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TEXAS A&M UNIVERSITY
College Station, Texas

ROTORDYNAMIC ANALYSIS OF THE SSME
TURBOPUMPS USING REDUCED MODELS

prepared for

George C. Marshall
Space Flight Center
Alabama 35812
under
CONTRACT NAS8 - 34505

Principal Investigator
Sherif T. Noah
Mechanical Engineering Department
Texas A&M University
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ABSTRACT

Alternative methods for the rotor—dynamic and sensitivity analysis of large rotor systems are examined. The methods are assessed for their ability to utilize accurate models of reduced size along with effective procedures for describing the dynamic behavior of the systems.

Frequency response-based techniques are developed for determining the steady state response to imbalance of the SSME turbopumps and the related eigenvalue problem. In these techniques, the rotor and housing are represented by reduced receptances associated with their coupling points. The housing may be described by all of its normal modes within a frequency range of interest. The effects of truncated higher and lower modes are accounted for in an approximate manner. A reduced model for the rotor can be formed through an exact dynamic reduction to the degrees of freedom at the connection points to the housing. Alternatively, the reduction can be accomplished using modal representation of the nonspinning rotor. The reduced impedences (or receptances) of the rotor and housing are assembled through the impedances of the coupling elements to form the turbopump system, The size of the resulting system impedance (or receptance) is that of the number of the degrees of freedom (or forces) at the coupling points.

A procedure is described for determining the sensitivity of the coupling forces to changes in the coupling elements and rotor speed of the turbopump systems. In addition, an eigenvalue sensitivity analysis technique is adopted for application to the systems.

Computer programs were developed for the numerical implementation of the impedance and eigenvalue sensitivity formulated in this study. Finally, recommendations are made concerning further developments and requirements for other types of analysis.

LIST OF FIGURES

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NOMENCLATURE

List of the main symbols

SUPERSCRIPTS

* the conjugate of a complex quantity

' indicates modified from a given variable

$\bar{}$ a bar over a quantity indicates complex quantity

$\tilde{}$ belongs to seals

$\dot{}$ a dot over a quantity indicates differentiation with respect to time

T the transponse of a matrix

SUBSCRIPTS

b combined effects at bearings

B refers to bearings

c refers to turbopump case (housing)

R refers to turbopump rotor

I refers to intermediate (coupling) components, such as seals, bearing, etc.

j particular element of a vector

Imb imbalance locations on rotor

k kept

s for seals

r reduced out

X,Y components in the X and Y directions of the inertial coordinate system XYZ

x,y components of a vector in the x and y directions of the rotor-fixed x, y axes

XZ, YZ elements in the XZ and YZ planes

$V,_p = \frac{\partial V}{\partial p}$

Brackets

$[\ ]$	square or rectangular matrix
$[\diagdown]$	diagonal matrix
$\{\ \}$, $(\ )$	column matrix
$[\ ]^{-1}$	inverse of a matrix
$[\ ]^{T}$	transpose of a matrix

VARIABLES, PARAMETERS AND CONSTANTS

$\{a\}$	position vectors of imbalance masses on rotor
$[A]$	matrix of undamped normal modes
$[C]$	damping matrix
$[\diagdown C_c \diagdown]$	damping ratios of free housing
$[D_R]$	matrix of velocity dependent coefficients for the rotor model
$[\diagdown E_R \diagdown]$	damping ratios of free-free rotor
$\{F\}$	force vector
$[G]$	flexibility matrix
i	imaginary number $\sqrt{-1}$
$[K]$	stiffness matrix
$[M]$	mass matrix
$\{q\}$	vector of generalized coordinates
$\{P_R\}$	vector of imbalance forces
$\{R\}$	physical displacements vector
$[T]$	transformation matrix to reduce coupling forces on housing and rotor in terms of those on the rotor
$[Y]$	receptance (dynamic flexibility) or generalized receptance matrix
$[Z]$	impedance (dynamic stiffness) matrix
λ	complex eigenvalue of coupled system

$[\lceil \Lambda \rfloor]$	matrix of frequencies of undamped housing or nonspinning rotor
$\dot{\phi}$	rotor spin speed about Z axis, rad/sec.
$\{\psi\}$	complex eigenvector of turbopump system
$\{R_R\}$	displacement vector of the coupling points on the rotor
$\{R_R'\}$	displacement vector of the coupling points, and of the points of application of the imbalance forces on the rotor

1. INTRODUCTION

1.1 Background

The dynamic analysis of models of large rotor systems, such as the turbopumps of the SSME (Space Shuttle Main Engine), may involve excessive computations and large round-off errors. Use of reduced size models for the systems is, therefore, highly desireable. To that end, the more frequently used methods include modal representation [1], substructure techniques [2] and static reduction techniques [3].

For large complex rotor systems, various modeling and analysis techniques vary in their ability co accurately describe the systems' behavior. This ability depends mainly on the configuration of the systems analyzed and the forcing conditions as well as the particular results sought. For the analysis of the turbopumps of the SSME, certain distinct characteristics should be considered. Accurate modeling of the housings requires representation by relatively large number of degrees of freedom. Also, the housing is unsymmetrical about the spin axis of the rotor. In addition, engine test results have tended to show the presence of significant coupled rotor/housing modes. The coupling occurs through nonconservative velocity dependent gyroscopic and viscous forces and couples. Finally, experience has shown that predicted onset speeds of instability tend to be remarkably sensitive to small changes in the modal representation of the housing.

Clearly, a strong need exists for a more thorough and careful consideration of the representation of the housing structural dynamics model and of the rotor/housing coupling procedure. For stability analysis the ability to predict how changes in the coupling parameters effect the complex eigenvalues of the assembled rotor/housing turbopump is equally important. An understanding of the nature of changes in the system eigenvalues due to changes in the model parameters can be helpful in quiding an experimental/analytical effort toward resolving any discrepancies between test and analysis results. Calculated parameter sensitivities would also be useful in evaluating design alternatives and in handling uncertainties in the input data.

1.2 Objectives and Scope

This study was mainly initiated for the purpose of developing improved dynamic analysis and modeling of large rotor systems. Specifically, the main objectives of the study were to:

(1) address the problem of structural dynamic modeling of the large housing of the SSME turbopumps and its coupling to the rotor.

(2) develop accurately reduced models and efficient procedures for determining the response and stability of the turbopumps of the SSME.

(3) develop and/or apply selected sensitivity analysis techniques to determine the effect of variations in coupling parameters on the stability and forced response of the turbopumps.

The main results of the study are outlined in this report. Following a comprehensive assessment of modeling and analysis methods for large rotor systems, alternative techniques were developed for the study of the steady state response and stability as well as parameter sensitivity of the SSME turbopump systems. The analysis is limited to being linear and the response considered only concerns steady state conditions under rotor imbalance.

2. REVIEW OF ROTORDYNAMIC ANALYSIS OF LARGE SYSTEMS

2.1 Modeling and Analysis

Various Analysts have proposed and utilized different procedures in obtaining the response and stability of large order rotor systems. The motion of the systems was described by physical coordinates, generalized coordinates or combinations of both.

Analysis strategies may be recognized as falling under one of two basic classes. Those using the complete system and those using the individual components of the system, together with appropriate coupling procedures. For large systems, both strategies call for reduction of the size of the models involved. An assessment of existing modeling and coupling techniques as related to analysis procedures used is presented in what follows. Also, a brief account is made of the associated eigenvalue and response sensitivity to changes in system parameters. The review is restricted to linear rotor systems.

2.1a Complete system-based methods

Starting with a full scale model of the total system, several methods have been devised for reducing the model's size for analysis purposes. A static reduction technique was introduced by Guyan [4] in which a given system is described by a selected subset of "master" degrees of freedom. The remaining degrees of freedom, called "slaves" are eliminated. The elimination is achieved through a transformation matrix formed from the associated static problem. The masters are capable of accurate representation of the lower modes of the system. However, the master coordinates must be carefully selected otherwise some of the lower eigenvalues of the system will be lost. Henshell and Ong [5] proposed an automatic technique for choosing the master coordinates. The criterion for the choice is based upon the corresponding ratios of the diagonal elements of the stiffness and mass matrices. A slave degree of freedom corresponds to a large stiffness to mass ratio. This is so since in a Guyan reduction it is implicitly assumed that the mass terms corresponding to slave degrees of freedom have negligible effect on the mode shapes. This leads to the conclusion that either the corresponding masses are small or that the stiffnesses are large.

Rouch and Kao [3] extended the Guyan static condensation technique to the analysis of simple rotor-bearing systems by accounting for gyroscopic matrices. The selection of master degrees of freedom was made taking advantage of the knowledge of characteristic beam bending modes of the rotor. It appears, as concluded by the authors, that the rotor-bearing systems are amenable to this type of reduction because of their nominally one-dimensional configuration. No assessment of the resulting accuracy with various types of analysis was given. Nordmann [6] attempted to minimize the uncertainties in the selection of the master degrees of freedom in the static condensation technique. This was achieved by applying the reduction technique to an arbitrarily divided rotor system and then assemble the reduced substructures to form a reduced system. The procedure is very laborious and no guarantee of accuracy is apparent.

Downs [7] proposed a reduction method which produces frequency dependent mass matrices. Strategies for selection of master coordinates were given which allow for progressive improvement in the selection. The method was used to obtain the eigenvalues of a given system up to an upper frequency limit.

A different approach for the reduction was recently advanced by several investigators. The main characteristic of the approach is to perform an exact reduction by replacing the stiffness matrices in the static approach by that of the dynamic stiffness, or impedance matrices. In an eigenvalue analysis (or steady state response analysis), the reduced system will be dependent on the eigenvalue (or the forcing frequency) under consideration. Iterative procedures are then used with this "dynamic condensation" method in order to calculate the eigenvalue of the system. No approximations are made in arriving at the reduced model. This reduction however is achieved at the expense of requiring of more involved computation. Several authors [8,9] proposed calculation procedures for determining the eigenvalues of undamped, conservatives structural systems.

In an attempt to alleviate the problem of frequency dependence of the dynamically reduced models, Fricker [10] devised a method in which the frequency-dependent terms are retained implicitly by using dynamic stiffness matrices defined at a number of fixed frequencies. The dynamic stiffness matrices may be condensed efficiently to a relatively small number of "master" coordinates using a front solution algorithm. The method appears of promise, although it is not apparent how to utilize it in conjunction with nonconservative gyroscopic systems.

A different approach for the analysis of mechanical structures is to describe the structures by their modal coordinates. A significant advantage of representing a structure by modal coordinates is that the representation forms the basis for size reduction of the structure's model.

This may be achieved by retaining only a small number of modes in a specified frequency range, usually selected from among the lower modes. However, the particular modes which would influence the behavior of a given system depend on the type and location of the external loads. No general quidelines are available in that connection.

Childs [11] used undamped normal modes of a given rotor to perform transient rotordynamic analysis. The modes selected are those of the lower modes of the complete system involved. No allowance was made for the truncated modes. Choy et al. [12] used complex modal analysis to determine the unbalance response of damped rotor systems. For simplified cases considered, the complex modes of the system were obtained from those of the undamped normal modes of the system.

2.1b <u>Subsystem-based methods</u>

A reduced size model can be obtained for a given system by first performing the reduction on the subsystems and then assemble them to form a reduced system.

Hou [13] devised a scheme in which the subsystems involved are represented by truncated sets of their free-interface modes. A transformation, derived from the displacement compatibility at the connection points of the substructures, is then applied to their modal equations of motion to arrive at a description of the motion of the assembled system. This and similar procedures are labelled modal synthesis techniques. Li and Gunter [14] utilized Hou type approach to calculate the vibrations of large multi-component rotor systems. They also conducted an evaluation of the number of modes required to yield acceptable accuracy for the unbalance response of a two-spool gas turbine engine [15]. In general, depending on the type of structure, a significant number of modes might be necessary to yield response information of acceptable accuracy at specified locations of interest on the structure. These locations could be those where coupling or interface to other structural components occurs.

Several methods exist in the literature which offer procedures for the improvement of analysis accuracy for a given number of retained modes. Some of these techniques utilize modal coordinates obtained with points of interest on the structure free (free-interface methods), and provide means for accomodating the effects of truncated higher modes in approximate manners. MacNeal [16] suggested a method in which the static contribution of the truncated modes is incorporated. This accounts for the missing flexibility effects (residual flexibility) due to truncation. The method can lead to significant improvement in the convergence in the solution for the dynamical problem. Childs and Bates [17] applied this technique in determining the transient response of a rotor and reported significant increase in solution accuracy for a given number of modes.

