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Abstract

In this paper we consider the Chandrasekhar equations arising in optimal control problems for linear distributed parameter systems. The equations are derived via approximation theory. This approach is used to obtain existence, uniqueness and strong differentiability of the solutions and provides the basis for a convergent computation scheme for approximating feedback gain operators. A numerical example is presented to illustrate these ideas.

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Introduction

It has been noted by a number of authors [4], [7], [10], [15] that Chandrasekhar type algorithms can significantly reduce the computations necessary to calculate optimal feedback gains for linear quadratic control problems when the number of inputs and outputs is small, relative to the dimension of the state space. These algorithms were initally developed for finite dimensional linear time invariant systems [7], [8], [15], [16] and later extended to various infinite dimensional systems [4], [10], [19], [20]. It has been observed [4], [10], [18], [20] that the Chandrasekhar algorithm when applied to certain distributed parameter systems affords a significant computational reduction, often even greater than in the finite dimensional case. Therefore, it is worthwhile to consider the numerical aspects connected with the solution of the Chandrasekhar equations in infinite dimensional spaces.

Before one can develop an approximation theory for these equations, it is necessary to first examine certain basic questions such as existence, uniqueness and regularity of solutions. The question of existence has been addressed by Casti and Ljung [10] for general time invariant systems (including certain boundary control problems) and extended to time varying systems by Baras and Lainiotis [4]. Sorine [18], [19], [20] developed existence, uniqueness and differentiability results for a special class of parabolic systems and noted in Reference [20] that uniqueness is much more difficult to obtain than existence. All of the papers cited above utilize the variational framework of J. L. Lions and formulated the Chandrasekhar equations in differential form. Distributional derivatives were used to define the equations and solutions which often can complicate convergence analysis of numerical schemes.

In the present paper we present an approach to the Chandrasekhar equations that is based on approximation theory. We restrict our attention to time invariant control systems with bounded input and bounded output operators. As a result we obtain existence, uniqueness and smoothness of solutions to integral versions of Chandrasekhar equations for distributed parameter control systems that include delay and hyperbolic systems not covered by Sorine's results [20]. Moreover, sufficient conditions for convergence of general approximation schemes is estabilished.

2. The Chandrasekhar Equations

Let H, U and Λ be Hilbert spaces. We denote by $L(X,Y)$ the Banach space of bounded linear operators between the Hilbert spaces X and Y, endowed with the uniform operator topology. Throughout this paper we assume that A is the generator of a C_0-semigroup $S(t)$ on H, $B \in L(U,H)$, $V \in L(H,\Lambda)$ and $R \in L(U,U)$ is self-adjoint and satisfies $\|R\| \geq m > 0$. Let $Q \in L(H,H)$ be defined by $Q = V^*V$ where V^* is the adjoint of the operator V.

The linear time invariant quadratic optimal control problem is to choose $u(\cdot) \in L_2(0,T;U)$ that minimizes the cost functional

$$J = \int_0^T (\|y(s)\|_\Lambda^2 + \langle Ru(s),u(s)\rangle)ds \tag{1}$$

subject to the constraint that

$$\dot{z}(t) = Az(t) + Bu(t) \tag{2}$$

$$z(0) = z_0 \tag{3}$$

with output

$$y(t) = Vz(t). \tag{4}$$

Solutions to (2), (3) are always defined to be mild solutions given by

$$z(t) = S(t)z_0 + \int_0^t S(t - s)\mathcal{B}u(s)ds \tag{5}$$

Under the assumptions stated above, it is known (see References [11], [13]) that there exists a unique $\overline{u}(\cdot) \in L_2(0,T;\mathcal{U})$ that minimizes J. Moreover, the optimal control is given by

$$\overline{u}(t) = -\mathcal{R}^{-1}\mathcal{B}^*\Pi(t)\overline{z}(t) \tag{6}$$

where $\Pi(t)$ is the unique solution to the operator evolution equation

$$\Pi(t) = \int_t^T S^*(\eta - t)[Q - \Pi(\eta)\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi(\eta)]S(\eta - t)d\eta \tag{7}$$

and $\overline{z}(\cdot)$ is the optimal trajectory generated by $\overline{u}(\cdot)$ (see References [13], [14] for details). It is shown in References [11], [13], [14] that the operator $[A - \mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi(t)]$ generates an evolution operator $U(t,s)$ on $\mathcal{H}$ and

$$\Pi(t) = \int_t^T U^*(\eta,t)[Q + \Pi(\eta)\mathcal{B}\mathcal{R}^{-1}\mathcal{B}^*\Pi(\eta)]U(\eta,t)d\eta \tag{8}$$

A formal differentiation of equation (7) (or (8)) yields a Riccati operator differential equation. However, the meaning of such a differentiated form of the equation must be precisely defined. In the papers by Casti and Ljung [10] and Baras and Lainiotis [4] distributional derivatives are used to obtain strong solutions. Sorine [20] used a similar approach. However, Sorine was also able to obtain regularity of strong solutions for the special case that A generates an analytic semigroup. This is an important matter since the derivation of the Chandrasekhar equations presented in References [10] and [20] make heavy use of the differentiability of $\Pi(t)$. We shall avoid many of these difficulties by concentrating on an integral version of the Chandrasekhar equations. The special form of the equations can be exploited to obtain existence, uniqueness and some regularity properties of the gain operators.

