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A Buckling Analysis for Rectangular Orthotropic Plates With Centrally Located Cutouts

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Contents

Introduction

An analysis for predicting the buckling load of a compression-loaded orthotropic plate with a centrally located cutout is presented in reference 1. In this reference, the classical two-dimensional analysis for determining the plate buckling load is converted to an approximately equivalent one-dimensional analysis by approximating the plate displacements with kinematically admissible series. The accuracy of the one-dimensional analysis is a function of the number of terms included in these series. The objectives of this report are to describe the one-dimensional analysis for a particular displacement series and to describe a computer program, BUCKO, associated with this analysis. The computer program is both easy and inexpensive to use.

The main body of this report presents an overview of the basic assumptions used in the analysis, the derivation and solution of the governing equations, and a description of computer program BUCKO. The details of the analysis are presented in the appendixes along with a user's guide and sample problems. Symbols appearing in the main body and appendixes are defined in appendix A. The person interested in simply using the computer program needs to read only the main body of the report and the appendix containing the user's guide and sample problems. The remaining sections of this report present the details required to modify the computer program.

Analysis

The assumptions made in the one-dimensional analysis are outlined as follows. The plate is assumed to be a balanced, symmetric laminate with uniform thickness. The unloaded edges of the plate are simply supported and the loaded edges are simply supported or are clamped. The cutout is centrally located with two mutually orthogonal axes of symmetry that coincide with the longitudinal and transverse axes of the plate. The geometry and the loading conditions used in the analysis are shown in figures 1 and 2, respectively. Loading is applied by either uniformly displacing or uniformly stressing two opposite edges of the plate. The plate has length $L = 2c$ and width $W = 2b$. The origin of the coordinate system is at the center of the plate. The x-axis is parallel to the loading direction and the y-axis is perpendicular to the loading direction. The unloaded edges are free to expand in-plane in the y-direction. The plate is assumed to deform into a symmetric shape having mutually orthogonal axes of symmetry that also coincide with the plate axes. In-plane and out-of-plane displacements are represented by truncated kinematically admissible series which reflect the above assumptions.

In reference 1 a one-dimensional formulation of the classical two-dimensional buckling analysis was derived following the Kantorovich method (ref. 2, pp. 304–327). The one-dimensional analysis consists of two parts: calculation of the in-plane stress distribution prior to buckling, hereafter referred to as the "prebuckling problem," and calculation of the plate displacements and load at buckling, hereafter referred to as the "buckling problem." The results from the prebuckling problem are used with the principle of minimum potential energy and the Trefftz criterion (ref. 3, pp. 365–367) to obtain the stability equations for the buckling problem.

Prebuckling

As mentioned above, the in-plane displacements of the middle surface of the plate, u^o and v^o, are approximated by truncated kinematically admissible series. These series contain terms that are products of trigonometric functions in the y-direction and unknown generalized displacement functions in the x-direction. The series for the in-plane displacements are

$$\left.\begin{aligned} u^o(x,y) &= \lambda\left\{u_0(x) + \sum_{k=1}^{N} u_{2k-1}(x)\cos\left[(2k-1)\frac{\pi y}{2b}\right]\right\} \\ v^o(x,y) &= \lambda\left\{v_0 y + \sum_{k=1}^{N} v_{2k-1}(x)\sin\left[(2k-1)\frac{\pi y}{2b}\right]\right\}\end{aligned}\right\} \quad (1)$$

where λ is the loading parameter and N is the number of trigonometric terms used. Substituting these series into the membrane potential energy functional and integrating over y reduces the potential energy to a one-dimensional form with the generalized in-plane displacements $u_0(x)$, $u_{2k-1}(x)$, and $v_{2k-1}(x)$ appearing as unknowns. The constant v_0 is a parameter which is selected to make the residual normal force on the unloaded edges of the plate equal to zero. The one-dimensional equilibrium equations are obtained by applying the principle of minimum potential energy. These equations constitute a system of simultaneous linear second-order ordinary differential equations with variable coefficients. The second-order system of equations is solved for the generalized displacements and their derivatives, and these values are used to calculate the prebuckling stress distribution. The derivation of the one-dimensional prebuckling equations for $N = 3$ is presented in appendix B. The $N = 3$ analysis is used in computer program BUCKO.

Buckling

The results from the prebuckling problem are used to determine the buckling load of the plate. The prebuckling stress distribution is substituted into the functional corresponding to the second variation of the potential energy due to membrane and bending action of the plate. The out-of-plane deflection of the plate

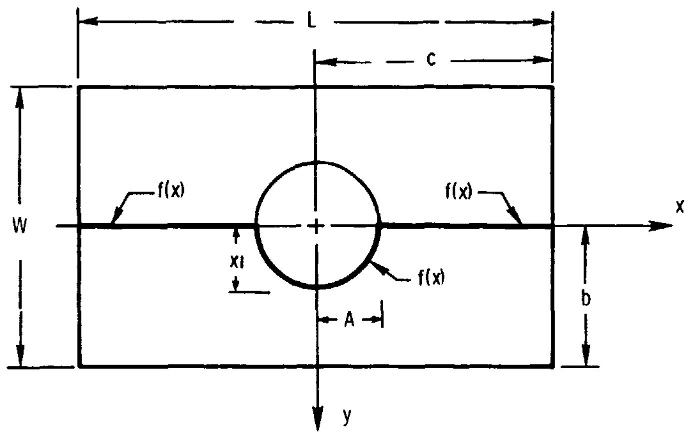
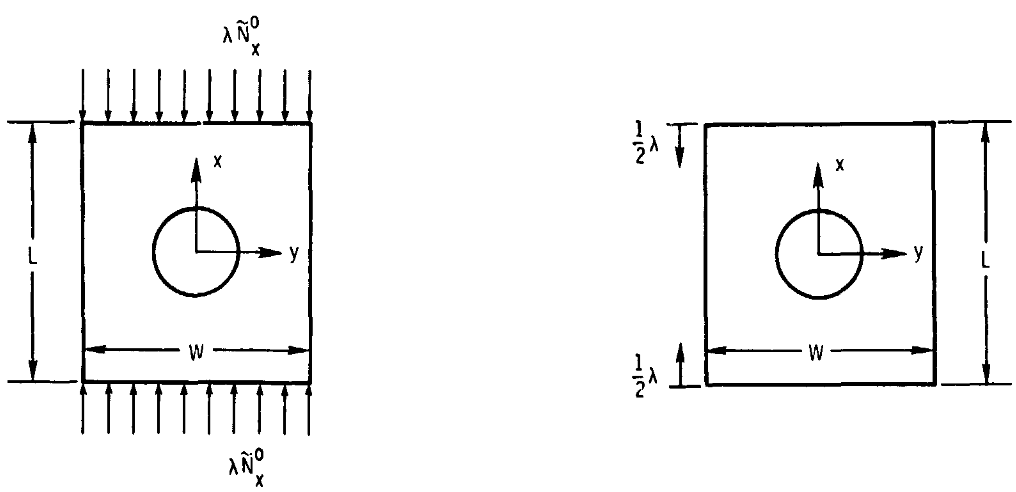


Figure 1. Geometry and coordinate system for a rectangular plate containing a centrally located cutout.



(a) Uniform compressive edge stress.

(b) Uniform compressive edge displacement.

Figure 2. Loading conditions.

middle surface w^o is approximated by the truncated kinematically admissible series

$$w^o(x,y) = \sum_{k=1}^{S} w_{2k-1}(x) \cos\left[(2k-1)\frac{\pi y}{2b}\right] \qquad (2)$$

where S is the number of trigonometric terms used. Substituting this series into the functional and integrating over y produces a one-dimensional form in which the generalized out-of-plane displacements at buckling $w_{2k-1}(x)$ appear as unknowns. The one-dimensional stability equations are obtained using the Trefftz criterion. These equations constitute a system of simultaneous linear fourth-order ordinary differential equations. The corresponding analysis for $S = 3$ is given in appendix C.

Numerical Solution of the Differential Equations

The prebuckling and buckling equations are solved numerically to determine the buckling load of the plate. The prebuckling analysis yields $2N + 1$ simultaneous second-order homogeneous differential equations and $4N + 2$ boundary conditions. Some of the equations for the boundary conditions are nonhomogeneous. Computer program BUCKO uses a subroutine called PASVAR (ref. 4) to solve the prebuckling differential equations. This subroutine applies to both linear and nonlinear systems of simultaneous first-order ordinary differential equations subject to two-point boundary conditions. To use PASVAR, the system of second-order differential equations is converted to the corresponding first-order system of equations. An example of this conversion is also presented in appendix B for the prebuckling analysis associated with $N = 3$.

The buckling analysis yields S simultaneous fourth-order ordinary differential equations. These differential equations are linear and homogeneous and have $4S$ homogeneous boundary conditions. These homogeneous equations and boundary conditions constitute an eigenvalue problem for the loading parameter λ in equation (1). The smallest value of λ for which a nontrivial solution exists is taken as the critical value of the loading parameter λ_{cr}. The corresponding critical load P_{cr} is given by the value of the applied loading scaled by the critical value of the loading parameter. The subroutine PASVAR cannot solve eigenvalue problems and is not used to solve the differential equations for the buckling problem. Central finite differences are applied to the buckling differential equations and the boundary conditions. This algebraic eigenvalue problem is solved for the critical buckling load of the plate. In the following discussion, the algebraic eigenvalue problem for the buckling analysis is presented. Also, the modifications required to use PASVAR in solving the prebuckling equations of the stress loading case are discussed.

Algebraic eigenvalue problem for buckling. The differential equations of the one-dimensional analysis can be expressed as

$$\sum_{n=1}^{S}\left\{L_{2m-1,2n-1}\left[w_{2n-1}(x)\right] - \lambda H_{2m-1,2n-1}\left[w_{2n-1}(x)\right]\right\} = 0 \qquad (m = 1,2,\ldots,S) \qquad (3)$$

where $L_{2m-1,2n-1}(\cdot)$ and $H_{2m-1,2n-1}(\cdot)$ are the differential operators and $w_{2n-1}(x)$ are the generalized out-of-plane displacements at buckling. Using $L_{rs}(\cdot)$ to represent any $L_{2m-1,2n-1}(\cdot)$ operator, $H_{rs}(\cdot)$ to represent any $H_{2m-1,2n-1}(\cdot)$ operator, and $w(x)$ to represent any generalized buckling displacement, the form of these operators can be expressed as

$$L_{rs}(w) = f_1(x)w(x) + [f_2(x)w'(x)]' + [f_3(x)w(x)]'' + f_4(x)w''(x) + [f_5(x)w''(x)]'' \qquad (4)$$

and

$$H_{rs}(w) = p_1(x)w(x) + [p_2(x)w'(x)]' + [p_3(x)w(x)]' + p_4(x)w'(x) \qquad (5)$$

where $f_k(x)$ and $p_k(x)$ are coefficients appearing in the differential operators and primes denote differentiation with respect to x.

In deriving the finite difference expressions for these operators, the definition of the ith station derivative based on equally spaced half central differences is used, with the exception of the two terms in equation (5) multiplied by $p_3(x)$ and $p_4(x)$. The definition of the ith station derivative based on equally spaced full station central differences is used for these terms, since they are the only terms in equations (4) and (5) from which $w^{i+1/2}$ and $w^{i-1/2}$ unknowns would arise. Using this scheme for obtaining the finite difference expressions for higher derivatives and derivatives of composition functions, the discretized form of the differential operators given in equations (4) and (5) are

$$\begin{aligned}
[L_{rs}(w)]^i = &+ w^{i-2}\left(f_5^{i-1}/\Delta^4\right) \\
&+ w^{i-1}\left[\left(f_2^{i-1/2} + f_3^{i-1} + f_4^i\right)/\Delta^2 - 2\left(f_5^i + f_5^{i-1}\right)/\Delta^4\right] \\
&+ w^i\left[f_1^i - \left(f_2^{i+1/2} + f_2^{i-1/2} + 2f_3^i + 2f_4^i\right)/\Delta^2 + \left(f_5^{i+1} + 4f_5^i + f_5^{i-1}\right)/\Delta^4\right] \\
&+ w^{i+1}\left[\left(f_2^{i+1/2} + f_3^{i+1} + f_4^i\right)/\Delta^2 - 2\left(f_5^{i+1} + f_5^i\right)/\Delta^4\right] \\
&+ w^{i+2}\left(f_5^{i+1}/\Delta^4\right)
\end{aligned} \qquad (6)$$

$$
\begin{aligned}
[H_{rs}(w)]^{i} = &+ w^{i-1}\left[p_2^{i-1/2}\Big/\Delta^2 - \left(p_3^{i-1} + p_4^{i}\right)\Big/2\Delta\right] \\
&+ w^{i}\left[p_1^{i} - \left(p_2^{i+1/2} + p_2^{i-1/2}\right)\Big/\Delta^2\right] \\
&+ w^{i+1}\left[p_2^{i+1/2}\Big/\Delta^2 + \left(p_3^{i+1} + p_4^{i}\right)\Big/2\Delta\right]
\end{aligned}
\tag{7}
$$

where the superscripts refer to the finite difference station. Applying equations (6) and (7) to the operators appearing in equation (3) at each interior station and applying the basic difference expressions for the derivatives to the boundary conditions yields the algebraic eigenvalue problem

$$[K]\{w\} = \lambda[g]\{w\} \tag{8}$$

The matrices $[K]$ and $[g]$ are square matrices of order $S(M-2)$, where M is the total number of difference stations. The vector $\{w\}$ contains the out-of-plane displacements at the interior difference stations. Because of the self-adjointness of the differential operators of equation (3), the matrices $[K]$ and $[g]$ are symmetric. In addition, since the bending energy is positive definite, the matrix $[K]$ is also positive definite. The smallest value of λ for which equation (3) has a nontrivial solution is the critical value of the loading parameter.

The finite difference and algebraic eigenvalue problem formulations for the one-dimensional buckling analysis corresponding to $S = 3$ are presented in appendix D.

Modifications for the stress loading case. The prebuckling displacement field for the stress loading case can be determined only to within an arbitrary constant. This constant represents a rigid body displacement. In the present analysis, the Jacobian matrix for the prebuckling differential equations (used in subroutine PASVAR) is singular because the displacement field is not unique. A procedure that corresponds to eliminating rigid body displacements from the displacement field is to reduce the number of prebuckling differential equations and to rearrange the unknown generalized displacements. Appendix E describes these modifications to the prebuckling equations for the stress loading case for $N = 3$.

Description of Computer Program

Computer program BUCKO is written in FORTRAN V for the Control Data Corporation (CDC) CYBER 170-series computers and is operational on the Network Operating System (NOS) level 1.4 at the NASA Langley Research Center. The computer program is composed of a main program, 10 subroutines, and 107 function subprograms, totaling approximately 3600 lines of computer code. The main program, program subroutines, and function subprograms are described in this section.

The purposes of the main program are to read the input data file and to assemble the arrays used in the prebuckling and buckling problems. To minimize storage requirements, the arrays are stored in a master vector. The locations of the variables contained in the master vector are determined in the main program.

The subroutine PREBUK is the major subroutine used in the prebuckling problem. This subroutine computes the approximate prebuckling stress distribution. Subroutines JACOB1, JACOB2, F1, and F2 are used in subroutine PREBUK. Subroutines JACOB1 and JACOB2 determine the Jacobian matrices for the system of differential equations corresponding to the displacement loading case and to the stress loading case, respectively. Subroutines F1 and F2 calculate the coefficients of the differential equations corresponding to the two loading cases. Function subprograms OMG1 through OMG16 calculate the variable coefficients in the differential equations. The coefficients are functions of the cutout shape $f(x)$. The cutout shape is determined by the function subprogram R which is presently set up for elliptical and rectangular shapes and may be modified easily to include other shapes. Subroutine PREBUK calls subroutine PASVAR, which performs the actual numerical solution of the prebuckling equations.

The major subroutine used in the buckling solution phase is BUCKL. This subroutine computes the finite difference coefficients, assembles the finite difference equations, and computes the buckling load. Actual calculation of the eigenvalues is performed by subroutine SYMGEP, which is obtained from the NOS mathematics library. From the eigenvalues, the critical value of the applied loading is computed. For the displacement loading case, the buckling load is obtained from the average normal stress acting at the loaded edges of the plate. For the stress loading case, the buckling displacement is taken as the average axial displacement of the loaded edges of the plate. Other output generated by the program are the prebuckling stress distribution, the buckling mode shape, the buckling coefficient, and average axial and transverse strains at buckling. A user's guide and program sample problems are described in appendix F.

Concluding Remarks

This report presented an approximate analysis for predicting the buckling load of a rectangular orthotropic plate with a centrally located cutout. The analysis is applicable to plates that are compressed uniaxially by either uniform edge displacement or uniform edge normal stress. The boundary conditions considered are simply supported unloaded edges and clamped or simply supported loaded edges. A computer program, BUCKO, associated with this analysis was described. Buckling results generated by the computer program are values of the critical axial load, critical end shortening, buckling coefficient, buckling mode shape, critical average axial strain, and critical average transverse strain.