A method, analogous to that of MacNeal, is also outlined in the literature to account for the truncated lower modes of frequencies below the frequency range of interest. The technique is described by Klosterman and McClelland [18], in which an approximate representation of the lower mode contribution to the structure response is proposed and labelled "inertia restraint". Further improvement of Hou's technique was introduced by Rubin [19]. Rubin proposed a method in which the first order approximation of MacNeal is utilized to construct a pseudo-static response of a substructure, acting within a structure, which includes inertial and dissipative contributions. Noah [20] has shown that inclusion of these second-order effects may drastically improve: (i) the modal representation of a substructure for the modal synthesis of a structure, and (ii) the damping synthesis for the assembled system.

A further reduction of a modal description of a subsystem, in addition to that achieved by imposing lower and upper cut-off frequencies, can still be accomplished by selecting certain modes from among those within the specified frequency range. Tolani [21] proposed a criterion based on the strain energy of a subsystem, whereby an upper limit on strain energy is set and only the modes having strain energy below this value are considered. An approach for selecting the modes of a subsystem that are important with respect to the motion of another given subsystem is due to Morosow and Abbott [22]. Unfortunately, the method is only applicable to weakly coupled systems.

An alternative to using free-interface modal coordinates is that using "fixed-interface" methods. In this method, the structure is represented by a truncated set of lower modes obtained with points of interest (usually those of the coupling points to other subsystems) on the structure completely fixed. The retained modes are complemented by static "constraint modes" obtained as influence coefficients corresponding to the points of interest on the structure. Glasgow and Nelson [23] extended this technique (introduced in [24] and [25]) to nonconservative systems and applied it to the dynamic substructure coupling of various rotordynamic systems. A disadvantage of this type of approach is that for systems with large number of coupling points among the components, the approach suffers the problem of introducing higher frequencies resulting from excessive number of constraints imposed at the coupling (or boundary) points. In a transient analysis, this will necessarily result in much smaller time increments and consequently will lead to excessive computational time and larger round-off errors.

Complete treatment for a generalization of free-interface methods to nonconservative systems, analogeous to that of the fixed-interface method, [23], is still lacking. Craig and Chung [26] utilized first order state space formulation of the equations of motion to represent a given component by a truncated set of its complex modes. Wu and Greif [27], in a novel approach, utilized two successive transformations based upon consecutive use of free interface modes in the existing physical coordinates. The set of modes for the first transformation are those of the undamped, free interface subsystems. The second transformation is based on the damped fixed-interface subsystem in generalized co-ordinates which freezes the generalized interfaces. The second set of generalized coordinates is truncated and is supplemented by the fixed-interface coupled coordinates. The system eigenvalue problem is solved, after physical interface compatibility is invoked in coupling the various subsystems. Further investigation of this technique is warranted.

Other methods applied to nonconservative systems include those using receptances of the subsystems. Palazzolo et al. [28,29] used hybrid representation of the receptances in terms of a truncated set of complex modes supplemented by correction matrices involving mass, stiffness and damping properties of the subsystems. The method was used to determine the steady state response of rotor-system components as well as in efficient eigenvalue reanalyses for the components. Applications to the synthesis of certain type of rotating machinery train was also made. The general applicability of the method as a component mode synthesis technique was not demonstrated, however.

The Dynamic reduction method discussed earlier [9] was also utilized by several analysts to reduce the size of the components of a system, prior to their coupling, to form a reduced system. Berman [30] presented a substructure coupling method that make use of frequency dependent response of the substructures (represented by reduced impedances) to interface and excitation forces. Leung [31] further developed his dynamic condensation technique for use with substructures. His method uses physical coordinates of the subsystems to satisfy compatibility conditions. The subsystems are described by a few lowest fixed-interface modes in conjunction with the static constraint modes of Hurty [24]. The size of the system matrices involved is equal to the number of degrees of freedom, or masters, for each subsystem. A similar approach to that of Berman [30] was presented by Geering [32]. Geering applied his method, based on what he termed dynamic elasticity transfer matrix, to obtain the transient response to periodic loads and the response spectra to stationary random loads. The eigenvalue problem was eluded to in his paper.

A totally fresh subsystem analysis technique was introduced by Hale and Meirovitch [33,34]. Admissible functions were employed for the representation of the substructures involved. The approach results in low order polynomial representation that simplifies computation. The admissible functions are not necessarily those of the substructures' eigenvectors. The geometric compatibility conditions are approximately enforced by the method of weighted residuals. In the author's opinion this method is a powerful and general technique for conducting dynamic substructure analysis. However, the method would not allow use of experimental data at the substructure level. Also, no apparent means exists for applying the technique to damped nonconservative systems.

2.2 Sensitivity Analysis

An important aspect of the modal representation of a linear dynamic system is the inclusion of accurate estimation of those system parameters to which the system behavior is most sensitive. The literature offers several studies and suggested procedures concerning such sensitivity analyses.

Fox and Kapoor [35] obtained expressions for the first derivatives of the eigenvalues and eigenvectors with respect to design parameters of the original self-adjoint systems. Plaut and Huseyin [36] derived general expressions for the derivatives of eigenvalues and eigenvectors in non-self-adjoint systems. Their results can be useful in examining the stability behavior of nonconservative systems. Other sensitivity analysis techniques have also been reported which differ in their computational efficiency, type of dynamic system considered, and order of the derivatives of the eigenparameters involved. In reference [36], the determination of any one of the derivatives of the eigenvectors of a nonself-adjoint system requires use of all of the left-hand and right-hand eigenvectors. This was avoided in a formulation given by Garg [37], in which the calculation of derivatives is reduced to solving two sets of simultaneous linear algebraic equations for each eigensolution of interest. Rudisill [38] developed an alternative procedure to that of Garg which could be extended to find any order of derivative of the eigenvalues and eigenvectors, provided they exit.

Nelson [39] discussed the computational requirements for obtaining the derivatives of the eigenvectors using the two approaches of [36] and [37]. He pointed out that the first approach, in which the eigenvector derivative is expressed as a sum of all the eigenvectors, although analytically simple becomes prohibitively expensive for large systems. On the other hand, the second approach requires only the specified eigenvalue and eigenvector but calls for the premultiplication of an (n+1) x n matrix by its transpose to form an nth order system of linear equations. Nelson presented an alternative simplified procedure for the sensitivity analysis, based on the second approach.

An interesting paper, in a series of publications by Simpson [40], discusses means for readily obtaining eigenvalue and response sensitivities using simple modification of the basic eigenvalue method of Kron [41]. The method employs a subsystem approach and provides for means of studying transmissibility of vibration between the system components. A recent study by Yoshimura [42] also addresses the need of developing efficient sensitivity analysis of frequency response. A method was presented for determining design sensitivity coefficients of receptance-frequency response evaluative functions. The method was applied to machine-tool structures.

Several sensitivity studies related to rotordynamic applications have been reported recently. Lund [43] presented a method to calculate sensitivities of the critical speeds of a conservative rotor system to changes in the design. Fritzen and Nordmann [44], on the other hand, reported on a sensitivity method applicable to nonconservative rotor system (involving representation of seals, oilfilm bearing, etc.). The method is based on Taylor's expansion for complex eigenvalues in terms of system parameters in which linear, quadratic, or higher-order formulas are obtained, depending on the order of the derivatives retained in the expansion. The method was demonstrated using several rotordynamic applications.

3. DEVELOPMENT OF ANALYSIS METHODS

An SSME turbopump is a large nonconservative mechanical structure consisting of a large component (housing) which is connected to a gyroscopic damped subsystem (rotor) through nonconservative coupling elements (bearing, seals, etc.). The motion of the main components of housing and rotor is highly coupled. Simulation of the system's dynamic behavior is sensitive to the choice of the model, its size and to the parameters of the coupling elements.

For this study, it is desired to develop significantly reduced size models for the housing which are sufficiently accurate, along with effective coupling and analysis procedures. The developed procedures would allow determining steady state response stability and sensitivity type of analysis. Based on a careful examination of existing techniques (of which an account is included in the previous section 2), alternative modeling and coupling techniques suitable for application to the SSME turbopumps are developed and are presented in this section.

In these methods the steady state complex response of a given turbopump to the coupling forces is determined for cases of forced response to imbalance. The coupling forces are those at a given constant rotor spin. For stability analysis, the system will be moving in one of the modes of the system with a given complex eigenvalue. The coupling of the subsystem is accomplished using one of two approaches. The first approach uses superposition of the complex impedances of the individual components to obtain the impedance of the complete system. The system's impedance can then be used to determine the dynamic behavior of the system. The other approach uses the receptances of the housing and rotor together with the impedances of the coupling elements to couple the system. This method is termed generalized receptance method since the final system's equations are in terms of the coupling forces.

The impedance, or alternatively the receptance technique, is used to develop a procedure for determining the response sensitivity to changes in the coupling parameters.

3.1 The Models

3.1a The rotor reduced model

The equations of motion for the spinning rotor of a given turbopump, while coupled to the housing, can be written as [45], (see Figures 1 and 2 as applied to the HPOTP)

$$\begin{bmatrix} \lceil M_R \rfloor & \\ & \lceil J_d \rfloor \end{bmatrix} \begin{Bmatrix} \ddot{R}_{XR} \\ \ddot{\beta}_{YR} \end{Bmatrix} + \begin{bmatrix} K_{XZR} \end{bmatrix} \begin{Bmatrix} R_{XR} \\ \beta_{YR} \end{Bmatrix} = \begin{Bmatrix} f_{XR} \\ M_{YR} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lceil \dot{\phi} J_p \rfloor \end{bmatrix} \begin{Bmatrix} \dot{R}_{YR} \\ \dot{\beta}_{XR} \end{Bmatrix} + \dot{\phi}^2 \begin{Bmatrix} (ma_X) \\ J_{XZ} \end{Bmatrix} + \ddot{\phi} \begin{Bmatrix} (ma_Y) \\ J_{YZ} \end{Bmatrix} \quad (1a)$$

$$\begin{bmatrix} \lceil M_R \rfloor & \\ & \lceil J_d \rfloor \end{bmatrix} \begin{Bmatrix} \ddot{R}_{YR} \\ \ddot{\beta}_{XR} \end{Bmatrix} + \begin{bmatrix} K_{YZR} \end{bmatrix} \begin{Bmatrix} R_{YR} \\ \beta_{XR} \end{Bmatrix} = \begin{Bmatrix} f_{YR} \\ M_{XR} \end{Bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \lceil \dot{\phi} J_p \rfloor \end{bmatrix} \begin{Bmatrix} \dot{R}_{XR} \\ \dot{\beta}_{YR} \end{Bmatrix} + \dot{\phi}^2 \begin{Bmatrix} (ma_Y) \\ -J_{YZ} \end{Bmatrix} + \ddot{\phi} \begin{Bmatrix} -(ma_X) \\ J_{XZ} \end{Bmatrix} \quad (1b)$$

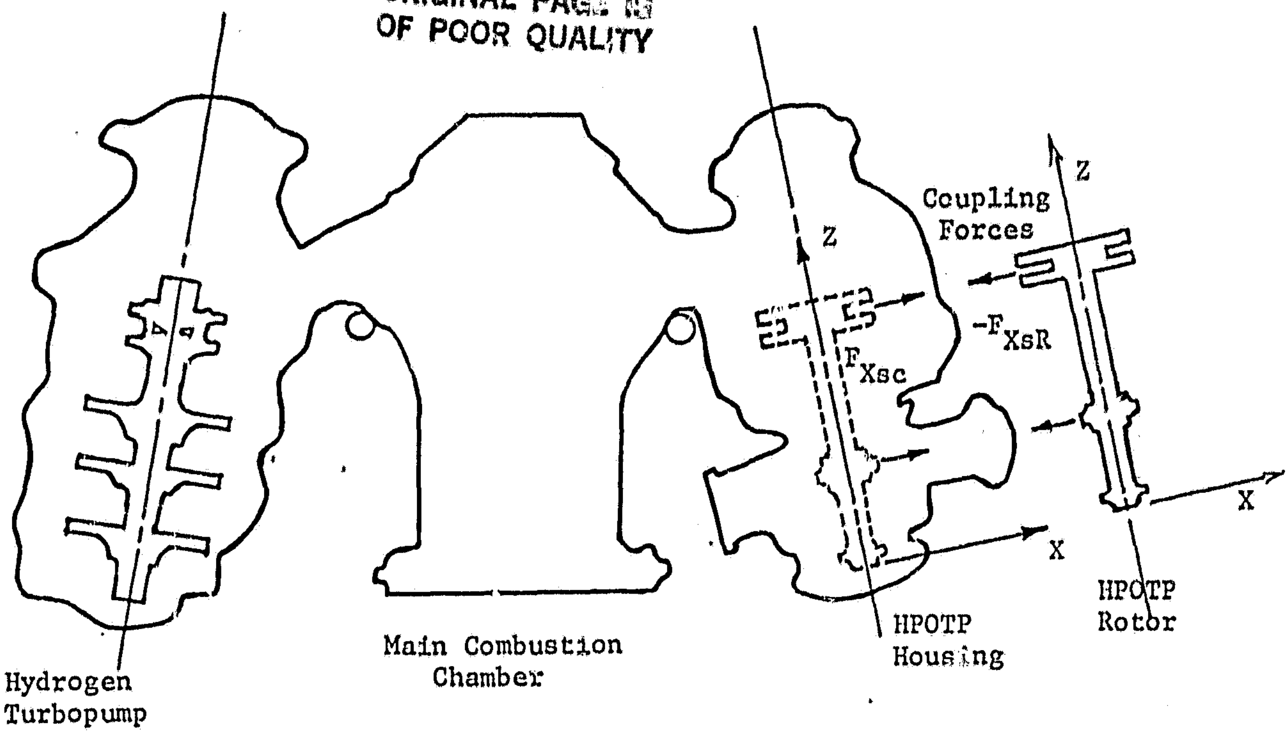

Figure 1. The SSME Engine System for the Analysis of the HPOTP.