Let $\hat{K}(t) = \mathcal{R}^{-1}\mathcal{B}^*\Pi(t)$ where $\Pi(t)$ satisfies (7) (or (8)). The goal is to derive a set of equations that allows one to solve for $\hat{K}(t)$ directly without first solving for $\Pi(t)$. We state here without proof the following theorem.

Theorem 1. There exists $K(t) \in \mathcal{L}(\mathcal{H},\mathcal{U})$ and $L(t) \in \mathcal{L}(\mathcal{H},\Lambda)$ such that for each $z \in \mathcal{H}$

$$K(t)z = \int_t^T \mathcal{R}^{-1}\mathcal{B}^* L^*(\eta)L(\eta)z\,d\eta \tag{9}$$

$$L(t)z = VS(T-t)z - \int_t^T L(\eta)\mathcal{B}K(\eta)S(\eta - t)z\,d\eta \tag{10}$$

for $0 \leq t \leq T$. Moreover, $\hat{K}(t)$ and $\hat{L}(t) = VU(T,t)$ are the unique strongly continuous solutions to (9), (10).

The representation (9), (10) can be exploited to establish the following result.

Theorem 2. If $z \in \mathcal{H}$, then $\hat{K}(t)z$ is continuously differentiable. Moreover, if $z \in \mathcal{D}(\mathcal{A})$, then $\hat{L}(t)z$ is differentiable and

$$\frac{d}{dt} \hat{K}(t)z = - \mathcal{R}^{-1}\mathcal{B}^{*}\hat{L}^{*}(t)\hat{L}(t)z \tag{11}$$

$$\frac{d}{dt} \hat{L}(t)z = \hat{L}(t)[\mathcal{A} + \mathcal{B}\hat{K}(t)]z \tag{12}$$

for $0 \leq t \leq T$.

Note that (9) can also be used to establish the differentiability of the Riccati operator. Under the additional assumption that $z \in \mathcal{D}(\mathcal{A})$ it can be shown that $\Pi(t)z$ is a continuously differentiable solution to the differentiated form of the Riccati equation. Although detailed proofs of these theorems will appear elsewhere, it is worthwhile to outline the approach in order to make a few observations about computational algorithms for approximating $\hat{K}(t)$.

<u>Definition 1.</u> A <u>strong approximating sequence</u> for the control problem defined by equations (1)-(4) is a sequence $(\mathcal{A}^N,\mathcal{B}^N,V^N,\mathcal{R}^N)$ such that $\mathcal{A}^N,\mathcal{B}^N,\mathcal{R}^N$ and $Q^N = [V^N]^* V^N$ are bounded operators satisfying $\|\mathcal{R}^N\| \geq m > 0$, $Q^N \geq 0$, $\mathcal{B}^N \to \mathcal{B}$ strongly, $\mathcal{R}^N \to \mathcal{R}$ strongly, $Q^N \to Q$ strongly and the operators $\mathcal{A}^N$ generate C_0-semigroups $S^N(t)$ satisfying

$$S^N(t)z \to S(t)z \quad \text{and} \quad [S^N(t)]^* z \to S^*(t)z$$

for all $z \in H$, uniformly for $t \in [0,T]$.

Observe that the assumptions on (A^N, B^N, V^N, R^N) in the definition of a strong approximation sequence are precisely those conditons used by Gibson [13], [14] (except for the boundness assumptions on A^N) to establish strong convergence of solutions to approximting Riccati equations to the solution $\Pi(t)$ of (7). For the case of a finite dimensional control space U, this strong convergence implies uniform convergence of the gain operators (see Theorem 6.2 in Reference [14]). However, in order to take full advantage of the factorization of $Q = V^*V$ additional assumptions on the convergence of V^N will be needed. Therefore, we state an additional condition.

<u>Hypothesis 1</u>. The operators V^N converge strongly to V.

Note that strong convergence of Q^N to Q does not imply strong convergence of V^N to V and, conversely, Hypothesis 1 does not imply strong convergence of Q^N to Q. In most examples we have considered, Hypothesis 1 is easily established.

Consider the approximating system of Chandrasekhar equations

$$K^N(t) = \int_t^T [R^N]^{-1} [B^N]^* [L^N(\eta)]^* L^N(\eta)d\eta \tag{13}$$

$$L^N(t) = V^N S^N(T - t) - \int_t^T L^N(\eta) B^N K^N(\eta) S^N(\eta - t)d\eta \tag{14}$$

where (A^N, B^N, V^N, R^N) is a strong approximating sequence. Since A^N is bounded it is straightforward to show that (13), (14) has a unique strongly continuous solution. We state the following lemma.