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Appendix A

Symbols

A	cutout dimension defined in figure 1
$A_{11}, A_{22}, A_{12}, A_{66}$	membrane stiffness coefficients of plate
b	rectangular plate half-width
c	rectangular plate half-length
$[C], C_{ij}$	coefficients defined by equations (C32)
d	circular cutout diameter
Den1,Den2	terms defined by equations (B59)
D_j^i	finite difference coefficients defined by equations (D13), (D14), and (D15)
$D_{11}, D_{22}, D_{12}, D_{66}$	bending stiffness coefficients of plate
E_j^i	finite difference coefficients defined by equations (D4), (D5), and (D6)
$f(x)$	function representing cutout shape (see fig. 1)
$f_k(x)$	coefficients defined by equation (4)
F	integrand defined by equation (C35)
F_{B1}, F_{B2}	integrals defined by equations (C33)
F_j^i	finite difference coefficients defined by equations (D7), (D8), and (D9)
$F_{x0}, \ldots, F_{x5}$ $F_{y0}, \ldots, F_{y5}$ $F_{xy1}, \ldots, F_{xy5}$	integrals defined by equations (C14), (C15), and (C16)
$[g]$	system geometric stiffness matrix
$[\tilde{G}]$	system geometric stiffness matrix before application of the boundary conditions
$[G], G_{ij}$	coefficients defined by equations (C7)
G_{B1}, G_{B2}	integrals defined by equations (C33)
G_j^i	finite difference coefficients defined by equations (D16), (D17), and (D18)
$G_{x0}, \ldots, G_{x5}$ $G_{y0}, \ldots, G_{y5}$ $G_{xy1}, \ldots, G_{xy5}$	integrals defined by equations (C8), (C9), and (C10)
H_{B1}, H_{B2}	integrals defined by equations (C33)
$H_{ij}(\cdot)$	differential operators defined by equations (C40)
H_j^i	finite difference coefficients defined by equations (D19), (D20), and (D21)
$[H_{rs}(w)]^i$	discretized differential operators defined by equation (7)
$H_{x0}, \ldots, H_{x5}$ $H_{y0}, \ldots, H_{y5}$ $H_{xy1}, \ldots, H_{xy5}$	integrals defined by equations (C11), (C12), and (C13)
$[K]$	system stiffness matrix
$[\tilde{K}]$	system stiffness matrix before application of the boundary conditions
L	rectangular plate length
$L_{ij}(\cdot)$	differential operators defined by equations (C30)
$[L_{rs}(w)]^i$	discretized differential operators defined by equation (6)
M	number of finite difference stations
N	number of terms in the prebuckling displacement series
N_x, N_y, N_{xy}	membrane stress resultants
$\tilde{N}_x^o$	applied uniform stress loading
$N_{x0}, \ldots, N_{x5}$ $N_{y0}, \ldots, N_{y5}$ $N_{xy1}, \ldots, N_{xy5}$	stress resultants defined by equations (B39), (B40), and (B41)
$p_k(x)$	coefficients defined by equation (5)
P_{B1}, P_{B12}, P_{B21}	integrals defined by equations (C33)
P_j^i	finite difference coefficients defined by equations (D10), (D11), and (D12)
P_x	applied axial force
P_y	residual force on the unloaded edges
$P_{x0}, \ldots, P_{x5}$ $P_{y0}, \ldots, P_{y5}$ $P_{xy1}, \ldots, P_{xy5}$ $P_{yx1}, \ldots, P_{yx5}$	integrals defined by equations (C17), (C18), (C19), and (C20)
Q_{B1}, Q_{B12}, Q_{B21}	integrals defined by equations (C33)
$Q_{x0}, \ldots, Q_{x5}$ $Q_{y0}, \ldots, Q_{y5}$ $Q_{xy1}, \ldots, Q_{xy5}$ $Q_{yx1}, \ldots, Q_{yx5}$	integrals defined by equations (C21), (C22), (C23), and (C24)
R_{B1}, R_{B12}, R_{B21}	integrals defined by equations (C33)
$R_{x0}, \ldots, R_{x5}$ $R_{y0}, \ldots, R_{y5}$ $R_{xy1}, \ldots, R_{xy5}$ $R_{yx1}, \ldots, R_{yx5}$	integrals defined by equations (C25), (C26), (C27), and (C28)

APPENDIX A

S	number of terms in the buckling displacement series
U_B	plate bending energy
U_{IS}	plate initial stress energy
$U_m, \hat{U}_m$	membrane strain energy and membrane strain energy density
u^o, v^o	prebuckling displacements in the x- and y-direction, respectively
u_0, u_{2k-1}, v_{2k-1}	generalized displacements for the prebuckling problem
v_0	displacement series parameter
$\{w\}$	buckling displacement vector
w^o	out-of-plane displacement
w_{2k-1}	generalized displacements for the buckling problem
w_1, w_3, w_5	values of the generalized displacements
W	rectangular plate width
W_e	external work
XI	cutout dimension defined in figure 1
x, y, z	Cartesian coordinates
$y_1, \ldots, y_{14}$	variables defined by equations (B42) through (B49)
y_{88}	a constant, see equation (E5)
$\tilde{y}_0, \ldots, \tilde{y}_{12}$	variables defined by equations (E1)
γ	displacement series parameter, $\pi/2b$
Δ	distance between equally spaced finite difference stations
δ	variational operator
$\varepsilon_x^o, \varepsilon_y^o, \gamma_{xy}^o$	midplane membrane strains
$\varsigma_1, \ldots, \varsigma_{10}$	coefficients defined by equations (B58)
$\theta_1, \ldots, \theta_{23}$	coefficients defined by equations (B60)
λ	loading parameter
$\Omega_1, \ldots, \Omega_{16}$	integrals defined by equations (B6)
$(\ )'$	differentiation of ( ) with respect to x

Appendix B

Prebuckling Analysis for $N = 3$

The in-plane displacements prior to buckling are assumed to be

$$\left.\begin{aligned} u^o(x,y) &= \lambda\left\{u_0(x) + \sum_{k=1}^{3} u_{2k-1}(x)\cos\left[(2k-1)\gamma y\right]\right\} \\ v^o(x,y) &= \lambda\left\{v_0 y + \sum_{k=1}^{3} v_{2k-1}(x)\sin\left[(2k-1)\gamma y\right]\right\} \end{aligned}\right\} \quad \text{(B1)}$$

where λ is a loading parameter, $\gamma = \pi/2b$, and the functions $u_0(x)$, $u_1(x)$, $u_3(x)$, $u_5(x)$, $v_1(x)$, $v_3(x)$, and $v_5(x)$ are the generalized displacements to be determined. The parameter v_0 is a constant which is determined by requiring that the unloaded edges have zero resultant forces acting on them. Substituting these expressions into the linear strain-displacement relations from two-dimensional theory of elasticity yields

$$\left.\begin{aligned} \varepsilon_x^o &= \lambda\left\{u_0'(x) + \sum_{k=1}^{3} u_{2k-1}'(x)\cos\left[(2k-1)\gamma y\right]\right\} \\ \varepsilon_y^o &= \lambda\left\{v_0 + \gamma\sum_{k=1}^{3}(2k-1)v_{2k-1}(x)\cos\left[(2k-1)\gamma y\right]\right\} \\ \gamma_{xy}^o &= \lambda\left\{\sum_{k=1}^{3}\left[v_{2k-1}'(x) - (2k-1)\gamma u_{2k-1}(x)\right]\sin\left[(2k-1)\gamma y\right]\right\} \end{aligned}\right\} \quad \text{(B2)}$$

where primes denote differentiation with respect to x. The membrane strain energy for the plate, which is assumed to deform symmetrically, is

$$U_m = \int_{-c}^{+c}\int_{f(x)}^{b}\left[A_{11}(\varepsilon_x^o)^2 + A_{22}(\varepsilon_y^o)^2 + 2A_{12}(\varepsilon_x^o\varepsilon_y^o) + A_{66}(\gamma_{xy}^o)^2\right]\,dy\,dx \quad \text{(B3)}$$

where $f(x)$ is the curve shown in figure 1 and the A_{ij} are the orthotropic membrane stiffness coefficients. Substituting the expressions for the strains into the membrane strain energy and performing some algebraic manipulations yields

$$U_m = \lambda^2\int_{-c}^{+c}\hat{U}_m\left(u_0, u_0', u_1, u_1', u_3, u_3', u_5, u_5', v_1, v_1', v_3, v_3', v_5, v_5'\right)\,dx \quad \text{(B4)}$$

where

$$\begin{aligned} \hat{U}_m = {} & \left[A_{11}(u_0')^2 + A_{22}v_o^2 + 2A_{12}u_0'v_0\right]\Omega_1 \\ & + \left[A_{11}(u_1')^2 + A_{22}\gamma^2 v_1^2 + 2A_{12}\gamma u_1' v_1\right]\Omega_5 \\ & + \left[A_{11}(u_3')^2 + 9A_{22}\gamma^2 v_3^2 + 6A_{12}\gamma u_3' v_3\right]\Omega_6 \\ & + \left[A_{11}(u_5')^2 + 25A_{22}\gamma^2 v_5^2 + 10A_{12}\gamma u_5' v_5\right]\Omega_{13} \\ & + 2\left[A_{11}u_0'u_1' + A_{22}\gamma v_0 v_1 + A_{12}\left(\gamma u_0' v_1 + v_0 u_1'\right)\right]\Omega_2 \\ & + 2\left[A_{11}u_0'u_3' + 3A_{22}\gamma v_0 v_3 + A_{12}\left(3\gamma u_0' v_3 + v_0 u_3'\right)\right]\Omega_3 \\ & + 2\left[A_{11}u_0'u_5' + 5A_{22}\gamma v_0 v_5 + A_{12}\left(5\gamma u_0' v_5 + v_0 u_5'\right)\right]\Omega_{10} \\ & + 2\left[A_{11}u_1'u_3' + 3A_{22}\gamma^2 v_1 v_3 + A_{12}\gamma\left(u_3' v_1 + 3u_1' v_3\right)\right]\Omega_4 \\ & + 2\left[A_{11}u_3'u_5' + 15A_{22}\gamma^2 v_3 v_5 + A_{12}\gamma\left(3u_5' v_3 + 5u_3' v_5\right)\right]\Omega_{11} \end{aligned}$$

Equation (B5) continued on next page

$$+\,2\left[A_{11}u_1'u_5' + 5A_{22}\gamma^2 v_1 v_5 + A_{12}\gamma\,(u_5' v_1 + 5u_1' v_5)\right]\Omega_{12}$$
$$+\left[A_{66}(v_1' - \gamma u_1)^2\right]\Omega_8$$
$$+\left[A_{66}(v_3' - 3\gamma u_3)^2\right]\Omega_9$$
$$+\left[A_{66}(v_5' - 5\gamma u_5)^2\right]\Omega_{16}$$
$$+\,2\left[A_{66}(v_1' - \gamma u_1)(v_3' - 3\gamma u_3)\right]\Omega_7$$
$$+\,2\left[A_{66}(v_3' - 3\gamma u_3)(v_5' - 5\gamma u_5)\right]\Omega_{14}$$
$$+\,2\left[A_{66}(v_1' - \gamma u_1)(v_5' - 5\gamma u_5)\right]\Omega_{15} \tag{B5}$$

and

$$\Omega_1(x) = \int_{f(x)}^{b} dy = b - f(x) \tag{B6a}$$

$$\Omega_2(x) = \int_{f(x)}^{b} \cos\gamma y\,dy = \frac{1}{\gamma}\{1 - \sin[\gamma f(x)]\} \tag{B6b}$$

$$\Omega_3(x) = \int_{f(x)}^{b} \cos 3\gamma y\,dy = -\frac{1}{3\gamma}\{1 + \sin[3\gamma f(x)]\} \tag{B6c}$$

$$\Omega_4(x) = \int_{f(x)}^{b} \cos\gamma y\cos 3\gamma y\,dy = -\frac{1}{4\gamma}\left\{\sin[2\gamma f(x)] + \frac{1}{2}\sin[4\gamma f(x)]\right\} \tag{B6d}$$

$$\Omega_5(x) = \int_{f(x)}^{b} \cos^2\gamma y\,dy = \frac{1}{2}[b - f(x)] - \frac{1}{4\gamma}\sin[2\gamma f(x)] \tag{B6e}$$

$$\Omega_6(x) = \int_{f(x)}^{b} \cos^2 3\gamma y\,dy = \frac{1}{2}[b - f(x)] - \frac{1}{12\gamma}\sin[6\gamma f(x)] \tag{B6f}$$

$$\Omega_7(x) = \int_{f(x)}^{b} \sin\gamma y\sin 3\gamma y\,dy = -\frac{1}{4\gamma}\left\{\sin[2\gamma f(x)] - \frac{1}{2}\sin[4\gamma f(x)]\right\} \tag{B6g}$$

$$\Omega_8(x) = \int_{f(x)}^{b} \sin^2\gamma y\,dy = \frac{1}{2}[b - f(x)] + \frac{1}{4\gamma}\sin[2\gamma f(x)] \tag{B6h}$$

$$\Omega_9(x) = \int_{f(x)}^{b} \sin^2 3\gamma y\,dy = \frac{1}{2}[b - f(x)] + \frac{1}{12\gamma}\sin[6\gamma f(x)] \tag{B6i}$$

$$\Omega_{10}(x) = \int_{f(x)}^{b} \cos 5\gamma y\,dy = \frac{1}{5\gamma}\{1 - \sin[5\gamma f(x)]\} \tag{B6j}$$

$$\Omega_{11}(x) = \int_{f(x)}^{b} \cos 3\gamma y\cos 5\gamma y\,dy = -\frac{1}{4\gamma}\left\{\sin[2\gamma f(x)] + \frac{1}{4}\sin[8\gamma f(x)]\right\} \tag{B6k}$$

$$\Omega_{12}(x) = \int_{f(x)}^{b} \cos\gamma y\cos 5\gamma y\,dy = -\frac{1}{4\gamma}\left\{\frac{1}{2}\sin[4\gamma f(x)] + \frac{1}{3}\sin[6\gamma f(x)]\right\} \tag{B6l}$$

$$\Omega_{13}(x) = \int_{f(x)}^{b} \cos^2 5\gamma y\,dy = \frac{1}{2}[b - f(x)] - \frac{1}{20\gamma}\sin[10\gamma f(x)] \tag{B6m}$$

$$\Omega_{14}(x) = \int_{f(x)}^{b} \sin 3\gamma y\sin 5\gamma y\,dy = -\frac{1}{4\gamma}\left\{\sin[2\gamma f(x)] - \frac{1}{4}\sin[8\gamma f(x)]\right\} \tag{B6n}$$

$$\Omega_{15}(x) = \int_{f(x)}^{b} \sin\gamma y\sin 5\gamma y\,dy = -\frac{1}{4\gamma}\left\{\frac{1}{2}\sin[4\gamma f(x)] - \frac{1}{3}\sin[6\gamma f(x)]\right\} \tag{B6o}$$

$$\Omega_{16}(x) = \int_{f(x)}^{b} \sin^2 5\gamma y\,dy = \frac{1}{2}[b - f(x)] + \frac{1}{20\gamma}\sin[10\gamma f(x)] \tag{B6p}$$

APPENDIX B

When the loading is a uniform compressive edge displacement, no external work is done to the system. However, when the loading is a uniform compressive edge stress, that is,

$$N_x(\pm c, y) = -\lambda \tilde{N}_x^o \tag{B7}$$

the external work is

$$W_e = -\lambda \tilde{N}_x^o \int_{-b}^{+b} [u^o(x,y)]_{x=-c}^{x=+c}\ dy \tag{B8}$$

The parameter $\tilde{N}_x^o$ is the applied stress shown in figure 2, and λ is the loading parameter previously defined. Substituting the displacements given by equations (B1) into equation (B8) for the external work yields

$$W_e = -\lambda^2 \tilde{N}_x^o \int_{-b}^{+b} [u_0(x) + u_1(x)\cos\gamma y + u_3(x)\cos 3\gamma y + u_5(x)\cos 5\gamma y]_{x=-c}^{x=+c}\ dy \tag{B9}$$

The symmetry of the problem allows the external work to be expressed as

$$W_e = -2\lambda^2 \tilde{N}_x^o \int_{0}^{b} [u_0(x) + u_1(x)\cos\gamma y + u_3(x)\cos 3\gamma y + u_5(x)\cos 5\gamma y]_{x=-c}^{x=+c}\ dy \tag{B10}$$

By noting that

$$\left.\begin{aligned}
\Omega_1(c) &= \int_0^b dy = \Omega_1(-c)\\
\Omega_2(c) &= \int_0^b \cos\gamma y\ dy = \Omega_2(-c)\\
\Omega_3(c) &= \int_0^b \cos 3\gamma y\ dy = \Omega_3(-c)\\
\Omega_{10}(c) &= \int_0^b \cos 5\gamma y\ dy = \Omega_{10}(-c)
\end{aligned}\right\} \tag{B11}$$

the external work can be expressed as

$$W_e = -2\lambda^2 \tilde{N}_x^o [\Omega_1(x)u_0(x) + \Omega_2(x)u_1(x) + \Omega_3(x)u_3(x) + \Omega_{10}(x)u_5(x)]_{x=-c}^{x=+c} \tag{B12}$$

The total potential energy of the system is

$$V = U_m - W_e \tag{B13}$$

which can be expressed as

$$V = \lambda^2 \int_{-c}^{+c} \hat{U}_m\,(u_0, u_0', u_1, u_1', u_3, u_3', u_5, u_5', v_1, v_1', v_3, v_3', v_5, v_5')\ dx + 2\lambda^2 \tilde{N}_x^o (\Omega_1 u_0 + \Omega_2 u_1 + \Omega_3 u_3 + \Omega_{10} u_5)_{x=-c}^{x=+c} \tag{B14}$$

The following ordinary differential equations and boundary conditions are obtained using the Euler-Lagrange equations of variational calculus on equation (B14):

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Differential equations (on $-c \leq x \leq +c$)

$$\left.\begin{aligned}
&\frac{\partial \hat{U}_m}{\partial u_0} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial u_0'}\right) = 0 \qquad && \frac{\partial \hat{U}_m}{\partial v_1} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial v_1'}\right) = 0 \\
&\frac{\partial \hat{U}_m}{\partial u_1} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial u_1'}\right) = 0 && \frac{\partial \hat{U}_m}{\partial v_3} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial v_3'}\right) = 0 \\
&\frac{\partial \hat{U}_m}{\partial u_3} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial u_3'}\right) = 0 && \frac{\partial \hat{U}_m}{\partial v_5} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial v_5'}\right) = 0 \\
&\frac{\partial \hat{U}_m}{\partial u_5} - \frac{d}{dx}\left(\frac{\partial \hat{U}_m}{\partial u_5'}\right) = 0
\end{aligned}\right\} \tag{B15}$$

Boundary conditions (at $x = \pm c$)

$$\left.\begin{aligned}
&\left(\frac{\partial \hat{U}_m}{\partial u_0'} + 2\Omega_1 \tilde{N}_x^o\right)\delta u_0 = 0 \qquad && \left(\frac{\partial \hat{U}_m}{\partial v_1'}\right)\delta v_1 = 0 \\
&\left(\frac{\partial \hat{U}_m}{\partial u_1'} + 2\Omega_2 \tilde{N}_x^o\right)\delta u_1 = 0 && \left(\frac{\partial \hat{U}_m}{\partial v_3'}\right)\delta v_3 = 0 \\
&\left(\frac{\partial \hat{U}_m}{\partial u_3'} + 2\Omega_3 \tilde{N}_x^o\right)\delta u_3 = 0 && \left(\frac{\partial \hat{U}_m}{\partial v_5'}\right)\delta v_5 = 0 \\
&\left(\frac{\partial \hat{U}_m}{\partial u_5'} + 2\Omega_{10} \tilde{N}_x^o\right)\delta u_5 = 0
\end{aligned}\right\} \tag{B16}$$

Substituting the expression for $\hat{U}_m$ given by equation (B5) into equations (B15) and (B16) gives