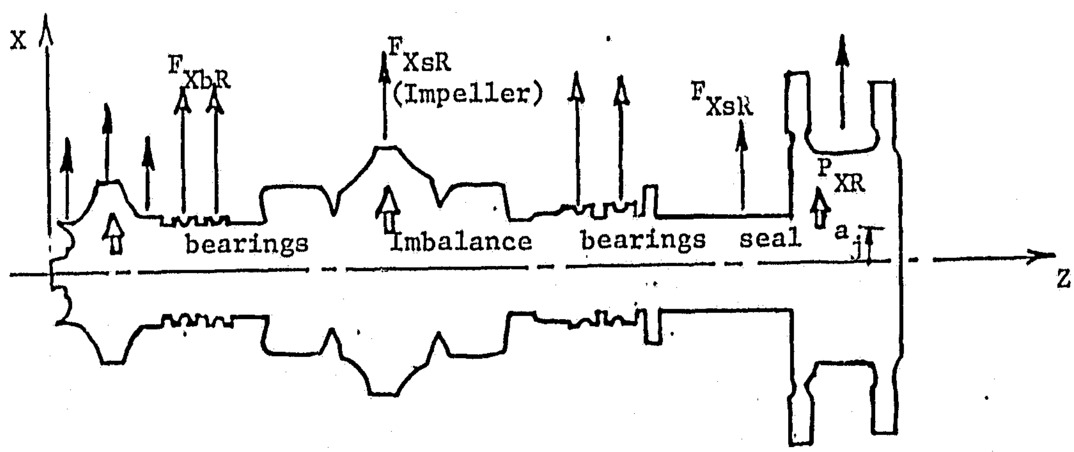

Figure 2. The HPOTP rotor under coupling and imbalance forces.

A dot in equations (1a) and (1b) denotes differentiation with respect to time, t. Also

X, Y, Z	=	Inertial coordinates system
x, y, z	=	Rotating coordinates system, x and y are fixed to the rotor
$\{R_{XR}\}$ $\{R_{YR}\}$	=	Displacement vectors of rigid bodies on rotor
$\{\beta_{XR}\}$ $\{\beta_{YR}\}$	=	Small rotation vectors of rigid bodies about X and Y axis, respectively
$[K_{XZR}]$, $[K_{YZR}]$	=	Stiffness matrices corresponding to motions in the X-Z and Y-Z planes, respectively
$\{f_{XR}\}$, $\{f_{YR}\}$, $\{M_{XR}\}$, $\{M_{YR}\}$	=	External forces and moments, including coupling forces
$\{J_{XZ}\}$ $\{J_{YZ}\}$	=	Products of inertia
ϕ, $\dot{\phi}$, $\ddot{\phi}$	=	Rotor's spin angle, velocity and acceleration
$\{a_X\}$	=	$\{a_x\}\cos\dot{\phi}t - \{a_y\}\sin\dot{\phi}t$
$\{a_Y\}$	=	$\{a_x\}\sin\dot{\phi}t + \{a_y\}\cos\dot{\phi}t$
$\{a_x\}$, $\{a_y\}$	=	Vectors of coordinates of the imbalance masses measured in the rotor fixed axes x, and y at the respective rigid bodies
$\{m\}$	=	imbalance masses
$[\diagdown J_d \diagdown]$	=	diametral moments of inertia for rigid bodies on rotor

$[\diagdown J_p \diagdown]$ = polar moments of inertia for rigid body about the Z-axis

$[\diagdown M_R \diagdown]$ = mass matrix of rotor associated with displacement degrees of freedom

A reduced size model for the rotor, appropriate for steady state and stability analyses, can be obtained by representing the rotor by its displacements at the coupling and imbalance locations. To that end, two procedure can be used, (i) Modal representation, or, (ii) Dynamic reduction of the equations of motion in physical coordinates.

(i) Modal representation of reduced rotor impedance:

The following modal transformations are employed in conjunction with the equations of motion (1a) and (1b)

$$\begin{Bmatrix} R_{XR} \\ \beta_{YR} \end{Bmatrix} = [A_{XZR}] \{q_{XR}\} \tag{2a}$$

$$\begin{Bmatrix} R_{YR} \\ \beta_{XR} \end{Bmatrix} = [A_{YZR}] \{q_{YR}\} \tag{2b}$$

where the matrices $[A_{XZR}]$ and $[A_{YZR}]$ are those of the planar modes in the X-Z and Y-Z planes, respectively. The modal matrices are obtained by solving the eigenvalue problem associated with equations (1a) and (1b) with their right hand sides equated to zero. The equations in this case would represent the free vibration of the undamped, nonspinning rotor. The vectors $\{q_{XR}\}$ and $\{q_{YR}\}$ are the generalized (modal coordinates) in the X-Z and Y-Z planes, respectively. The eigenvectors of equations (2a) and (2b) are normalized with respect to the inertia matrix of the rotor, or

$$[A_{XZR}]^T \begin{bmatrix} \lceil M_R \rfloor & \\ & \lceil J_d \rfloor \end{bmatrix} [A_{XZR}] = [A_{YZR}]^T \begin{bmatrix} \lceil M_R \rfloor & \\ & \lceil J_d \rfloor \end{bmatrix} [A_{YZR}] = \lceil I \rfloor \tag{3}$$

in which $\lceil I \rfloor$ is the unit matrix. Assuming symmetry about the axis of rotation, Z, one can write

$$\lceil \Lambda_{XR} \rfloor = [A_{XZR}]^T [K_{XZR}] [A_{XZR}] = [A_{YZR}]^T [K_{YZR}] [A_{YZR}] = \lceil \Lambda_{YR} \rfloor \tag{4}$$

where $\lceil \Lambda_{XR} \rfloor$ and $\lceil \Lambda_{YR} \rfloor$ are equal diagonal matrices whose elements are the square of the natural frequencies of the free undamped, nonspinning rotor.

For steady state motion with constant spinning speed $\dot{\phi}$, the acceleration $\ddot{\phi}$ vanishes, hence the last term in both of equations (1a) and (1b) will also vanish, leading to the following equations

$$\lceil I \rfloor \{\ddot{q}_{XR}\} + \lceil \Lambda_{XR} \rfloor \{q_{XR}\} = [A_{XZR}]^T \begin{Bmatrix} f_{XR} \\ M_{YR} \end{Bmatrix}$$

$$+ [A_{XZR}]^T \begin{bmatrix} 0 & 0 \\ 0 & \lceil \dot{\phi} J_p \rfloor \end{bmatrix} [A_{YZR}] \{\dot{q}_{YR}\}$$

$$+ \quad \dot{\phi}^2 [A_{XZR}]^T \begin{Bmatrix} \{ma_X\} \\ J_{XZ} \end{Bmatrix} \tag{5a}$$

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and

$$[\ulcorner I \lrcorner]\{\ddot{q}_{YR}\} + [\ulcorner \Lambda_{YR} \lrcorner]\{q_{YR}\} = [A_{YZR}]^T \begin{Bmatrix} f_{YR} \\ M_{XR} \end{Bmatrix}$$

$$- [A_{YZR}]^T \begin{bmatrix} 0 & 0 \\ 0 & [\ulcorner \dot{\phi} J_p \lrcorner] \end{bmatrix} [A_{XZR}]\{\dot{q}_{XR}\}$$

$$+ \dot{\phi}^2 [A_{YZR}]^T \begin{Bmatrix} (ma_Y) \\ -J_{YZ} \end{Bmatrix} \tag{5b}$$

Let

$$\{q_R\} = \begin{Bmatrix} q_{XR} \\ q_{YR} \end{Bmatrix}, \quad \{F_{XR}\} = \begin{Bmatrix} f_{XR} \\ M_{YR} \end{Bmatrix}, \quad \{F_{YR}\} = \begin{Bmatrix} f_{YR} \\ M_{XR} \end{Bmatrix}$$

$$\{P_{XR}\} = \dot{\phi}^2 \begin{Bmatrix} (ma_X) \\ J_{XZ} \end{Bmatrix}, \quad \{P_{YR}\} = \dot{\phi}^2 \begin{Bmatrix} (ma_Y) \\ -J_{YZ} \end{Bmatrix},$$

$$[J_p'] = \begin{bmatrix} 0 & 0 \\ 0 & [\ulcorner \dot{\phi} J_p \lrcorner] \end{bmatrix} \tag{6}$$

Equations (5a) and (5b) may then be put in a block matrix form which includes modal damping; the modal damping is taken as in reference [45]

$$\{\ddot{q}_R\} + \begin{bmatrix} [\ulcorner C_{XR} \lrcorner] & 0 \\ 0 & [\ulcorner C_{YR} \lrcorner] \end{bmatrix} \{\dot{q}_R\} + \begin{bmatrix} [\ulcorner \Lambda_{XR} \lrcorner] & \dot{\phi}[\ulcorner C_{XR} \lrcorner] \\ -\dot{\phi}[\ulcorner C_{YR} \lrcorner] & [\ulcorner \Lambda_{YR} \lrcorner] \end{bmatrix} \{q_R\}$$

$$= \begin{bmatrix} [A_{XZR}]^T & \\ & [A_{YZR}]^T \end{bmatrix} \begin{Bmatrix} F_{XR} \\ F_{YR} \end{Bmatrix} + \begin{bmatrix} 0 & [A_{XZR}]^T [J_p'] [A_{YZR}] \\ -[A_{YZR}]^T [J_p'] [A_{XZR}] & 0 \end{bmatrix} \{\dot{q}_R\}$$

$$+ \begin{bmatrix} [A_{XZR}]^T & 0 \\ 0 & [A_{YZR}]^T \end{bmatrix} \begin{Bmatrix} P_{XR} \\ P_{YR} \end{Bmatrix}, \qquad (7)$$

in which due to rotor symmetry

$$[\diagdown C_{XR} \diagdown] = [\diagdown C_{YR} \diagdown] = [\diagdown C_R \diagdown] = [\diagdown 2\, E_R \Lambda_R{}^{1/2} \diagdown]$$

$$(8)$$

also $\qquad [\diagdown \Lambda_{XR} \diagdown] = [\diagdown \Lambda_{YR} \diagdown] = [\diagdown \Lambda_R \diagdown]$

and $\qquad [\diagdown E_R \diagdown]$ is the matrix of modal damping ratios

If the case under consideration is that of a steady state response to imbalance at a rotor speed of $\dot{\phi}$, the imbalance forces can be written in terms of forward and backward whirling as

$$\{P_R(t)\} = \{\bar{P}_R\}\, e^{i\dot{\phi}t} + \{\bar{P}_R^*\}\, e^{-i\dot{\phi}t};\ i = \sqrt{-1} \qquad (9)$$

where $\{\bar{P}_R^*\}$ is the conjugate of the vector of complex amplitudes $\{\bar{P}_R\}$. This is so since $\{P_R\}$ is a vector of real variables. As an example, the imbalance forces, associated with the HPOTP of the SSME [46] are given by

$$\begin{aligned} \{P_{XR}\} &= \dot{\phi}^2\, \{ma_x\} \cos \dot{\phi}t\ , \text{ since } \{ma_y\} = \{0\} \\ &= \left\{\tfrac{1}{2}\, \dot{\phi}^2\, ma_x\right\} e^{i\dot{\phi}t} + \left\{\tfrac{1}{2}\, \dot{\phi}^2\, ma_x\right\} e^{-i\dot{\phi}t} \\ &= \{\bar{P}_{XR}\}\, e^{i\dot{\phi}t} + \{\bar{P}_{XR}^*\}\, e^{-i\dot{\phi}t} \end{aligned} \qquad (10)$$

