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NASA TECHNICAL MEMORANDUM NASA TM- 77557

MEASUREMENT OF THE INERTIAL CONSTANTS OF A RIGID OR FLEXIBLE STRUCTURE OF ARBITRARY SHAPE THROUGH A VIBRATION TEST

D. Engrand, J. Cortial

Translation of "Mesure des constantes d'inertie d'une structure indéformable ou non, à l'aide d'un essai de vibrations," La Recherche Aérospatiale, 1970, No. 4 (July-Aug), pp. 213-219

N85-11381

(NASA-TM-77557) MEASUREMENT OF THE INERTIAL CONSTANTS OF A RIGID OR FLEXIBLE STRUCTURE OF ARBITRARY SHAPE THROUGH A VIBRATION TEST (National Aeronautics and Space Administration) 23 p HC A02/MF A01 CSCL 20K G3/39

Unclas
24863

NATIONAL AERONATICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 MARCH 1983

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STANDARD TITLE PAGE

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM- 77557		
4. Title and Subtitle MEASUREMENT OF THE INERTIAL CONSTANTS OF A RIGID OR FLEXIBLE STRUCTURE OF ARBITRARY SHAPE THROUGH A VIBRATION TEST		5. Report Date March 1983 6. Performing Organization Code
7. Author(s) D. Engrand and J. Cortial		8. Performing Organization Report No. 10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, CA 94063		11. Contract or Grant No. NASw-3199 13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, C.D. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "Mesure des constantes d'inertie d'une structure indéformable ou non, à l'aide d'un essai de vibrations," La Recherche Aérospatiale, 1970, No. 4 (July-Aug), pp. 213-219.

16. Abstract An original method, developed by R. Kappus, aims at defining the inertial constants of an aircraft or a rocket, or of any other structure, even flexible, without materializing any rotating axis. This paper presents a new synthesis of the method. The necessary equipment is very similar to that used normally for ground vibration tests. The only new device provided is an elastic suspension for obtaining the total natural modes corresponding to the motions of the structure as a solid. From the measurements of the generalized masses of these modes it is possible to compute the inertial constants: center of inertia, tensor of inertia, mass. When the structure is not strictly rigid a purification process, based on the mean square method makes it possible to "rigidify" it at the price of some approximations and a few more measurements. Lastly, eventual additional masses, that are not parts of the structure, can be taken into account.

17. Key Words (Selected by Author(s))	18. Distribution Statement
Inertia, Moments of inertia, Constants, Structures, Aircraft structures, Missile structures, Rigid structures, Spacecraft, Mass distribution, structures, Fexible bodies, Vibration mode, Center of gravity,	Unclassified-Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22.
Unclassified	Unclassified		

SUMMARY

An original method, developed by R. Kappus, aims at defining the inertial constants of an aircraft or a rocket, or any other structure, even flexible, without materializing any rotating axis. This paper presents a new synthesis of the method. The necessary equipment is very similar to that used normally for ground vibration tests. The only new device to be provided is an elastic suspension for obtaining the total natural modes corresponding to the movements of the structures as a solid. From the measurement of the generalized masses of these modes it is possible to compute the inertial constants: center of inertia, tensor of inertia, mass. When the structure is not strictly rigid a purification process, based on the mean square method, makes it possible to "rigidify" it at the price of some approximations and a few more measurements. Lastly, eventual additional masses, that are not parts of the structure, can be taken into account.

MEASUREMENT OF THE INERTIAL CONSTANTS OF A RIGID OR FLEXIBLE STRUCTURE OF ARBITRARY SHAPE THROUGH A VIBRATION TEST

D. Engrand, J. Cortial

INTRODUCTION /214*

A ground test is performed using usual equipment to determine the inertial constants of the structure of an aircraft, missile, or of any other body whether flying or not. In many cases, the structure may be considered rigid, but potential deformations inevitably occur, and they are sometimes quite large. In R. Kappus's report [1], the general case of a deformable structure is first considered, and the resolution is obtained directly by methods of approximation (least squares), by taking into account the spurious masses introduced by suspending the structure.

As a user of this method, and we are especially addressing other potential users, we have considered the problem in a somewhat simpler form and we sometimes offer a different viewpoint than that of the author: in the first phase, the structure is assumed to be perfectly rigid, suspended to a rigid frame using springs (linear) without mass. This somewhat simplistic scheme easily demonstrates that if we know the generalized masses of the system (structure + suspension), and a convenient number of amplitude measurements, we can find all of the inertial characteristics of the structure: mass, coordinates at the center of gravity, moments and products of inertia in a given reference.

