}^{-1} S_{22}, \quad C_{12} = S_{11} S_{21}^{-1}, \qquad (3.4/35)$$
$$C_{21} = - S_{21}^{-1} S_{22}, \quad C_{22} = S_{21}^{-1}.$$

The submatrices C_{KL}--to the extent that they exist--will be as square as are the submatrices S_{KL}, and of m rows. The wave-chain form 3.4/34 can conversely be transformed into the scattering form, if only C_{22} is regular. Since, however, C_{22} is the inverse of S_{21}, C_{22} will be regular when S_{21} is regular, i.e., when the transformation from the scattering form to the wave-chain form was possible. The submatrices S_{KL} are then obtained in a corresponding manner from the submatrices C_{KL}:

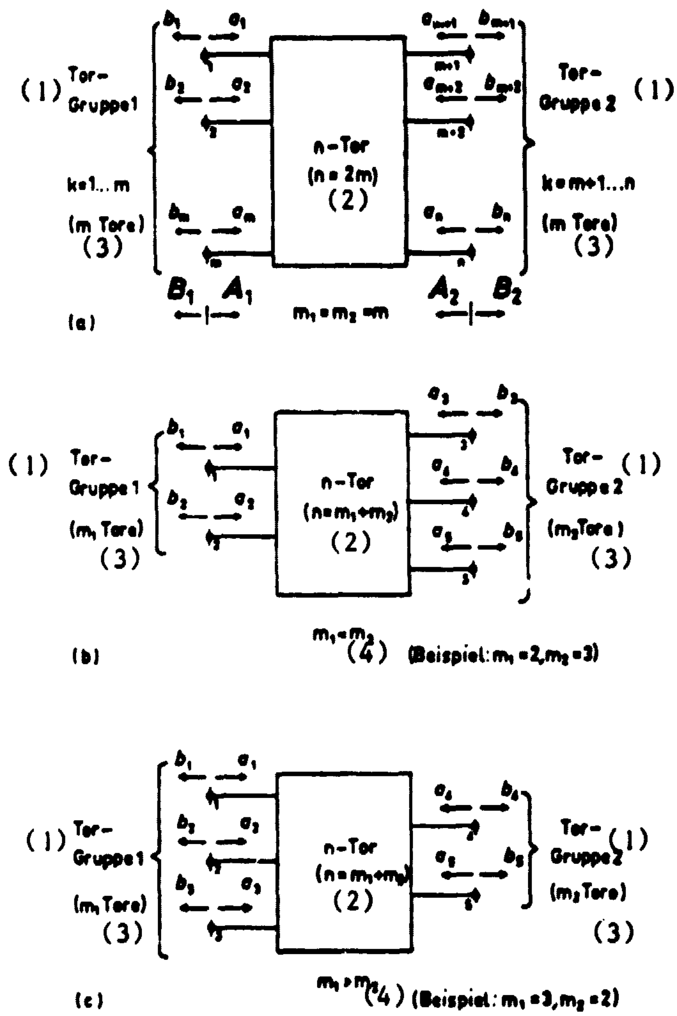
$$S_{11} = C_{12} C_{22}^{-1}, \quad S_{12} = C_{11} - C_{12} C_{22}^{-1} C_{21}, \qquad (3.4/36)$$
$$S_{21} = C_{22}^{-1}, \quad S_{22} = - C_{22}^{-1} C_{21}.$$

To conclude, we now want to consider two more cases, in which the port numbers in the two port groups are not equal. Here the transmission matrices of the scattering form S_{12} and S_{21} as well as all submatrices C_{KL} of a wave-chain form, are rectangular.

In the first place we shall consider an n-port where port number m_1 of group 1 is smaller than the port number m_2 of group 2 ($m_1 < m_2$). In the example sketched in Figure 3/10b (page 23), $m_1 = 2$ and $m_2 = 3$. The submatrix S_{21} then has m_1 columns and m_2 rows and is rectangular on end. If S_{21} is also column-regular[9], then equation 3.4/33b can be transformed into equation 3.4/33c by means of the so-called left-inverse[9] $G^{-1}S_{21}^T$ and thus we obtain the wave-chain form from the scattering form. In equation 3.4/35 the inverse S_{21}^{-1} must be replaced by the left-inverse $G^{-1}S_{21}^T$. Here, $G = S_{21}^T S_{21}$ is the Gaussian transformation of S_{21}; it has m_1 rows, is square and not singular. All C_{KL} have then m_1 rows and m_2 columns. All told, the C matrix then has $4m_1m_2$ coefficients, compared to the $(M_1 + m_2)^2$ coefficients of the S matrix, where $4m_1m_2 < (m_1+m_2)^2$.

[9] Cf., for instance, [B3].

With the transformation $S \to C$, an information loss is thus incurred. Since we had assumed S_{21} as column regular, its left inverse $G^{-1}S^{T}_{21} = C_{22}$ is row regular[9] (see footnote page 22).



ORIGINAL PAGE IS OF POOR QUALITY

(1) port group (2) n-port (3) m ports (4) example

Figure 3/10. For the construction of chain-wave forms of n-ports with 2 port groups:
a) number of ports of group 1 = number of ports of group 2
b) number of ports of group 1 < number of ports of group 2
c) number of ports of group 1 > number of ports of group 2

In order to now obtain the scattering form, again, from the cascade form of equation 3.4/34, we would have to solve equation 3.4/34b for B_2 and in the process also eliminate C_{22} by left-multiplication with its left inverse. Since, however, for a row-regular, rectangular matrix C_{22} only a right inverse' (see footnote page 22) exists, this transformation process is not possible. Hence, the back-transformation $C \to S$ is not possible in the case $m_1 < m_2$.

Finally, we shall consider the case of port number m_1 being greater than port number m_2. Figure 3/10c (page 23) provides an example with $m_1 = 3$ and $m_2 = 2$. To begin with, we submit that a cascade form exists. All C_{KL} then have m_1 rows and m_2 columns. In this case, the rectangular matrix would be rectangular on end. Assuming that C_{22} is column regular, it would be possible to perform a $C \to S$ transformation with a corresponding left inversion $(C_{22}^T C_{22})^{-1} C_{22}^T = S_{21}$. The rectangular matrix S_{21} with m_1 columns and m_2 rows would then be row regular and since for it only a right inverse exists, in this case the transformation $S \to C$ would not be possible. On the other hand, however, every physically feasible n-port has a corresponding scattering matrix S. From this we conclude that in the case $m_1 > m_2$ the cascade matrix does not exist. This is immediate for the case $m_2 = 1$: if we formally construct the left inverse for the single row matrix S_{21}, for the $S \to C$ transformation, then we obtain, as the Gaussian transformation $S_{21}^T S_{21}$, a diadic product that is singular and hence, not inversible.

These difficulties in the construction of the C matrix or its transformation into an S matrix show that the wave-chain form is not suited to the description of n-ports with two port groups with different numbers of ports. For the treatment of cascade switchings for such ports-number asymmetrical n-ports, we shall thus have to develop methods in Chapter 4 in the wave representation that only build upon the universally applicable scattering matrix.