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16. Abstract

The present paper deals with the investigation of the helical flow in an annulus between two coaxial cylinders a regards its stability against the formation of helical vortices of the type known as Taylor's annulár vortices. Assuming the annulus to be small and the velocities to vary linearly with radius, it is shown that the problem can be reduced to the classical case of flow between two rotating cylinders. An appropriate stability criterion for helical flows is derived from Rayleigh's stability criterion applicable to such flows.

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STABILITY OF FLUID FLOW IN A CYLINDRICAL ANNULUS

Hubert Ludwieg

1. INTRODUCTION

Consider a fully developed rotationally symmetric flow in a cylindrical annulus. In such a flow, there is no radial component of the velocity, and the longitudinal and tangential velocity components do not vary longitudinally or tangentially but depend only on the radius. Thus, only two types of instabilities are important. /135*

The first type of instability arises due to separation of the flow on the inner boundary of the cylindrical annulus and gives rise to the so-called dead water. This instability has been discussed by several authors [1,2] and will not be considered here.

The second type of instability is fairly well known for the special case of a fully developed rotationally symmetric flow in an annulus in which the longitudinal velocity component vanishes. This type of flow occurs in the annulus between rotating coaxial cylinders. The case of inviscid flow has already been discussed by Lord Rayleigh [3]. It was shown that the circulation is unstable when the tangential velocity decreases

*Numbers in right margin indicate foreign pagination.

faster than 1/r. The instability of similar flow when friction is considered has been treated by G. I. Taylor [4] where it has been shown that friction stabilizes the flow, i.e., the stability limit is extended into the previously unstable fluid flow region. For large Reynolds numbers (where the effect of friction goes to zero), Rayleigh stability limit is reached again.

In this paper, the stability of a fully developed rotationally symmetric annular fluid flow is studied for the general case of non-zero longitudinal velocity component. Since we shall apply it to a fluid flow with large Reynolds numbers, we shall consider only the simple case of inviscid flow.

2. STABILITY CRITERION

Let us consider an inviscid fluid flow in a cylindrical annulus between two coaxial cylinders which is fully developed and rotationally symmetric. The circulation is assumed to be resulting from the fact that a fluid whose axial and tangential components depend only on r flows from a fixture V into the annulus R (Figure 1). A radial equilibrium is considered to exist in this flow at

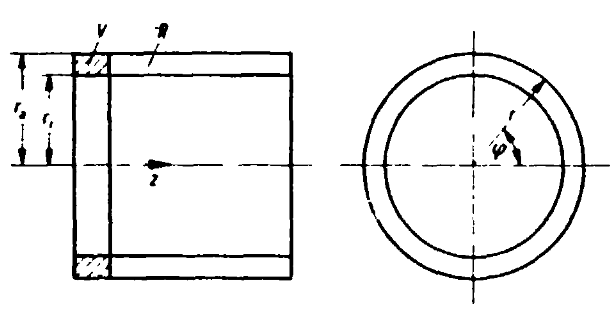

Figure 1. Cylindrical Annulus

egress (centrifugal force = force due to radial pressure gradient). The fluid medium thus flows into the annulus without a change in velocity. We shall investigate this flow for its stability. (If the flow at the time of discharge from V is not in radial equilibrium, it is reached in a short distance from V).

We shall use cylindrical coordinates r, ϕ, z and denote the corresponding velocity components by V_r, V_ϕ, V_z. Thus,

$$V_r = 0, \quad V_\phi = V_\phi(r), \quad V_z = V_z(r). \tag{1}$$

Because the flow is considered to be frictionless (inviscid), V_q and V_z need not be zero on the walls $r = r_i$ and $r = r_a$. For brevity, we set

$$\frac{r_i + r_a}{2} = r_0$$

and denote the velocities V_ϕ and V_z at the point $r = r_0$ as $V_{\phi 0}$ and V_{z0}. For simplicity, we further assume that /136
$r_0 - r_i = \Delta r \ll r_0$ and that V_ϕ and V_z are linear in r. We can, then, write that

$$\begin{cases} V_\phi = V_{\phi 0} + c_\phi (r - r_0), \\ V_z = V_{z0} + c_z (r - r_0). \end{cases} \tag{2}$$

In spite of these constraints, the fluid flow described by Equation (2) is quite general. The flow is not irrotational and is not governed by constant pressure.