In the second phase, the problem of a flexible structure is presented, by showing that it is possible, to a large extent, to include the possible deformability of the structure if we know its first specific modes (for the type of boundary conditions imposed by the suspension).

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*Numbers in the margin indicate pagination in the original text.

In the last phase, corrections are made by accounting for the "spurious masses" introduced by the non-negligible masses of the suspension. This problem is presented very quickly in the appendix, for the simple purpose of providing a principle for eliminating the disturbances introduced by the suspension. Details of the operations required for this elimination are not indispensable for the understanding of this report; they were therefore limited to a few general characteristics.

In closing this introduction, let us simply point out that we have we have attempted nothing more than to offer a summary, by sometimes adopting a slightly different viewpoint than that of R. Kappus, while retaining a system of notations very close to his. Specifically, the first three paragraphs are often quite different from the reference document, while the last two are more direcly extracted from it. In the last paragraph (appendix), we have not included the effective caclulation of spurious masses (matrix 'I). so as to avoid needlessly cluttering a report whose purpose is simply to summarize the essential points of the reference document.

I. - SIMPLIFIED DESCRIPTION OF THE PROBLEM FOR A RIGID STRUCTURE

I.1 - Hypothesis

-The center of gravity of the structure under study is assumed to be known.

-The rigid structure S is suspended by a convenient number of negligible mass springs to an infinitely rigid and perfectly fixed frame. Let G_0 be the position of the center of gravity G of S at rest.

-The small motions of S are studied in the vicinity of the position of equilibrium, in a system of ortho-normed axes $G_0 \vec{x}\vec{y}\vec{z}$.

The solid thus suspended possess six degrees of freedom at the most. The motion will be described by a vector where x, y, z are the coordinates in $G_o\vec{x}\vec{y}\vec{z}$, and 0x, 0y, 0z are the rotations of S which are assumed to occur around $G\vec{x}, G\vec{y}, G\vec{z}$ (hypothesis of small motions). l is an arbitrary length intended to render the components of $\vec{X}$ homogeneous.

$$\vec{X} = \begin{bmatrix} x/l \\ \theta_y \\ z/l \\ \theta_x \\ y/l \\ \theta_z \end{bmatrix} \qquad (1.1.1)$$

1.2. - Establishing The Equations

The kinetic energy of the system in motion is expressed:

$$2T = m\left[\vec{V}(G)\right]^2 + \vec{\Omega}.\mathcal{J}\vec{\Omega} \qquad (1.2.1)$$

with:

$$\vec{V}(G) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad \vec{\Omega} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \qquad (1.2.2)$$

and:

$$\mathcal{J} = \begin{bmatrix} I_{XX} & -I_{XY} & -i_{XZ} \\ -I_{XY} & I_{YY} & -I_{YZ} \\ -I_{XZ} & -I_{YZ} & I_{ZZ} \end{bmatrix}. \qquad (1.2.3)$$

(tensor of inertia of S in G)

The "linearized" expression (1.2.1) (i.e. omitting the terms whose order is greater than 2) is:

$$\begin{aligned} 2T = {} & m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + I_{XX}\dot{\theta}_x^2 + I_{YY}\dot{\theta}_y^2 + I_{ZZ}\dot{\theta}_z^2 \\ & - 2I_{XY}\dot{\theta}_x\dot{\theta}_y - 2I_{YZ}\dot{\theta}_y\dot{\theta}_z - 2I_{ZX}\dot{\theta}_z\dot{\theta}_x. \end{aligned} \qquad (1.2.4)$$

If we let U be the quadratic form representing the "linearized" function of force from which the elastic suspension stresses are derived, the Lagrange equations are expressed:

(1,2,5)

$$\begin{bmatrix} ml^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{YY} & 0 & -I_{XY} & 0 & -I_{YZ} \\ 0 & 0 & ml^2 & 0 & 0 & 0 \\ 0 & -I_{XY} & 0 & I_{XX} & 0 & -I_{XZ} \\ 0 & 0 & 0 & 0 & ml^2 & 0 \\ 0 & -I_{YZ} & 0 & -I_{XZ} & 0 & I_{ZZ} \end{bmatrix} \begin{bmatrix} \ddot{x}\, l \\ \ddot{\theta}_y \\ \ddot{z} \\ \ddot{\theta}_x \\ \ddot{y}\, l \\ \ddot{\theta}_z \end{bmatrix} = - \begin{bmatrix} l\frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial \theta_y} \\ l\frac{\partial U}{\partial z} \\ \frac{\partial U}{\partial \theta_x} \\ l\frac{\partial U}{\partial y} \\ \frac{\partial U}{\partial \theta_z} \end{bmatrix} \qquad (1.2.5)$$

The system (1.2.5) may therefore be expressed in the conventional form:

$$\mathcal{M}\ddot{X} + \mathcal{K}\dot{X} = 0 \qquad (1.2.6)$$

with $\mathcal{K}$ being the stiffness matrix of the suspension system and /215
$\mathcal{M}$ the matrix of inertia explained in (1.2.5). $\mathcal{M}$, $\mathcal{K}$ are the matrices defined as positive, symmetrical. We know that matrix $\mathcal{M}^{-1}\mathcal{K}$ is, under these conditions, always diagonisable. Let Q be the matrix of its specific vectors. Matrices $Q^T\mathcal{M}$ and $Q^T\mathcal{K}Q$ in which Q^T is transposed by Q, are the diagonal matrices (orthogonality relationships of the specific forms), and (1.2.6) may be expressed in the form:

$$\mu\ddot{\vec{Y}} + \Omega^2\gamma\vec{Y} = 0 \qquad (1.2.7)$$

with $\mu = Q^T\mathcal{M}Q,\ \gamma = Q^T\mathcal{K}Q,\ \Omega^2 = Q^{-1}\mathcal{M}^{-1}\mathcal{K}Q$

mu is the matrix of generalized masses, gamma the matrix of generalized stiffnesses, corresponding to the normalization selected for the specific vectors of $\mathcal{M}^{-1}\mathcal{K}$.

If we know Q, we see that it is possible to know $\mathcal{M}$, and therefore the inertias of S, by inverting according to relationship:

$$\mathcal{M} = (Q^T)^{-1}\mu Q^{-1}. \tag{1.2.8}$$

Generally speaking, a vibration test makes it possible to determine matrix mu. The operation for obtaining Q often presents more difficulties, as the structure is rarely rigid. Furthermore, the suspension springs are rarely without mass (or negligible mass with respect to that of S), and the position of the center of gravity G of S is not always easy to determine on planes.

We shall see in the next few paragraphs that it is not necessary to know point G a priori, and that it is possible, using a few approximations to overcome the experimental difficulties.

II - GENERAL EXPRESSION OF THE MATRIX $\mathcal{M}$

(The position of the center of gravity G of S is not given).

I.1. Equations

-Let O be any point of the structure. Its position O_E at equilibrium will be adopted as the origin of a fixed reference $O_E\vec{x}\vec{y}\vec{z}$ in which the motion of S will be described.

We will define a second reference $O_{\vec{X}\vec{Y}\vec{Z}}$, correlated with S, and which, at rest, is indistinguishable from $O_{\vec{x}\vec{y}\vec{z}}$.

The kinetic energy of the system is given by:

$$2T = \int_S [\vec{V}(M)]^2 \, dm\,(M) \tag{II.1.1}$$

where M represents a running point or the solid, and we have:

$$\vec{V}(M) = \vec{V}(O) + \vec{\Omega} \wedge \overrightarrow{OM} \tag{II.1.2}$$

($\vec{\Omega}$ being the vector for the rotation speed of S).

We therefore have:

$$\begin{aligned} 2T = \int_S [\vec{V}(M)]^2 \, dm = \int_S [\vec{V}(O)]^2 \, dm &+ \\ + 2\int_S \vec{V}(O).(\vec{\Omega} \wedge \overrightarrow{OM}) \, dm &+ \\ + \int (\vec{\Omega} \wedge \overrightarrow{OM}).(\vec{\Omega} \wedge \overrightarrow{OM}) \, dm&. \end{aligned} \tag{II.1.3}$$