Similarly

$$\{P_{YR}\} = \dot{\phi}^2 \{ma_x\} \sin \dot{\phi} t$$

$$= \left\{\frac{-i}{2} \dot{\phi}^2 \, ma_x\right\} e^{i\dot{\phi}t} + \left\{\frac{i}{2} \dot{\phi}^2 \, ma_x\right\} e^{-i\dot{\phi}t}$$

$$= \{\overline{P}_{YR}\} e^{i\dot{\phi}t} + \{\overline{P}^*_{YR}\} e^{-i\dot{\phi}t} \tag{11}$$

It follows that the coupling forces and generalized coordinates in equation (7) can be put in the form

$$\{F_R\} = \{\overline{F}_R\} e^{i\dot{\phi}t} + \{\overline{F}^*_R\} e^{-i\dot{\phi}t} \tag{12}$$

$$\{q_R\} = \{\overline{q}_R\} e^{i\dot{\phi}t} + \{\overline{q}^*_R\} e^{-i\dot{\phi}t} \tag{13}$$

Equation (7) can then be written as coefficients of $e^{i\dot{\phi}t}$ and $e^{-i\dot{\phi}t}$, or

$$e^{i\dot{\phi}t}: \left[-\dot{\phi}^2 \lceil I_R \rfloor + i\dot{\phi} [D_R] + [\Omega_R]\right] \{\overline{q}_R\} = [A_R]^T \{\overline{F}_R + \overline{P}_R\} \tag{14a}$$

$$e^{i\dot{\phi}t}: \left[-\dot{\phi}^2 \lceil I_R \rfloor + i\dot{\phi} [D_R] + [\Omega_R]\right] \{\overline{q}^*_R\} = [A_R]^T \{\overline{F}^*_R + \overline{P}^*_R\} \tag{14b}$$

where

$$[A_R] = \begin{bmatrix} [A_{XZR}] & \\ & [A_{YZR}] \end{bmatrix},$$

$$[D_R] = \begin{bmatrix} \lceil C_R \rfloor & -[A_{XZR}]^T [J'_p] \, [A_{YZR}] \\ [A_{YZR}]^T [J'_p] [A_{XZR}] & \lceil C_R \rfloor \end{bmatrix}$$

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$$[\Omega_R] = \begin{bmatrix} [\ulcorner \Lambda_R \lrcorner] & \dot{\phi}[\ulcorner C_R \lrcorner] \\ -\dot{\phi}[\ulcorner C_R \lrcorner] & [\ulcorner \Lambda_R \lrcorner] \end{bmatrix}$$

But $\{R'_R\} = [A_R]\{q_R\}$ (15)

where $\{R'_R\}$ are the displacements of the coupling points and of the locations at which the imbalance forces are applied

$$\{\overline{R}'_R\} e^{i\dot{\phi}t} + \{\overline{R}^{*\prime}_R\} e^{-i\dot{\phi}t} = [A_R] \left(\{\overline{q}_R\} e^{i\dot{\phi}t} + \{\overline{q}^*_R\} e^{-i\dot{\phi}t}\right)$$

hence

$$\{\overline{R}'_R\} = [A_R]\{\overline{q}_R\} \quad (16a)$$

$$\{\overline{R}^{*\prime}_R\} = [A_R]\{\overline{q}^*_R\} \quad (16b)$$

Equations (14a) and (16a) yield the reduced model

$$\{\overline{R}'_R\} = [A_R]\left[-\dot{\phi}^2 [\ulcorner I_R \lrcorner] + i\dot{\phi}[D_R] + [\Omega_R]\right]^{-1} [A_R]^T \{\overline{F}_R + \overline{P}_R\}$$

or

$$\{\overline{R}'_R\} = [Y_R]\{\overline{F}_R + \overline{P}_R\} \quad (17)$$

Similarly, equations (14b) and (16b) lead to the conjugate representation

$$\{\overline{R}^{*\prime}_R\} = [Y^*_R]\{\overline{F}^*_R + \overline{P}^*_R\} \quad (18)$$

However, only the reduced model of equation (17) need be determined. The conjugate representation given by equation (18) can then be obtained directly.

In equation (17), the reduced impedance matrix of the rotor associated with the forward whirl speed $(+\dot{\phi})$ is given by

$$[Z_R] = [Y_R]^{-1} = \left[[A_R] \left[-\dot{\phi}^2 [\ulcorner I_R \lrcorner] + i\dot{\phi}[D_R] + [\Omega_R] \right]^{-1} [A_R]^T \right]^{-1} \quad (19)$$

in which $[Y_R]$ is the reduced receptance matrix of the rotor.

(ii) Direct dynamic reduction:

An alternative approach can be utilized to construct the reduced impedance matrix for the rotor. This can be achieved by applying Gau elimination only to the upper partition of the complete impedance matrix of the rotor corresponding to the unwanted degrees of freedom. The "kept" coordinates will be those of the coupling points and those at which imbalance forces are applied. To that end, equations (1a) and (1b) are arranged as

$$[\ulcorner M_R' \lrcorner] \begin{Bmatrix} \ddot{R}_{Rr} \\ \ddot{\beta}_{Rr} \\ \ddot{R}'_{Rk} \end{Bmatrix} + [C_R'] \begin{Bmatrix} \dot{R}_{Rr} \\ \dot{\beta}_{Rr} \\ \dot{R}'_{Rk} \end{Bmatrix} + [K_R'] \begin{Bmatrix} R_{Rr} \\ \beta_{Rr} \\ R'_{Rk} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ F_R \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ P_R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \quad (20)$$

where the subscripts "r" and "k" denote reduced out and kept coordinates, respectively.

At a given rotor spin speed $\dot{\phi}$, the steady state variables may be written, as before, in the form

$$\{F\} = \{\overline{F}\} e^{i\dot{\phi}t} + \{\overline{F}^*\} e^{-i\dot{\phi}t} \quad (21)$$

and

$$\begin{Bmatrix} R_{Rr} \\ \beta_{Rr} \\ R'_{Rk} \end{Bmatrix} = \begin{Bmatrix} \bar{R}_{Rr} \\ \bar{\beta}_{Rr} \\ \bar{R}'_{Rk} \end{Bmatrix} e^{i\dot{\phi}t} + \begin{Bmatrix} \bar{R}^*_{Rr} \\ \bar{\beta}^*_{Rr} \\ \bar{R}^{*\prime}_{Rk} \end{Bmatrix} e^{-i\dot{\phi}t} \tag{22}$$

The coefficients of the forward whirl $e^{+i\dot{\phi}t}$ lead to

$$\begin{bmatrix} [-\dot{\phi}^2 M_{rr} + i\dot{\phi}C_{rr} + K_{rr}] & [-\dot{\phi}^2 M_{rk} + i\dot{\phi}C_{rk} + K_{rk}] \\ [-\dot{\phi}^2 M_{kr} + i\dot{\phi}C_{kr} + K_{kr}] & [-\dot{\phi}^2 M_{kk} + i\dot{\phi}C_{kk} + K_{kk}] \end{bmatrix} \begin{Bmatrix} S_{Rr} \\ R'_{Rk} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F} \end{Bmatrix} \tag{23}$$

where

$$\{\bar{S}_r\} = \begin{Bmatrix} \bar{R}_{Rr} \\ \bar{\beta}_{Rr} \end{Bmatrix}$$

For convenience, equation (23) is written as

$$\begin{bmatrix} Z_{rr} & Z_{rk} \\ Z_{kr} & Z_{kk} \end{bmatrix} \begin{Bmatrix} \bar{S}_{Rr} \\ \bar{R}'_{Rk} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F} \end{Bmatrix} \tag{24}$$

applying Gauss elimination procedure <u>only</u> to the upper partition of the impedance matrix, one gets [47]

$$\begin{bmatrix} Z'_{rr} & Z'_{rk} \\ 0 & Z'_{kk} \end{bmatrix} \begin{Bmatrix} \bar{S}_{Rr} \\ \bar{R}'_{Rk} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{F} \end{Bmatrix} \tag{25}$$

The rotor can now be expressed in terms of the reduced impedance matrix (reduced to the kept coordinates) as follows

$$[Z'_{kk}] \{\overline{R}'_{Rk}\} = \{\overline{F}\} \tag{26}$$

where $[Z'_{kk}]$ is the reduced impedance of the rotor at a given rotor speed associated with the coordinates and forces of coupling to the housing and of imbalance. Equation (26) can be used for coupling the rotor to the housing model.

3.1b The casing (housing) reduced model

The equations of motion of the SSME housing while connected to the rotor, in terms of the displacements $\{R_{SS}\}$ of selected points, can be written as

$$[\diagdown M_{SS} \diagdown] \{\ddot{R}_{SS}\} + [C_{SS}] \{\dot{R}_{SS}\} + [K_{SS}] \{R_{SS}\} = \{F_{SS}(t)\} \tag{27}$$

where the subscripts "SS" refer to the space shuttle main engine model in absence of the rotor of the turbopump under consideration but involving the nonspinning rotor of the other turbopump (see Figure 1). The forces $\{F_{SS}\}$ are those of the unknown interaction or coupling forces with the excluded rotor. In other words, the missing rotor is represented by the coupling forces yet to be determined.

A modal matrix of selected eigenvectors of the undamped, free vibration of the SSME (with $[C_{SS}] = \{F_{SS}\}$ set to zero in equation (27)) is used to represent the displacements in terms of the associated truncated set of generalized coordinates $\{q_c\}$

$$\{R_{SS}(t)\} = [A_{SS}] \{q_c(t)\} \tag{28}$$

where

$$[A_{SS}]^T \left\lceil M_{SS} \right\rfloor [A_{SS}] = \left\lceil I_C \right\rfloor \tag{29}$$

use of the relations (28) and (29) with equation (27) yields

$$\left\lceil I_c \right\rfloor \{\ddot{q}_c\} + \left\lceil 2\, C_c\, \Lambda_c^{1/2} \right\rfloor \{\dot{q}_c\} + \left\lceil \Lambda_c \right\rfloor \{q_c\}$$
$$= [A_{SS}]^T \{F_{SS}\} = [A_c]^T \{F_c\} \tag{30}$$

where $[A_c]$ are sub-eigenvectors corresponding to the coupling points to the rotor, and $\{F_c\}$ are the coupling forces.

In equation (30), $\left\lceil \Lambda_c \right\rfloor$ is a diagonal matrix whose elements are the squares of the undamped natural frequencies of the SSME associated with the retained modes. The matrix $\left\lceil C_c \right\rfloor$ is the diagonal matrix of modal damping ratios.

To obtain a reduced representation of a turbopump casing, only the coordinates $\{R_c\}$ at the coupling locations to the rotor are considered. In addition, if the retained generalized coordinates correspond to a range of excitation frequencies, the truncated higher and lower modes can be approximated using the associated residual flexibility [16] and inertia restraint [18] terms, respectively. The motion of the casing may now be described as

$$\{R_c(t)\} = \underbrace{[G_{inert.}]}_{\substack{\text{inertia} \\ \text{restraint}}} \{F_c(t)\} + \underbrace{[A_c]\ \{q_c(t)\}}_{\substack{\text{modal} \\ \text{representation}}} + \underbrace{[G_{resid.}]}_{\substack{\text{residual} \\ \text{flexibility}}} \{F_c(t)\} \tag{31}$$

and is associated with equation (30).

It can be shown that the residual flexibility matrix $[G_{resid.}]$ can be expressed by [20]

$$[G_{resid.}] = [A_c]_{du} \left\lceil \Lambda_c \right\rfloor^{-1}_{du} [A_c]^T_{du} \tag{32}$$

in which "du" stands for "deleted upper" modes. Also, the inertia restraint matrix can be shown to take the form

$$[G_{inert}] = \frac{1}{\lambda^2} [A_c]_{dL} [A_c]^T_{dL} \tag{33}$$

where "dL" denotes "deleted lower" modes, and λ is the unknown eigenvalues of the turbopump system (coupled housing/rotor) or, in case of forced (imbalance) response, $\lambda^2 = -\dot{\phi}^2$.