Lemma 1. If there exists a strong approximating sequence (A^N, B^N, V^N, R^N) such that Hypothesis 1 is satisfied, then the Chandrasekhar equations (9), (10) have a unique strongly continuous solution $\hat{K}(t)$, $\hat{L}(t)$ and $K^N(t) \to \hat{K}(t)$ strongly, $L^N(t) \to \hat{L}(t)$ strongly.

This result can be established by a slight extension of Gibson's results in Reference [13]. Lemma 1 not only provides existence and uniqueness, it provides sufficient conditions for the convergence of numerical approximation schemes. The only remaining "hole" in proof of Theorem 1 is to establish that there always exists strong approximating sequences that satisfy Hypothesis 1. Defining A^N to be the Yosida approximation $A^N = NA(NI - A)^{-1}$ (for N sufficiently large so that $N \varepsilon \rho(A)$) and letting $V^N = V$, $B^N = B$, $R^N = R$, the resulting sequence satisfies all the conditions of Lemma 1. It is important to note that Yosida approximates provide a tool for establishing Theorem 1 and lead to a numerical scheme that is convegent. However as a practical matter, Yosida approximates do not generally lead to efficient numerical algorithms. Therefore, in practice it is worthwhile to develop other approximating schemes that satisfy the sufficient conditions in Lemma 1.

3. A Numerical Example

We illustrate the power of the Chandrasekhar algorithm by applying the averaging approximation scheme (see References [2], [3], [14]) to a simple delay-differential equation model of the two-dimensional airfoil shown in Figure 1. We note that the averaging scheme has been shown to satisfy the sufficient conditions for convergence (see References [2], [14]). In order to make efficient use of the structure of the problem, the Chandrasekhar algorithm was combined with F-reduction techniques [12] ,[17] to obtain reduced order approximations.

A complete dynamic model for the system in which the elastic motions of the structure are coupled with the motions of the surrounding fluid results in a functional differential equation of neutral type (see Reference [6]). However, for this paper we shall use a simplified model based on the generalized Jones type approximations of the Wagner function described in [5]. The parameters used for the numerical examples below were obtained by applying the parameter identification scheme developed in [5] to experimental wind tunnel data. The resulting model is a five-dimensional delay-differential equation of the form

$$\dot{x}(t) = A_0x(t) + A_1x(t - r) + Bu(t) \tag{15}$$

$$x(s) = \phi(s) \qquad -r \leq s \leq 0 \tag{16}$$

with output

$$y(t) = Cx(t). \tag{17}$$

In this model $x = \mathrm{col}(\dot{h},\dot{\alpha},h,\alpha,\Gamma)$, where h is the plunge, α is the pitch angle and Γ represents a generalized aerodydnamic "lag state." The initial data was taken to be constant on $[-r,0]$. There is one control so that B is a 5×1 matrix and $C = \mathrm{diag}(c_1,c_2,c_3,c_4,c_5)$. The matrices A_0 and A_1 are 5×5 with the only nonzero entry in A_1 in the last row and last column. In particular, the time delay $r = .05$ and

$$A_0 = \begin{bmatrix} -4.2106 & -31.2446 & -4473.27 & -3704.37 & 3.06111 \\ .5302 & -8.0098 & 563.315 & -7391.14 & -1.15822 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -2.1187 & 1262.16 & -2250.91 & -32863.5 & -256.508 \end{bmatrix}$$

$$A_1(5,5) = -47.00, \quad A_1(i,j) = 0, \quad (i,j) \neq (5,5)$$

$$B = [-81.6087 \quad 192.589 \quad 0.0 \quad 0.0 \quad 678.182]^T$$

$$R = 10.0$$

$$C = \mathrm{diag}[\sqrt{50.0} \quad \sqrt{50.0} \quad \sqrt{10.0} \quad \sqrt{10.0} \quad 1].$$

The initial state is the constant function

$$\phi(s) \equiv [-.80 \quad .50 \quad .055 \quad .029 \quad 50.0]$$

and the control chosen represents a downward force applied at a point along the airfoil. All integrations were performed using a standard fourth order Runge-Kutta method with a fixed step size of $h = .001$. The Chandrasekhar algorithm was applied to the approximating system to obtain the gains $K^N(t)$. The resulting closed-loop system was integrated forward to calculate the optimal control and response. The system was initially solved on the interval $[0,.25]$. The results of the closed loop response (continuous line graph) and the unforced system (Δ graph) appear in Figures 2-6, and of particular note is that all closed loop responses approach zero as $t \to .25$. In Reference [1] where a similar problem was treated, high frequency oscillations were obtained near $t = .25$ which may be due to numerical instabilities.

The strength of the Chandrsekhar algorithm and F-reduction is revealed in a simple count of equations. The averaging scheme used divides $[-r,0]$ into N subintervals and approximates the "history" of the equation by piecewise constant functions (see References [2] and [14] for details). Simulations of the above model were performed with $N = 2,4,8,16,20$ and convergence was obtained at $N = 16$. (The results in Figures 2-7 are for $N = 16$). The averaging scheme results in an ordinary differential equation model that has 85 states. The gain, $K^N(t)$, is usually calculated by solving a matrix Riccati differential equation, and for $N = 16$ this necessitates solving 3655 equations. However, when the Chandrasekhar algorithm and F-reduction are applied, the number of states is reduced to 21, and it is necessary to solve only 126 equations to obtain $K^N(t)$. In this case, the calculation of the gain, state, and control took approximately 11 seconds on an IBM 3081.