Let us next consider the simple case of $V_z = 0$. In this case, the flow is only in the tangential direction and so the Rayleigh stability criterion is valid. This criterion states that the fluid flow is stable when V_ϕ decreases slower than 1/r or perhaps increases, for increasing radius.

Thus, in our notation, the flow is stable for

$$\frac{c_\phi r_0}{V_{\phi 0}} > -1 \tag{3}$$

For,

$$\frac{c_\phi r_0}{V_{\phi 0}} < -1$$

perturbations known as Taylor ring vortices appear and lead to a quick change in velocity profile.

The occurrence of the Rayleigh stability limit V_ϕ proportional to $1/r$ can be easily understood when a ring shaped fluid element coaxial with the annulus is considered. Due to rotational symmetry, parts of this fluid element are dispersed radially, so that a ring shaped element having a larger radius is formed. From the law of conservation of angular momentum, this displacement results in a change in tangential velocity V_ϕ which is proportional to $1/r$. If the velocity V_ϕ of the outer fluid in the neighboring region decreases faster than $1/r$, the centrifugal force occurring in the ring dominates the restoring force due to pressure gradients of the surrounding fluid flow and thus causes instability. If V_ϕ in the outward direction decreases slower than $1/r$ or even increases, the force resulting from pressure gradients exceeds the centrifugal force, and the flow is stable.

The case $V_z=V_{z0}$, i.e., $c_z = 0$, does not lead to anything new compared to the results of $V_z = 0$. We need only to consider the flow phenomenon in a reference frame moving with a velocity V_{z0}. It is clear immediately, that we have the same flow as in the case of $V_{z0}=0$, and therefore, have the same stability criterion.

We now consider the case $c_z \neq 0$. We could use, here, the usual technique to derive a stability criterion, i.e., apply a suitable perturbation to the flow and with the help of the theory of small oscillations study the conditions under which the perturbation increases with time. Relative to Rayleigh case, a complication arises that, in addition to the generalization of the flow, it is no longer justified to apply purely ring vortices as perturbation (Taylor vortices). We need to also consider perturbation due to helical vortices. These calculations can, however, be performed economically if we could, by making a good approximation, possibly reduce the problem to the Rayleigh case.

We consider the flow in a reference frame moving with a constant velocity U along the z-axis. In this reference system, we have once again a fully developed rotationally symmetric flow in a ring annulus, having velocities and velocity-gradients

$$V_{\varphi 0},\ \tilde{V}_{z0} = V_{z0} - U,\ c_\varphi,\ c_z$$

respectively. The velocity U is chosen such that the slope of all helically shaped streamlines of this flow is independent of the radius r. The planes containing the streamlines which are normal to the cylindrical surface are then purely helical. The condition for U is thus given by

$$\frac{V_{z0} - U}{V_{\varphi 0}} \cdot 2\pi r_0 = \frac{V_{z0} - U + c_z \Delta r}{V_{\varphi 0} + c_\varphi \Delta r} \cdot 2\pi (r_0 + \Delta r)$$

By approximating the solution to be linear, we obtain for $U/V_{\varphi 0}$ and $\tilde{V}_{z0}/V_{\varphi 0}$

$$\frac{U}{V_{\varphi 0}} = \frac{V_{z0}}{V_{\varphi 0}} + \frac{c_z r_0/V_{\varphi 0}}{1 - c_\varphi r_0/V_{\varphi 0}}, \tag{4a}$$

$$\operatorname{tg} \delta = \frac{\tilde{V}_{z0}}{V_{\varphi 0}} = - \frac{c_z r_0/V_{\varphi 0}}{1 - c_\varphi r_0/V_{\varphi 0}}. \tag{4b}$$

where δ is the angular slope of the streamlines relative to the radius r_0 in the moving frame.

Consider now a tube of streamlines which is formed from two neighboring helical planes with slope δ and the walls of the cylinder. It can be seen that by changing the reference system, an almost planar flow is obtained. This tube of streamlines is different from those from the Rayleigh case only by the fact this tube is also curved in a direction perpendicular to the streamlines. And, as we advance along the direction of flow, this tube of constant cross-section slowly rotates around its axis. It should be realized that the stability or instability of the Rayleigh flow arises due to the interplay between centrifugal force and the pressure gradient.