For steady state under imbalance forces on the rotor at the constant speed $\dot{\phi}$, the motion and coupling forces take the form

$$\{R_c(t)\} = \{\bar{R}_c\} e^{i\dot{\phi}t} + \{\bar{R}^*_c\} e^{-i\dot{\phi}t}; \tag{34a}$$

$$\{F_c(t)\} = \{\bar{F}_c\} e^{i\dot{\phi}t} + \{\bar{F}^*_c\} e^{-i\dot{\phi}t} \tag{34b}$$

in which the complex amplitudes $\{\bar{R}_c\}$ and $\{\bar{R}^*_c\}$ are conjugates. Similarly

$$\{q_c(t)\} = \{\bar{q}_c\} e^{i\dot{\phi}t} + \{\bar{q}^*_c\} e^{-i\dot{\phi}t} \tag{35}$$

This leads, using equation (30), to

$$\{\bar{q}_c\}, \{\bar{q}^*_c\} = \left[-\dot{\phi}^2 \left\lceil I_c \right\rfloor \pm i\dot{\phi}\left\lceil 2\, C_c\, \Lambda_c^{1/2} \right\rfloor + \left\lceil \Lambda_c \right\rfloor\right]^{-1} \cdot [A_c]^T \left(\{\bar{F}_c\}, \{\bar{F}^*_c\}\right) \tag{36}$$

Similarly, using equation (34), equation (31) yields

$$\{\bar{R}_c\} = \Big[[G_{inert.}] + [Y_c'] + [G_{resid.}]\Big] \{\bar{F}_c\} \qquad (37a)$$

and

$$\{\bar{R}_c^*\} = \Big[[G_{inert.}] + [Y_c'^*] + [G_{resid.}]\Big] \{\bar{F}_c^*\}, \qquad (37b)$$

in which $[Y_c'^*]$ is the conjugate of $[Y_c']$.

For the purpose of determining the steady state response to imbalance of the rotor/housing system, equation (37a) is written as

$$\{\bar{R}_c\} = [Y_c] \{\bar{F}_c\} \,, \qquad (38a)$$

$$[Y_c]=[G_{inert.}]+[A_c]\left[-\dot{\phi}^2[\ulcorner I_c \lrcorner]+i\phi[\ulcorner 2\, C_c\, \Lambda_c{}^{1/2} \lrcorner]+[\ulcorner \Lambda_c \lrcorner]\right]^{-1}[A_c]^T+[G_{resid.}] \qquad (38b)$$

where $[Y_c]$ is the reduced receptance of the housing (case) associated with the coupling points to the rotor. The vectors $\{\bar{R}_c\}$and $\{\bar{F}_c\}$ are the complex amplitudes of the physical coordinates and forces at the coupling points, respectively, associated with the $e^{+i\phi t}$ part of the solution. No computation involving equation (37b) is necessary since it would lead to conjugate amplitudes associated with $e^{-i\phi t}$. This amplitudes can be constructed from the first part, equation (37a) without resorting to extra calculations.

The reduced impedance of the housing is given by

$$[Z_c] = [Y_c]^{-1} \qquad (39)$$

so that

$$\{\bar{F}_c\} = [Z_c] \{\bar{R}_c\} \qquad (40)$$

3.1c The coupling elements:

The rotor and housing are interacting through bearings, seals and impeller reaction forces. These coupling forces can be related to the displacements and velocities at the connection points to the rotor and housing as follows (see Figure 3).

Bearing and local stiffness at the housing:

In the X-Z plane:

$$-\begin{Bmatrix} F_{Xbc} \\ F_{XbR} \end{Bmatrix} = [K_b] \begin{Bmatrix} R_{Xc} \\ R_{XR} \end{Bmatrix} + [C_b] \begin{Bmatrix} \dot{R}_{Xc} \\ \dot{R}_{XR} \end{Bmatrix} \tag{41}$$

Similar equation may be written for the Y-Z plane, so that the combined relations for the X-Z and Y-Z planes are

$$-\begin{Bmatrix} F_{Xbc} \\ F_{Ybc} \\ R_{XbR} \\ F_{YbR} \end{Bmatrix} = [K_b'] \begin{Bmatrix} R_{Xc} \\ R_{Yc} \\ R_{XR} \\ R_{YR} \end{Bmatrix} + [C_b'] \begin{Bmatrix} \dot{R}_{Xc} \\ \dot{R}_{Yc} \\ \dot{R}_{XR} \\ \dot{R}_{YR} \end{Bmatrix} \tag{42}$$

Seal and impeller Forces:

The motion and resulting forces in the X-Z, Y-Z planes are coupled. For a given seal or impeller, j, the forces due to displacements of the housing and rotor can be expressed as

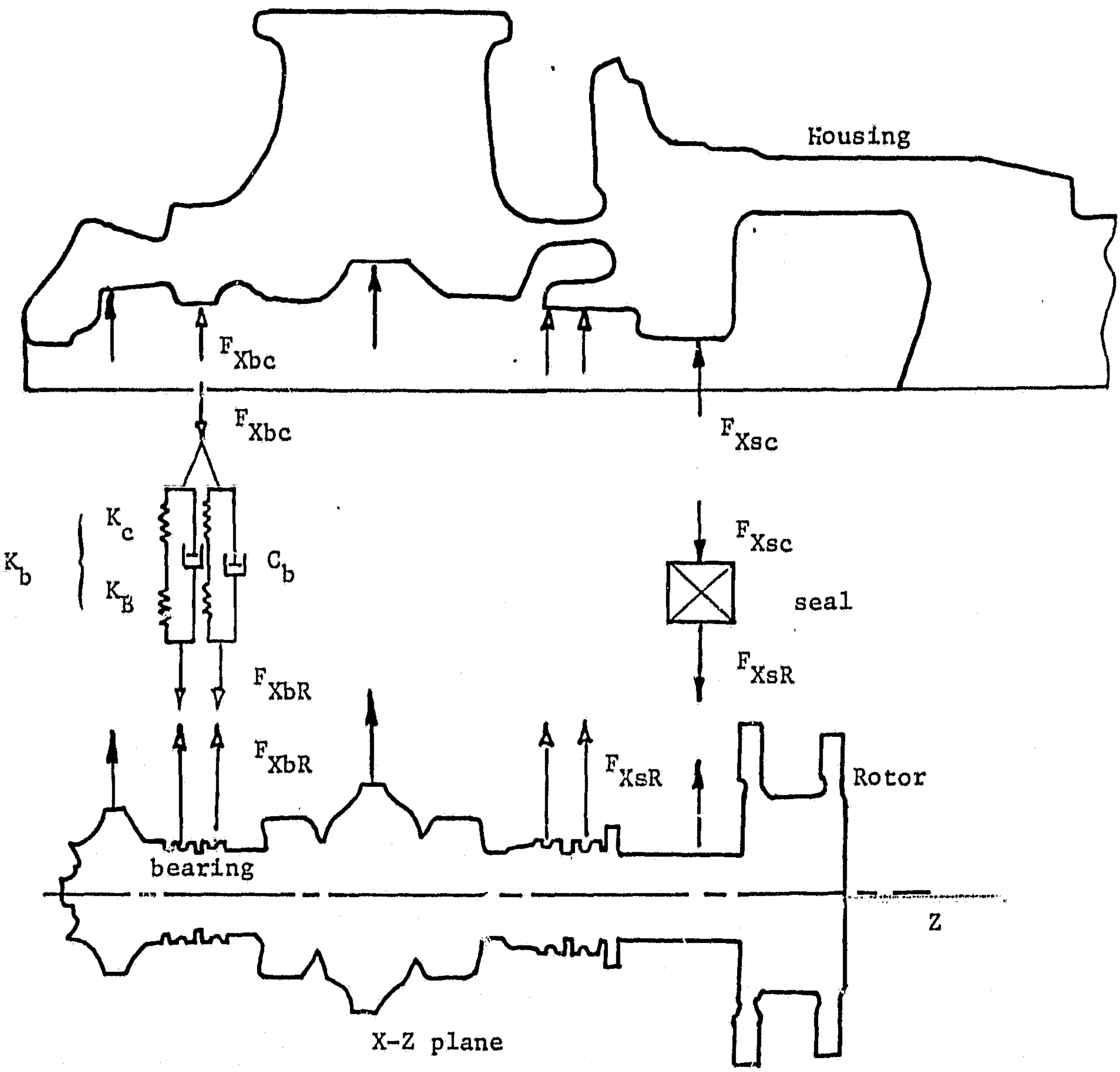

Figure 3. Coupling elements and forces in the HPOTP.

$$-\begin{Bmatrix} F_{jXsc} \\ F_{jYsc} \end{Bmatrix} = \begin{bmatrix} \tilde{K}_j & \tilde{k}_j \\ -\tilde{k}_j & \tilde{K}_j \end{bmatrix} \begin{Bmatrix} R_{jXc} - R_{jXR} \\ R_{jYc} - R_{jYR} \end{Bmatrix} + \begin{bmatrix} \tilde{C} & \tilde{c} \\ -\tilde{c} & \tilde{C} \end{bmatrix} \begin{Bmatrix} \dot{R}_{jXc} - \dot{R}_{jXR} \\ \dot{R}_{jYc} - \dot{R}_{jYR} \end{Bmatrix} \tag{43}$$

or

$$-\begin{Bmatrix} F_{jXsc} \\ F_{jYsc} \end{Bmatrix} = [K_{js}] \begin{Bmatrix} R_{jXc} \\ R_{jYc} \\ R_{jXR} \\ R_{jYR} \end{Bmatrix} + [C_{js}] \begin{Bmatrix} \dot{R}_{jXc} \\ \dot{R}_{jYc} \\ \dot{R}_{jXR} \\ \dot{R}_{jYR} \end{Bmatrix} \tag{44}$$

Similar relations as those of equation (44) can be written for the forces at the rotor. Combining the forces on the case and housing due to all seals and impeller forces, one can express the resulting relations in a matrix form as follows.

$$-\begin{Bmatrix} F_{Xsc} \\ F_{Ysc} \\ F_{XsR} \\ F_{YsR} \end{Bmatrix} = [K_s'] \begin{Bmatrix} R_{Xc} \\ R_{Yc} \\ R_{XR} \\ R_{YR} \end{Bmatrix} + [C_s'] \begin{Bmatrix} \dot{R}_{Xc} \\ \dot{R}_{Yc} \\ \dot{R}_{XR} \\ \dot{R}_{YR} \end{Bmatrix} \tag{45}$$

Combining the relations for the bearings and seals, yields

$$-\begin{Bmatrix} F_{Ic} \\ F_{IR} \end{Bmatrix} = [K_I] \begin{Bmatrix} R_{Ic} \\ R_{IR} \end{Bmatrix} + [C_I] \begin{Bmatrix} \dot{R}_{Ic} \\ \dot{R}_{IR} \end{Bmatrix} \tag{46}$$

where $[K_I]$ and $[C_I]$ are the stiffness and damping matrices for the coupling or intermediate components, i.e. the bearings and seals.

For the steady state response to imbalance, at the spinning speed $\dot{\phi}$, the solution can be written as

$$\begin{Bmatrix} R_{Ic} \\ R_{IR} \end{Bmatrix} = \begin{Bmatrix} \bar{R}_{Ic} \\ \bar{R}_{IR} \end{Bmatrix} e^{i\dot{\phi}t} + \begin{Bmatrix} \bar{R}^*_{Ic} \\ \bar{R}^*_{IR} \end{Bmatrix} e^{-i\dot{\phi}t} \tag{47}$$

A similar form can be written for the coupling forces, leading to the following relations for the coefficients of $e^{i\dot{\phi}t}$

$$-\begin{Bmatrix} \bar{F}_{Ic} \\ \bar{F}_{IR} \end{Bmatrix} = \Big[[K_I] + i\dot{\phi}\,[C_I] \Big] \begin{Bmatrix} \bar{R}_{Ic} \\ \bar{R}_{IR} \end{Bmatrix} \tag{48}$$

$$= [Z_I] \begin{Bmatrix} \bar{R}_{Ic} \\ \bar{R}_{IR} \end{Bmatrix}$$

where $[Z_I]$ is the impedance of the intermediate components for a given $\dot{\phi}$. The impedance matrix corresponding to $e^{-i\dot{\phi}t}$, $[Z^*_I]$, is the conjugate of $[Z_I]$.

3.2 The Analysis:

The system model may now be assembled from the individual representation of its components. The assembled model can be used for a given $\dot{\phi}$ to determine the steady state response to imbalance or the eigen-parameters of the turbopump systems.

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3.2a Steady state response

Two alternative procedures may be used to form the system's equations. The first approach is to (i) assemble the system impedance matrix from the individual impedances of the components. This can be identified as an impedance (dynamic stiffness) method. Another approach is to (ii) utilize the receptances (dynamic flexibilities) of the rotor and case together with the impedance of the coupling components to construct a reduced generalized receptance matrix for the coupled system. These two variations are outlined in what follows.