We comment here that the Chandrasekhar equations were also integrated on the interval $[0,1]$ to obtain the steady-state gain. In this case excellent performance was also obtained when the loop was closed.

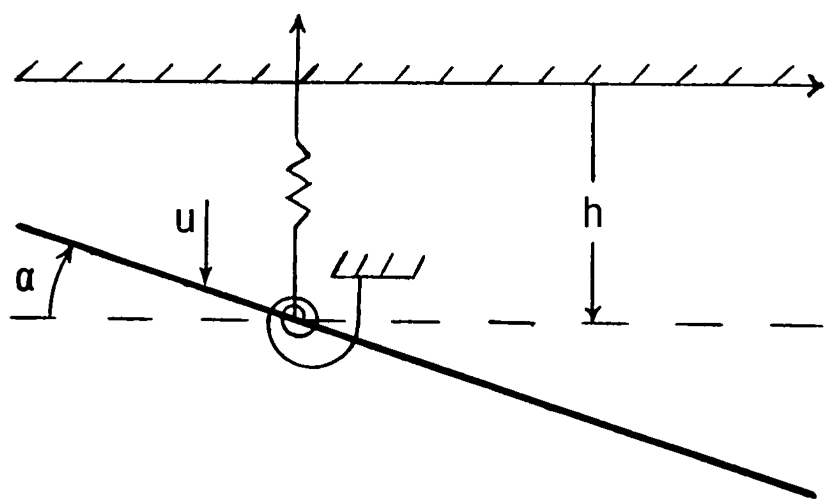