Differential equations ($-c \leq x \leq +c$)

$$\begin{aligned}
&\frac{d}{dx}\left[\left(A_{11}u_0' + A_{12}v_0\right)\Omega_1 + \left(A_{11}u_1' + A_{12}\gamma v_1\right)\Omega_2 \right.\\
&\qquad \left. + \left(A_{11}u_3' + 3A_{12}\gamma v_3\right)\Omega_3 + \left(A_{11}u_5' + 5A_{12}\gamma v_5\right)\Omega_{10}\right] = 0
\end{aligned} \tag{B17}$$

$$\begin{aligned}
&A_{66}\gamma\left[\left(v_3' - 3\gamma u_3\right)\Omega_7 + \left(v_1' - \gamma u_1\right)\Omega_8 + \left(v_5' - 5\gamma u_5\right)\Omega_{15}\right] \\
&\qquad + \frac{d}{dx}\left[\left(A_{11}u_0' + A_{12}v_0\right)\Omega_2 + \left(A_{11}u_3' + 3A_{12}\gamma v_3\right)\Omega_4 \right.\\
&\qquad \left. + \left(A_{11}u_5' + 5A_{12}\gamma v_5\right)\Omega_{12} + \left(A_{11}u_1' + A_{12}\gamma v_1\right)\Omega_5\right] = 0
\end{aligned} \tag{B18}$$

$$\begin{aligned}
&3A_{66}\gamma\left[\left(v_1' - \gamma u_1\right)\Omega_7 + \left(v_3' - 3\gamma u_3\right)\Omega_9 + \left(v_5' - 5\gamma u_5\right)\Omega_{14}\right] \\
&\qquad + \frac{d}{dx}\left[\left(A_{11}u_0' + A_{12}v_0\right)\Omega_3 + \left(A_{11}u_1' + A_{12}\gamma v_1\right)\Omega_4 \right.\\
&\qquad \left. + \left(A_{11}u_3' + 3A_{12}\gamma v_3\right)\Omega_6 + \left(A_{11}u_5' + 5A_{12}\gamma v_5\right)\Omega_{11}\right] = 0
\end{aligned} \tag{B19}$$

$$\begin{aligned}
&5A_{66}\gamma\left[\left(v_1' - \gamma u_1\right)\Omega_{15} + \left(v_3' - 3\gamma u_3\right)\Omega_{14} + \left(v_5' - 5\gamma u_5\right)\Omega_{16}\right] \\
&\qquad + \frac{d}{dx}\left[\left(A_{11}u_0' + A_{12}v_0\right)\Omega_{10} + \left(A_{11}u_1' + A_{12}\gamma v_1\right)\Omega_{12} \right.\\
&\qquad \left. + \left(A_{11}u_3' + 3A_{12}\gamma v_3\right)\Omega_{11} + \left(A_{11}u_5' + 5A_{12}\gamma v_5\right)\Omega_{13}\right] = 0
\end{aligned} \tag{B20}$$

$$\gamma\left[(A_{12}u_0' + A_{22}v_0)\,\Omega_2 + (A_{12}u_1' + A_{22}\gamma v_1)\,\Omega_5 + (A_{12}u_3' + 3A_{22}\gamma v_3)\,\Omega_4 + (A_{12}u_5' + 5A_{22}\gamma v_5)\,\Omega_{12}\right] - \frac{d}{dx}\left\{A_{66}\left[(v_1' - \gamma u_1)\,\Omega_8 + (v_3' - 3\gamma u_3)\,\Omega_7 + (v_5' - 5\gamma u_5)\,\Omega_{15}\right]\right\} = 0 \tag{B21}$$

$$3\gamma\left[(A_{12}u_0' + A_{22}v_0)\,\Omega_3 + (A_{12}u_1' + A_{22}\gamma v_1)\,\Omega_4 + (A_{12}u_3' + 3A_{22}\gamma v_3)\,\Omega_6 + (A_{12}u_5' + 5A_{22}\gamma v_5)\,\Omega_{11}\right] - \frac{d}{dx}\left\{A_{66}\left[(v_1' - \gamma u_1)\,\Omega_7 + (v_3' - 3\gamma u_3)\,\Omega_9 + (v_5' - 5\gamma u_5)\,\Omega_{14}\right]\right\} = 0 \tag{B22}$$

$$5\gamma\left[(A_{12}u_0' + A_{22}v_0)\,\Omega_{10} + (A_{12}u_1' + A_{22}\gamma v_1)\,\Omega_{12} + (A_{12}u_3' + 3A_{22}\gamma v_3)\,\Omega_{11} + (A_{12}u_5' + 5A_{22}\gamma v_5)\,\Omega_{13}\right] - \frac{d}{dx}\left\{A_{66}\left[(v_1' - \gamma u_1)\,\Omega_{15} + (v_3' - 3\gamma u_3)\,\Omega_{14} + (v_5' - 5\gamma u_5)\,\Omega_{16}\right]\right\} = 0 \tag{B23}$$

Boundary conditions ($x = \pm c$)

$$\left\{\left[(A_{11}u_0' + A_{12}v_0)\,\Omega_1 + (A_{11}u_1' + A_{12}\gamma v_1)\,\Omega_2 + (A_{11}u_3' + 3A_{12}\gamma v_3)\,\Omega_3 + (A_{11}u_5' + 5A_{12}\gamma v_5)\,\Omega_{10} + \Omega_1 \tilde{N}_x^o\right]\,\delta u_0\right\}_{-c}^{+c} = 0 \tag{B24}$$

$$\left\{\left[(A_{11}u_0' + A_{12}v_0)\,\Omega_2 + (A_{11}u_1' + A_{12}\gamma v_1)\,\Omega_5 + (A_{11}u_3' + 3A_{12}\gamma v_3)\,\Omega_4 + (A_{11}u_5' + 5A_{12}\gamma v_5)\,\Omega_{12} + \Omega_2 \tilde{N}_x^o\right]\,\delta u_1\right\}_{-c}^{+c} = 0 \tag{B25}$$

$$\left\{\left[(A_{11}u_0' + A_{12}v_0)\,\Omega_3 + (A_{11}u_1' + A_{12}\gamma v_1)\,\Omega_4 + (A_{11}u_3' + 3A_{12}\gamma v_3)\,\Omega_6 + (A_{11}u_5' + 5A_{12}\gamma v_5)\,\Omega_{11} + \Omega_3 \tilde{N}_x^o\right]\,\delta u_3\right\}_{-c}^{+c} = 0 \tag{B26}$$

$$\left\{\left[(A_{11}u_0' + A_{12}v_0)\,\Omega_{10} + (A_{11}u_1' + A_{12}\gamma v_1)\,\Omega_{12} + (A_{11}u_3' + 3A_{12}\gamma v_3)\,\Omega_{11} + (A_{11}u_5' + 5A_{12}\gamma v_5)\,\Omega_{13} + \Omega_{10} \tilde{N}_x^o\right]\,\delta u_5\right\}_{-c}^{+c} = 0 \tag{B27}$$

$$\{[(v_1' - \gamma u_1)\,\Omega_8 + (v_3' - 3\gamma u_3)\,\Omega_7 + (v_5' - 5\gamma u_5)\,\Omega_{15}]\;\delta v_1\}_{-c}^{+c} = 0 \tag{B28}$$

$$\{[(v_1' - \gamma u_1)\,\Omega_7 + (v_3' - 3\gamma u_3)\,\Omega_9 + (v_5' - 5\gamma u_5)\,\Omega_{14}]\;\delta v_3\}_{-c}^{+c} = 0 \tag{B29}$$

$$\{[(v_1' - \gamma u_1)\,\Omega_{15} + (v_3' - 3\gamma u_3)\,\Omega_{14} + (v_5' - 5\gamma u_5)\,\Omega_{16}]\;\delta v_5\}_{-c}^{+c} = 0 \tag{B30}$$

The boundary conditions for the two loading cases investigated in the present study are given by

Uniform compressive edge displacement

$$u^o(\pm c, y) = \mp\frac{1}{2}\lambda \qquad N_{xy}(\pm c, y) = 0 \tag{B31}$$

Uniform compressive edge stress

$$N_x(\pm c, y) = -\lambda \tilde{N}_x^o \qquad N_{xy}(\pm c, y) = 0 \tag{B32}$$

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Substituting the displacement series (eqs. (B1)) into the displacement boundary conditions given in equations (B31) and collecting like terms gives the following conditions:

$$\left.\begin{aligned} u_0(\pm c) &= \mp\frac{1}{2}\lambda \qquad & u_3(\pm c) &= 0 \\ u_1(\pm c) &= 0 & u_5(\pm c) &= 0 \end{aligned}\right\} \tag{B33}$$

Using the constitutive relations of two-dimensional orthotropic elasticity and the strain-displacement equations given by equations (B2) yields

$$\begin{aligned} N_x = \lambda\, [&(A_{11}u_0' + A_{12}v_0) + (A_{11}u_1' + A_{12}\gamma v_1)\cos\gamma y \\ &+ (A_{11}u_3' + 3A_{12}\gamma v_3)\cos 3\gamma y + (A_{11}u_5' + 5A_{12}\gamma v_5)\cos 5\gamma y] \end{aligned} \tag{B34}$$

$$N_{xy} = \lambda A_{66}\,[(v_1' - \gamma u_1)\sin\gamma y + (v_3' - 3\gamma u_3)\sin 3\gamma y + (v_5' - 5\gamma u_5)\sin 5\gamma y] \tag{B35}$$

Substituting these expressions into equations (B31) and (B32) yields the following conditions for the prescribed normal stress boundary condition in equations (B32),

$$\left.\begin{aligned} A_{11}u_0'(\pm c) + A_{12}v_0 &= -\tilde{N}_x^o \\ A_{11}u_1'(\pm c) + A_{12}\gamma v_1(\pm c) &= 0 \\ A_{11}u_3'(\pm c) + 3A_{12}\gamma v_3(\pm c) &= 0 \\ A_{11}u_5'(\pm c) + 5A_{12}\gamma v_5(\pm c) &= 0 \end{aligned}\right\} \tag{B36}$$

and for the shear stress boundary conditions in equations (B31) and (B32),

$$\left.\begin{aligned} v_1'(\pm c) - \gamma u_1(\pm c) &= 0 \\ v_3'(\pm c) - 3\gamma u_3(\pm c) &= 0 \\ v_5'(\pm c) - 5\gamma u_5(\pm c) &= 0 \end{aligned}\right\} \tag{B37}$$

Inspection of the conditions given by equations (B33), (B36), and (B37) reveals that these conditions are admissible since they satisfy the boundary conditions derived from the variational procedure.

Upon solution of the system of ordinary differential equations for a specific loading case, the stress resultants can be obtained from the following equations:

$$\left.\begin{aligned} N_x &= \lambda\,(N_{x0} + N_{x1}\cos\gamma y + N_{x3}\cos 3\gamma y + N_{x5}\cos 5\gamma y) \\ N_y &= \lambda\,(N_{y0} + N_{y1}\cos\gamma y + N_{y3}\cos 3\gamma y + N_{y5}\cos 5\gamma y) \\ N_{xy} &= \lambda\,(N_{xy1}\sin\gamma y + N_{xy3}\sin 3\gamma y + N_{xy5}\sin 5\gamma y) \end{aligned}\right\} \tag{B38}$$

where

$$\left.\begin{aligned} N_{x0} &= A_{11}u_0' + A_{12}v_0 \\ N_{x1} &= A_{11}u_1' + A_{12}\gamma v_1 \\ N_{x3} &= A_{11}u_3' + 3A_{12}\gamma v_3 \\ N_{x5} &= A_{11}u_5' + 5A_{12}\gamma v_5 \end{aligned}\right\} \tag{B39}$$

$$\left.\begin{aligned} N_{y0} &= A_{12}u_0' + A_{22}v_0 \\ N_{y1} &= A_{12}u_1' + A_{22}\gamma v_1 \\ N_{y3} &= A_{12}u_3' + 3A_{22}\gamma v_3 \\ N_{y5} &= A_{12}u_5' + 5A_{22}\gamma v_5 \end{aligned}\right\} \tag{B40}$$

$$\left.\begin{aligned} N_{xy1} &= A_{66}(v_1' - \gamma u_1) \\ N_{xy3} &= A_{66}(v_3' - 3\gamma u_3) \\ N_{xy5} &= A_{66}(v_5' - 5\gamma u_5) \end{aligned}\right\} \tag{B41}$$

The differential equations given by equations (B17) through (B23) constitute a system of linear second-order homogeneous ordinary differential equations with variable coefficients, and these equations are solved numerically for the generalized displacements and their derivatives. To make use of existing computer subroutines, the second-order system is converted to an equivalent first-order system. The undetermined functions appearing in the displacement series are redefined as follows:

$$\left.\begin{aligned} u_0(x) &= y_1 & v_1(x) &= y_5 \\ u_1(x) &= y_2 & v_3(x) &= y_6 \\ u_3(x) &= y_3 & v_5(x) &= y_7 \\ u_5(x) &= y_4 \end{aligned}\right\} \tag{B42}$$

In addition, the following functions are defined as

$$y_8 = (A_{11}y_1' + A_{12}v_0)\Omega_1 + (A_{11}y_2' + A_{12}\gamma y_5)\Omega_2 + (A_{11}y_3' + 3A_{12}\gamma y_6)\Omega_3 + (A_{11}y_4' + 5A_{12}\gamma y_7)\Omega_{10} \tag{B43}$$

$$y_9 = (A_{11}y_1' + A_{12}v_0)\Omega_2 + (A_{11}y_2' + A_{12}\gamma y_5)\Omega_5 + (A_{11}y_3' + 3A_{12}\gamma y_6)\Omega_4 + (A_{11}y_4' + 5A_{12}\gamma y_7)\Omega_{12} \tag{B44}$$

$$y_{10} = (A_{11}y_1' + A_{12}v_0)\Omega_3 + (A_{11}y_2' + A_{12}\gamma y_5)\Omega_4 + (A_{11}y_3' + 3A_{12}\gamma y_6)\Omega_6 + (A_{11}y_4' + 5A_{12}\gamma y_7)\Omega_{11} \tag{B45}$$

$$y_{11} = (A_{11}y_1' + A_{12}v_0)\Omega_{10} + (A_{11}y_2' + A_{12}\gamma y_5)\Omega_{12} + (A_{11}y_3' + 3A_{12}\gamma y_6)\Omega_{11} + (A_{11}y_4' + 5A_{12}\gamma y_7)\Omega_{13} \tag{B46}$$

$$y_{12} = A_{66}\left[(y_6' - 3\gamma y_3)\,\Omega_7 + (y_7' - 5\gamma y_4)\,\Omega_{15} + (y_5' - \gamma y_2)\,\Omega_8\right] \tag{B47}$$

$$y_{13} = A_{66}\left[(y_5' - \gamma y_2)\,\Omega_7 + (y_6' - 3\gamma y_3)\,\Omega_9 + (y_7' - 5\gamma y_4)\,\Omega_{14}\right] \tag{B48}$$

$$y_{14} = A_{66}\left[(y_5' - \gamma y_2)\,\Omega_{15} + (y_6' - 3\gamma y_3)\,\Omega_{14} + (y_7' - 5\gamma y_4)\,\Omega_{16}\right] \tag{B49}$$

Substituting expressions (B43) through (B49) into the governing differential equations yields

$$y_8' = 0 \tag{B50}$$

$$y_9' = -\gamma y_{12} \tag{B51}$$

$$y_{10}' = -3\gamma y_{13} \tag{B52}$$

$$y_{11}' = -5\gamma y_{14} \tag{B53}$$

$$\begin{aligned} &\gamma\left[(A_{12}y_1' + A_{22}v_0)\,\Omega_2 + (A_{12}y_2' + A_{22}\gamma y_5)\,\Omega_5 \right. \\ &\qquad \left. + (A_{12}y_3' + 3\gamma A_{22}y_6)\,\Omega_4 + (A_{12}y_4' + 5\gamma A_{22}y_7)\,\Omega_{12}\right] - y_{12}' = 0 \end{aligned} \tag{B54}$$

$$\begin{aligned} &3\gamma\left[(A_{12}y_1' + A_{22}v_0)\,\Omega_3 + (A_{12}y_2' + A_{22}\gamma y_5)\,\Omega_4 \right. \\ &\qquad \left. + (A_{12}y_3' + 3A_{22}\gamma y_6)\,\Omega_6 + (A_{12}y_4' + 5A_{22}\gamma y_7)\,\Omega_{11}\right] - y_{13}' = 0 \end{aligned} \tag{B55}$$

$$\begin{aligned} &5\gamma\left[(A_{12}y_1' + A_{22}v_0)\,\Omega_{10} + (A_{12}y_2' + A_{22}\gamma y_5)\,\Omega_{12} \right. \\ &\qquad \left. + (A_{12}y_3' + 3A_{22}\gamma y_6)\,\Omega_{11} + (A_{12}y_4' + 5A_{22}\gamma y_7)\,\Omega_{13}\right] - y_{14}' = 0 \end{aligned} \tag{B56}$$

The equivalent first-order system of differential equations is obtained by solving equations (B43) through (B46) simultaneously for y_1', y_2', y_3', and y_4'; solving equations (B47) through (B49) simultaneously for y_5', y_6', and y_7';

and then substituting these values into equations (B54) through (B56) to obtain y'_{12}, y'_{13}, and y'_{14}. The resulting first-order system is given by

$$y'_1 = -\frac{A_{12}}{A_{11}}v_0 + \frac{\varsigma_1 y_8 + \varsigma_2 y_9 + \varsigma_3 y_{10} + \varsigma_4 y_{11}}{A_{11}\mathrm{Den1}} \tag{B57a}$$

$$y'_2 = -\frac{A_{12}}{A_{11}}\gamma y_5 + \frac{\varsigma_2 y_8 + \varsigma_5 y_9 + \varsigma_6 y_{10} + \varsigma_7 y_{11}}{A_{11}\mathrm{Den1}} \tag{B57b}$$

$$y'_3 = -\frac{3A_{12}}{A_{11}}\gamma y_6 + \frac{\varsigma_3 y_8 + \varsigma_6 y_9 + \varsigma_8 y_{10} + \varsigma_9 y_{11}}{A_{11}\mathrm{Den1}} \tag{B57c}$$

$$y'_4 = -\frac{5A_{12}}{A_{11}}\gamma y_7 + \frac{\varsigma_4 y_8 + \varsigma_7 y_9 + \varsigma_9 y_{10} + \varsigma_{10} y_{11}}{A_{11}\mathrm{Den1}} \tag{B57d}$$

$$y'_5 = \gamma y_2 + \frac{\theta_{15} y_{12} + \theta_{16} y_{13} + \theta_{17} y_{14}}{A_{66}\mathrm{Den2}} \tag{B57e}$$

$$y'_6 = 3\gamma y_3 + \frac{\theta_{16} y_{12} + \theta_{18} y_{13} + \theta_{19} y_{14}}{A_{66}\mathrm{Den2}} \tag{B57f}$$

$$y'_7 = 5\gamma y_4 + \frac{\theta_{17} y_{12} + \theta_{19} y_{13} + \theta_{20} y_{14}}{A_{66}\mathrm{Den2}} \tag{B57g}$$

$$y'_8 = 0 \tag{B57h}$$

$$y'_9 = -\gamma y_{12} \tag{B57i}$$

$$y'_{10} = -3\gamma y_{13} \tag{B57j}$$

$$y'_{11} = -5\gamma y_{14} \tag{B57k}$$

$$y'_{12} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}\gamma\left[\Omega_2 v_0 + \gamma\left(\Omega_5 y_5 + 3\Omega_4 y_6 + 5\Omega_{12} y_7\right)\right] + \frac{A_{12}}{A_{11}}\gamma y_9 \tag{B57l}$$

$$y'_{13} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}3\gamma\left[\Omega_3 v_0 + \gamma\left(\Omega_4 y_5 + 3\Omega_6 y_6 + 5\Omega_{11} y_7\right)\right] + \frac{3A_{12}}{A_{11}}\gamma y_{10} \tag{B57m}$$

$$y'_{14} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}5\gamma\left[\Omega_{10} v_0 + \gamma\left(\Omega_{12} y_5 + 3\Omega_{11} y_6 + 5\Omega_{13} y_7\right)\right] + \frac{5A_{12}}{A_{11}}\gamma y_{11} \tag{B57n}$$