Noting that:

$$\int_S [\vec{V}(O)]^2 \, dm = m[\vec{V}(O)]^2 \tag{II.1.4}$$

$$\begin{aligned} 2\int_S \vec{V}(O).(\vec{\Omega} \wedge \overrightarrow{OM}) \, dm &= \\ = 2(\vec{V}(O) \wedge \vec{\Omega}).\int_S \overrightarrow{OM} \, dm &= \\ = 2m(\vec{V}(O) \wedge \vec{\Omega}).\overrightarrow{OG}& \end{aligned} \tag{II.1.5}$$

$$\int_S (\vec{\Omega} \wedge \overrightarrow{OM}).(\vec{\Omega} \wedge \overrightarrow{OM}) \, dm = \vec{\Omega}.\mathfrak{I}(O)\vec{\Omega} \tag{II.1.6}$$

$\mathfrak{I}(O)$ being the tensor of inertia in O of S, we may express:

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$$2T = m\left[\vec{V}(O)\right]^2 + 2m\left(\vec{V}(O) \wedge \vec{\Omega}\right) . \vec{OG} + \vec{\Omega}.J(O)\vec{\Omega}. \quad \text{(II.1.7)}$$

Note also that $\vec{OG}$ is a vector of constant length involved in the motion of the solid. We have:

$$\text{(II.1,8)} \quad \dot{\vec{OG}} = \dot{\vec{OO}}_E + \dot{\vec{O_E G}} = \dot{\vec{OO}}_E + \dot{\vec{O_E G}}_E + \vec{\omega} \wedge \vec{O_E G}_E \quad \text{(II.1.8)}$$

with:

G_E : position of G at equilibrium (unknown)
$\vec{\omega}$: geometric rotation vector $\vec{\omega} = \begin{bmatrix}\theta_x\\ \theta_y\\ \theta_z\end{bmatrix}$.

If we explain the expression (II.1.7) for kinetic energy by omitting the terms of order greater than 2, we derive:

$$\begin{aligned} = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) &+ 2m(\dot{y}\dot{\theta}_z - \dot{z}\dot{\theta}_y)x_G + \dot{\theta}_x^2 I_{XX} - 2\dot{\theta}_z\dot{\theta}_y I_{YZ} \\ &+ 2m(\dot{z}\dot{\theta}_x - \dot{x}\dot{\theta}_z)y_G + \dot{\theta}_y^2 I_{YY} - 2\dot{\theta}_x\dot{\theta}_z I_{XZ} \\ &+ 2m(\dot{x}\dot{\theta}_y - \dot{y}\dot{\theta}_x)z_G + \dot{\theta}_z^2 I_{ZZ} - 2\dot{\theta}_x\dot{\theta}_y I_{XY}. \end{aligned} \quad \text{(II.1.9)}$$

With the following notations:

x, y, z :Coordinates of O (in reference $O_E\vec{x}\vec{y}\vec{z}$.
x_G, y_G, z_G :Coordinates of G_E (

Lagrange's equations may therefore be expressed:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = -\frac{\partial U}{\partial q_i}. \quad \text{(II.1.10)}$$

The first member having the following expression:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= m\ddot{x} + m\left(\ddot{\theta}_y z_G - \ddot{\theta}_z y_G\right)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m\ddot{y} + m\left(\ddot{\theta}_z x_G - \text{[illegible]}\right)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) &= m\ddot{z} + m\left(\ddot{\theta}_x y_G - \ddot{\theta}_y x_G\right)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_x}\right) &= I_{XX}\ddot{\theta}_x - I_{XY}\ddot{\theta}_y - I_{XZ}\ddot{\theta}_z + m\left(\ddot{z} y_G - \ddot{y} z_G\right)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_y}\right) &= I_{YY}\ddot{\theta}_y - I_{XY}\ddot{\theta}_x - I_{YZ}\ddot{\theta}_z + m\left(\ddot{x} z_G - \ddot{z} x_G\right)\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_z}\right) &= I_{ZZ}\ddot{\theta}_z - I_{YZ}\ddot{\theta}_y - I_{XZ}\ddot{\theta}_x + m\left(\ddot{y} x_G - \ddot{x} y_G\right).
\end{aligned}
\tag{II.1.11}
$$