Figure 1

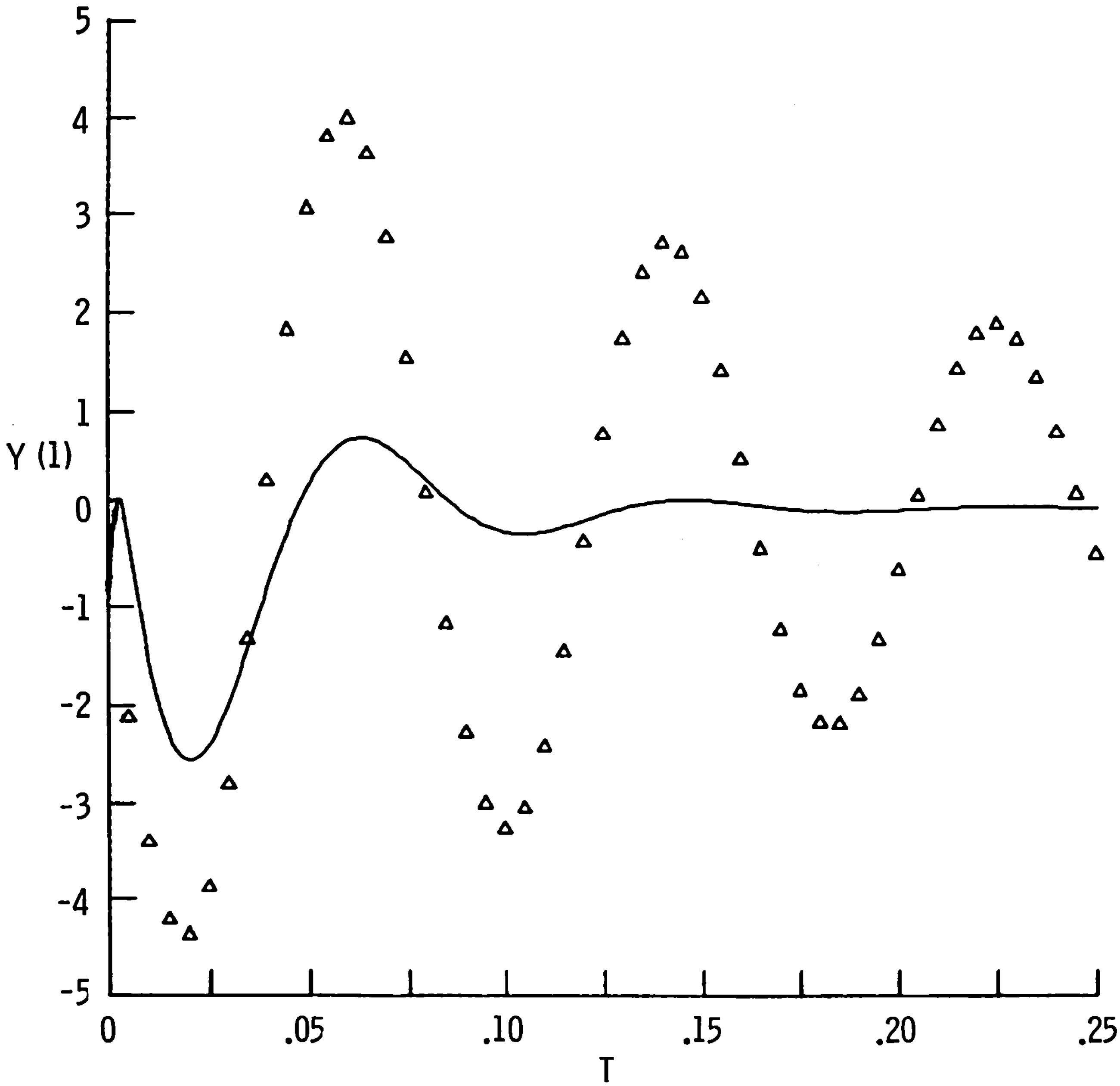

Figure 2

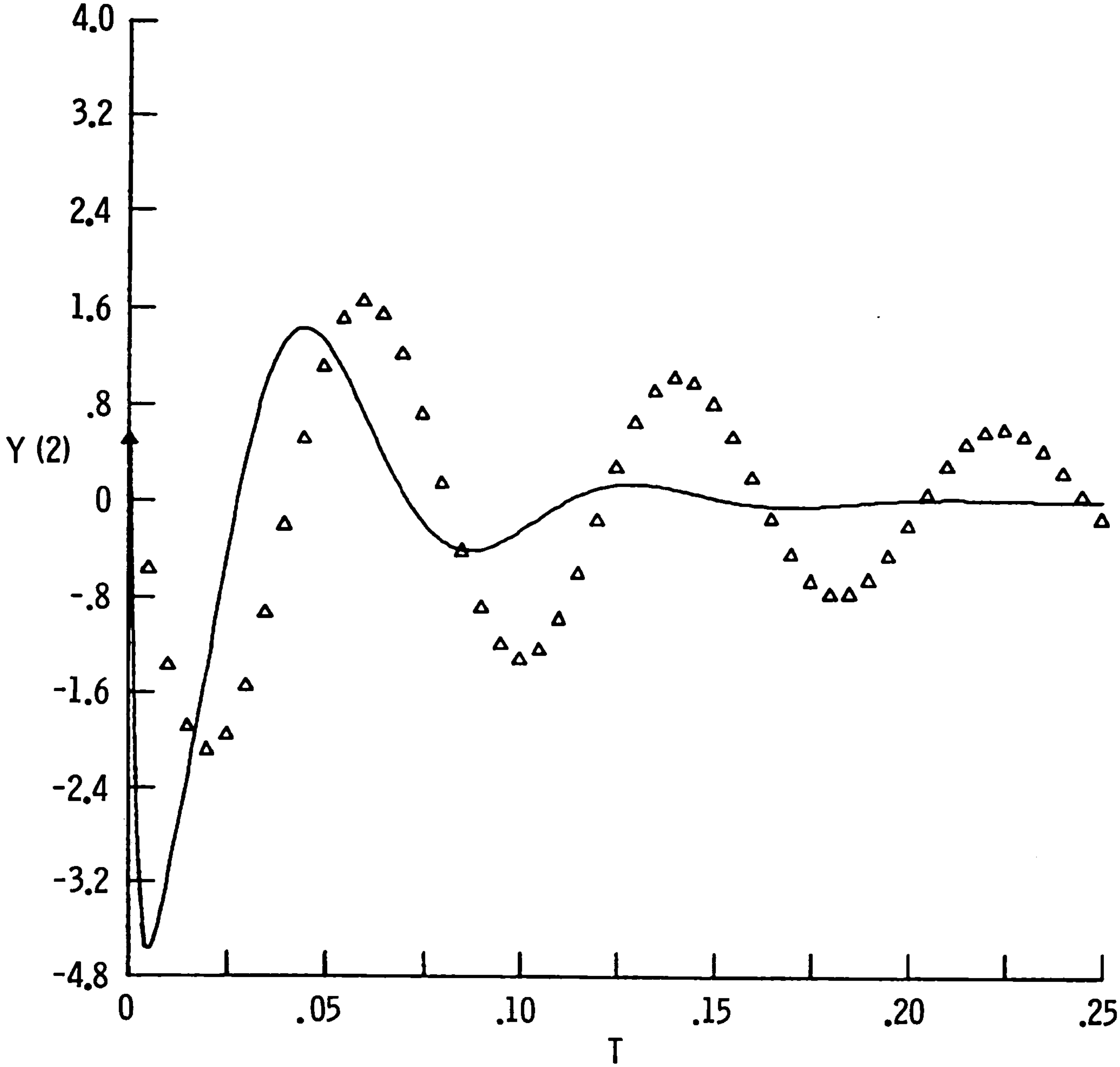

Figure 3