It follows immediately that, under the assumption of narrow annulus ($\Delta r/r_0 \ll 1$), only the curvature along the streamlines is important for stability and that both deviations from the Rayleigh case discussed above have no influence on the stability. We can, therefore, apply the Rayleigh stability criterion, and introduce the following effective quantities

$$r_{0\,\mathrm{eff}} = \frac{r_0}{\cos^2\delta},$$

$$V_{\varphi 0\,\mathrm{eff}} = \sqrt{V_{\varphi 0}^2 + \tilde{V}_{z0}^2} = \frac{V_{\varphi 0}}{\cos\delta},$$

$$c_{\varphi\,\mathrm{eff}} = c_\varphi \cos\delta + c_z \sin\delta .$$

By setting these quantities in Equation (3), the following condition is obtained

$$\frac{c_{\varphi\,\mathrm{eff}}\, r_{0\,\mathrm{eff}}}{V_{\varphi\,\mathrm{eff}}} = \frac{r_0\,(c_\varphi + c_z \operatorname{tg}\delta)}{V_{\varphi 0}} > -1 .$$

Using the value of tangent δ from Equation (4b), we obtain as stability criterion of our flow*: /137

$$\frac{c_\varphi r_0}{V_{\varphi 0}} - \frac{(c_z r_0/V_{\varphi 0})^2}{1 - c_\varphi r_0/V_{\varphi 0}} > -1 . \qquad (5)$$

3. DISCUSSION (OF THE RESULT)

For reduction of the variables, we substitute for $V_{\phi 0}/r_0$ in our stability calculation, the dimensionless velocity gradients

$$\tilde{c}_\varphi = \frac{c_\varphi r_0}{V_{\varphi 0}}, \quad \tilde{c}_z = \frac{c_z r_0}{V_{\varphi 0}}$$

The stability condition, then, becomes

$$\tilde{c}_\varphi - \frac{\tilde{c}_z^2}{1 - \tilde{c}_\varphi} > -1 . \qquad (6)$$

---

*Subsequently, it can also be verified numerically from the differential equations for a friction free flow that helical vortices which correspond to Taylor ring vortices, having velocity given by Equation (4a) and slope given by Equation (4b) can occur in the fluid flow when condition (5) is not satisfied. This will be shown in the appendix.

The resulting stable and unstable regions are drawn in a $\tilde{c}_\varphi, \tilde{c}_z$-plane in Figure 2. We see immediately that for $c_z = 0$,

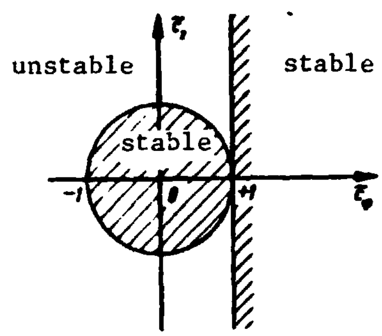

Figure 2. Stable and unstable regions in $\tilde{c}_\varphi, \tilde{c}_z$ plane.

the Rayleigh condition $\tilde{c}_\varphi > -1$ is obtained again. For C_z values different from zero, the condition $\tilde{c}_\varphi > -1$ (weaker than potential vortex) is insufficient for stability. Thus, an additional z-component of the velocity gradient C_z functions always as a destabilizer, irrespective of its sign. The stability limit for C_z values for which $\tilde{c}_\varphi > -1$, is given by the straight line $\tilde{c}_\varphi = 1$, is shown in Figure 2. In this case, the neutral equilibrium is reached by a distribution of the tangential velocity component V_ϕ, which corresponds to a rigid body rotation. (V_ϕ is proportional to r.)

Let us consider the form of the unstable perturbation. It is clear immediately by differentiation that it follows the direction of velocity flow in the reference system moving with velocity U. We thus have a stationary helically shaped vortex whose slope is given by Equation (4b). In a $\tilde{c}_\varphi, \tilde{c}_z$-diagram (Figure 3), the direction of a helical vortex corresponding to the point P having value $\tilde{c}_\varphi, \tilde{c}_z$- and consistent with Equation (4b) is given by a straight line through the points P and R (coordinates 1,0), if the ϕ direction is considered as $\tilde{c}_\phi$ -axis and the z-axis as $\tilde{c}_z$. In a stationary reference system, these helical vortices move with a velocity U along the z-axis.