(i) Impedance method:

For coupling the system components, equations (17), (40) and (48) are used to form the system equations as

$$\begin{bmatrix} [Z'_{I_{11}}] + [Z_c] & [Z'_{I_{12}}] \\ [Z'_{I_{21}}] & [Z'_{I_{22}}] + [Z_R] \end{bmatrix} \begin{Bmatrix} \bar{R}_c \\ \bar{R}_R + \bar{R}_{Imb} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} \qquad (49)$$

where the submatrices $[Z'_{I_{ij}}]$ are partitions of the impedance matrix $[Z'_I]$ of the coupling components, $[Z'_{I_{21}}] = [Z'_{I_{12}}]^T$ and $\{\bar{P}_R\}$ are the imbalance forces acting on the rotor. The Submatrices $[Z'_{I_{ij}}]$ would be inflated with columns and rows of zero elements if necessary to make them consistent with $\{\bar{R}_R\}$ + $\{\bar{R}_{Imb}\}$ which include displacements at imbalance forces, $\{\bar{R}_{Imb}\}$. Equation (49) relates the complex amplitudes of the displacements at the coupling points of the base and rotor, associated with $e^{i\dot{\phi}t}$ part of the solution, to those of the imbalance forces. The conjugate associated with $e^{-i\dot{\phi}t}$ does not need to be computed, rather can be immediately deduced from that of equation (49). The system of Equations (49) can now be solved to yield the steady state response.

The resultant maximum bearing reaction at a given bearing j can be determined as follows. Express

$$\begin{aligned} \Delta X_j &= R_{jXc} - R_{jXR} \\ &= [\bar{R}_{jXc} - \bar{R}_{jXR}]\, e^{i\dot{\phi}t} + [\bar{R}^*_{jXc} - \bar{R}^*_{jXR}]\, e^{-i\dot{\phi}t} \\ &= \Delta\bar{X}_j\, e^{i\dot{\phi}t} + \Delta\bar{X}^*_j\, e^{-i\dot{\phi}t} \\ &= (\Delta\bar{X}_j + \Delta\bar{X}^*_j) \cos \dot{\phi}t + i(\Delta\bar{X}_j - \Delta\bar{X}^*_j) \sin \dot{\phi}t \end{aligned} \tag{50}$$

since the R_j's are physical displacements, the coefficients of cos $\dot{\phi}$ t and sin $\dot{\phi}$ t must be real.

$$\text{Let } A_j = (\Delta\bar{X}_j + \Delta\bar{X}^*_j),\ B_j = (\Delta\bar{X}_j - \Delta\bar{X}^*_j) \tag{51}$$

Similarly,

$$\Delta Y_j = C_j \cos \dot{\phi}t + D_j \sin \dot{\phi}t \tag{52}$$

in which

$$C_j = (\Delta\bar{Y}_j + \Delta\bar{Y}^*_j)\ ,\ D_j = (\Delta\bar{Y}_j - \Delta\bar{Y}^*_j) \tag{53}$$

The magnitude, ΔU_j, of the relative displacement between the casing and rotor at the jth bearing location can be shown to be

$$(\Delta U_j)^2 = \Delta X_j^2 + \Delta Y_j^2 = L_j + V_j \cos 2\,\dot{\phi}t + H_j \sin 2\,\dot{\phi}t \tag{54}$$

where

$$L_j = \frac{1}{2} (A_j^2 + B_j^2 + C_j^2 + D_j^2)$$

$$V_j = \frac{1}{2} (A_j^2 - B_j^2 + C_j^2 - D_j^2)$$

$$H_j = A_j B_j + C_j D_j$$

in which the maximum magnitude of the relative displacement is

$$(\Delta U_j)_{max} = \left(L_j + (V_j^2 + H_j^2)^{1/2} \right)^{1/2} \tag{55}$$

The maximum bearing reaction force of the jth bearing follows directly as

$$(F_{B_j})_{max} = (K_{b_j}) \, (\Delta U_j)_{max} \tag{56}$$

where K_{b_j} is the combined stiffness at the jth bearing.

(ii) Generalized receptance method:

The case and rotor are represented by their receptances associated with the forward whirl $e^{+i\dot{\phi}t}$, and reduced to their degrees of freedom at the coupling points.

Using equations (17) and (38), the reduced models are

$$\{\overline{R}_R + \overline{R}_{Imb}\} = [Y_R] \{\overline{F}_R + \overline{P}_R\} = \{\overline{R}_R'\} \tag{57}$$

and

$$\{\overline{R}_c\} = [Y_c] \{\overline{F}_c\} \tag{58}$$

Equations (58) and (57) can be put in a block matrix form as

$$\begin{Bmatrix} \bar{R}_c \\ \\ \bar{R}_R + \bar{R}_{Imb} \end{Bmatrix} = \begin{bmatrix} [Y_c] & 0 \\ \\ 0 & [Y_R] \end{bmatrix} \begin{Bmatrix} \bar{F}_c \\ \\ \bar{F}_R' + \bar{P}_R \end{Bmatrix} \tag{58}$$

On the other hand, the displacement-force relations for the coupling components may be written in terms of their impedances, at a given rotor speed $\dot{\phi}$, as

$$-\begin{Bmatrix} \bar{F}_c \\ \\ \bar{F}_R' \end{Bmatrix} = [Z_I'] \begin{Bmatrix} \bar{R}_c \\ \\ \bar{R}_R + \bar{R}_{Imb} \end{Bmatrix} \tag{59}$$

in which $\{\bar{F}_R'\}$ is inflated with null values in locations corresponding to $\{\bar{P}_R\}$, and $[Z_I']$ is inflated for compatibility with the mobility matrix in equation (58). Substitution from (58) into (59) yields

$$\begin{aligned} -\begin{Bmatrix} \bar{F}_c \\ \\ \bar{F}_R' \end{Bmatrix} &= [Z_I'] \begin{bmatrix} [Y_c] & \\ \\ & [Y_R] \end{bmatrix} \begin{Bmatrix} \bar{F}_c \\ \\ \bar{F}_R' \end{Bmatrix} \\ &+ [Z_I'] \begin{bmatrix} [Y_c] & 0 \\ 0 & [Y_R] \end{bmatrix} \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} \end{aligned} \tag{60}$$

or

$$\begin{aligned} &\left[\lceil I \rfloor + [Z_I'] \begin{bmatrix} [Y_c] & 0 \\ 0 & [Y_R] \end{bmatrix} \right] \begin{Bmatrix} \bar{F}_c \\ \bar{F}_R' \end{Bmatrix} \\ &= -[Z_I'] \begin{bmatrix} [Y_c] & 0 \\ \\ 0 & [Y_R] \end{bmatrix} \begin{Bmatrix} 0 \\ \\ \bar{P}_R \end{Bmatrix} \end{aligned} \tag{61}$$

This may be expressed as

$$[Y_p] \begin{Bmatrix} \bar{F}_c \\ \bar{F}'_R \end{Bmatrix} = - [E_p] \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} \tag{62}$$

The coupling forces $\{\bar{F}_c\}$ and $\{\bar{F}'_R\}$ may be obtained from the above equation (62). However, since these forces are related, a more efficient procedure can be adopted by which the problem is stated only in terms of, say, $\{\bar{F}_R\}$ To that end, introduce the following transformation

$$\begin{Bmatrix} \bar{F}_c \\ \bar{F}'_R \end{Bmatrix} = [T] \ \{\bar{F}_R\} \tag{63}$$

substituting in equation (62) and premultiplying by $[T]^T$, yield

$$[T]^T [Y_p][T] \{\bar{F}_R\} = - [T]^T [E_p] \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} \tag{64}$$

or

$$[Y] \{\bar{F}_R\} = \{\bar{P}'_R\} \tag{65}$$

The above generalized receptance relations (65) may now be solved to obtain the coupling forces at the rotor. The bearing reactions can then be obtained by calculating $\{\bar{F}_c\}$ from (63), $\{\bar{R}_c\}$ from equation (38) and $\{\bar{R}_R\}$ from equation (57). The maximum resultant bearing forces follows as before, using equation (56).

3.2b <u>Stability analysis-system eigenvalues</u>

The dynamic stability of the turbopump non-conservative system can be determined using either the impedance or the generalized receptance formulations. For a given rotor speed $\dot{\phi}$, it is desired to determine the eigenvalues, λ's, of the coupled rotor/housing system. The forms of displacements and internal coupling forces become (replacing equations (15), (34), (12) and (34b))

$$\{R_R(t)\} = \{\bar{\bar{R}}_R\} e^{\lambda t} \quad , \{R_c(t)\} = \{\bar{\bar{R}}_c\} e^{\lambda t} \tag{66a}$$

$$\{F_R(t)\} = \{\bar{\bar{F}}_R\} e^{\lambda t} \quad , \{F_c(t)\} = \{\bar{\bar{F}}_c\} e^{\lambda t} \tag{66b}$$

where "=" above any of the variables represents complex amplitude and λ is the complex eigenvalue whose real part determines the stability of the turbopump system.

Equation(65) is best suited for the eigenvalue analysis since the matrices involved are of small size. To that end, with the right hand side of the equation equals to zero, the coefficient matrix on the left hand side becomes the λ-matrix whose determinant vanishes for values of λ equal to the eigenvalues of the system. The λ-matrix can be written, using equation (65), as

$$\det [Y(\lambda)] = 0 \tag{67}$$

in which

$$[Y(\lambda)] = [T]^T [Y_p(\lambda)] [T]$$

where

$$[Y_p(\lambda)] = \left[[\diagdown I \diagdown] + [Z_I'(\lambda)] \begin{bmatrix} [Y_c(\lambda)] & 0 \\ 0 & [Y_R(\lambda)] \end{bmatrix} \right] \tag{68}$$

In expression (68), the matrices of the right hand side are obtained from equations (48), (37a) and (19), with λ replacing $(i\dot{\phi})$.

Conceptually, equation (67) can be solved possibly by using iterative procedures or other techniques.

Another approach applicable specifically to the analysis of the SSME turbopumps can be formulated as follows. The number of retained free-free planar normal modes of the nonspinning rotor are selected so that it is equal to the number of degrees of freedom in the plane of the modes at the coupling points. This will result in square modal matrices $[A_R]$ in equation (19). The impedance matrix of the rotor corresponding to a given eigenvalue, λ, of the rotor/housing system takes the form

$$[Z_R] = [A_R]^{-T}_{\text{square}} \left[\lambda^2 [\setminus I_R \setminus] + \lambda\, [D_R] + [\Omega_R]\right] [A_R]^{-1}_{\text{square}} \qquad (69)$$

The inverses $[A_R]^{-T}_{\text{square}}$ and $[A_R]^{-1}_{\text{square}}$ need be calculated only once for a given rotor.

Now, equation (49) is written, in absence of imbalance forces, in the form

$$\left[[Z_I] + \begin{bmatrix} [Z_c] & 0 \\ 0 & [Z_R] \end{bmatrix} \right] \begin{Bmatrix} \bar{R}_c \\ \bar{R}_R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (70)$$

premultiply both sides by $\begin{bmatrix} [Y_c] & 0 \\ 0 & [\setminus I_R \setminus] \end{bmatrix}$ one obtains

$$\left[\begin{bmatrix} [Y_c] & 0 \\ 0 & [\setminus I_R \setminus] \end{bmatrix} [Z_I] + \begin{bmatrix} [\setminus I_c \setminus] & 0 \\ 0 & [Z_R] \end{bmatrix} \right] \begin{Bmatrix} \bar{R}_c \\ \bar{R}_R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (71)$$

Equation (71) is now more amenable to manipulation for determining the eigenvalues. This is the case since the eigenvalues, λ, appear in the submatrices $[Y_c]$, $[Z_I]$ and $[Z_R]$ in an explicit fashion. In addition, this formulation makes solution methods, including those of iterative procedure, computationally efficient. Methods such as those of reference [48] can be used in this case.

It should be noted that similar formulation to that just described can be utilized for determining the steady state imbalance response. This would be advantageous as long as a square modal matrix, $[A_R]_{square}$, is employed. In case of forced response, equation (71) would take the form

$$\left[\begin{bmatrix} [Y_c] & 0 \\ 0 & [\diagdown I_R \diagdown] \end{bmatrix} [Z_I'] + \begin{bmatrix} [\diagdown I_c \diagdown] & 0 \\ 0 & [Z_R] \end{bmatrix}\right] \begin{Bmatrix} \bar{R}_c \\ \bar{R}_R + \bar{R}_{Imb} \end{Bmatrix} = \begin{bmatrix} [Y_c] & 0 \\ 0 & [\diagdown I_R \diagdown] \end{bmatrix} \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{P}_R \end{Bmatrix} \tag{72}$$

3.2c <u>Sensitivity analysis</u>

The sensitivity analysis in this study concerns predicting the variations in the turbopumps' eigenvalues and steady state imbalance response due to changes in the rotor/housing coupling parameters.