where

$$\left.\begin{aligned}
\varsigma_1 &= \Omega_{11}\theta_7 + \Omega_{12}\theta_8 + \Omega_{13}\theta_1 & \varsigma_6 &= \Omega_{10}\theta_{12} + \Omega_{11}\theta_{13} + \Omega_{13}\theta_5\\
\varsigma_2 &= \Omega_{13}\theta_2 - \Omega_{11}\theta_9 - \Omega_{10}\theta_8 & \varsigma_7 &= -\Omega_{10}\theta_2 - \Omega_{11}\theta_5 - \Omega_{12}\theta_4\\
\varsigma_3 &= \Omega_{13}\theta_3 + \Omega_{12}\theta_9 - \Omega_{10}\theta_7 & \varsigma_8 &= \Omega_{10}\theta_{14} - \Omega_{12}\theta_{13} + \Omega_{13}\theta_6\\
\varsigma_4 &= -\Omega_{10}\theta_1 - \Omega_{11}\theta_3 - \Omega_{12}\theta_2 & \varsigma_9 &= -\Omega_{10}\theta_3 - \Omega_{11}\theta_6 - \Omega_{12}\theta_5\\
\varsigma_5 &= \Omega_{10}\theta_{10} + \Omega_{11}\theta_{11} + \Omega_{13}\theta_4 & \varsigma_{10} &= \Omega_3\theta_3 + \Omega_4\theta_5 + \Omega_6\theta_6
\end{aligned}\right\} \tag{B58}$$

$$\left.\begin{aligned}
\mathrm{Den1} &= \theta_6\theta_{21} + \theta_3\theta_{22} + \theta_5\theta_{23} - \Omega_{11}\left(\theta_2\Omega_{10} + \theta_4\Omega_{12}\right) - \Omega_{10}\left(\theta_2\Omega_{12} + \theta_1\Omega_{10}\right)\\
\mathrm{Den2} &= \theta_{20}\Omega_{16} + \theta_{17}\Omega_{15} + \theta_{19}\Omega_{14}
\end{aligned}\right\} \tag{B59}$$

$$\left.\begin{aligned}
\theta_1 &= \Omega_5\Omega_6 - \Omega_4^2 & \theta_9 &= \Omega_3\Omega_{12} - \Omega_2\Omega_{11} & \theta_{17} &= \Omega_7\Omega_{14} - \Omega_9\Omega_{15}\\
\theta_2 &= \Omega_3\Omega_4 - \Omega_2\Omega_6 & \theta_{10} &= \Omega_3\Omega_{11} - \Omega_6\Omega_{10} & \theta_{18} &= \Omega_8\Omega_{16} - \Omega_{15}^2\\
\theta_3 &= \Omega_2\Omega_4 - \Omega_3\Omega_5 & \theta_{11} &= \Omega_3\Omega_{10} - \Omega_1\Omega_{11} & \theta_{19} &= \Omega_7\Omega_{15} - \Omega_8\Omega_{14}\\
\theta_4 &= \Omega_1\Omega_6 - \Omega_3^2 & \theta_{12} &= \Omega_4\Omega_{10} - \Omega_3\Omega_{12} & \theta_{20} &= \Omega_8\Omega_9 - \Omega_7^2\\
\theta_5 &= \Omega_2\Omega_3 - \Omega_1\Omega_4 & \theta_{13} &= \Omega_1\Omega_{12} - \Omega_2\Omega_{10} & \theta_{21} &= \Omega_6\Omega_{13} - \Omega_{11}^2\\
\theta_6 &= \Omega_1\Omega_5 - \Omega_2^2 & \theta_{14} &= \Omega_2\Omega_{12} - \Omega_5\Omega_{10} & \theta_{22} &= \Omega_3\Omega_{13} - 2\Omega_{10}\Omega_{11}\\
\theta_7 &= \Omega_4\Omega_{12} - \Omega_5\Omega_{11} & \theta_{15} &= \Omega_{16}\Omega_9 - \Omega_{14}^2 & \theta_{23} &= \Omega_4\Omega_{13} - 2\Omega_{11}\Omega_{12}\\
\theta_8 &= \Omega_4\Omega_{11} - \Omega_6\Omega_{12} & \theta_{16} &= \Omega_{14}\Omega_{15} - \Omega_7\Omega_{16} & &
\end{aligned}\right\} \tag{B60}$$

The boundary conditions for the two loading cases are obtained from equations (B33), (B36), and (B37) and are summarized as follows:

Uniform compressive edge displacement

$$y_1(\pm c) = \mp\frac{1}{2} \tag{B61a}$$

$$y_2(\pm c) = 0 \tag{B61b}$$

$$y_3(\pm c) = 0 \tag{B61c}$$

$$y_4(\pm c) = 0 \tag{B61d}$$

$$\theta_{15}(\pm c)y_{12}(\pm c) + \theta_{16}(\pm c)y_{13}(\pm c) + \theta_{17}(\pm c)y_{14}(\pm c) = 0 \tag{B61e}$$

$$\theta_{16}(\pm c)y_{12}(\pm c) + \theta_{18}(\pm c)y_{13}(\pm c) + \theta_{19}(\pm c)y_{14}(\pm c) = 0 \tag{B61f}$$

$$\theta_{17}(\pm c)y_{12}(\pm c) + \theta_{19}(\pm c)y_{13}(\pm c) + \theta_{20}(\pm c)y_{14}(\pm c) = 0 \tag{B61g}$$

Uniform compressive edge stress

$$\varsigma_1(\pm c)y_8(\pm c) + \varsigma_2(\pm c)y_9(\pm c) + \varsigma_3(\pm c)y_{10}(\pm c) + \varsigma_4(\pm c)y_{11}(\pm c) = -\tilde{N}_x^o\text{Den1}(\pm c) \tag{B62a}$$

$$\varsigma_2(\pm c)y_8(\pm c) + \varsigma_5(\pm c)y_9(\pm c) + \varsigma_6(\pm c)y_{10}(\pm c) + \varsigma_7(\pm c)y_{11}(\pm c) = 0 \tag{B62b}$$

$$\varsigma_3(\pm c)y_8(\pm c) + \varsigma_6(\pm c)y_9(\pm c) + \varsigma_8(\pm c)y_{10}(\pm c) + \varsigma_9(\pm c)y_{11}(\pm c) = 0 \tag{B62c}$$

$$\varsigma_4(\pm c)y_8(\pm c) + \varsigma_7(\pm c)y_9(\pm c) + \varsigma_9(\pm c)y_{10}(\pm c) + \varsigma_{10}(\pm c)y_{11}(\pm c) = 0 \tag{B62d}$$

$$\theta_{15}(\pm c)y_{12}(\pm c) + \theta_{16}(\pm c)y_{13}(\pm c) + \theta_{17}(\pm c)y_{14}(\pm c) = 0 \tag{B62e}$$

$$\theta_{16}(\pm c)y_{12}(\pm c) + \theta_{18}(\pm c)y_{13}(\pm c) + \theta_{19}(\pm c)y_{14}(\pm c) = 0 \tag{B62f}$$

$$\theta_{17}(\pm c)y_{12}(\pm c) + \theta_{19}(\pm c)y_{13}(\pm c) + \theta_{20}(\pm c)y_{14}(\pm c) = 0 \tag{B62g}$$

For the uniform compressive stress loading, the differential equations and boundary conditions must be modified to make use of subroutine PASVAR. The modifications are given in appendix E. After numerical solution of the first-order system of equations, the stress resultants are given by equations (B38) and equations (B39) through (B41). In terms of the first-order variables, equations (B39) through (B41) become

$$N_{x0} = \frac{\varsigma_1 y_8 + \varsigma_2 y_9 + \varsigma_3 y_{10} + \varsigma_4 y_{11}}{\text{Den1}} \tag{B63a}$$

$$N_{x1} = \frac{\varsigma_2 y_8 + \varsigma_5 y_9 + \varsigma_6 y_{10} + \varsigma_7 y_{11}}{\text{Den1}} \tag{B63b}$$

$$N_{x3} = \frac{\varsigma_3 y_8 + \varsigma_6 y_9 + \varsigma_8 y_{10} + \varsigma_9 y_{11}}{\text{Den1}} \tag{B63c}$$

$$N_{x5} = \frac{\varsigma_4 y_8 + \varsigma_7 y_9 + \varsigma_9 y_{10} + \varsigma_{10} y_{11}}{\text{Den1}} \tag{B63d}$$

$$N_{y0} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} v_0 + \frac{A_{12}}{A_{11}} N_{x0} \tag{B63e}$$

$$N_{y1} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma y_5 + \frac{A_{12}}{A_{11}} N_{x1} \tag{B63f}$$

$$N_{y3} = 3\frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma y_6 + \frac{A_{12}}{A_{11}} N_{x3} \tag{B63g}$$

$$N_{y5} = 5\frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma y_7 + \frac{A_{12}}{A_{11}} N_{x5} \tag{B63h}$$

$$N_{xy1} = \frac{\theta_{15}y_{12} + \theta_{16}y_{13} + \theta_{17}y_{14}}{\text{Den2}} \tag{B63i}$$

$$N_{xy3} = \frac{\theta_{16}y_{12} + \theta_{18}y_{13} + \theta_{19}y_{14}}{\text{Den2}} \tag{B63j}$$

$$N_{xy5} = \frac{\theta_{17}y_{12} + \theta_{19}y_{13} + \theta_{20}y_{14}}{\text{Den2}} \tag{B63k}$$

Appendix C

Buckling Analysis for $S = 3$

The prebuckling stress resultants used in this analysis are

$$\left.\begin{aligned} N_x(x,y) &= \lambda \left\{ N_{x0}(x) + \sum_{k=1}^{3} N_{x(2k-1)}(x)\cos[(2k-1)\gamma y] \right\} \\ N_y(x,y) &= \lambda \left\{ N_{y0}(x) + \sum_{k=1}^{3} N_{y(2k-1)}(x)\cos[(2k-1)\gamma y] \right\} \\ N_{xy}(x,y) &= \lambda \left\{ \sum_{k=1}^{3} N_{xy(2k-1)}(x)\sin[(2k-1)\gamma y] \right\} \end{aligned}\right\} \tag{C1}$$

where λ is a loading parameter, $\gamma = \pi/2b$, and the functions $N_{x0}(x)$, $N_{x1}(x)$, $N_{x3}(x)$, $N_{x5}(x)$, $N_{y0}(x)$, $N_{y1}(x)$, $N_{y3}(x)$, $N_{y5}(x)$, $N_{xy1}(x)$, $N_{xy3}(x)$, and $N_{xy5}(x)$ are determined from the prebuckling analysis given in appendix B (see eqs. (B33) through (B41)). The series representation used for the out-of-plane buckling displacement is

$$w^o(x,y) = \sum_{j=1}^{3} w_{2j-1}(x)\cos[(2j-1)\gamma y] \tag{C2}$$

where the prescribed trigonometric functions satisfy both the kinematic and natural boundary conditions for a simply supported edge at $y = \pm b$. The initial stress energy (contribution of the prebuckling stress distribution to the second variation of the potential energy) for the plate, which is assumed to deform symmetrically, is

$$U_{IS} = \int_{-c}^{+c} \int_{f(x)}^{b} \left[N_x \left(w^o_{,x}\right)^2 + N_y \left(w^o_{,y}\right)^2 + 2N_{xy}\left(w^o_{,x} w^o_{,y}\right) \right] dy\, dx \tag{C3}$$

where $f(x)$ is the curve shown in figure 1 and $(\ )_{,x}$ and $(\ )_{,y}$ denote partial differentiation with respect to the x- and y-coordinates, respectively. Substituting the series expressions for the stress resultants (eqs. (C1)) and buckling displacement (eq. (C2)) into the initial stress energy (eq. (C3)) and integrating over the y-coordinate yields

$$U_{IS} = \lambda \int_{-c}^{+c} \{w\}^T [G] \{w\}\, dx \tag{C4}$$

where

$$\{w\}^T = \left[w_1(x) w_1'(x) w_1''(x) \ \middle|\ w_3(x) w_3'(x) w_3''(x) \ \middle|\ w_5(x) w_5'(x) w_5''(x) \right] \tag{C5}$$

$$[G] = \begin{bmatrix} G_{11} & G_{12} & 0 & G_{14} & G_{15} & 0 & G_{17} & G_{18} & 0 \\ & G_{22} & 0 & G_{24} & G_{25} & 0 & G_{27} & G_{28} & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & G_{44} & G_{45} & 0 & G_{47} & G_{48} & 0 \\ & & & & G_{55} & 0 & G_{57} & G_{58} & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & G_{77} & G_{78} & 0 \\ & & & & & & & G_{88} & 0 \\ \text{Symmetric} & & & & & & & & 0 \end{bmatrix} \tag{C6}$$

Primes denote differentiation with respect to x. The nonzero expressions appearing in $[G]$ above are given by

$$G_{11} = N_{y0}G_{y0} + N_{y1}G_{y1} + N_{y3}G_{y3} + N_{y5}G_{y5} \tag{C7a}$$
$$G_{12} = N_{xy1}G_{xy1} + N_{xy3}G_{xy3} + N_{xy5}G_{xy5} \tag{C7b}$$
$$G_{14} = N_{y0}P_{y0} + N_{y1}P_{y1} + N_{y3}P_{y3} + N_{y5}P_{y5} \tag{C7c}$$
$$G_{15} = N_{xy1}P_{yx1} + N_{xy3}P_{yx3} + N_{xy5}P_{yx5} \tag{C7d}$$
$$G_{17} = N_{y0}Q_{y0} + N_{y1}Q_{y1} + N_{y3}Q_{y3} + N_{y5}Q_{y5} \tag{C7e}$$
$$G_{18} = N_{xy1}Q_{yx1} + N_{xy3}Q_{yx3} + N_{xy5}Q_{yx5} \tag{C7f}$$
$$G_{22} = N_{x0}G_{x0} + N_{x1}G_{x1} + N_{x3}G_{x3} + N_{x5}G_{x5} \tag{C7g}$$
$$G_{24} = N_{xy1}P_{xy1} + N_{xy3}P_{xy3} + N_{xy5}P_{xy5} \tag{C7h}$$
$$G_{25} = N_{x0}P_{x0} + N_{x1}P_{x1} + N_{x3}P_{x3} + N_{x5}P_{x5} \tag{C7i}$$
$$G_{27} = N_{xy1}Q_{xy1} + N_{xy3}Q_{xy3} + N_{xy5}Q_{xy5} \tag{C7j}$$
$$G_{28} = N_{x0}Q_{x0} + N_{x1}Q_{x1} + N_{x3}Q_{x3} + N_{x5}Q_{x5} \tag{C7k}$$
$$G_{44} = N_{y0}H_{y0} + N_{y1}H_{y1} + N_{y3}H_{y3} + N_{y5}H_{y5} \tag{C7l}$$
$$G_{45} = N_{xy1}H_{xy1} + N_{xy3}H_{xy3} + N_{xy5}H_{xy5} \tag{C7m}$$
$$G_{47} = N_{y0}R_{y0} + N_{y1}R_{y1} + N_{y3}R_{y3} + N_{y5}R_{y5} \tag{C7n}$$
$$G_{48} = N_{xy1}R_{yx1} + N_{xy3}R_{yx3} + N_{xy5}R_{yx5} \tag{C7o}$$
$$G_{55} = N_{x0}H_{x0} + N_{x1}H_{x1} + N_{x3}H_{x3} + N_{x5}H_{x5} \tag{C7p}$$
$$G_{57} = N_{xy1}R_{xy1} + N_{xy3}R_{xy3} + N_{xy5}R_{xy5} \tag{C7q}$$
$$G_{58} = N_{x0}R_{x0} + N_{x1}R_{x1} + N_{x3}R_{x3} + N_{x5}R_{x5} \tag{C7r}$$
$$G_{77} = N_{y0}F_{y0} + N_{y1}F_{y1} + N_{y3}F_{y3} + N_{y5}F_{y5} \tag{C7s}$$
$$G_{78} = N_{xy1}F_{xy1} + N_{xy3}F_{xy3} + N_{xy5}F_{xy5} \tag{C7t}$$
$$G_{88} = N_{x0}F_{x0} + N_{x1}F_{x1} + N_{x3}F_{x3} + N_{x5}F_{x5} \tag{C7u}$$

where

$$G_{x0} = \int_{f(x)}^{b} \cos^2 \gamma y \, dy = \frac{1}{2}[b - f(x)] - \frac{1}{4\gamma} \sin[2\gamma f(x)] \tag{C8a}$$
$$G_{x1} = \int_{f(x)}^{b} \cos^3 \gamma y \, dy = \frac{1}{3\gamma} \left(2 - \sin[\gamma f(x)] \left\{2 + \cos^2[\gamma f(x)]\right\}\right) \tag{C8b}$$
$$G_{x3} = \int_{f(x)}^{b} \cos^2 \gamma y \cos 3\gamma y \, dy = \frac{1}{15\gamma} \left(2 - \sin[\gamma f(x)] \left\{2 + cos^2[\gamma f(x)] + 12\cos^4[\gamma f(x)]\right\}\right) \tag{C8c}$$
$$G_{x5} = \int_{f(x)}^{b} \cos^2 \gamma y \cos 5\gamma y \, dy = \frac{-1}{420\gamma} \left\{8 + 35\sin[3\gamma f(x)] + 42\sin[5\gamma f(x)] + 15\sin[7\gamma f(x)]\right\} \tag{C8d}$$

$$G_{y0} = \int_{f(x)}^{b} \gamma^2 \sin^2 \gamma y \, dy = \frac{\gamma^2}{2} \left\{[b - f(x)] + \frac{1}{2\gamma} \sin[2\gamma f(x)]\right\} \tag{C9a}$$
$$G_{y1} = \int_{f(x)}^{b} \gamma^2 \sin^2 \gamma y \cos \gamma y \, dy = \frac{\gamma}{3} \left\{1 - \sin^3[\gamma f(x)]\right\} \tag{C9b}$$
$$G_{y3} = \int_{f(x)}^{b} \gamma^2 \sin^2 \gamma y \cos 3\gamma y \, dy = \frac{\gamma}{15} \left(7 - \sin^3[\gamma f(x)] \left\{7 + 12\cos^2[\gamma f(x)]\right\}\right) \tag{C9c}$$

$$G_{y5} = \int_{f(x)}^{b} \gamma^2 \sin^2 \gamma y \cos 5\gamma y \; dy = \frac{1}{420} \{92 + 35 \sin [3\gamma f(x)] - 42 \sin [5\gamma f(x)] + 15 \sin [7\gamma f(x)]\} \qquad \text{(C9d)}$$

$$G_{xy1} = -\int_{f(x)}^{b} \gamma \cos \gamma y \sin^2 \gamma y \; dy = \frac{1}{3} \{\sin^3 [\gamma f(x)] - 1\} \qquad \text{(C10a)}$$

$$G_{xy3} = -\int_{f(x)}^{b} \gamma \cos \gamma y \sin \gamma y \sin 3\gamma y \; dy = \frac{1}{4} \{\sin [\gamma f(x)] - 1\} - \frac{1}{20} \{\sin [5\gamma f(x)] - 1\} \qquad \text{(C10b)}$$

$$G_{xy5} = -\int_{f(x)}^{b} \gamma \cos \gamma y \sin \gamma y \sin 5\gamma y \; dy = \frac{1}{84} \{4 + 7 \sin [3\gamma f(x)] - 3 \sin [7\gamma f(x)]\} \qquad \text{(C10c)}$$

$$H_{x0} = \int_{f(x)}^{b} \cos^2 3\gamma y \; dy = \frac{1}{2}[b - f(x)] - \frac{1}{12\gamma} \sin[6\gamma f(x)] \qquad \text{(C11a)}$$