As in the first case, the linear system forming small motions is expressed: /216

$$\mathcal{M}\ddot{\vec{X}} + \mathcal{K}\vec{X} = O. \tag{II.1.12}$$

With, in conformity with (II.1.11):

$$
\mathcal{M} = \begin{vmatrix}
ml^2 & mlz_G & O & O & O & -mly_G\\
mlz_G & I_{YY} & -mlx_G & -I_{XY} & O & -I_{YZ}\\
O & -mlx_G & ml^2 & mly_G & O & O\\
O & -I_{XY} & mly_G & I_{XX} & -mlz_G & -I_{XZ}\\
O & O & O & -mlz_G & ml^2 & mlx_G\\
-mly_G & -I_{YZ} & O & -I_{YZ} & mlx_G & I_{ZZ}
\end{vmatrix} \tag{II.1.13}
$$

$$
\vec{X} = \begin{bmatrix} x/l\\ \theta_y\\ z/l\\ \theta_x\\ y/l\\ \theta_z \end{bmatrix}. \tag{II.1.14}
$$

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(Remember that l is the arbitrary length that we can choose according to the dimensions of the structure, and that it does not serve to render the equations homogeneous).

Therefore, if we can return to matrix from matrix mu (generalized masses), all of the inertial characteristics of the structure S are known: tensor of inertia in O and G coordinates in reference $O_{\vec{x}\vec{y}\vec{z}}$ and thus also the tensor of inertia in G in the same system of axes, and total mass m of S.

III. - OBTAINING THE MATRIX FROM THE SPECIFIC FORMS Q

The hypotheses are still those of the preceding paragraph.

II.1. - Preliminary Remarks

-A point O of S being selected arbitrarily, the motion will be described by six generalized parameters that we will use below as amplitudes of harmonic motions. These six parameters form the components of the vector $\vec{\sigma}$:

$$\vec{\sigma} = \begin{bmatrix} x/l \\ \theta_y \\ z/l \\ \theta_x \\ y/l \\ \theta_z \end{bmatrix} \quad \text{(III.1.1)}$$

(vector $\vec{\sigma}$ still relates to point)).

Still under the hypothesis of small motions, any point P of S, of coordinates X_p, Y_p, Z_p, in reference $O_{\vec{x}\vec{y}\vec{z}}$ will have, in harmonic motions, a displacement given by:

$$\vec{d}_p = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} O & Z_p & -Y_p \\ -Z_p & O & X_p \\ Y_p & -X_p & O \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \quad \text{(III.1.2)}$$

We see that if we know x_p, y_p, and z_p, as well as the rotation $\left(\vec{\omega} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}\right)$ we can determine the motion of O.

Therefore, in principle, if we can measure six independent amplitudes (two in each of the three directions $\vec{Ox}, \vec{Oy}, \vec{Oz}$ in the different points) it is possible to determine the motion of O.

In practice, we will do N measurements, for each of the six modes; at least two of the measurements will be performed in each of the directions $\vec{Ox}, \vec{Oy}, \vec{Oz}$. Thus, for the k^{th} mode, we will perform N measurements in N' points $P_1, \ldots, P_{N'}$, (N' is less than or equal to N), with a maximum of three measurements (independent) made on each point along the three axes of reference $O\vec{x}\vec{y}\vec{z}$. These measurements will be stored on a $d_{(k)}$ vector:

$$\vec{d}_{(k)} = \begin{bmatrix} x_{P_1} \\ x_{P_2} \\ \vdots \\ y_{P_3} \\ y_{P_4} \\ \vdots \\ z_{P_5} \\ z_{P_6} \\ \vdots \end{bmatrix} \qquad \text{(III.1.3)}$$

we might of course have in column $\vec{d}_{(k)}$, for a given point P_i, the three components x_{P_i}, y_{P_i}, z_{P_i}, or only two of these instead of one. It should be stressed that the measuring points must be the same for the six modes.

We may therefore express, in a similar manner to (III.1.2):

$$\vec{d}_{(k)} = T\vec{\sigma}_{(k)} \qquad \text{(III.1.4)}$$

or, by explaining:

(matrix T having N lines and columns).

$$\begin{bmatrix} x_{P_1} \\ x_{P_2} \\ \vdots \\ y_{P_3} \\ y_{P_4} \\ \vdots \\ z_{P_5} \\ z_{P_n} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & Z_{P_1} & 0 & 0 & 0 & -Y_{P_1} \\ 1 & Z_{P_2} & 0 & 0 & 0 & -Y_{P_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -Z_{P_3} & 1 & X_{P_3} \\ 0 & 0 & 0 & -Z_{P_4} & 1 & X_{P_4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -X_{P_5} & 1 & Y_{P_5} & 0 & 0 \\ 0 & -X_{P_n} & 1 & Y_{P_n} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x/l \\ \theta_y \\ z/l \\ \theta_x \\ y/l \\ \theta_z \end{bmatrix}$$

Note that matrix T does not depend on the measuring points selected.