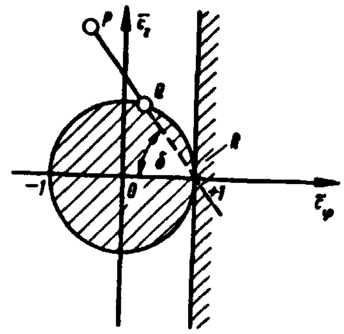

Figure 3. Change in $\tilde{c}_\varphi, \tilde{c}_z$ values for an unstable flow shown at point P.

Let us now consider an unstable flow which flows from a fixture V into an apparatus, as shown in Figure 1. Then, the perturbations will grow very rapidly and produce a momentum exchange which will give rise to an effective velocity gradient $c_{\varphi\,eff}$. The components of the gradient vector $(\tilde{c}_\varphi, \tilde{c}_z)$ along the direction of the vortex decrease whereas those perpendicular to it remain unchanged. The latter do not correspond to shear but only to a rotation of the fluid. In $\tilde{c}_\varphi, \tilde{c}_z$-diagram (Figure 3), the motion is therefore along the straight line through points P and R.

At the stability circle (point Q), the momentum exchange becomes stationary, and the velocity profile is now stable. However, by some combination of normal turbulence, particularly for large Reynold numbers, the velocity profile may possibly change. In addition, because of the frictional effect of the walls which cannot be switched off, the velocity profile will change slowly due to presence of the walls, even after reaching a stable point.

4. FLOW STABILITY IN WIDE ANNULUS

The calculations presented here are valid under the assumption that the annulus is small $(\Delta r/r_0 \ll 1)$, and that the velocity components can be written as linear functions of r. When these conditions do not hold, our stability criterion is no longer directly applicable. In this case, we can, however, divide the cylindrical annulus into a number of small ones of differential width dr and apply our stability criterion to each of them. The question, then, arises if it is possible to make a statement concerning the total flow. It can be safely stated that our stability condition is definitely a necessary condition for each radius of the annulus. If this is not the case for an annulus of given radius, then the flow in that region, as observed from a suitable reference system moving along the axial direction, is approximately planar, and unstable as shown for the Rayleigh case. It may, also, be assumed that this condition is sufficient, since for all perturbations involving a vortex of small radial extension, the flow is stable when it is satisfied.

For perturbation due to vortices of large radial width, there /138
is a displacement in each of the small cylinders. This, in turn, releases restoring forces as seen for the Rayleigh case, so that it may be assumed that the complete flow is stable with respect to these vortices. Formally stated, it implies that the flow is stable, if and only if, the condition

$$\frac{dV_\varphi}{dr}\frac{r}{V_\varphi} - \frac{\left(\frac{dV_z}{dr}\right)^2\left(\frac{r}{V_\varphi}\right)^2}{1-\frac{dV_\varphi}{dr}\frac{r}{V_\varphi}} > -1. \tag{7}$$

is valid for every radius r.

5. POSSIBLE EXPERIMENTAL VERIFICATION

In order to verify the stability criterion experimentally, it can be so arranged that stable and unstable flows are generated in an apparatus of Figure 1 and then the development of flow profile along z-direction is measured. For a stable flow, the velocity profile must remain stationary until disturbed by a growth of the boundary layer on the walls. On the other hand, a quick immediate change in velocity profile occurs over the entire annulus for an unstable flow. Eventually, a stable final position, as shown in Figure 3, is also determined from the fact that after reaching this distribution, there is no change or only a slow change in flow profile.

We could now think of directly observing the appearance of helical vortices, and thereby, verify the stability criterion. This would, however, fail for the apparatus shown in Figure 1, because after the desired flow is generated through the fixture V, a number of unstable disturbances occur immediately. These perturbations grow concurrently and, therefore, do not lead to the development of a distinct helical vortex.