$$H_{x1} = \int_{f(x)}^{b} \cos \gamma y \cos^2 3\gamma y \; dy = \frac{1}{140\gamma}\{72 - 5 \sin[7\gamma f(x)] - 7 \sin[5\gamma f(x)] - 70 \sin[\gamma f(x)]\} \qquad \text{(C11b)}$$

$$H_{x3} = \int_{f(x)}^{b} \cos^3 3\gamma y \; dy = \frac{1}{9\gamma}(\sin[3\gamma f(x)]\{\sin^2[3\gamma f(x)] - 3\} - 2) \qquad \text{(C11c)}$$

$$H_{x5} = \int_{f(x)}^{b} \cos^2 3\gamma y \cos 5\gamma y \; dy = \frac{1}{220\gamma}\{72 - 55 \sin[\gamma f(x)] - 22 \sin[5\gamma f(x)] - 5 \sin[11\gamma f(x)]\} \qquad \text{(C11d)}$$

$$H_{y0} = \int_{f(x)}^{b} 9\gamma^2 \sin^2 3\gamma y \; dy = 9\gamma^2 \left\{\frac{1}{2}[b - f(x)] + \frac{1}{12\gamma} \sin[6\gamma f(x)]\right\} \qquad \text{(C12a)}$$

$$H_{y1} = \int_{f(x)}^{b} 9\gamma^2 \sin^2 3\gamma y \cos \gamma y \; dy = \frac{9}{560\gamma}\{68 + 5 \sin[7\gamma f(x)] + 7 \sin[5\gamma f(x)] - 70 \sin[\gamma f(x)]\} \qquad \text{(C12b)}$$

$$H_{y3} = \int_{f(x)}^{b} 9\gamma^2 \sin^2 3\gamma y \cos 3\gamma y \; dy = -\gamma\{1 + \sin^3[3\gamma f(x)]\} \qquad \text{(C12c)}$$

$$H_{y5} = \int_{f(x)}^{b} 9\gamma^2 \sin^2 3\gamma y \cos 5\gamma y \; dy = \frac{-9\gamma}{220}\{28 - 55 \sin[\gamma f(x)] + 22 \sin[5\gamma f(x)] - 5 \sin[11\gamma f(x)]\} \qquad \text{(C12d)}$$

$$H_{xy1} = -\int_{f(x)}^{b} 3\gamma \sin \gamma y \sin 3\gamma y \cos 3\gamma y \; dy = \frac{-3}{35}\left(3 + \frac{1}{4}\{5 \sin[7\gamma f(x)] - 7 \sin[5\gamma f(x)]\}\right) \qquad \text{(C13a)}$$

$$H_{xy3} = -\int_{f(x)}^{b} 3\gamma \sin^2 3\gamma y \cos 3\gamma \; dy = \frac{1}{3}\{1 + \sin^3[3\gamma f(x)]\} \qquad \text{(C13b)}$$

$$H_{xy5} = -\int_{f(x)}^{b} 3\gamma \sin 3\gamma y \cos 3\gamma y \sin 5\gamma y \; dy = \frac{-3}{44}\{12 - 11 \sin[\gamma f(x)] + \sin[11\gamma f(x)]\} \qquad \text{(C13c)}$$

$$F_{x0} = \int_{f(x)}^{b} \cos^2 5\gamma y \; dy = \frac{1}{2}[b - f(x)] - \frac{1}{20\gamma} \sin[10\gamma f(x)] \qquad \text{(C14a)}$$

$$F_{x1} = \int_{f(x)}^{b} \cos \gamma y \cos^2 5\gamma y \; dy = \frac{1}{396\gamma}\{300 - 198 \sin[\gamma f(x)] - 11 \sin[9\gamma f(x)] - 9 \sin[11\gamma f(x)]\} \qquad \text{(C14b)}$$

$$F_{x3} = \int_{f(x)}^{b} \cos 3\gamma y \cos^2 5\gamma y \; dy = \frac{-1}{1092\gamma}\{200 + 182 \sin[3\gamma f(x)] - 39 \sin[7\gamma f(x)] - 21 \sin[13\gamma f(x)]\} \qquad \text{(C14c)}$$

$$F_{x5} = \int_{f(x)}^{b} \cos^3 5\gamma y \, dy = \frac{1}{15\gamma} \{2 - 3\sin[5\gamma f(x)] + \sin^3[5\gamma f(x)]\} \qquad \text{(C14d)}$$

$$F_{y0} = \int_{f(x)}^{b} 25\gamma^2 \sin^2 5\gamma y \, dy = \frac{25\gamma^2}{2} \left\{ b - f(x) + \frac{1}{10\gamma} \sin[10\gamma f(x)] \right\} \qquad \text{(C15a)}$$

$$F_{y1} = \int_{f(x)}^{b} 25\gamma^2 \cos\gamma y \sin^2 5\gamma y \, dy = \frac{25\gamma}{396} \{294 - 198\sin[\gamma f(x)] + 11\sin[9\gamma f(x)] + 9\sin[11\gamma f(x)]\} \qquad \text{(C15b)}$$

$$F_{y3} = \int_{f(x)}^{b} 25\gamma^2 \cos 3\gamma y \sin^2 5\gamma y \, dy = \frac{25\gamma}{1092} \{-164 - 182\sin[3\gamma f(x)] + 39\sin[7\gamma f(x)] + 2\sin[13\gamma f(x)]\} \qquad \text{(C15c)}$$

$$F_{y5} = \int_{f(x)}^{b} 25\gamma^2 \sin^2 5\gamma y \cos 5\gamma y \, dy = \frac{5\gamma}{3} \{1 - \sin^3[5\gamma f(x)]\} \qquad \text{(C15d)}$$

$$F_{xy1} = -\int_{f(x)}^{b} 5\gamma \sin\gamma y \sin 5\gamma y \cos 5\gamma y \, dy = \frac{-5}{396} \{30 - 11\sin[9\gamma f(x)] + 9\sin[11\gamma f(x)]\} \qquad \text{(C16a)}$$

$$F_{xy3} = -\int_{f(x)}^{b} 5\gamma \sin 3\gamma y \sin 5\gamma y \cos 5\gamma y \, dy = \frac{5}{364} \{20 + 13\sin[7\gamma f(x)] - 7\sin[13\gamma f(x)]\} \qquad \text{(C16b)}$$

$$F_{xy5} = -\int_{f(x)}^{b} 5\gamma \sin^2 5\gamma y \cos 5\gamma y \, dy = \frac{1}{3} \{\sin^3[5\gamma f(x)] - 1\} \qquad \text{(C16c)}$$

$$P_{x0} = \int_{f(x)}^{b} \cos\gamma y \cos 3\gamma y \, dy = \frac{-1}{4\gamma} \left\{ \sin[2\gamma f(x)] + \frac{1}{2}\sin[4\gamma f(x)] \right\} \qquad \text{(C17a)}$$

$$P_{x1} = \int_{f(x)}^{b} \cos^2\gamma y \cos 3\gamma y \, dy = \frac{2}{15\gamma} \left(1 - \frac{1}{8}\{3\sin[5\gamma f(x)] + 10\sin[3\gamma f(x)] + 15\sin[\gamma f(x)]\} \right) \qquad \text{(C17b)}$$

$$P_{x3} = \int_{f(x)}^{b} \cos\gamma y \cos^2 3\gamma y \, dy = \frac{1}{70\gamma} \left(36 - \frac{1}{2}\{5\sin[7\gamma f(x)] + 7\sin[5\gamma f(x)] + 70\sin[\gamma f(x)]\} \right) \qquad \text{(C17c)}$$

$$P_{x5} = \int_{f(x)}^{b} \cos\gamma y \cos 3\gamma y \cos 5\gamma y \, dy = \frac{1}{252\gamma} \{40 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] - 7\sin[9\gamma f(x)]\} \qquad \text{(C17d)}$$

$$P_{y0} = \int_{f(x)}^{b} 3\gamma^2 \sin\gamma y \sin 3\gamma y \, dy = \frac{-3\gamma}{4} \left\{ \sin[2\gamma f(x)] - \frac{1}{2}\sin[4\gamma f(x)] \right\} \qquad \text{(C18a)}$$

$$P_{y1} = \int_{f(x)}^{b} 3\gamma^2 \cos\gamma y \sin\gamma y \sin 3\gamma y \, dy = \frac{3\gamma}{5} \left(1 + \frac{1}{4}\{\sin[5\gamma f(x)] - 5\sin[\gamma f(x)]\} \right) \qquad \text{(C18b)}$$

$$P_{y3} = \int_{f(x)}^{b} 3\gamma^2 \sin\gamma y \sin 3\gamma y \cos 3\gamma y \, dy = \frac{3\gamma}{35} \left(3 + \frac{1}{4}\{5\sin[7\gamma f(x)] - 7\sin[5\gamma f(x)]\} \right) \qquad \text{(C18c)}$$

$$P_{y5} = \int_{f(x)}^{b} 3\gamma^2 \sin\gamma y \sin 3\gamma y \cos 5\gamma y \, dy = \frac{\gamma}{84} \{-100 + 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \qquad \text{(C18d)}$$

$$P_{xy1} = -\int_{f(x)}^{b} 3\gamma \cos\gamma y \sin\gamma y \sin 3\gamma y \, dy = -\frac{3}{5}\left(1 + \frac{1}{4}\{\sin[5\gamma f(x)] - 5\sin[\gamma f(x)]\}\right) \quad \text{(C19a)}$$

$$P_{xy3} = -\int_{f(x)}^{b} 3\gamma \cos\gamma y \sin^2 3\gamma y \, dy = -\frac{3}{35}\left(17 + \frac{1}{4}\{5\sin[7\gamma f(x)] + 7\sin[5\gamma f(x)] - 70\sin[\gamma f(x)]\}\right) \quad \text{(C19b)}$$

$$P_{xy5} = -\int_{f(x)}^{b} 3\gamma \cos\gamma y \sin 3\gamma y \sin 5\gamma y \, dy$$
$$= -\frac{1}{84}\{44 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] + 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C19c)}$$

$$P_{yx1} = -\int_{f(x)}^{b} \gamma \sin^2\gamma y \cos 3\gamma y \, dy = \frac{1}{15}\left(7 - \frac{1}{4}\{3\sin[5\gamma f(x)] - 10\sin[3\gamma f(x)] + 15\sin[\gamma f(x)]\}\right) \quad \text{(C20a)}$$

$$P_{yx3} = -\int_{f(x)}^{b} \gamma \sin\gamma y \sin 3\gamma y \cos 3\gamma y \, dy = -\frac{1}{35}\left(3 + \frac{1}{4}\{5\sin[7\gamma f(x)] - 7\sin[5\gamma f(x)]\}\right) \quad \text{(C20b)}$$

$$P_{yx5} = -\int_{f(x)}^{b} \gamma \sin\gamma y \cos 3\gamma y \sin 5\gamma y \, dy$$
$$= \frac{-1}{252}\{68 - 63\sin[\gamma f(x)] + 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C20c)}$$

$$Q_{x0} = \int_{f(x)}^{b} \cos\gamma y \cos 5\gamma y \, dy = -\frac{1}{24\gamma}\{3\sin[4\gamma f(x)] + 2\sin[6\gamma f(x)]\} \quad \text{(C21a)}$$

$$Q_{x1} = \int_{f(x)}^{b} \cos^2\gamma y \cos 5\gamma y \, dy = \frac{-1}{420\gamma}\{8 + 35\sin[3\gamma f(x)] + 42\sin[5\gamma f(x)] + 15\sin[7\gamma f(x)]\} \quad \text{(C21b)}$$

$$Q_{x3} = \int_{f(x)}^{b} \cos\gamma y \cos 3\gamma y \cos 5\gamma y \, dy$$
$$= \frac{1}{252\gamma}\{40 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] - 7\sin[9\gamma f(x)]\} \quad \text{(C21c)}$$

$$Q_{x5} = \int_{f(x)}^{b} \cos\gamma y \cos^2 5\gamma y \, dy = \frac{1}{396\gamma}\{200 - 198\sin[\gamma f(x)] - 11\sin[9\gamma f(x)] - 9\sin[11\gamma f(x)]\} \quad \text{(C21d)}$$

$$Q_{y0} = \int_{f(x)}^{b} 5\gamma^2 \sin\gamma y \sin 5\gamma y \, dy = \frac{5\gamma}{24}\{2\sin[6\gamma f(x)] - 3\sin[4\gamma f(x)]\} \quad \text{(C22a)}$$

$$Q_{y1} = \int_{f(x)}^{b} 5\gamma^2 \sin\gamma y \cos\gamma y \sin 5\gamma y \, dy = -\frac{5\gamma}{84}\{4 + 7\sin[3\gamma f(x)] - 3\sin[7\gamma f(x)]\} \quad \text{(C22b)}$$

$$Q_{y3} = \int_{f(x)}^{b} 5\gamma^2 \sin\gamma y \cos 3\gamma y \sin 5\gamma y \, dy$$
$$= \frac{5\gamma}{252}\{68 - 63\sin[\gamma f(x)] + 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C22c)}$$

$$Q_{y5} = \int_{f(x)}^{b} 5\gamma^2 \sin\gamma y \sin 5\gamma y \cos 5\gamma y \, dy = \frac{5\gamma}{396}\{20 - 11\sin[9\gamma f(x)] + 9\sin[11\gamma f(x)]\} \quad \text{(C22d)}$$

$$Q_{xy1} = -\int_{f(x)}^{b} 5\gamma \cos\gamma y \sin\gamma y \sin 5\gamma y \, dy = \frac{5}{84}\{4 + 7\sin[3\gamma f(x)] - 3\sin[7\gamma f(x)]\} \quad \text{(C23a)}$$

$$Q_{xy3} = -\int_{f(x)}^{b} 5\gamma \cos\gamma y \sin 3\gamma y \sin 5\gamma y \; dy$$

$$= \frac{-5}{252}\{44 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] + 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C23b)}$$

$$Q_{xy5} = -\int_{f(x)}^{b} 5\gamma \cos\gamma y \sin^2 5\gamma y \; dy = \frac{-5}{396}\{196 - 198\sin[\gamma f(x)] + 11\sin[9\gamma f(x)] + 9\sin[11\gamma f(x)]\} \quad \text{(C23c)}$$

$$Q_{yx1} = -\int_{f(x)}^{b} \gamma \sin^2\gamma y \cos 5\gamma y \; dy = \frac{-1}{420}\{92 + 35\sin[3\gamma f(x)] - 42\sin[5\gamma f(x)] + 15\sin[7\gamma f(x)]\} \quad \text{(C24a)}$$

$$Q_{yx3} = -\int_{f(x)}^{b} \gamma \sin\gamma y \sin 3\gamma y \cos 5\gamma y \; dy$$

$$= \frac{1}{252}\{100 - 63\sin[\gamma f(x)] + 21\sin[3\gamma f(x)] + 9\sin[7\gamma f(x)] - 7\sin[9\gamma f(x)]\} \quad \text{(C24b)}$$

$$Q_{yx5} = -\int_{f(x)}^{b} \gamma \sin\gamma y \sin 5\gamma y \cos 5\gamma y \; dy = \frac{-1}{396}\{20 - 11\sin[9\gamma f(x)] + 9\sin[11\gamma f(x)]\} \quad \text{(C24c)}$$

$$R_{x0} = \int_{f(x)}^{b} \cos 3\gamma y \cos 5\gamma y \; dy = \frac{-1}{16\gamma}\{4\sin[2\gamma f(x)] + \sin[8\gamma f(x)]\} \quad \text{(C25a)}$$

$$R_{x1} = \int_{f(x)}^{b} \cos\gamma y \cos 3\gamma y \cos 5\gamma y \; dy$$

$$= \frac{1}{252\gamma}\{40 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] - 7\sin[9\gamma f(x)]\} \quad \text{(C25b)}$$

$$R_{x3} = \int_{f(x)}^{b} \cos^2 3\gamma y \cos 5\gamma y \; dy$$

$$= \frac{1}{220\gamma}\{72 - 55\sin[\gamma f(x)] - 22\sin[5\gamma f(x)] - 5\sin[11\gamma f(x)]\} \quad \text{(C25c)}$$

$$R_{x5} = \int_{f(x)}^{b} \cos 3\gamma y \cos^2 5\gamma y \; dy = \frac{-1}{1092\gamma}\{200 + 182\sin[3\gamma f(x)] + 39\sin[7\gamma f(x)] + 21\sin[13\gamma f(x)]\} \quad \text{(C25d)}$$

$$R_{y0} = \int_{f(x)}^{b} 15\gamma^2 \sin 3\gamma y \sin 5\gamma y \; dy = \frac{15\gamma}{16}\{\sin[8\gamma f(x)] - 4\sin[2\gamma f(x)]\} \quad \text{(C26a)}$$

$$R_{y1} = \int_{f(x)}^{b} 15\gamma^2 \cos\gamma y \sin 3\gamma y \sin 5\gamma y \; dy$$

$$= \frac{5\gamma}{84}\{44 - 63\sin[\gamma f(x)] - 21\sin[3\gamma f(x)] + 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C26b)}$$

$$R_{y3} = \int_{f(x)}^{b} 15\gamma^2 \sin 3\gamma y \cos 3\gamma y \sin 5\gamma y \; dy = \frac{15\gamma}{44}\{12 - 11\sin[\gamma f(x)] + \sin[11\gamma f(x)]\} \quad \text{(C26c)}$$

$$R_{y5} = \int_{f(x)}^{b} 15\gamma^2 \sin 3\gamma y \sin 5\gamma y \cos 5\gamma y \; dy = \frac{-15\gamma}{364}\{20 + 13\sin[7\gamma f(x)] - 7\sin[13\gamma f(x)]\} \quad \text{(C26d)}$$

$$R_{xy1} = -\int_{f(x)}^{b} 5\gamma \sin\gamma y \cos 3\gamma y \sin 5\gamma y \; dy$$

$$= \frac{-5}{252}\{68 - 63\sin[\gamma f(x)] + 21\sin[3\gamma f(x)] - 9\sin[7\gamma f(x)] + 7\sin[9\gamma f(x)]\} \quad \text{(C27a)}$$

$$R_{xy3} = -\int_{f(x)}^{b} 5\gamma \sin 3\gamma y \cos 3\gamma y \sin 5\gamma y \; dy = \frac{-5}{44}\{12 \quad 11 \sin[\gamma f(x)] + \sin[11\gamma f(x)]\} \tag{C27b}$$

$$R_{xy5} = -\int_{f(x)}^{b} 5\gamma \cos 3\gamma y \sin^2 5\gamma y \; dy = \frac{5}{1092}\{164 + 182 \sin[3\gamma f(x)] - 39 \sin[7\gamma f(x)] - 21 \sin[13\gamma f(x)]\} \tag{C27c}$$

$$R_{yx1} = -\int_{f(x)}^{b} 3\gamma \sin \gamma y \sin 3\gamma y \cos 5\gamma y \; dy = \frac{1}{84}\{25 - 63 \sin[\gamma f(x)] + 21 \sin[3\gamma f(x)] + 9 \sin[7\gamma f9x)] - 7 \sin[9\gamma f(x)]\} \tag{C28a}$$

$$R_{yx3} = -\int_{f(x)}^{b} 3\gamma \sin^2 3\gamma y \cos 5\gamma y \; dy = \frac{3}{220}\{28 - 55 \sin[\gamma f(x)] + 22 \sin[5\gamma f(x)] - 5 \sin[11\gamma f(x)]\} \tag{C28b}$$

$$R_{yx5} = -\int_{f(x)}^{b} 3\gamma \sin 3\gamma y \sin 5\gamma y \cos 5\gamma y \; dy = \frac{3}{364}\{20 + 13 \sin[7\gamma f(x)] - 7 \sin[13\gamma f(x)]\} \tag{C28c}$$