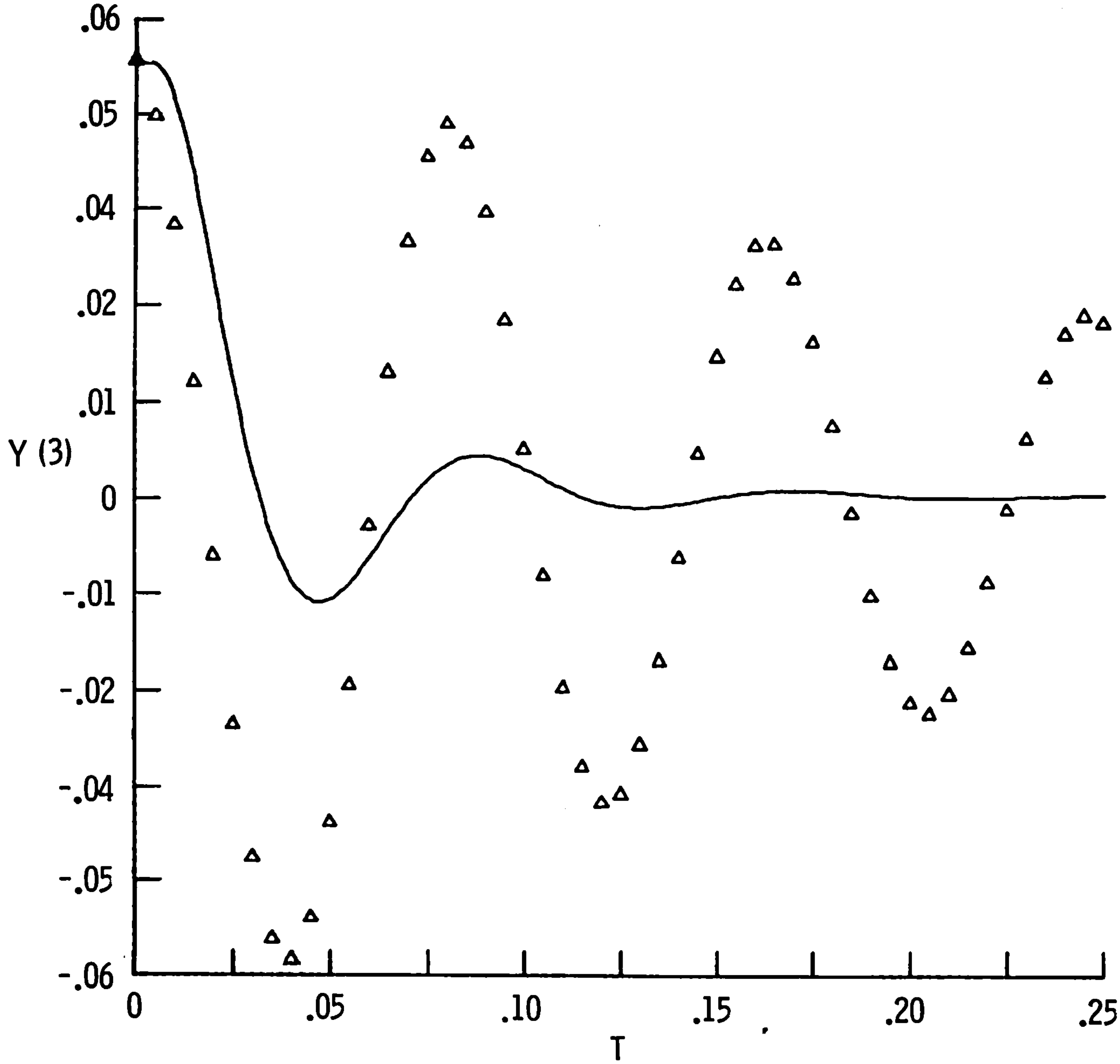

Figure 4

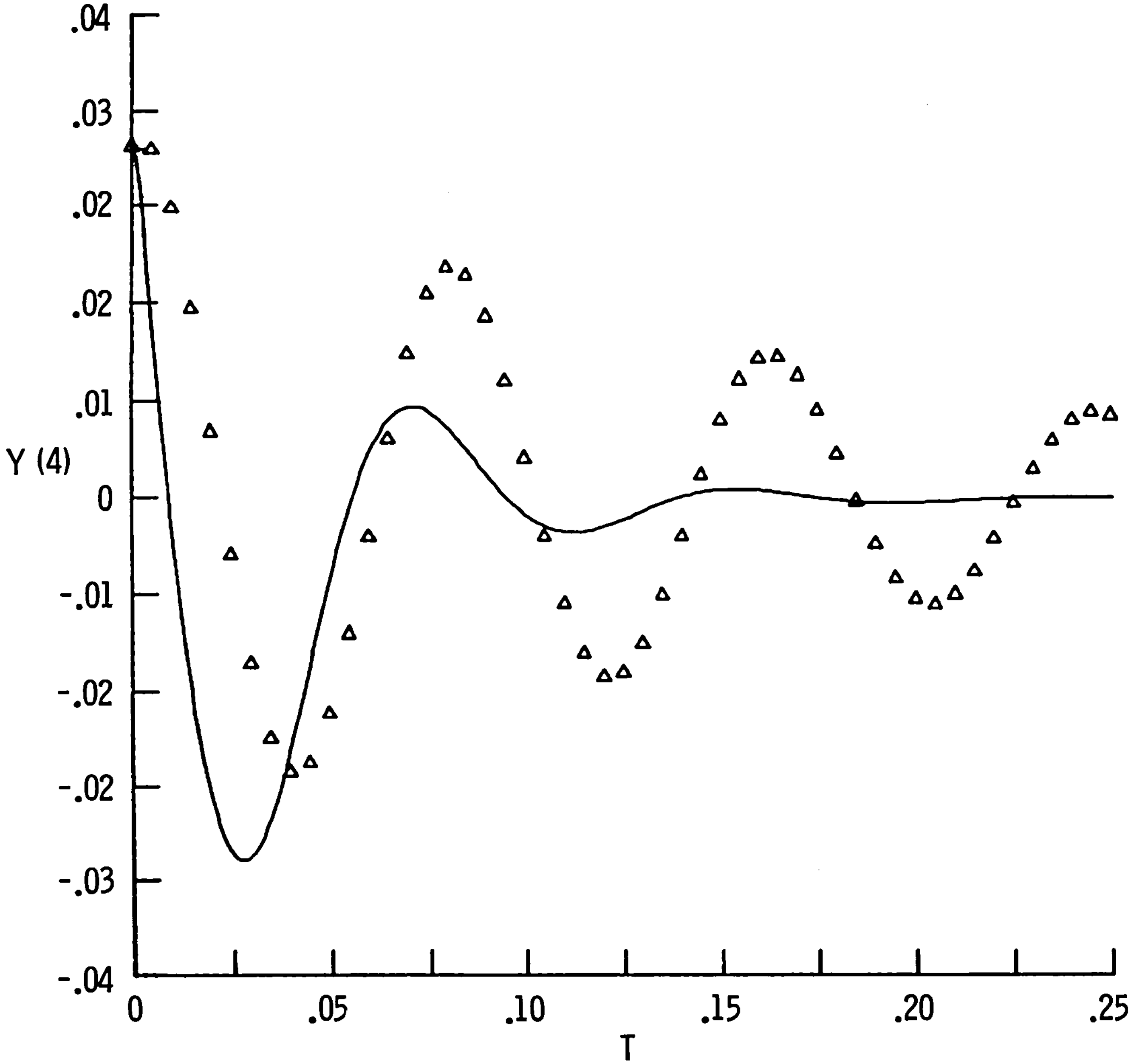

Figure 5

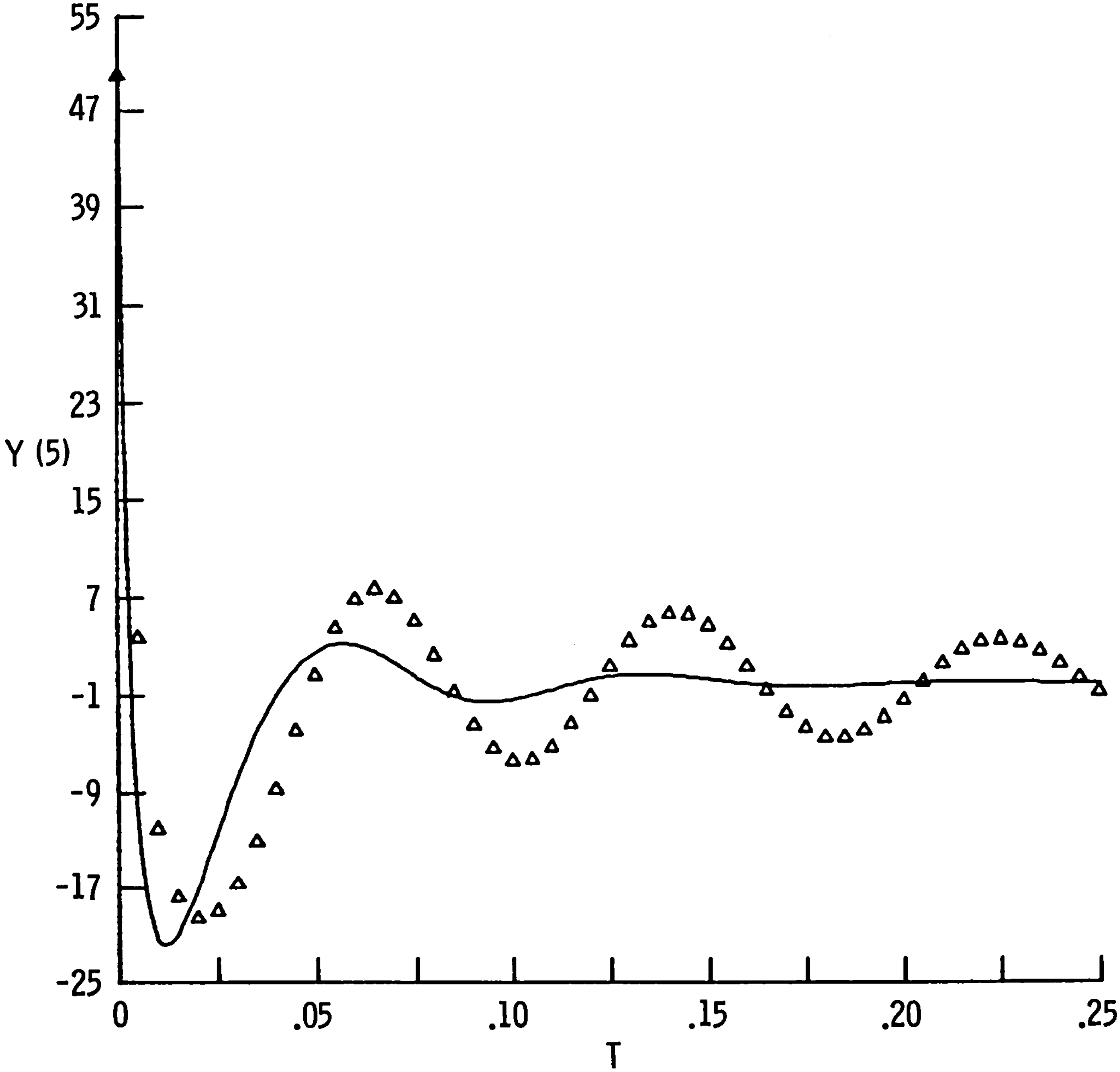