The stability criterion can yet be tested in a completely different way. We can use two sufficiently long coaxial cylinders which rotate independently of each other, and can, in addition, move relative to each other along the axial direction. In the middle part of the aperture, a fluid of suitable viscosity is introduced such that no turbulence occurs. The desired flow pattern having several values of $\dot{c}_q, \dot{c}_z$ is obtained by a suitable choice of

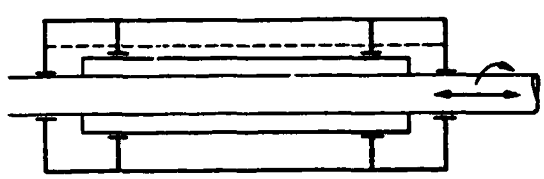

Figure 4. Proposed apparatus for generating helically shaped ring vortices

tangential and displacement velocities. The Reynolds number is chosen high enough that it has very little influence on the flow properties. The remaining effect of the viscosity on the flow profile between rotating cylinders can be easily estimated by Taylor computation [4] because these results can be carried over in our case in the same manner as for the Rayleigh case. For the flow shown in Figure 4, the vortices should be observable just like the Taylor vortices between two rotating cylinders. By calculating the slope of the helical vortices and their displacement velocity U, we can confirm the validity of our considerations. Furthermore, the appearance of unstable fluid flow should also be recognizable by a quick change in velocity profile.

6. APPENDIX

The Euler differential equations for the flow in cylindrical coordinates are given by

$$
\begin{aligned}
&\frac{\partial w_r}{\partial t} + w_r\frac{\partial w_r}{\partial r} + w_\varphi\frac{\partial w_r}{r\,\partial\varphi} - \frac{w_\varphi^2}{r} + w_z\frac{\partial w_r}{\partial z} = -\frac{1}{\varrho}\frac{\partial p'}{\partial r},\\
&\frac{\partial w_\varphi}{\partial t} + w_r\frac{\partial w_\varphi}{\partial r} + w_\varphi\frac{\partial w_\varphi}{r\,\partial\varphi} + \frac{w_r w_\varphi}{r} + w_z\frac{\partial w_\varphi}{\partial z} = -\frac{1}{\varrho}\frac{\partial p'}{r\,\partial\varphi},\\
&\frac{\partial w_z}{\partial t} + w_r\frac{\partial w_z}{\partial r} + w_\varphi\frac{\partial w_z}{r\,\partial\varphi} + w_z\frac{\partial w_z}{\partial z} = -\frac{1}{\varrho}\frac{\partial p'}{\partial z},\\
&\frac{\partial w_r}{\partial r} + \frac{w_r}{r} + \frac{\partial w_\varphi}{r\,\partial\varphi} + \frac{\partial w_z}{\partial z} = 0.
\end{aligned}
$$

Superposition of the basic flow given by Equation (1) with a perturbation having velocity components v_r, v_φ, v_z, yields

$$
\begin{aligned}
&w_r = v_r(r,\varphi,z,t),\\
&v_\varphi = V_\varphi(r) + v_\varphi(r,\varphi,z,t),\\
&w_z = V_z(r) + v_z(r,\varphi,z,t),\\
&p' = P(r) + p(r,\varphi,z,t).
\end{aligned}
$$

By inserting these components in the above differential equations and taking into account that the basic fluid flow satisfies these differential equations, and ignoring all quadratic terms in v_r, v_φ, v_z, we obtain

$$
\left|\;
\begin{aligned}
&\frac{\partial v_r}{\partial t} + V_\varphi\frac{\partial v_r}{r\,\partial\varphi} - \frac{2V_\varphi}{r}v_\varphi + V_z\frac{\partial v_r}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial r},\\
&\frac{\partial v_\varphi}{\partial t} + \frac{\partial V_\varphi}{\partial r}v_r + V_\varphi\frac{\partial v_\varphi}{r\,\partial\varphi} + \frac{V_\varphi}{r}v_r + V_z\frac{\partial v_\varphi}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{r\,\partial\varphi},\\
&\frac{\partial v_z}{\partial t} + \frac{\partial V_z}{\partial r}v_r + V_\varphi\frac{\partial v_z}{r\,\partial\varphi} + V_z\frac{\partial v_z}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial z},\\
&\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_\varphi}{r\,\partial\varphi} + \frac{\partial v_z}{\partial z} = 0.
\end{aligned}
\right.
\tag{8}
$$