The strain energy of the plate due to the out-of-plane bending action is

$$U_B = \int_{-c}^{+c} \int_{f(x)}^{b} \left[D_{11}\left(w^o_{,xx}\right)^2 + D_{22}\left(w^o_{,yy}\right)^2 + 2D_{12} w^o_{,xx} w^o_{,yy} + 4D_{66}\left(w^o_{,xy}\right)^2\right] dy \, dx \tag{C29}$$

Substituting the buckling displacement series (eq. (C2)) into this expression and integrating over y yields

$$U_B = \int_{-c}^{+c} \{w\}^T [C] \{w\} \, dx \tag{C30}$$

where

$$[C] = \begin{bmatrix} C_{11} & 0 & C_{13} & C_{14} & 0 & C_{16} & C_{17} & 0 & C_{19} \\ & C_{22} & 0 & 0 & C_{25} & 0 & 0 & C_{28} & 0 \\ & & C_{33} & C_{34} & 0 & C_{36} & C_{37} & 0 & C_{39} \\ & & & C_{44} & 0 & C_{46} & C_{47} & 0 & C_{49} \\ & & & & C_{55} & 0 & 0 & C_{58} & 0 \\ & & & & & C_{66} & C_{67} & 0 & C_{69} \\ & & & & & & C_{77} & 0 & C_{79} \\ & & & & & & & C_{88} & 0 \\ \text{Symmetric} & & & & & & & & C_{99} \end{bmatrix} \tag{C31}$$

The nonzero expressions appearing in $[C]$ are given by

$$\left.\begin{array}{lll} C_{11} = D_{22} G_{B1} & C_{33} = D_{11} G_{x0} & C_{55} = 4D_{66} H_{y0} \\ C_{13} = D_{12} G_{B2} & C_{34} = D_{12} P_{B12} & C_{58} = 4D_{66} R_{y0} \\ C_{14} = D_{22} P_{B1} & C_{36} = D_{11} P_{x0} & C_{66} = D_{11} H_{x0} \\ C_{16} = D_{12} P_{B21} & C_{37} = D_{12} Q_{B12} & C_{67} = D_{12} R_{B12} \\ C_{17} = D_{22} Q_{B1} & C_{39} = D_{11} Q_{x0} & C_{69} = D_{11} R_{x0} \\ C_{19} = D_{12} Q_{B21} & C_{44} = D_{22} H_{B1} & C_{77} = D_{22} F_{B1} \\ C_{22} = 4D_{66} G_{y0} & C_{46} = D_{12} H_{B2} & C_{79} = D_{12} F_{B2} \\ C_{25} = 4D_{66} P_{y0} & C_{47} = D_{22} R_{B1} & C_{88} = 4D_{66} F_{y0} \\ C_{28} = 4D_{66} Q_{y0} & C_{49} = D_{12} R_{B21} & C_{99} = D_{11} F_{x0} \end{array}\right\} \tag{C32}$$

where

$$G_{B1} = \int_{f(x)}^{b} \gamma^4 \cos^2 \gamma y \, dy = \frac{\gamma^4}{2} \left\{ [b - f(x)] - \frac{1}{2\gamma} \sin[2\gamma f(x)] \right\} \tag{C33a}$$

$$G_{B2} = -\int_{f(x)}^{b} \gamma^2 \cos^2 \gamma y \, dy = \frac{-\gamma^2}{2} \left\{ [b - f(x)] - \frac{1}{2\gamma} \sin[2\gamma f(x)] \right\} \tag{C33b}$$

$$H_{B1} = \int_{f(x)}^{b} 81\gamma^4 \cos^2 3\gamma y \, dy = \frac{81\gamma^4}{2} \left\{ [b - f(x)] - \frac{1}{6\gamma} \sin[6\gamma f(x)] \right\} \tag{C33c}$$

$$H_{B2} = -\int_{f(x)}^{b} 9\gamma^2 \cos^2 3\gamma y \, dy = \frac{-9\gamma^2}{2} \left\{ [b - f(x)] - \frac{1}{6\gamma} \sin[6\gamma f(x)] \right\} \tag{C33d}$$

$$F_{B1} = \int_{f(x)}^{b} 625\gamma^4 \cos^2 5\gamma y \, dy = \frac{625\gamma^4}{2} \left\{ b - f(x) - \frac{1}{10\gamma} \sin[10\gamma f(x)] \right\} \tag{C33e}$$

$$F_{B2} = -\int_{f(x)}^{b} 25\gamma^2 \cos^2 5\gamma y \, dy = \frac{-25\gamma^2}{2} \left\{ b - f(x) - \frac{1}{10\gamma} \sin[10\gamma f(x)] \right\} \tag{C33f}$$

$$P_{B1} = \int_{f(x)}^{b} 9\gamma^4 \cos \gamma y \cos 3\gamma y \, dy = \frac{-9\gamma^3}{8} \{ \sin[4\gamma f(x)] + 2\sin[2\gamma f(x)] \} \tag{C33g}$$

$$P_{B12} = -\int_{f(x)}^{b} 9\gamma^2 \cos \gamma y \cos 3\gamma y \, dy = \frac{9\gamma}{8} \{ \sin[4\gamma f(x)] + 2\sin[2\gamma f(x)] \} \tag{C33h}$$

$$P_{B21} = -\int_{f(x)}^{b} \gamma^2 \cos \gamma y \cos 3\gamma y \, dy = \frac{\gamma}{8} \{ \sin[4\gamma f(x)] + 2\sin[2\gamma f(x)] \} \tag{C33i}$$

$$R_{B1} = \int_{f(x)}^{b} 225\gamma^4 \cos 3\gamma y \cos 5\gamma y \, dy = \frac{-225\gamma^3}{16} \{ 4\sin[2\gamma f(x)] + \sin[8\gamma f(x)] \} \tag{C33j}$$

$$R_{B12} = -\int_{f(x)}^{b} 25\gamma^2 \cos 3\gamma y \cos 5\gamma y \, dy = \frac{25\gamma}{16} \{ 4\sin[2\gamma f(x)] + \sin[8\gamma f(x)] \} \tag{C33k}$$

$$R_{B21} = -\int_{f(x)}^{b} 9\gamma^2 \cos 3\gamma y \cos 5\gamma y \, dy = \frac{9\gamma}{16} \{ 4\sin[2\gamma f(x)] + \sin[8\gamma f(x)] \} \tag{C33l}$$

$$Q_{B1} = \int_{f(x)}^{b} 25\gamma^4 \cos \gamma y \cos 5\gamma y \, dy = \frac{-25\gamma^3}{24} \{ 3\sin[4\gamma f(x)] + 2\sin[6\gamma f(x)] \} \tag{C33m}$$

$$Q_{B12} = -\int_{f(x)}^{b} 25\gamma^2 \cos \gamma y \cos 5\gamma y \, dy = \frac{25\gamma}{24} \{ 3\sin[4\gamma f(x)] + 2\sin[6\gamma f(x)] \} \tag{C33n}$$

$$Q_{B21} = -\int_{f(x)}^{b} \gamma^2 \cos \gamma y \cos 5\gamma y \, dy = \frac{\gamma}{24} \{ 3\sin[4\gamma f(x)] + 2\sin[6\gamma f(x)] \} \tag{C33o}$$

As in reference 1, the stability problem can be expressed by

$$\delta(U_{IS} + U_B) = 0 \tag{C34}$$

This variational statement can be expressed as a functional containing the unknown functions in the buckling displacement and their derivatives, that is,

$$\delta(U_{IS} + U_B) = \delta \int_{-c}^{+c} F(w_1, w_1', w_1'', w_3, w_3', w_3'', w_5, w_5', w_5'') \, dx = 0 \tag{C35}$$

APPENDIX C

The following ordinary differential equations and boundary conditions are obtained using the Euler-Lagrange equations of variational calculus on equation (C35):

Differential equations (on $-c \le x \le c$)

$$\left.\begin{aligned} \frac{\partial F}{\partial w_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_1'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_1''}\right) = 0 \\ \frac{\partial F}{\partial w_3} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_3'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_3''}\right) = 0 \\ \frac{\partial F}{\partial w_5} - \frac{d}{dx}\left(\frac{\partial F}{\partial w_5'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial w_5''}\right) = 0 \end{aligned}\right\} \tag{C36}$$

Boundary conditions (at $x = \pm c$)

$$\left.\begin{aligned} \left(\frac{\partial F}{\partial w_1'} - \frac{d}{dx}\frac{\partial F}{\partial w_1''}\right)\delta w_1 = 0 \qquad \left(\frac{\partial F}{\partial w_1''}\right)\delta w_1' = 0 \\ \left(\frac{\partial F}{\partial w_3'} - \frac{d}{dx}\frac{\partial F}{\partial w_3''}\right)\delta w_3 = 0 \qquad \left(\frac{\partial F}{\partial w_3''}\right)\delta w_3' = 0 \\ \left(\frac{\partial F}{\partial w_5'} - \frac{d}{dx}\frac{\partial F}{\partial w_5''}\right)\delta w_5 = 0 \qquad \left(\frac{\partial F}{\partial w_5''}\right)\delta w_5' = 0 \end{aligned}\right\} \tag{C37}$$

Alternately, the system of ordinary differential equations can be written in operator form as

$$\begin{bmatrix} L_{11}(\cdot) & L_{13}(\cdot) & L_{15}(\cdot) \\ L_{31}(\cdot) & L_{33}(\cdot) & L_{35}(\cdot) \\ L_{51}(\cdot) & L_{53}(\cdot) & L_{55}(\cdot) \end{bmatrix} \begin{Bmatrix} w_1(x) \\ w_3(x) \\ w_5(x) \end{Bmatrix} = \lambda \begin{bmatrix} H_{11}(\cdot) & H_{13}(\cdot) & H_{15}(\cdot) \\ H_{31}(\cdot) & H_{33}(\cdot) & H_{35}(\cdot) \\ H_{51}(\cdot) & H_{53}(\cdot) & H_{55}(\cdot) \end{bmatrix} \begin{Bmatrix} w_1(x) \\ w_3(x) \\ w_5(x) \end{Bmatrix} \tag{C38}$$

where

$$\left.\begin{aligned} L_{11}(\cdot) &= [C_{13}(\cdot) + C_{33}(\cdot)'']'' - [C_{22}(\cdot)']' + [C_{11}(\cdot) + C_{13}(\cdot)''] \\ L_{13}(\cdot) &= [C_{34}(\cdot) + C_{36}(\cdot)'']'' - [C_{25}(\cdot)']' + [C_{14}(\cdot) + C_{16}(\cdot)''] \\ L_{15}(\cdot) &= [C_{37}(\cdot) + C_{39}(\cdot)'']'' - [C_{28}(\cdot)']' + [C_{17}(\cdot) + C_{19}(\cdot)''] \\ L_{31}(\cdot) &= [C_{16}(\cdot) + C_{36}(\cdot)'']'' - [C_{25}(\cdot)']' + [C_{14}(\cdot) + C_{34}(\cdot)''] \\ L_{33}(\cdot) &= [C_{46}(\cdot) + C_{66}(\cdot)'']'' - [C_{55}(\cdot)']' + [C_{44}(\cdot) + C_{46}(\cdot)''] \\ L_{35}(\cdot) &= [C_{67}(\cdot) + C_{69}(\cdot)'']'' - [C_{58}(\cdot)']' + [C_{47}(\cdot) + C_{49}(\cdot)''] \\ L_{51}(\cdot) &= [C_{19}(\cdot) + C_{39}(\cdot)'']'' - [C_{28}(\cdot)']' + [C_{17}(\cdot) + C_{37}(\cdot)''] \\ L_{53}(\cdot) &= [C_{49}(\cdot) + C_{69}(\cdot)'']'' - [C_{58}(\cdot)']' + [C_{47}(\cdot) + C_{67}(\cdot)''] \\ L_{55}(\cdot) &= [C_{79}(\cdot) + C_{99}(\cdot)'']'' - [C_{88}(\cdot)']' + [C_{77}(\cdot) + C_{79}(\cdot)''] \end{aligned}\right\} \tag{C39}$$

and

$$\left.\begin{aligned} H_{11}(\cdot) &= [G_{12}(\cdot) + G_{22}(\cdot)']' - [G_{11}(\cdot) + G_{12}(\cdot)'] \\ H_{13}(\cdot) &= [G_{24}(\cdot) + G_{25}(\cdot)']' - [G_{14}(\cdot) + G_{15}(\cdot)'] \\ H_{15}(\cdot) &= [G_{27}(\cdot) + G_{28}(\cdot)']' - [G_{17}(\cdot) + G_{18}(\cdot)'] \\ H_{31}(\cdot) &= [G_{15}(\cdot) + G_{25}(\cdot)']' - [G_{14}(\cdot) + G_{24}(\cdot)'] \\ H_{33}(\cdot) &= [G_{45}(\cdot) + G_{55}(\cdot)']' - [G_{44}(\cdot) + G_{45}(\cdot)'] \\ H_{35}(\cdot) &= [G_{57}(\cdot) + G_{58}(\cdot)']' - [G_{47}(\cdot) + G_{48}(\cdot)'] \\ H_{51}(\cdot) &= [G_{18}(\cdot) + G_{28}(\cdot)']' - [G_{17}(\cdot) + G_{27}(\cdot)'] \\ H_{53}(\cdot) &= [G_{48}(\cdot) + G_{58}(\cdot)']' - [G_{47}(\cdot) + G_{57}(\cdot)'] \\ H_{55}(\cdot) &= [G_{78}(\cdot) + G_{88}(\cdot)']' - [G_{77}(\cdot) + G_{78}(\cdot)'] \end{aligned}\right\} \tag{C40}$$

APPENDIX C

The specific boundary conditions considered in this study are given in terms of the unknown functions as follows:

Simply supported edges at $x = \pm c$

$$\left.\begin{array}{ll} w_1(\pm c) = 0 & w_1''(\pm c) = 0 \\ w_3(\pm c) = 0 & w_3''(\pm c) = 0 \\ w_5(\pm c) = 0 & w_5''(\pm c) = 0 \end{array}\right\} \tag{C41}$$

Clamped edges at $x = \pm c$

$$\left.\begin{array}{ll} w_1(\pm c) = 0 & w_1'(\pm c) = 0 \\ w_3(\pm c) = 0 & w_3'(\pm c) = 0 \\ w_5(\pm c) = 0 & w_5'(\pm c) = 0 \end{array}\right\} \tag{C42}$$

Appendix D

Finite Difference Equations for $S = 3$

For the case of $S = 3$, the differential equations of the buckling problem are

$$\sum_{j=1}^{3} \{L_{2i-1,2j-1}\left[w_{2j-1}(x)\right] - \lambda H_{2i-1,2j-1}\left[w_{2j-1}(x)\right]\} = 0 \qquad (i = 1, 2, \ldots, S) \tag{D1}$$

Expressions for the differential operators are given by equations (C39) and (C40). Applying the difference expressions for the operators given by equations (6) and (7) to equations (C39) and (C40) yields the following finite difference equations (superscript i refers to the ith difference station):

$$\left.\begin{aligned}
[L_{11}(w_1)]^i &= E_1^i w_1^{i-2} + E_2^i w_1^{i-1} + E_3^i w_1^i + E_4^i w_1^{i+1} + E_5^i w_1^{i+2} \\
[L_{13}(w_3)]^i &= E_6^i w_3^{i-2} + E_7^i w_3^{i-1} + E_8^i w_3^i + E_9^i w_3^{i+1} + E_{10}^i w_3^{i+2} \\
[L_{15}(w_5)]^i &= E_{11}^i w_5^{i-2} + E_{12}^i w_5^{i-1} + E_{13}^i w_5^i + E_{14}^i w_5^{i+1} + E_{15}^i w_5^{i+2} \\
[L_{31}(w_1)]^i &= F_1^i w_1^{i-2} + F_2^i w_1^{i-1} + F_3^i w_1^i + F_4^i w_1^{i+1} + F_5^i w_1^{i+2} \\
[L_{33}(w_3)]^i &= F_6^i w_3^{i-2} + F_7^i w_3^{i-1} + F_8^i w_3^i + F_9^i w_3^{i+1} + F_{10}^i w_3^{i+2} \\
[L_{35}(w_5)]^i &= F_{11}^i w_5^{i-2} + F_{12}^i w_5^{i-1} + F_{13}^i w_5^i + F_{14}^i w_5^{i+1} + F_{15}^i w_5^{i+2} \\
[L_{51}(w_1)]^i &= P_1^i w_1^{i-2} + P_2^i w_1^{i-1} + P_3^i w_1^i + P_4^i w_1^{i+1} + P_5^i w_1^{i+2} \\
[L_{53}(w_3)]^i &= P_6^i w_3^{i-2} + P_7^i w_3^{i-1} + P_8^i w_3^i + P_9^i w_3^{i+1} + P_{10}^i w_3^{i+2} \\
[L_{55}(w_5)]^i &= P_{11}^i w_5^{i-2} + P_{12}^i w_5^{i-1} + P_{13}^i w_5^i + P_{14}^i w_5^{i+1} + P_{15}^i w_5^{i+2}
\end{aligned}\right\} \tag{D2}$$

and

$$\left.\begin{aligned}
[H_{11}(w_1)]^i &= D_2^i w_1^{i-1} + D_3^i w_1^i + D_4^i w_1^{i+1} \\
[H_{13}(w_3)]^i &= D_7^i w_3^{i-1} + D_8^i w_3^i + D_9^i w_3^{i+1} \\
[H_{15}(w_5)]^i &= D_{12}^i w_5^{i-1} + D_{13}^i w_5^i + D_{14}^i w_5^{i+1} \\
[H_{31}(w_1)]^i &= G_2^i w_1^{i-1} + G_3^i w_1^i + G_4^i w_1^{i+1} \\
[H_{33}(w_3)]^i &= G_7^i w_3^{i-1} + G_8^i w_3^i + G_9^i w_3^{i+1} \\
[H_{35}(w_5)]^i &= G_{12}^i w_5^{i-1} + G_{13}^i w_5^i + G_{14}^i w_5^{i+1} \\
[H_{51}(w_1)]^i &= H_2^i w_1^{i-1} + H_3^i w_1^i + H_4^i w_1^{i+1} \\
[H_{53}(w_3)]^i &= H_7^i w_3^{i-1} + H_8^i w_3^i + H_9^i w_3^{i+1} \\
[H_{55}(w_5)]^i &= H_{12}^i w_5^{i-1} + H_{13}^i w_5^i + H_{14}^i w_5^{i+1}
\end{aligned}\right\} \tag{D3}$$