Figure 6

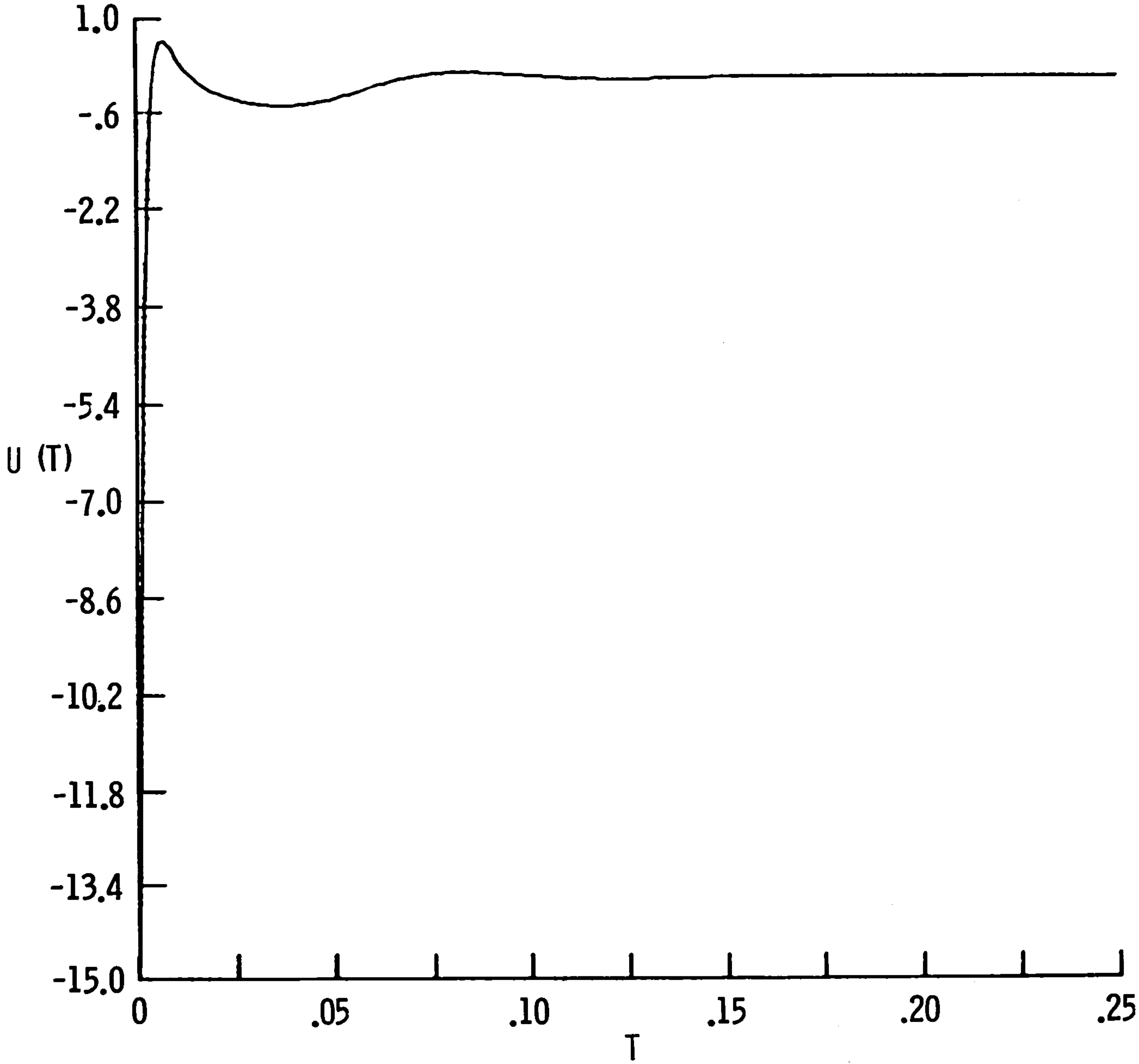

Figure 7

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