The eigenvalue sensitivity analysis was developed at an earlier stage of this study and is based on the formulations of the equations of motion of Childs [45]. In his approach, Childs utilized the modal parameters of the rotor/housing system coupled <u>only</u> through the stiffness at the bearing to form the full dynamic models of the turbopump involved. A computer program was developed by Childs based on formulations of [45] for determining the stability and forced response of the HTOTP and HTFTP of the SSME.

This computer program was modified to incorporate a procedure for determining the eigenvalue sensitivity of turbopump systems. On the other hand, a response sensitivity method was developed based on the impedance and generalized receptance techniques described in previous sections of this report. An eigenvalue sensitivity analysis method can also be developed based on the receptance techniques but was not attempted in this study.

(i) Eigenvalue sensitivity:

Formulations of reference [45] for the equations of motion of the turbopump are used for the following development of eigenvalue analysis procedure. In this development, the notations used in conjunction with formulations of [45] are basically retained for ease of reference.

The equations describing the motion of the coupled housing/rotor of a given turbopump in terms of their respective free, undamped modes and the stiffness at the bearings can be shown to take the form

$$[\diagdown I \diagdown]\begin{Bmatrix}\ddot{q}_R\\ \ddot{q}_c\end{Bmatrix}+\begin{bmatrix}[\diagdown \Lambda_R \diagdown] + [BAE_R]^T[\diagdown K_b \diagdown][BAE_R] & -[BAE_R]^T[\diagdown K_b \diagdown][BAE_c]\\ -[BAE_c]^T[\diagdown K_b \diagdown][BAE_R] & [\diagdown \Lambda_c \diagdown]+[BAE_c]^T[\diagdown K_b \diagdown][BAE_c]\end{bmatrix}\begin{Bmatrix}q_R\\ q_c\end{Bmatrix}=0 \tag{73}$$

As an approximation, two uncoupled equations of the form of equation (73) were assumed for the X-Z and Y-Z planes.

$[\diagdown \Lambda_R \diagdown]$ = eigenvalues of the undamped, free-free nonspinning rotor as in equation (8), and

$[\diagdown \Lambda_{c} \diagdown]$ = eigenvalues of the undamped free housing of the turbopump considered in presence of the nonspinning rotor of the other turbopump, as defined previously in connection with equation (30)

$[BAE_R]$ = modal sub-matrix of the free-free rotor corresponding to the coupling degrees of freedom in either the X-Z or the Y-Z plane

$[BAE_c]$ = equivalent to $[BAE_R]$ but for the housing in either the X-Z or the Y-Z plane

$[\diagdown K_b \diagdown]$ = total stiffness at bearings due to bearing and local stiffness at housing.

Equation (73) for the coupled rotor/housing motion is solved for the eigenvalues $[\diagdown \Lambda_X \diagdown]$ and modal matrix $[\Psi_{XZ}]$ and from a similar equation in the Y-Z plane for $[\diagdown \Lambda_Y \diagdown]$ and $[\Psi_{YZ}]$.

Utilizing the above modal parameters, the coupled equations of free motion of the complete turbopump system including damping, seals and impeller forces can be shown to take the form [45].

$$[\diagdown I \diagdown]\begin{Bmatrix} \ddot{q}_X \\ \ddot{q}_Y \end{Bmatrix} + \begin{bmatrix} [CX] + [SCX] & \dot{\phi}[CM] + [S_c] \\ -(\dot{\phi}[CM]^T + [S_c]^T) & [CY] + [SCY] \end{bmatrix}\begin{Bmatrix} \dot{q}_X \\ \dot{q}_Y \end{Bmatrix}$$

$$+ \begin{bmatrix} [\diagdown \Lambda_X \diagdown] + [SKX] & \dot{\phi}[\bar{c}] + [S_k] \\ -(\dot{\phi}[\bar{c}]^T + [S_k]^T) & [\Lambda_Y] + [SKY] \end{bmatrix}\begin{Bmatrix} q_X \\ q_Y \end{Bmatrix} = \{0\}, \quad (74)$$

where the various submatrices are defined in [45] and reflect the coupling parameters, structural damping, spin rotor speed and gyroscopic terms.

It is desired to determine the sensitivity of the eigenvalues λ_n associated with equation (74) to the various parameters of the coupling elements such as the bearing stiffness. To that end, the eigenvalue problem associated with equation (74) is expressed as

$$\left[\lambda^2[\diagdown \underline{M} \diagdown] + \lambda[\underline{C}] + [\underline{K}]\right]\{q\} = \{0\} \tag{75}$$

Following the method described in reference [44], the sensitivity of a particular eigenvalue λ_n to a parameter p is determined to a first order approximation by the eigenvalue derivative.

$$\lambda_{n,p} = \frac{\partial \lambda_n}{\partial p} \tag{76}$$

The derivative was shown in reference [44] to be

$$\lambda_{n,p} = - \{\ell_n\}^T \; [\lambda_n^2 \; \underline{M},p + \lambda_n \; \underline{C}, \; p + \underline{K},p] \; \{h_n\} \tag{77}$$

where $\{\ell_n\}$ and $\{h_n\}$ are the nth left and right eigenvectors for the complex eigenvalue problem normalized such that

$$\{\ell_n\}^T \; [2\lambda_n \; \underline{M} + \underline{C}] \; \{h_n\} = 1 \tag{78}$$

To a first order approximation in a Taylor series, a new eigenvalue λ_n corresponding to a change of Δp in the parameter p is

$$\lambda_n = \lambda_{no} + \lambda_{n,p}\Big|_0 \cdot \Delta p \tag{79}$$

or

$$\lambda_n = \lambda_{no} + \Delta\lambda_n \tag{80}$$

In order to assess the degree of change in λ_n, the variables involved may be normalized, so that comparison can be made between the following

$$\text{Relative change is } p = \frac{\Delta p}{p} \tag{81}$$

$$\text{Relative change in } \lambda_n = \frac{\Delta\lambda_n}{\lambda_n} \tag{82}$$

By way of example, suppose that it is desired to determine the sensitivity of the eigenvalue (or stability) of the complete rotor system to the magnitude of the equivalent stiffness at the ith bearings. Equation (77) in this case yields for the nth eigenvalue

$$\lambda_{n,K_{B_j}} = -\{\ell_n\}^T \left[\lambda_n^2 \underline{M},_{K_{B_j}} + \lambda_n \underline{C},_{K_{B_j}} + \underline{K},_{K_{B_j}} \right] \{h_n\} \tag{83}$$

The derivatives of the matrices in equation (83) may be determined taking into consideration equation (74). The result is

$$\underline{M},_{K_{B_j}} = 0 \ , \ \underline{C},_{K_{B_j}} = 0 \tag{84}$$

and

$$\underline{K},_{K_{B_j}} = \begin{bmatrix} [\ulcorner \Lambda_X \lrcorner],_{K_{B_j}} & \\ 0 & [\ulcorner \Lambda_Y \lrcorner],_{K_{B_j}} \end{bmatrix}$$

In equations (84), the indicated derivatives of $[\ulcorner \Lambda_X \lrcorner]$ and $[\ulcorner \Lambda_Y \lrcorner]$ can be determined from equation (73) using the method developed by Fox and Kapoor [35] in which the eigenvalue problem involves symmetric matrices K and M, or

$$[\bar{K} - \mu \bar{M}] \{\Psi_n\} = 0 \tag{85}$$

The derivative of the nth eigenvalue, μ_n, is

$$\mu_{n,p} = \{\Psi_n\}^T [\bar{K}_{,p} - \mu_n \bar{M}_{,p}] \{\Psi_n\} \tag{86}$$

in the case considered here. Using equation (73), this leads to

$$[\bar{M}]_{,K_{B_j}} = 0$$

$$[\bar{K}]_{,K_{B_j}} = \begin{bmatrix} [BAE_R]^T \lceil K_b \rfloor_{,K_{B_j}} [BAE_R] & -[BAE_R]^T \lceil K_b \rfloor_{,K_{B_j}} [BAE_c] \\ -[BAE_c]^T \lceil K_b \rfloor_{,K_{B_j}} [BAE_R] & [BAE_c]^T \lceil K_b \rfloor_{,K_{B_j}} [BAE_c] \end{bmatrix} \tag{87}$$

Hence

$$\lceil \Lambda_X \rfloor_{,K_{B_j}} = \{\Psi\}_{XZ}^T [\bar{K}]_{,K_{B_j}} \{\Psi\}_{XZ} \tag{88}$$

and

$$\lceil \Lambda_Y \rfloor_{,K_{B_j}} = \{\Psi\}_{YZ}^T [\bar{K}]_{,K_{B_j}} \{\Psi\}_{YZ} \tag{89}$$

in which the $\{\Psi\}$'s are the eigenvectors associated with equation (73).
Therefore

$$\lceil \Lambda \rfloor_{,K_{B_j}} = - [L]^T \begin{bmatrix} \lceil \Lambda_X \rfloor_{,K_{B_j}} & 0 \\ 0 & \lceil \Lambda_Y \rfloor_{,K_{B_j}} \end{bmatrix} [H] \tag{90}$$

where [L] and [H] are the matrices of left and right hand eigenvectors of the original problem (74). This yields the desired new eigenvalues λ_n's as

$$\lambda_n = \lambda_{no} + \lambda_{n,K_{B_j}} \cdot \Delta K_{B_j} \tag{91}$$

(ii) Response sensitivity:

The sensitivity of steady state imbalance response to the various coupling parameters can best be determined using the impedance or the generalized receptance methods outlined earlier.

Suppose that the sensitivity of the coupling forces exerted on the rotor is to be determined. To that end, equation (64) is rewritten in the form

$$[T]^T \left[[\diagdown I \diagdown] + [Z_I'] \begin{bmatrix} [Y_c] & 0 \\ 0 & [Y_R] \end{bmatrix} \right] [T] \{\bar{F}_R\} = \{\bar{F}_R'\} \tag{92}$$

where $[Z_I']$ is the influted impedance matrix of the coupling components and is described in conjunction with equation (49). Since $[Z_I']$ is a function of the coupling parameters (bearings, seals and impellers force coefficients), one expresses equation (92) as

$$[U(k_j)] \{\bar{F}_R\} = \{\bar{F}_R'\} \tag{93}$$

in which $U(k_j)$ denotes the functional dependence of [U] on the coupling parameters of magnitude k_j. These constants, k_j, can be the stiffness of a bearing, the cross-coupled damping of a seal, etc.

the derivatives of the coupling forces $\{\bar{F}_R\}$ with respect to a given parameter of magnitude k_j is calculated as follows

$$\{\bar{F}_R\}_{,k_j} = \left([U]^{-1}\ \{\bar{P}'_R\}\right)_{,k_j} \tag{94}$$

$$= \left(\frac{\partial}{\partial k_j}\ [U]^{-1}\right)\{\bar{P}'_R\}$$

$$\{\bar{F}_R\}_{,k_j} = -\ [U]^{-1}\ [U]_{,k_j}\ [U]^{-1}\ \{\bar{P}'_R\} \tag{95}$$

The derivatives of the coupling forces for a given $\dot{\phi}$ and coupling parameters, as given by equation (95), indicate their sensitivity to changes in the magnitude of the coupling parameter k_j.

Similarly, sensitivity of the steady state imbalance response as given by the impedance approach, equation (49), may be determined in the same fashion. Sensitivity of the acceleration response at the coupling points or the reaction forces at these points could be derived directly from those of the corresponding displacement responses.

4. NUMERICAL IMPLEMENTATION AND EXAMPLE ANALYSIS

4.1 Computer Programs

Two Computer Programs were developed based on the analysis methods presented in section 3 of this report. These two programs, written in fortran 77, are as follows:

(i) A modified version was developed of the computer program (ESTAB2), 1983 update, which was developed by D. Childs of Texas A&M University. The modified program, (EIGNSENS) was developed by U.J. Fan also of Texas A&M. The program uses the equations of motion developed in [45] together with the sensitivity analysis of reference [44] to determine the complex eigenvalue derivatives for the complete SSME Turbopump system. The theoretical background was described in section 3.2c.

(ii) A new computer program (ESTABIMP) was written, mainly by U.J. Fan,to calculate the maximum resultant bearing forces of the SSME turbopumps in a steady state response to rotor imbalance. The program can also yield the maximum acceleration levels at selected locations on the housing. The program, which is highly efficient, is based on the impedance and generalized receptance methods presented in section 3.2 of this report.