where

$$\left.\begin{aligned}
E_1^i &= C_{33}^{i-1}/\Delta^4 \\
E_2^i &= \left(C_{13}^i + C_{13}^{i-1} - C_{22}^{i-1/2}\right)/\Delta^2 - 2\left(C_{33}^i + C_{33}^{i-1}\right)/\Delta^4 \\
E_3^i &= C_{11}^i - \left(4C_{13}^i - C_{22}^{i+1/2} - C_{22}^{i-1/2}\right)/\Delta^2 + \left(C_{33}^{i-1} + 4C_{33}^i + C_{33}^{i+1}\right)/\Delta^4 \\
E_4^i &= \left(C_{13}^i + C_{13}^{i+1} - C_{22}^{i+1/2}\right)/\Delta^2 - 2\left(C_{33}^i + C_{33}^{i+1}\right)/\Delta^4 \\
E_5^i &= C_{33}^{i+1}/\Delta^4
\end{aligned}\right\} \tag{D4}$$

$$\left.\begin{aligned}
E_6^i &= C_{36}^{i-1}/\Delta^4 \\
E_7^i &= \left(C_{16}^i + C_{34}^{i-1} - C_{25}^{i-1/2}\right)/\Delta^2 - 2\left(C_{36}^i \quad C_{36}^{i-1}\right)/\Delta^4 \\
E_8^i &= C_{14}^i - \left(2C_{16}^i + 2C_{34}^i - C_{25}^{i+1/2} - C_{25}^{i-1/2}\right)/\Delta^2 + \left(C_{36}^{i+1} + 4C_{36}^i + C_{36}^{i-1}\right)/\Delta^4 \\
E_9^i &= \left(C_{16}^i + C_{34}^{i+1} - C_{25}^{i+1/2}\right)/\Delta^2 - 2\left(C_{36}^i + C_{36}^{i+1}\right)/\Delta^4 \\
E_{10}^i &= C_{36}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D5)}$$

$$\left.\begin{aligned}
E_{11}^i &= C_{39}^{i-1}/\Delta^4 \\
E_{12}^i &= \left(C_{19}^i + C_{37}^{i-1} - C_{28}^{i-1/2}\right)/\Delta^2 - 2\left(C_{39}^i + C_{39}^{i-1}\right)/\Delta^4 \\
E_{13}^i &= C_{17}^i - \left(2C_{19}^i + 2C_{37}^i - C_{28}^{i-1/2} - C_{28}^{i+1/2}\right)/\Delta^2 + \left(C_{39}^{i+1} + 4C_{39}^i + C_{39}^{i-1}\right)/\Delta^4 \\
E_{14}^i &= \left(C_{19}^i + C_{37}^{i+1} - C_{28}^{i+1/2}\right)/\Delta^2 - 2\left(C_{39}^i + C_{39}^{i+1}\right)/\Delta^4 \\
E_{15}^i &= C_{39}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D6)}$$

$$\left.\begin{aligned}
F_1^i &= C_{36}^{i-1}/\Delta^4 \\
F_2^i &= \left(C_{34}^i + C_{16}^{i-1} - C_{25}^{i-1/2}\right)/\Delta^2 - 2\left(C_{36}^i + C_{36}^{i-1}\right)/\Delta^4 \\
F_3^i &= C_{14}^i - \left(2C_{34}^i + 2C_{16}^i - C_{25}^{i-1/2} - C_{25}^{i+1/2}\right)/\Delta^2 + \left(C_{36}^{i+1} + 4C_{36}^i + C_{36}^{i-1}\right)/\Delta^4 \\
F_4^i &= \left(C_{34}^i + C_{16}^{i+1} - C_{25}^{i+1/2}\right)/\Delta^2 - 2\left(C_{36}^i + C_{36}^{i+1}\right)/\Delta^4 \\
F_5^i &= C_{36}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D7)}$$

$$\left.\begin{aligned}
F_6^i &= C_{66}^{i-1}/\Delta^4 \\
F_7^i &= \left(C_{46}^i + C_{46}^{i-1} - C_{55}^{i-1/2}\right)/\Delta^2 - 2\left(C_{66}^i + C_{66}^{i-1}\right)/\Delta^4 \\
F_8^i &= C_{44}^i - \left(4C_{46}^i - C_{55}^{i+1/2} - C_{55}^{i-1/2}\right)/\Delta^2 + \left(C_{66}^{i+1} + 4C_{66}^i + C_{66}^{i-1}\right)/\Delta^4 \\
F_9^i &= \left(C_{46}^i + C_{46}^{i+1} - C_{55}^{i+1/2}\right)/\Delta^2 - 2\left(C_{66}^i + C_{66}^{i+1}\right)/\Delta^4 \\
F_{10}^i &= C_{66}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D8)}$$

$$\left.\begin{aligned}
F_{11}^i &= C_{69}^{i-1}/\Delta^4 \\
F_{12}^i &= \left(C_{49}^i + C_{67}^{i-1} - C_{58}^{i-1/2}\right)/\Delta^2 - 2\left(C_{69}^i + C_{69}^{i-1}\right)/\Delta^4 \\
F_{13}^i &= C_{47}^i - \left(2C_{49}^i + 2C_{67}^i - C_{58}^{i-1/2} - C_{58}^{i+1/2}\right)/\Delta^2 + \left(C_{69}^{i+1} + 4C_{69}^i + C_{69}^{i-1}\right)/\Delta^4 \\
F_{14}^i &= \left(C_{49}^i + C_{67}^{i+1} - C_{58}^{i+1/2}\right)/\Delta^2 - 2\left(C_{69}^i + C_{69}^{i+1}\right)/\Delta^4 \\
F_{15}^i &= C_{69}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D9)}$$

$$\left.\begin{aligned}
P_1^i &= C_{39}^{i-1}/\Delta^4 \\
P_2^i &= \left(C_{37}^i + C_{19}^{i-1} - C_{28}^{i-1/2}\right)/\Delta^2 - 2\left(C_{39}^i + C_{39}^{i-1}\right)/\Delta^4 \\
P_3^i &= C_{17}^i - \left(2C_{37}^i + 2C_{19}^i - C_{28}^{i+1/2} - C_{28}^{i-1/2}\right)/\Delta^2 + \left(C_{39}^{i+1} + 4C_{39}^i + C_{39}^{i-1}\right)/\Delta^4 \\
P_4^i &= \left(C_{37}^i + C_{19}^{i+1} - C_{28}^{i+1/2}\right)/\Delta^2 - 2\left(C_{39}^i + C_{39}^{i+1}\right)/\Delta^4 \\
P_5^i &= C_{39}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D10)}$$

$$\left.\begin{aligned}
P_6^i &= C_{69}^{i-1}/\Delta^4\\
P_7^i &= \left(C_{67}^i + C_{49}^{i-1} - C_{58}^{i-1/2}\right)/\Delta^2 - 2\left(C_{69}^i + C_{69}^{i-1}\right)/\Delta^4\\
P_8^i &= C_{47}^i - \left(2C_{67}^i + 2C_{49}^i - C_{58}^{i+1/2} - C_{58}^{i-1/2}\right)/\Delta^2 + \left(C_{69}^{i+1} + 4C_{69}^i + C_{69}^{i-1}\right)/\Delta^4\\
P_9^i &= \left(C_{67}^i + C_{49}^{i+1} - C_{58}^{i+1/2}\right)/\Delta^2 - 2\left(C_{69}^i + C_{69}^{i+1}\right)/\Delta^4\\
P_{10}^i &= C_{69}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D11)}$$

$$\left.\begin{aligned}
P_{11}^i &= C_{99}^{i-1}/\Delta^4\\
P_{12}^i &= \left(C_{79}^i + C_{79}^{i-1} - C_{88}^{i-1/2}\right)/\Delta^2 - 2\left(C_{99}^i + C_{99}^{i-1}\right)/\Delta^4\\
P_{13}^i &= C_{77}^i - \left(4C_{79}^i - C_{88}^{i-1/2} - C_{88}^{i+1/2}\right)/\Delta^2 + \left(C_{99}^{i+1} + 4C_{99}^i + C_{99}^{i-1}\right)/\Delta^4\\
P_{14}^i &= \left(C_{79}^i + C_{79}^{i+1} - C_{88}^{i+1/2}\right)/\Delta^2 - 2\left(C_{99}^i + C_{99}^{i+1}\right)/\Delta^4\\
P_{15}^i &= C_{99}^{i+1}/\Delta^4
\end{aligned}\right\} \qquad \text{(D12)}$$

$$\left.\begin{aligned}
D_2^i &= G_{22}^{i-1/2}/\Delta^2\\
D_3^i &= -G_{11}^i + \left(G_{12}^{i+1/2} - G_{12}^{i-1/2}\right)/\Delta - \left(G_{22}^{i+1/2} + G_{22}^{i-1/2}\right)/\Delta^2\\
D_4^i &= G_{22}^{i+1/2}/\Delta^2
\end{aligned}\right\} \qquad \text{(D13)}$$

$$\left.\begin{aligned}
D_7^i &= G_{25}^{i-1/2}/\Delta^2 - \left(G_{24}^{i-1} - G_{15}^i\right)/2\Delta\\
D_8^i &= -G_{14}^i - \left(G_{25}^{i+1/2} + G_{25}^{i-1/2}\right)/\Delta^2\\
D_9^i &= G_{25}^{i+1/2}/\Delta^2 + \left(G_{24}^{i+1} - G_{15}^i\right)/2\Delta
\end{aligned}\right\} \qquad \text{(D14)}$$

$$\left.\begin{aligned}
D_{12}^i &= G_{28}^{i-1/2}/\Delta^2 - \left(G_{27}^{i-1} - G_{18}^i\right)/2\Delta\\
D_{13}^i &= -G_{17}^i - \left(G_{28}^{i+1/2} + G_{28}^{i-1/2}\right)/\Delta^2\\
D_{14}^i &= G_{28}^{i+1/2}/\Delta^2 + \left(G_{27}^{i+1} - G_{18}^i\right)/2\Delta
\end{aligned}\right\} \qquad \text{(D15)}$$

$$\left.\begin{aligned}
G_2^i &= G_{25}^{i-1/2}/\Delta^2 - \left(G_{15}^{i-1} - G_{24}^i\right)/2\Delta\\
G_3^i &= -G_{14}^i - \left(G_{25}^{i+1/2} + G_{25}^{i-1/2}\right)/\Delta^2\\
G_4^i &= G_{25}^{i+1/2}/\Delta^2 + \left(G_{15}^{i+1} - G_{24}^i\right)/2\Delta
\end{aligned}\right\} \qquad \text{(D16)}$$

$$\left.\begin{aligned}
G_7^i &= G_{55}^{i-1/2}/\Delta^2\\
G_8^i &= -G_{44}^i + \left(G_{45}^{i+1/2} - G_{45}^{i-1/2}\right)/\Delta - \left(G_{55}^{i+1/2} + G_{55}^{i-1/2}\right)/\Delta^2\\
G_9^i &= G_{55}^{i+1/2}/\Delta^2
\end{aligned}\right\} \qquad \text{(D17)}$$

$$\left.\begin{aligned}
G_{12}^i &= G_{58}^{i-1/2}/\Delta^2 - \left(G_{57}^{i-1} - G_{48}^i\right)/2\Delta\\
G_{13}^i &= -G_{47}^i - \left(G_{58}^{i+1/2} + G_{58}^{i-1/2}\right)/\Delta^2\\
G_{14} &= G_{58}^{i+1/2}/\Delta^2 + \left(G_{57}^{i+1} - G_{48}^i\right)/2\Delta
\end{aligned}\right\} \qquad \text{(D18)}$$

$$\left.\begin{aligned} H_2^i &= G_{28}^{i-1/2}/\Delta^2 - \left(G_{18}^{i-1} - G_{27}^i\right)/2\Delta \\ H_3^i &= -G_{17}^i - \left(G_{28}^{i+1/2} + G_{28}^{i-1/2}\right)/\Delta^2 \\ H_4^i &= G_{28}^{i+1/2}/\Delta^2 + \left(G_{18}^{i+1} - G_{27}^i\right)/2\Delta \end{aligned}\right\} \tag{D19}$$

$$\left.\begin{aligned} H_7^i &= G_{58}^{i-1/2}/\Delta^2 - \left(G_{48}^{i-1} - G_{57}^i\right)/2\Delta \\ H_8^i &= -G_{47}^i - \left(G_{58}^{i+1/2} + G_{58}^{i-1/2}\right)/\Delta^2 \\ H_9^i &= G_{58}^{i+1/2}/\Delta^2 + \left(G_{48}^{i+1} - G_{57}^i\right)/2\Delta \end{aligned}\right\} \tag{D20}$$

$$\left.\begin{aligned} H_{12}^i &= G_{88}^{i-1/2}/\Delta^2 \\ H_{13}^i &= -G_{77}^i + \left(G_{78}^{i+1/2} - G_{78}^{i-1/2}\right)/\Delta - \left(G_{88}^{i+1/2} + G_{88}^{i-1/2}\right)/\Delta^2 \\ H_{14}^i &= G_{88}^{i+1/2}/\Delta^2 \end{aligned}\right\} \tag{D21}$$

The C_{pq} and G_{pq} terms appearing in equations (D4) through (D21) are defined in appendix C by equations (C7) and (C32).

Applying the difference expressions for the differential operators given in equations (D2) and (D3) to the governing differential equations at each finite difference station on the interior of the one-dimensional domain results in a set of homogeneous algebraic equations given by

$$[\tilde{K}]\{w\} = \lambda[\tilde{G}]\{w\} \tag{D22}$$

The matrices $[\tilde{K}]$ and $[\tilde{G}]$ are of dimension $3(M-2)$ by $3(M+2)$, where M is the number of finite difference stations. To reduce the system of equations further and render them symmetric, the boundary conditions are applied. The boundary conditions of interest in this study are given in appendix C by equations (C41) and (C42). The finite difference expressions for the boundary conditions are obtained directly from the central difference expressions and are summarized as follows:

Simply supported edges at $x = \pm c$

$$\left.\begin{aligned} &w_1^1 = w_1^M = 0 \qquad w_1^0 = -w_1^2 \qquad w_1^{M+1} = w_1^{M-1} \\ &w_3^1 = w_3^M = 0 \qquad w_3^0 = -w_3^2 \qquad w_3^{M+1} = w_3^{M-1} \\ &w_5^1 = w_5^M = 0 \qquad w_5^0 = -w_5^2 \qquad w_5^{M+1} = w_5^{M-1} \end{aligned}\right\} \tag{D23}$$

Clamped edges at $x = \pm c$

$$\left.\begin{aligned} &w_1^1 = w_1^M = 0 \qquad w_1^0 = w_1^2 \qquad w_1^{M+1} = w_1^{M-1} \\ &w_3^1 = w_3^M = 0 \qquad w_3^0 = w_3^2 \qquad w_3^{M+1} = w_3^{M-1} \\ &w_5^1 = w_5^M = 0 \qquad w_5^0 = w_5^2 \qquad w_5^{M+1} = w_5^{M-1} \end{aligned}\right\} \tag{D24}$$

In the boundary conditions given by equations (D24), the first derivative is defined at full stations instead of at half stations to simplify implementation into the computer program.

APPENDIX D

After applying the boundary conditions, the resulting system of equations constitutes a real, symmetric generalized eigenvalue problem given by

$$\left[\begin{array}{c|c|c} K_{11} & K_{13} & K_{15} \\ \hline K_{31} & K_{33} & K_{35} \\ \hline K_{51} & K_{53} & K_{55} \end{array}\right] \left\{\begin{array}{c} \tilde{w}_1 \\ \hline \tilde{w}_3 \\ \hline \tilde{w}_5 \end{array}\right\} = \lambda \left[\begin{array}{c|c|c} g_{11} & g_{13} & g_{15} \\ \hline g_{31} & g_{33} & g_{35} \\ \hline g_{51} & g_{53} & g_{55} \end{array}\right] \left\{\begin{array}{c} \tilde{w}_1 \\ \hline \tilde{w}_3 \\ \hline \tilde{w}_5 \end{array}\right\} \tag{D25}$$

where

$$\left.\begin{array}{l} \tilde{w}_1^T = \left\{w_1^2 \; w_1^3 \; \cdots \; w_1^{M-1}\right\} \\ \tilde{w}_3^T = \left\{w_3^2 \; w_3^3 \; \cdots \; w_3^{M-1}\right\} \\ \tilde{w}_5^T = \left\{w_5^2 \; w_5^3 \; \cdots \; w_5^{M-1}\right\} \end{array}\right\} \tag{D26}$$

Because of the self-adjoint nature of the differential operators, it follows that

$$\left.\begin{array}{ll} K_{31} = K_{13}^T & g_{31} = g_{13}^T \\ K_{51} = K_{15}^T & g_{51} = g_{15}^T \\ K_{53} = K_{35}^T & g_{53} = g_{35}^T \end{array}\right\} \tag{D27}$$

Furthermore, as a result of the bending energy being positive definite, the matrix

$$\left[\begin{array}{c|c|c} K_{11} & K_{13} & K_{15} \\ \hline & K_{33} & K_{35} \\ \cline{2-3} \text{Symmetric} & & K_{55} \end{array}\right] \tag{D28}$$

is positive definite. The submatrices K_{ij} (after assembly of the difference equations) are described in the following matrix equations. In each of these matrices, the $\pm$ symbol refers to the boundary condition on the loaded edges: the plus sign corresponds to the loaded edges being clamped, and the minus sign corresponds to the loaded edges being simply supported.