We now make a trial solution for the pertubation velocities,

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$$\left|\begin{aligned} v_r &= \hat{v}_r(r)\, e^{i(\alpha z + \gamma\varphi - \beta t)},\\ v_\varphi &= \hat{v}_\varphi(r)\, e^{i(\alpha z + \gamma\varphi - \beta t)},\\ v_z &= \hat{v}_z(r)\, e^{i(\alpha z + \gamma\varphi - \beta t)},\\ p &= \hat{p}(r)\, e^{i(\alpha z + \gamma\varphi - \beta t)}. \end{aligned}\right. \tag{9}$$

Here α and γ are real whereas β can be complex. Thus, we apply a vortex of the type given by Taylor vortices as perturbation; however, it moves with a constant velocity and is helically shaped. Substituting Equation (9) in Equation (8) yields, /139

$$\left|\begin{aligned} &\left(-\beta i + V_\varphi \frac{\gamma}{r} i + V_z \alpha i\right)\hat{v}_r - \\ &\qquad -\frac{2V_\varphi}{r}\hat{v}_\varphi = -\frac{1}{\varrho}\frac{d\hat{p}}{dr},\\ &\left(-\beta i + V_\varphi \frac{\gamma}{r} i + V_z \alpha i\right)\hat{v}_\varphi + \\ &\qquad +\left(\frac{dV_\varphi}{dr} + \frac{V_\varphi}{r}\right)\hat{v}_r = -\frac{1}{\varrho}\frac{\gamma}{r} i\hat{p},\\ &\left(-\beta i + V_\varphi \frac{\gamma}{r} i + V_z \alpha i\right)\hat{v}_z + \\ &\qquad +\frac{dV_z}{dr}\hat{v}_r = -\frac{1}{\varrho}\alpha i\hat{p},\\ &\frac{d\hat{v}_r}{dr} + \frac{\hat{v}_r}{r} + \frac{\gamma}{r} i\hat{v}_\varphi + \alpha i\hat{v}_z = 0. \end{aligned}\right. \tag{10}$$

Let us first consider the Rayleigh case, i.e., a fluid flow without the z-component and a stationary ring-shaped vortex as perturbation. We can then set

$$V_z = 0,\quad \frac{dV_z}{dr} = 0,\quad \gamma = 0,\quad \Re(\beta) = \beta_\Re = 0$$

Further we can, under the assumption that the ring aperture is small ($\Delta r/r_0 \ll 1$), ignore $\hat{v}_r/r$ relative to other terms in the continuity equation and approximate

$$V_\varphi \approx V_{\varphi 0},\quad r \approx r_0,\quad \frac{dV_\varphi}{dr} \approx c_\varphi,\quad \frac{dV_z}{dr} \approx c_z$$

Thus, we obtain

$$\beta_{\mathfrak{i}}\hat{v}_r - \frac{2V_{\varphi 0}}{r_0}\hat{v}_\varphi = -\frac{1}{\varrho}\frac{d\hat{p}}{dr},$$

$$\beta_{\mathfrak{i}}\hat{v}_\varphi + \left(c_\varphi + \frac{V_{\varphi 0}}{r_0}\right)\hat{v}_r = 0,$$

$$\beta_{\mathfrak{i}}\hat{v}_z = -\frac{1}{\varrho}\alpha i \hat{p},$$

$$\frac{d\hat{v}_r}{dr} + \alpha i \hat{v}_z = 0.$$

By eliminating other quantities, the following differential equation is obtained for $\hat{v}_r$

$$\frac{d^2\hat{v}_r}{dr^2} - \alpha^2\left[1 + \frac{2}{\beta_{\mathfrak{i}}^2}\frac{V_{\varphi 0}}{r_0}\left(c_\varphi + \frac{V_{\varphi 0}}{r_0}\right)\right]\hat{v}_r = 0. \tag{11}$$

This differential equation yields the well-known sinusoidal solutions which vanish at the inner and outer cylinder, only when the second term in the angular bracket is negative. It follows, then, that the stability condition is given by

$$\frac{V_{\varphi 0}}{r_0}\left(c_\varphi + \frac{V_{\varphi 0}}{r_0}\right) > 0,$$

which is identical with the Rayleigh stability criterion.