III.2 - Obtaining the Specific Forms /217

The structure being excited in the appropriate manner on its kth mode, N measurement are performed which give the vector $\vec{d}_{(k)}$.

The specific form $\vec{h}_{(k)}$ will therefore be known if we are able to solve the system:

$$T\vec{h}_{(k)} = \vec{d}_{(k)}. \quad \text{(III.2.1)}$$

This system is generally superabundant. We perform its approached resolution using the least squares method.

Resolution of the System (III.2.1)

If we call u_j (j = 1, ..., 6) the components of $\vec{h}_{(k)}$ and V_i (i = 1, ... N) the components of $\vec{d}_{(k)}$, the system may be expressed, with the convention of adding the silent indices:

$$T_{ij}\, u_j = V_i. \quad \text{(III.2.3)}$$

The method consists of minimizing the quantity:

$$|R|^2 = (T_{ij}\, u_j - V_i)\,(T_{ij}\, u_j - V_i) \quad \text{(III.2.4)}$$

by cancelling its partial derivatives with respect to u_i (= 1, ..., 6)

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$$\frac{\partial|R|^2}{\partial u_l} = 2T_{ij}\frac{\partial u_j}{\partial u_l}(T_{ij}u_i - V_i) = 0$$

or:

$$2T_{il}(T_{ip}u_p - V_i) = 0.$$

(III.2.5)

If we vary i from 1 to 6, we obtain the new system:

(III.2.6)

$$T^T T\vec{h}_{(k)} = T^T \vec{d}_{(k)}.$$

Therefore:

$$\vec{h}_{(k)} = (T^T T)^{-1} T^T \vec{d}_{(k)}$$

(II.2.7)

(approached solution).

As the values of $\vec{h}_{(k)}$ are given for the six modes by (III.2.7), the matrix Q is known and we can return to matrix $\mathcal{M}$ if we know μ.

III.3 - Determination of μ

This is obviously done experimentally, using in particular the "complex power method" [2] or the "displaced frequencies" method [3]. R. Kappus [1] recommends several measurements using several methods in order to be able to cross-check the results in order to be as accurate as possible, as the generalized masses are sometimes difficult to measure accurately.

With the hypotheses of a perfectly undeformable solid, a suspension without mass, and an infinitely rigid frame, we thus see that it is easy to return to the inertial characteristics of the structure, by starting with a somewhat modified ground test [3]. This case is clearly somewhat too ideal, and we are left with considering cases where the solid is deformable, by taking into account the "spurious masses" and the possible deformation of the suspending frame.

IV. - CASE OF THE DEFORMABLE SOLID

IV.1 - Hypotheses

We are still assuming that the suspension is without mass, and that the frame is infinitely rigid, and perfectly fixed. In this paragraph, only the hypotheses made unti present of the undeformability of the structure S is discarded, and we are assuming that this structure is capable of underoing small elastic deformations.

The gound test provides in this case a certain number ν of specific modes (in finifite theory), wich will encompass both both the solid motions and structural deformations. Furthermore, matrix $\mathcal{M}$ can be determined only be means of the specific modes corresponding to the motions of pure solids. It is therefore necessary to "rigidify" the structure, whenever possible, either by strengthening it mechanically during the ground test, or artifically, by looking for the specific modes of a fictive structure S' (attached to an identical suspension) offering the same geometery and the same mass distribution as the real structure at rest, but which should be undeformable. The method proposed by Kappus [1], and successfully applied, consists of effecting a linear superposition of the specific modes recorded, by assigning participation coefficients to them that are selected in such a manner that this superposition leads to a motion (fictive) that is as close as possible to the motion of a solid.

In summary, this is a smoothing operation. The participation coefficents are obtained using the least squares method.