4.2 Example Analysis

4.2a Imbalance response of the HPOTP

Test runs using the newly developed (ESTABIMP) were made to obtain the steady state response to rotor imbalance of the HPOTP, at various rotor speeds, $\dot{\phi}$. The HPOTP model used is based on data given in reference [46]. In the turbopump configuration used, the rotor is that termed current rotor in Appendix A of [46] and is based on a model by B. Rowan. The housing is based on a 1982 Rocketdyne model. For both rotor and housing, a 0.5% of critical damping was assumed. This value replaces the diagonal elements of the matrices $[\smallsetminus E_R \smallsetminus]$ and $[\smallsetminus C_c \smallsetminus]$ of equations (8) and (30), respectively. The damping at the bearings was taken as 3.0 lb. sec/in. The stiffness of all the bearings were the same and of the constant value of $5x10^5$ lbs/in. The values used for the local stiffnesses of the housing at the bearing locations are given by equation (1) of [46]. the seals and impeller force - coefficients and imbalance distribution were also as given in reference [46]. The housing free-free modes are used in an approximate fashion as in program (ESTAB2). Both the impendance and the generalized receptance approaches were used in the computational procedure as follows.

The impedance matrices for the rotor and housing are formed using equations (19) and (39), respectively. However, no account was made of the residual flexibility and inertia restraint terms for the housing as in equation (38b). The assembled turbopump model as given by equation (49) was used, in conjunction with a modified Gauss elimination technique, to determine the response amplitudes of the connection points on the rotor and housing. These amplitudes were used with equations (50) through (56) to determine the resultant maximum bearing reactions at various values of the rotor speeds, $\dot{\phi}$. Table 1 shows a comparison of the results obtained for two selected values of $\dot{\phi}$ between the impedance method using (ESTABIMP) program and the method of reference [45], using the (ESTAB2) program. The slight differences noted in the results of the two methods can be attributed to the difference in the numerical implementation of two different procedures. The difference may also be due to the approximations involved

Table 1. Comparison of the maximum bearing reaction forces calculated by the impedance and reference [45] methods

Bearing Number	Reaction forces, lbs at 13,250 rpm		Rection forces, lbs at 24,000 rpm	
	Impedance	Ref. [45]	Impedance	Ref. [45]
1	257.	251.7	98.9	100.3
2	280.7	283.	114.5	117.3
3	490.1	471.2	44.	47.5
4	833.5	808.4	3.8	2.8

in handling the housing modes which influence the representation of the two models in different ways.

Experience with the programs showed that the impedance approach is more efficient than that of the approach of [45] for a given rotor speed. However, when the steady state response at various rotor speeds are required, the approach of [45] appears more beneficial to use. On the other hand, the generalized receptance method, although not fully tested, appears to be more efficient than both of the impedance method and the method of reference [45]. The receptance method requires only one inversion (see equation (17)) for assembling the system's generalized receptance matrix. In addition, the size of the assembled matrix is considerably smaller than that of the impedacne, as can be deduced from examination of equation (65).

4.2b Eigenvalue sensitivity of the HPOTP

Few test cases were run using the newly developed program (EIGNSENS). A case considered is that of determining the effect of changing the stiffness of the first bearing on the eigenvalues of the HPOTP. The configuration and data for the HPOTP are those discussed in section 4.2a. The rotor speed is taken as $\dot{\phi}$ = 30,000 rpm and the assumed change in the stiffness of the first bearing was 20%, or

$$K_{B_1} = 0.80 \times (K_{B_1})_o = 0.8 \times (5 \times 10^5)$$

$$= 4 \times 10^5 \text{ lbs/in} \qquad (95)$$

The bearings' stiffnesses are assumed to be in series with the local stiffnesses, K_c's at the housing. The equivalent stiffness at the first bearing location is therefore

$$K_{b_1} = \frac{K_{B_1} \cdot K_{c_1}}{K_{B_1} + K_{c_1}} ; \tag{96}$$

and

$$(K_{b_1})_{,K_{B_1}} = \frac{K_{c_1}^{\ 2}}{(K_{B_1} + K_{c_1})^2} \tag{97}$$

so that in equation (87) for one of the planes

$$[\ K_{b_1}\ ]_{,K_{B_1}} = \begin{bmatrix} (K_{b_1})_{,K_{B_1}} & & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{98}$$

The derivatives of the eigenvalues for the HPOTP System considered were then calculated using (EIGNSENS). Examination of the results can best be achieved by using the sensitivity measures proposed by Fritzen and Nordmann [44]. The eigenvalues of the complete turbopump system is expressed as

$$\lambda_n = \alpha_n + i\omega_n \tag{99}$$

A relative measure of the sensitivity can be constructed by defining the following nondimensional ratios, using the relations given by equations (91), so

$$\frac{\Delta\alpha_n/\alpha_{no}}{\Delta K_{B_1}/K_{B_1}} = \mathrm{Re}\ (\frac{\partial\lambda_n}{\partial K_{B_1}}) \cdot \frac{K_{B_1}}{\alpha_{no}} \tag{100a}$$

and

$$\frac{\Delta\omega_n/\omega_{no}}{\Delta K_{B_1}/K_{B_1}} = \mathrm{Im}\left(\frac{\partial\lambda_n}{\partial K_{B_1}}\right) \cdot \frac{K_{B_1}}{w_{no}} \tag{100b}$$

As discussed in reference [44], the designation of relations (100a) and (100b) would enable comparison of the extent by which modifications of several parameters may have on different eigenvalues. Large ratios of equations (100a) and (100b) indicate that changes in K_{B_1} will have a large effect on both the damping and frequencies of a given complex mode.

The results of the sensitivity analysis of the HPOTP eigenvalues for the case considered is presented in Table 2. The results were also checked by calculating directly the new eigenvalues using (ESTAB2), with the first bearing stiffness as given by equation (95). Close agreement was observed between the results of the direct calculation and those using sensitivity derivatives.

In Table 2, the relative change of stiffness is negative, or

$$\Delta K_{B_1}/K_{B_1} = -0.2$$

The reversed sign in the last column, of Table 2, therefore, indicates that the decrease in stiffness at the first bearing results in an increase in the damped frequency of the system. That increase is highest at the 13th and 14th conjugate modes as well as in the higher complex modes.

The relative change in α_{no} is of particular significance since it indicates the effect of stiffness change on a possible change from stable to unstable condition for the system. The results of Table 2 shows the highest sensitivity in this regard to be associated with modes number

TABLE 2. Relative changes in eigenvalues due to variation in bearing stiffness

Eigenvalues	α_o	ω_o	$\dfrac{\Delta\alpha_n/\alpha_{no}}{\Delta K_{B_1}/K_{B_1}}$	$\dfrac{\Delta\omega_n/\omega_{no}}{\Delta K_{B_1}/K_{B_1}}$
1,2	-1.3467	±275.81	$+1.86 \times 10^{-4}$	0
3,4	-1.4204	±284.07	0	0
5,6	-2.8753	±530.76	$+0.94 \times 10^{-2}$	$\mp 2.78 \times 10^{-4}$
7,8	-2.6884	±537.60	0	0
9,10	-4.6546	±690.71	$+1.91 \times 10^{-2}$	$\mp 2.12 \times 10^{-4}$
11,12	-3.4907	±700.08	-1.43×10^{-4}	0
13,14	-146.73	±1323.7	$+2.39 \times 10^{-3}$	$\mp 0.57 \times 10^{-2}$
15,16	-6.6063	±1865.8	-1.18×10^{-1}	$\mp 2.41 \times 10^{-3}$
17,18	-9.4490	±1888.0	$+1.59 \times 10^{-4}$	0
19,20	-5.5610	±1904.7	$+0.67 \times 10^{-1}$	$\mp 1.32 \times 10^{-3}$
21,22	-9.7791	±1948.4	-1.54×10^{-4}	0
23,24	-8.6708	±1961.3	$+2.68 \times 10^{-2}$	$\mp 0.51 \times 10^{-3}$
25,26	-11.093	±2207.3	$+1.81 \times 10^{-3}$	0
27,28	-11.167	±2209.4	$+0.58 \times 10^{-2}$	0
29,30	-21.747	±2681.5	$+2.09 \times 10^{-2}$	$\mp 0.56 \times 10^{-3}$
31,32	-13.863	±2715.1	$+0.61 \times 10^{-2}$	$\mp 1.84 \times 10^{-4}$
33,34	-26.206	±2878.8	$+0.89 \times 10^{-1}$	$\mp 4.52 \times 10^{-3}$
35,36	-14.735	±2943.5	$+1.02 \times 10^{-4}$	0
37,38	-16.792	±3058.5	$+1.79 \times 10^{-2}$	$\mp 4.91 \times 10^{-4}$
39,40	-15.648	±3068.0	$+1.12 \times 10^{-2}$	0
41,42	-60.200	±3170.3	$+1.94 \times 10^{-1}$	$\mp 1.44 \times 10^{-2}$
43,44	-2.1325	±3375.0	$+0.63 \times 10^{-1}$	0
45,46	-17.354	±3408.2	-6.15	$\mp 0.57 \times 10^{-1}$
47,48	-32.047	±3463.5	+1.11	$\mp 2.17 \times 10^{-2}$
49,50	-384.47	±4395.3	-1.51×10^{-2}	$\mp 1.66 \times 10^{-1}$
51,52	-184.90	±5069.9	-1.8×10^{-1}	$\mp 1.14 \times 10^{-1}$

15, 16, 19, 20, 33, 34 and the last eight modes. A negative sign associated with the relative change in an α_{no} indicates that the effective damping of the corresponding mode is reduced. It might be concluded that modes 15 and 16 could first become unstable.

5. CONCLUSIONS AND RECOMMENDATIONS

Based on the assessment of existing analysis procedures and on results obtained in the course of this study, the following conclusions can be made in relation to the SSME turbopumps.

5.1 Conclusions

(i) The frequency response methods developed in this study appear to constitute a highly effective tool for determining the steady state linear response of the SSME turbopumps. The methods, as formulated with a sub-system approach, demonstrate the following advantages:

(a) the dynamic model of the housing is reduced to the degrees of freedom of the connections points to the rotor. Since the number of these point is relatively small, the size of a housing model is drastically reduced.

(b) in cases where the housing is represented by a truncated set of its free eigenvalues, the methods will facilitate incorporating approximate representation of the truncated lower and higher modes of the housing. This is accomplished without affecting the size of the reduced model of the housing.

(c) a reduced model of the rotor may be obtained directly from its representation by physical coordinates. In that case, no approximations are made. The rotor may also be represented by <u>all</u> (if desired) of its real modes. The resulting reduced model will still be of the same size corresponding to the number of degrees of freedom at the connection points. This representation also has the advantage of being based on the <u>real</u> free-free undamped modes of the nonspinning rotor, avoiding the use of complex modes.

(d) the formulation adopted in this study, which uses individual models of the free components to form coupled systems, allows performing efficient re-analyses to determine effect of changes in the coupling elements.

(e) frequency response information for the rotor at its coupling points may be obtained directly from experimental test data.

(ii) The generalized receptance formulations can yield accurate, reduced Lamda matrices for determining all of the eigenvalues of the coupled rotor-housing systems. The accuracy of the eigenvalues will depend, however, on the accuracy of representation of the individual subsystems. If the modal coordinates were used in constructing the receptance matrices, the number of the system eigenvalues which can be determined will be equal to the total number of retained modes for rotor and housing less the number of degrees of freedom at the connection points. Procedures are yet to be further developed to render the method practically efficient.

(iii) The response and eigenvalue sensitivity relations developed here can be used effectively in guiding directions of design changes and in determining effects of estimated data errors on the analysis.

5.2 Recommendations

Further developments can be made based on some aspects of the procedures adopted for this study and on other representations of the components. In particular, it is recommended to

(i) Develop analogous procedures for transient and other analysis requirements. The procedures would utilize reductions of subsystems to their coordinates at the connection points and allow mixed representations of the subsystems.

(ii) Further develop and optimize the generalized receptance method. Actual test analyses using this method should be made using corrected modal housing model.

(iii) Pursue use of the dynamic reduction approach in developing accurate methods of determining eigenvalues and eigenvectors of coupled subsystems.

(iv) Develop an alternative procedure to the dynamic reduction approach as applied to all components of a rotor system. With the SSME turbopumps, the procedure could utilize static reduction method for the rotors and dynamic reduction of the housings. The relative merits of the two procedures can then be tested. The latter procedure can highly facilitate computation of the coupled rotor/housing eigenvalues.

(v) Extend the free-interface modal synthesis methods for application to nonconservative systems. These methods can be used in reducing the rotor's models for use in performing transient analyses.

(vi) To explore using Fourier and Laplace transformations in the analysis of reduced rotor systems. The impedance or receptance formulations might prove useful in this connection.

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