We now consider the general case with non-zero V_z. By using

$$V_\varphi = V_{\varphi 0} + c_\varphi (r - r_0),$$
$$V_z = V_{z0} + c_z (r - r_0)$$

the expression $-\beta i + V_\varphi (\gamma/r) i + V_z \alpha i$ appearing in (10) can be rewritten as

$$-\beta i + V_\varphi \frac{\gamma}{r} i + V_z \alpha i = \left(-\beta_{\mathfrak{r}} + V_{\varphi 0}\frac{\gamma}{r_0} + V_{z0}\alpha\right)i +$$
$$+\left(\frac{\gamma}{r_0}c_\varphi - \frac{V_{\varphi 0}}{r_0}\frac{\gamma}{r_0} + \alpha c_z\right)(r - r_0) i + \beta_{\mathfrak{i}}.$$

Setting

$$\left|\begin{array}{l} -\beta_{\mathfrak{r}} + V_{\varphi 0}\dfrac{\gamma}{r_0} + V_{z0}\alpha = 0, \\ \dfrac{\gamma}{r_0}c_\varphi - \dfrac{V_{\varphi 0}}{r_0}\dfrac{\gamma}{r_0} + \alpha c_z = 0, \end{array}\right. \tag{12}$$

corresponds to defining the displacement velocity β_λ/α and the slope of the helical vortex and leads to the same results as in Equations (4a) and (4b). Thus, we obtain

$$-\beta i + V_\varphi \frac{\gamma}{r} i + V_z \alpha i = \beta_\lambda. \tag{13}$$

By eliminating from (10) the quantities $\hat{p}, \hat{v}_\varphi, \hat{v}_z$ and again, omitting small higher order terms, and using (12) and (13), we obtain the differential equation for $\hat{v}_r$

$$\left\{ \begin{aligned} &\frac{d^2 \hat{v}_r}{dr^2} - \alpha^2 \left\{ \left[1 + \frac{c_z^2}{(c_\varphi - V_{\varphi 0}/r_0)^2} \right] + \right. \\ &\left. + \frac{2}{\beta_\lambda^2} \frac{V_{\varphi 0}/r_0}{c_\varphi - V_{\varphi 0}/r_0} \left[c_z^2 + c_\varphi^2 - \left(\frac{V_{\varphi 0}}{r_0} \right)^2 \right] \right\} \frac{d\hat{v}_r}{dr} = 0. \end{aligned} \right. \tag{14}$$

The stability criterion follows from (14) in a manner analogous to the Rayleigh case

$$\left\{ \begin{aligned} &\frac{1}{V_{\varphi 0}/r_0} \frac{1}{c_\varphi - V_{\varphi 0}/r_0} \left[c_z^2 + c_\varphi^2 - \left(\frac{V_{\varphi 0}}{r_0} \right)^2 \right] = \\ &\qquad = \frac{1}{\bar{c}_\varphi - 1} (\bar{c}_z^2 + \bar{c}_\varphi^2 - 1) > 0, \end{aligned} \right. \tag{15}$$

which is identical to the stability condition (6).

7. SUMMARY

Stability of an inviscid, helically shaped fluid flow in a cylindrical annulus is studied with respect to the formation of helically shaped vortices using the well-known method of Taylor ring vortices. Under the assumption of a small annular space, and tangential and axial components V_ϕ and V_z varying linearly with r, stability condition for the helically shaped flow is derived from the corresponding Rayleigh case for the flow along the tangential direction only. Denoting the average velocity along the tangential direction by $V_{\phi 0}$, the average radius of curvature of the annular space by r_0, and the dimensionless velocity gradients by $c_\varphi = dV_\varphi/dr$ and $c_z = dV_z/dr$ with $\bar{c}_\varphi$ and $\bar{c}_z$, the stability condition may be expressed as

$$\tilde{c}_\varphi - \frac{\tilde{c}_z^2}{1 - \tilde{c}_\varphi} > -1.$$

The stable and unstable regions given by this relation are drawn in a $\tilde{c}_\phi$, $\tilde{c}_z$ plane in Figure 2. The Rayleigh stability limit $\tilde{c}_\varphi = -1$ holds for $\tilde{c}_z = 0$ and corresponds to a tangential velocity component V_ϕ proportional to 1/r (potential vortex). A finite value for $\tilde{c}_z$ always degrades the stability. For $\tilde{c}_z > 1$, the stability limit is given by $\tilde{c}_\varphi = 1$ which corresponds to the tangential velocity component V_ϕ being proportional to r (rigid rotation).

The stability criterion is also given for a wide annulus. The validity of this criterion is, however, only made plausible, and has not been strictly derived.

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