IV.2 - DESCRIPTION OF THE METHOD

A number $\nu\ (N > \nu \geq 6)$ of specific modes are recorded during the test, each mode being represented by N measurements selected as discussed in the preceding paragraph, and arranged in a column $\vec{d}_{(k)}$ (k being the index of the mode). These columns, arranged side-by-side, form a matrix V that will be the "measured modal matrix" (dependent upon the measuring points):

$$V_{ik} = \vec{d}_{(k)_i} \quad = i^{th} \text{ component of } \vec{d}_{(k)}. \qquad \text{(IV.2.1)}$$

Note that V is a rectangualr matrix consisting of N lines and ν columns. We still have to find a conversion enabling us to pass from the measured V matrix to the T matrix of the corresponding "rigidified" body. This is accomplished by looking for the participation coefficients $\Lambda_{(k)}$ such as:

$$V_{ik} \Lambda_{kj} = T_{ij} \qquad \text{(IV.2.2)}$$

(with the convention of silent indices). This involves solving the system of matrices:

$$V \Lambda = T \qquad \text{(IV.2.3)}$$

where Λ is a matrix consisting of ν lines and six columns, which means that we have $6 \cdot N$ equations for $6 \cdot \nu$ unknowns, and therefore a superabundant system.

An approached resolution of this system can be easily obtained using the least squares method. As for (III.2.6), this leads to the resolution of the system:

$$V^T V \Lambda = V^T T. \qquad \text{(IV.2.4)}$$

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As the columns of V are independent, as far as specific modes are concerned, V^TV is a constant matrix and we have:

$$\Lambda = (V^TV)^{-1} V^T T. \qquad \text{(IV.2.5)}$$

While doing the calculations, it is of course useful to estimate the error introduced by the "least squares approximation", although it is the best possible method (reference [1], pp. 12 and 13). /218

We then have to return from matrix μ (that now has ν lines and ν columns) to matrix $\mathcal{K}$.

IV.3 - CALCULATION OF MATRIX $\mathcal{K}$

a) Correlation of Orthogonality; Matrix M

In any specific motion of the system, of angular velocity ω, represented by a vector $\vec{d}$ of which N components are the N amplitudes measured, the vibrations of the system are governed by a system with the expression:

$$\omega^2 M\vec{d} = K\vec{d} \qquad \text{(IV.3.1)}$$

M and K are the squares matrices of order N, representing the mass distribution and rigidities of the system (S + suspension). In theory, these matrices may be explained on the basis of the total energy of the system in any vibratory motion.

In fact, they will be precious intermediaries, but it will not be necessary to explain them.

-The correlations of orthogonality of the specific vectors will be expressed in this representation:

$$V^T M V = \mu = \begin{bmatrix} \mu_{11} & & 0 \\ & \mu_{22} \ddots & \\ 0 & & \mu_{\nu\nu} \end{bmatrix}. \qquad \text{(IV.3.2)}$$

For a give structure M, M depends only upon the selected measuring points.

b) Correlation Between M and $\mathcal{K}$.

For any virtual harmonic motion, of angular velocity ω in conformity with the undeformability of the "riqidified" structure, defined by:

$$\vec{\sigma}^* = \begin{bmatrix} x^*/l \\ \theta_y^* \\ z^*/l \\ \theta_x^* \\ y^*/l \\ \theta_z^* \end{bmatrix} \qquad \text{(IV.3.3)}$$

the kinetic energy E* is expressed:

$$E^* = \frac{1}{2}\omega^{*2}\vec{\sigma}^{*T}\mathcal{K}\vec{\sigma}^*. \qquad \text{(IV.3.4)}$$

The corresponding virtual "measurement vector" is expressed:

$$\vec{d}^* = T\vec{\sigma}^* \qquad \text{(IV.3.5)}$$

and the kinetic energy may also be expressed:

$$E^* = \frac{1}{2}\omega^{*2}\vec{d}^{*T}M\vec{d}^* \qquad \text{(IV.3.6)}$$

or

$$E^* = \frac{1}{2}\omega^{*2}\vec{\sigma}^{*T}T^T M T\vec{\sigma}^*. \qquad \text{(IV.3.7)}$$

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Through identification between (IV.3.4) and (IV.3.70, we therefore obtain:

$$\mathcal{M} = T^T M T \tag{IV.3.8}$$

By correlating (IV.2.3) and (IV.3.*), we therefore have:

$$\mathcal{M} = \Lambda^T V^T M V \Lambda. \tag{IV.3.9}$$

(We should still not forget that this equality is not perfectly accurate due to the approximation made in the calculation of Λ)..

By using the correlation of orthogonality (IV.3.2), we finally obtain:

$$\boxed{\mathcal{M} = \Lambda^T \mu \Lambda}. \tag{IV.3.10}$$

Note: This method of resolution is valid only if the deformations of S occur only in the form of spurious motions, which is often the case. It constitutes the main point of the method developed by R. Kappus [1], and is also its main advantage.

APPENDIX - ELIMINATION OF SPURIOUS MASSES

A1. -

For the report to be complete, it is necessary to bring up this problem, because it is not always of the secondary importance that we have described in the preceding paragraphs.

The expression"spurious masses" designates here all masses contributing to the motion and that do not belong to the structure S itself. We are dealing with suspension rods and, possibly, the frame when the latter is not rigid enough (situation that is good to avoid). Strictly speaking, the inclusion of these masses means that additional modes will have to be taken into consideration, but we will assume that these be only of a very local nature and that they have specific frequencies that are high with respect to those measured during the ground tests.

A2 - DESCRIPTION OF THE METHOD [1] pp. 6 to 8)

a) Measurements

N measuring points should be provided and distributed over the structure and suspension. N_o points (N_o is less than or equal to N) will be selected on the structure, $N - N_o$ on the suspension (possibly on the frame also). We will record ν specific modes (S is always assumed to be deformable, and therefore $\nu \cdot 6$). Therefore:

$$N \cdot N_o \quad \nu \cdot 6. \tag{A2.1}$$

The modal matrix V is relative to N measurements, as well as matrices M and K. We still have the correlation of orthogonality:

$$V^T M V = \mu. \tag{A2.2}$$

According to the assumptions made, μ is disturbed by masses foreign to S. The purpose of the following is to define a matrix mu_o that correlates only to S, using a process of elimination of the spurious masses.

b) Resolution of the Problem

It consists of subtracting from M a matrix M' representing the effects per unit of mass of the suspension rods (or possibly the frame), and of looking for a modal matrix V_o representing the measurements that we could have performed if the suspension was without mass. This matrix V_o verifies a theoretical relationship expressed: /219

$$(M - M')\,V_o = K V_o\,\Omega_0^{-2} \qquad \text{(A2.3)}$$

(Omega$_o$ being the unknown matrix (diagonal) of the specific angular velocities of the system without spurious masses).

The method proposed by R. Kappus consists of setting:

$$V_o = Vr \qquad \text{(A2.4)}$$

R being an unknown least squares matrix of order ν, (A2.4) becomes:

$$(M - M')\,VR = KVR\Omega_0^{-2}. \qquad \text{(A2.5)}$$

This is a superabundant system that we solve again using the least squares method:

$$(V^T M V - V^T M' V)\,R = (V^T K V)\,R\Omega_0^{-2}.$$

Accounting for the correlations of orthogonality, and by setting $V^T M V = \Psi$ (Ψ assumed to be known), we obtain:

$$(\mu - \Psi)\,R = \mu\Omega^2\,R\Omega_0^{-2} \qquad \text{(A2.6)}$$

(remember that omega is the matrix of the measured specific ν angular velocities).

The system (A2.6) has a conventional equation:

$$JR = IR\Omega_0^{-2}$$

where I and J are the symmetrical matrices defines as positive. We know that thesolution R of such a system is an orthogonal matrix in terms of J. In other words, the matrix:

$$\mu_o = R^T JR = R^T (\mu - \Psi) R$$

is a diagonal matrix. As R is thus determined, we have V_o = VR, and we now only have to truncate the matrix V_o by removing its lines corresponding to the measuring points foreign to S itself. Let V_{os} be this truncated matrix. It is also useful to define a matrix M_o (N X N) such that $\mu_o = V_{os}^T M_o V_{os}$.

We then simply have to finish solving the problem by replacing N by N_o, V by V_{os}, M by M_o, μ by μ_o in paragraph IV.

c) Obtaining Matrix M'

This matrix (symmetrical) must be determined by calculating on planes ([1] pp. 14 to 17), which may be relatively simple.

CONCLUSION

We have seen that, even in the most unfavorable cases, it is possible to determine the inertia constants (tensor of inertia, mass, center of gravity) of a structure by doing a few approximations on a simple ground test. Generally speaking the procedure to follow is:

1. Eliminate the spurious masses, if necessary.
2. Solve the problem according to the method in paragraph IV if the structure is deformable, or according to that of

paragraph III if it is found to be undeformable.

In any case, the resolution is always carried out using an exisitng automatic computer program at the ONERA computer center. Numerous applications [3] have made it possible to evaluate the method with considerable accuracy.

This document was submitted on June 4, 1970.

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