$$K_{11} = \begin{bmatrix} (E_3^2 \pm E_1^2) & E_4^2 & E_5^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & E_3^3 & E_4^3 & E_5^3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & E_3^4 & E_4^4 & E_5^4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & E_3^5 & E_4^5 & E_5^5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & E_3^{M-5} & E_4^{M-5} & E_5^{M-5} & 0 & 0 \\ & & & & & & & & & E_3^{M-4} & E_4^{M-4} & E_5^{M-4} & 0 \\ & & & & & & & & & & E_3^{M-3} & E_4^{M-3} & E_5^{M-3} \\ & & & & & & & & & & & E_3^{M-2} & E_4^{M-2} \\ \text{Symmetric} & & & & & & & & & & & & (E_3^{M-1} \pm E_5^{M-1}) \end{bmatrix} \tag{D29}$$

$$
K_{13} = \begin{bmatrix}
(E_8^2 \pm E_6^2) & E_9^2 & E_{10}^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_7^3 & E_8^3 & E_9^3 & E_{10}^3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_6^4 & E_7^4 & E_8^4 & E_9^4 & E_{10}^4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_6^5 & E_7^5 & E_8^5 & E_9^5 & E_{10}^5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & E_6^{M-5} & E_7^{M-5} & E_8^{M-5} & E_9^{M-5} & E_{10}^{M-5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & E_6^{M-4} & E_7^{M-4} & E_8^{M-4} & E_9^{M-4} & E_{10}^{M-4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & E_6^{M-3} & E_7^{M-3} & E_8^{M-3} & E_9^{M-3} & E_{10}^{M-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & E_6^{M-2} & E_7^{M-2} & E_8^{M-2} & E_9^{M-2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & E_6^{M-1} & E_7^{M-1} & (E_8^{M-1} \pm E_{10}^{M-1})
\end{bmatrix} \qquad \text{(D30)}
$$

$$
K_{15} = \begin{bmatrix}
(E_{13}^2 \pm E_{11}^2) & E_{14}^2 & E_{15}^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{12}^3 & E_{13}^3 & E_{14}^3 & E_{15}^3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{11}^4 & E_{12}^4 & E_{13}^4 & E_{14}^4 & E_{15}^4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E_{11}^5 & E_{12}^5 & E_{13}^5 & E_{14}^5 & E_{15}^5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & E_{11}^{M-5} & E_{12}^{M-5} & E_{13}^{M-5} & E_{14}^{M-5} & E_{15}^{M-5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & E_{11}^{M-4} & E_{12}^{M-4} & E_{13}^{M-4} & E_{14}^{M-4} & E_{15}^{M-4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & E_{11}^{M-3} & E_{12}^{M-3} & E_{13}^{M-3} & E_{14}^{M-3} & E_{15}^{M-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & E_{11}^{M-2} & E_{12}^{M-2} & E_{13}^{M-2} & E_{14}^{M-2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & E_{11}^{M-1} & E_{12}^{M-1} & (E_{13}^{M-1} \pm E_{15}^{M-1})
\end{bmatrix} \qquad \text{(D31)}
$$

$$
K_{33} = \begin{bmatrix}
(F_8^2 \pm F_6^2) & F_9^2 & F_{10}^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & F_8^3 & F_9^3 & F_{10}^3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & F_8^4 & F_9^4 & F_{10}^4 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & F_8^5 & F_9^5 & F_{10}^5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & F_8^{M-5} & F_9^{M-5} & F_{10}^{M-5} & 0 & 0 \\
 & & & & & & & & & F_8^{M-4} & F_9^{M-4} & F_{10}^{M-4} & 0 \\
 & & & & & & & & & & F_8^{M-3} & F_9^{M-3} & F_{10}^{M-3} \\
 & & & & & & & & & & & F_8^{M-2} & F_9^{M-2} \\
\text{Symmetric} & & & & & & & & & & & & (F_8^{M-1} \pm F_{10}^{M-1})
\end{bmatrix} \qquad \text{(D32)}
$$

$$
K_{35} = \begin{bmatrix}
(F_{13}^{2} \pm F_{11}^{2}) & F_{14}^{2} & F_{15}^{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{12}^{3} & F_{13}^{3} & F_{14}^{3} & F_{15}^{3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F_{11}^{4} & F_{12}^{4} & F_{13}^{4} & F_{14}^{4} & F_{15}^{4} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & F_{11}^{5} & F_{12}^{5} & F_{13}^{5} & F_{14}^{5} & F_{15}^{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & F_{11}^{M-5} & F_{12}^{M-5} & F_{13}^{M-5} & F_{14}^{M-5} & F_{15}^{M-5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & F_{11}^{M-4} & F_{12}^{M-4} & F_{13}^{M-4} & F_{14}^{M-4} & F_{15}^{M-4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & F_{11}^{M-3} & F_{12}^{M-3} & F_{13}^{M-3} & F_{14}^{M-3} & F_{15}^{M-3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & F_{11}^{M-2} & F_{12}^{M-2} & F_{13}^{M-2} & F_{14}^{M-2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & F_{11}^{M-1} & F_{12}^{M-1} & (F_{13}^{M-1} \pm F_{15}^{M-1})
\end{bmatrix} \quad \text{(D33)}
$$

$$
K_{55} = \begin{bmatrix}
(P_{13}^{2} \pm P_{11}^{2}) & P_{14}^{2} & P_{15}^{2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & P_{13}^{3} & P_{14}^{3} & P_{15}^{3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & P_{13}^{4} & P_{14}^{4} & P_{15}^{4} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & P_{13}^{5} & P_{14}^{5} & P_{15}^{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & P_{13}^{M-5} & P_{14}^{M-5} & P_{15}^{M-5} & 0 & 0 \\
 & & & & & & & & & P_{13}^{M-4} & P_{14}^{M-4} & P_{15}^{M-4} & 0 \\
 & & & & & & & & & & P_{13}^{M-3} & P_{14}^{M-3} & P_{15}^{M-3} \\
 & & & & & & & & & & & P_{13}^{M-2} & P_{14}^{M-2} \\
\text{Symmetric} & & & & & & & & & & & & (P_{13}^{M-1} \pm P_{15}^{M-1})
\end{bmatrix} . \quad \text{(D34)}
$$

The elements of the matrices are defined by equations (D4) through (D12) and M is the number of difference stations. The g_{ij} submatrices are similar to the K_{ij} submatrices with the E_i^j being replaced by D_i^j, the F_i^j being replaced by G_i^j, and the P_i^j being replaced by the H_i^j (see eqs. (D13) through (D21)).

Appendix E

Modifications for the Stress Loading Case

When specifying the stress loading case, differential equations (B57) and boundary conditions (B62) must be modified to eliminate any rigid body displacements of the plate. The modifications used in this study are presented in this appendix.

Examination of equations (B57) indicates that the first and eighth differential equations are independent of the others. From the eighth differential equation, $y_8' = 0$, the solution can be obtained directly as $y_8 = y_{88}$, a constant. To separate the system and make use of PASVAR, the equations are redefined as follows:

$$\left.\begin{array}{lll} \tilde{y}_0 = y_1 & & \\ \tilde{y}_1 = y_2 & \tilde{y}_5 = y_6 & \tilde{y}_9 = y_{11} \\ \tilde{y}_2 = y_3 & \tilde{y}_6 = y_7 & \tilde{y}_{10} = y_{12} \\ \tilde{y}_3 = y_4 & \tilde{y}_7 = y_9 & \tilde{y}_{11} = y_{13} \\ \tilde{y}_4 = y_5 & \tilde{y}_8 = y_{10} & \tilde{y}_{12} = y_{14} \end{array}\right\} \tag{E1}$$

Redefining the unknown variables leads to the following set of differential equations for PASVAR:

$$\tilde{y}_1' = -\frac{A_{12}}{A_{11}}\gamma\tilde{y}_4 + \frac{\varsigma_2 y_{88} + \varsigma_5\tilde{y}_7 + \varsigma_6\tilde{y}_8 + \varsigma_7\tilde{y}_9}{A_{11}\mathrm{Den1}} \tag{E2a}$$

$$\tilde{y}_2' = -\frac{3A_{12}}{A_{11}}\gamma\tilde{y}_5 + \frac{\varsigma_3 y_{88} + \varsigma_6\tilde{y}_7 + \varsigma_8\tilde{y}_8 + \varsigma_9\tilde{y}_9}{A_{11}\mathrm{Den1}} \tag{E2b}$$

$$\tilde{y}_3' = -\frac{5A_{12}}{A_{11}}\gamma\tilde{y}_6 + \frac{\varsigma_4 y_{88} + \varsigma_7\tilde{y}_7 + \varsigma_9\tilde{y}_8 + \varsigma_{10}\tilde{y}_9}{A_{11}\mathrm{Den1}} \tag{E2c}$$

$$\tilde{y}_4' = \gamma\tilde{y}_1 + \frac{\theta_{15}\tilde{y}_{10} + \theta_{16}\tilde{y}_{11} + \theta_{17}\tilde{y}_{12}}{A_{66}\mathrm{Den2}} \tag{E2d}$$

$$\tilde{y}_5' = 3\gamma\tilde{y}_2 + \frac{\theta_{16}\tilde{y}_{10} + \theta_{18}\tilde{y}_{11} + \theta_{19}\tilde{y}_{12}}{A_{66}\mathrm{Den2}} \tag{E2e}$$

$$\tilde{y}_6' = 5\gamma\tilde{y}_3 + \frac{\theta_{17}\tilde{y}_{10} + \theta_{19}\tilde{y}_{11} + \theta_{20}\tilde{y}_{12}}{A_{66}\mathrm{Den2}} \tag{E2f}$$

$$\tilde{y}_7' = -\gamma\tilde{y}_{10} \tag{E2g}$$

$$\tilde{y}_8' = -3\gamma\tilde{y}_{11} \tag{E2h}$$

$$\tilde{y}_9' = -5\gamma\tilde{y}_{12} \tag{E2i}$$

$$\tilde{y}_{10}' = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}\gamma(\Omega_2 v_0 + \gamma\Omega_5\tilde{y}_4 + 3\gamma\Omega_4\tilde{y}_5 + 5\gamma\Omega_{12}\tilde{y}_6) + \frac{A_{12}}{A_{11}}\gamma\tilde{y}_7 \tag{E2j}$$

$$\tilde{y}_{11}' = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}3\gamma(\Omega_3 v_0 + \gamma\Omega_4\tilde{y}_4 + 3\gamma\Omega_6\tilde{y}_5 + 5\gamma\Omega_{11}\tilde{y}_6) + \frac{3A_{12}}{A_{11}}\gamma\tilde{y}_8 \tag{E2k}$$

$$\tilde{y}_{12}' = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}}5\gamma(\Omega_{10} v_0 + \gamma\Omega_{12}\tilde{y}_4 + 3\gamma\Omega_{11}\tilde{y}_5 + 5\gamma\Omega_{13}\tilde{y}_6) + \frac{5A_{12}}{A_{11}}\gamma\tilde{y}_9 \tag{E2l}$$

Upon solution of this system of equations, the displacement component $\tilde{y}_0$ is obtained by direct integration of the following differential equation:

$$\tilde{y}_0' = -\frac{A_{12}}{A_{11}}v_0 + \frac{\varsigma_1 y_{88} + \varsigma_2\tilde{y}_7 + \varsigma_3\tilde{y}_8 + \varsigma_4\tilde{y}_9}{A_{11}\mathrm{Den1}} \tag{E3}$$

APPENDIX E

The expressions for the stress resultants given by equations (B63) become

$$N_{x0} = \frac{\varsigma_1 y_{88} + \varsigma_2 \tilde{y}_7 + \varsigma_3 \tilde{y}_8 + \varsigma_4 \tilde{y}_9}{\text{Den1}} \tag{E4a}$$

$$N_{x1} = \frac{\varsigma_2 y_{88} + \varsigma_5 \tilde{y}_7 + \varsigma_6 \tilde{y}_8 + \varsigma_7 \tilde{y}_9}{\text{Den1}} \tag{E4b}$$

$$N_{x3} = \frac{\varsigma_3 y_{88} + \varsigma_6 \tilde{y}_7 + \varsigma_8 \tilde{y}_8 + \varsigma_9 \tilde{y}_9}{\text{Den1}} \tag{E4c}$$

$$N_{x5} = \frac{\varsigma_4 y_{88} + \varsigma_7 \tilde{y}_7 + \varsigma_9 \tilde{y}_8 + \varsigma_{10} \tilde{y}_9}{\text{Den1}} \tag{E4d}$$

$$N_{y0} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} v_0 + \frac{A_{12}}{A_{11}} N_{x0} \tag{E4e}$$

$$N_{y1} = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma \tilde{y}_4 + \frac{A_{12}}{A_{11}} N_{x1} \tag{E4f}$$

$$N_{y3} = 3\frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma \tilde{y}_5 + \frac{A_{12}}{A_{11}} N_{x3} \tag{E4g}$$

$$N_{y5} = 5\frac{A_{11}A_{22} - A_{12}^2}{A_{11}} \gamma \tilde{y}_6 + \frac{A_{12}}{A_{11}} N_{x5} \tag{E4h}$$

$$N_{xy1} = \frac{\theta_{15}\tilde{y}_{10} + \theta_{16}\tilde{y}_{11} + \theta_{17}\tilde{y}_{12}}{\text{Den2}} \tag{E4i}$$

$$N_{xy3} = \frac{\theta_{16}\tilde{y}_{10} + \theta_{18}\tilde{y}_{11} + \theta_{19}\tilde{y}_{12}}{\text{Den2}} \tag{E4j}$$

$$N_{xy5} = \frac{\theta_{17}\tilde{y}_{10} + \theta_{19}\tilde{y}_{11} + \theta_{20}\tilde{y}_{12}}{\text{Den2}} \tag{E4k}$$

The boundary conditions for the system of differential equations (E2) are obtained using plate equilibrium. This procedure is described as follows. Using the expressions for the stress resultants given by equations (B39) through (B41) with equations (B43) and (E1) gives

$$y_{88} = N_{x0}\Omega_1 + N_{x1}\Omega_2 + N_{x3}\Omega_3 + N_{x5}\Omega_{10} \tag{E5}$$

As mentioned previously, y_{88} is a constant, and this expression represents the axial force divided by 2λ at any cross section of the plate. Thus, equating the applied force at the loaded edges with equation (E5) multiplied by 2λ yields

$$y_{88} = -b\tilde{N}_x^o \tag{E6}$$

The 14 boundary conditions corresponding to the uniform compressive stress loading, given by equations (B32), are

$$\left.\begin{array}{ll} N_{x0}(\pm c) = -\tilde{N}_x^o & N_{xy1}(\pm c) = 0 \\ N_{x1}(\pm c) = 0 & N_{xy3}(\pm c) = 0 \\ N_{x3}(\pm c) = 0 & N_{xy5}(\pm c) = 0 \\ N_{x5}(\pm c) = 0 & \end{array}\right\} \tag{E7}$$

The corresponding 12 boundary conditions for differential equations (E2) are obtained by eliminating two boundary conditions from equations (E7). The boundary conditions to be eliminated are obtained directly from equation (E5) by specifying

$$\left.\begin{aligned} N_{x0}(\pm c) &= \frac{y_{88}}{\Omega_1(\pm c)} \\ N_{x1}(\pm c) &= 0 \\ N_{x3}(\pm c) &= 0 \end{aligned}\right\} \tag{E8}$$

Equilibrium equation (E5) requires that $N_{x5}(\pm c) = 0$, since $\Omega_{10}(\pm c)$ is nonzero. Thus, the boundary conditions of differential equations (E2) are

$$\left.\begin{aligned} \varsigma_1\tilde{y}_{88} + \varsigma_2\tilde{y}_7 + \varsigma_3\tilde{y}_8 + \varsigma_4\tilde{y}_9 &= -\tilde{N}_x^o\mathrm{Den1} \\ \varsigma_2\tilde{y}_{88} + \varsigma_5\tilde{y}_7 + \varsigma_6\tilde{y}_8 + \varsigma_7\tilde{y}_9 &= 0 \\ \varsigma_3\tilde{y}_{88} + \varsigma_6\tilde{y}_7 + \varsigma_8\tilde{y}_8 + \varsigma_9\tilde{y}_9 &= 0 \\ \theta_{15}\tilde{y}_{10} + \theta_{16}\tilde{y}_{11} + \theta_{17}\tilde{y}_{12} &= 0 \\ \theta_{16}\tilde{y}_{10} + \theta_{18}\tilde{y}_{11} + \theta_{19}\tilde{y}_{12} &= 0 \\ \theta_{17}\tilde{y}_{10} + \theta_{19}\tilde{y}_{11} + \theta_{20}\tilde{y}_{12} &= 0 \end{aligned}\right\} \quad (x = \pm c) \tag{E9}$$

The remaining step in solving the stress loading case is to determine the value of the constant v_0. The correct value of v_0 is defined to be the value which makes the residual normal force acting on the unloaded edges negligible compared with the applied loading. The net residual normal force on the unloaded edges is

$$P_y(\pm b) = \lambda \int_{-c}^{+c} [A_{12}u_0'(x) + A_{22}v_0]\, dx \tag{E10}$$

The applied load is

$$P_x(\pm c) = 2\int_{f(x)}^{b} N_x(\pm c, y)\, dy = 2\lambda y_{88}(\pm c) \tag{E11}$$

The constant v_0 is related to the generalized displacements by

$$v_0 = -\frac{A_{12}}{A_{22}}\frac{1}{2c}[u_0(c) - u_0(-c)] \tag{E12}$$

When the plate is stress loaded, the displacements of the loaded edges, $u^o(\pm c)$, are not known a priori. However, the prebuckling problem is linear and v_0 can be obtained by interpolation.

Appendix F

User's Guide and Sample Problems

For a constant cutout size and plate width, the program can calculate the buckling load and mode shape for a plate for several aspect ratios. The buckling results for the plate having the initial half-length, STARTC, are calculated. Then, the plate half-length is increased by CINC, and the buckling results for the plate with the new aspect ratio are calculated. This incremental process continues until the buckling results for the plate having the final half-length, STOPC, are calculated. The program input data are simple and are contained on four card images. The data on each card image are described as follows:

Card 1: CONV, CONF, NSTATS

CONV, CONF convergence tolerances on the prebuckling displacements and stresses, respectively

NSTATS number of finite difference stations

The values of CONV and CONF represent the final maximum errors in the displacements and stresses calculated by PASVAR. These values do not have to be known exactly and should be specified in accordance with the magnitudes of the displacements and stresses expected. CONV is much smaller than CONF since the displacements are several orders of magnitude smaller than the stresses.

Card 2: A, XI, B, STARTC, CINC, STOPC, IPRINT, IBCON, ICTOUT, ILOAD

A, XI cutout dimensions (see fig. 1)

B plate half-width (see b on fig. 1)

STARTC initial value of the plate half-length (see c on fig. 1)

CINC plate half-length increment for multiple aspect ratio calculations

STOPC terminating value of the plate half-length

IPRINT output option:
1 minimum program output
2 maximum program output

IBCON boundary condition option:
1 loaded edges clamped
2 loaded edges simply supported

ICTOUT cutout shape option.
1 rectangular cutout given by $x = A$ and $y = XI$
2 elliptical cutout given by $(x/A)**2 + (y/XI)**2 = 1$

ILOAD loading option:
1 uniform edge displacement
2 uniform edge stress

Card 3: A11, A12, A22, A66

A11,A12, A22,A66 membrane stiffness coefficients for orthotropic plates

Card 4: D11, D12, D22, D66

D11,D12, D22.D66 bending stiffness coefficients for orthotropic plates

To execute multiple analyses, repeat cards 2, 3, and 4 for each additional problem. The program execution halts on reaching the end of input data.

To illustrate program use, two sample problems are presented.

Sample Problem 1

The buckling load, buckling coefficient, and critical end shortening for a square orthotropic plate with a centrally located circular cutout are calculated. The plate is 10 inches on a side, and multiple analyses are executed for ratios of cutout diameter to plate width from 0 to 0.6. The stiffness coefficients of the orthotropic plate are given by

$$[A_{ij}] = \begin{bmatrix} 18.698 & 0.566 & 0 \\ 0.566 & 1.617 & 0 \\ 0 & 0 & 0.832 \end{bmatrix} \times 10^5 \text{ lb/in.}$$

$$[D_{ij}] = \begin{bmatrix} 15.582 & 0.472 & 0 \\ 0.472 & 1.348 & 0 \\ 0 & 0 & 0.693 \end{bmatrix} \times 10^2 \text{ lb/in.}$$

All edges of the plates are simply supported. The loading is a uniform edge displacement. The corresponding input data are

Card 1: 1.0 6.2E+5 62
Card 2: 0.0 0.0 5.0 5.0 0.0 5.0 1 2 2 1
Card 3: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 4: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 5: 0.5 0.5 5.0 5.0 0.0 5.0 1 2 2 1
Card 6: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 7: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 8: 1.0 1.0 5.0 5.0 0.0 5.0 1 2 2 1
Card 9: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 10: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 11: 1.5 1.5 5.0 5.0 0.0 5.0 1 2 2 1
Card 12: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 13: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 14: 2.0 2.0 5.0 5.0 0.0 5.0 1 2 2 1
Card 15: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 16: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 17: 2.5 2.5 5.0 5.0 0.0 5.0 1 2 2 1
Card 18: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 19: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

Card 20: 3.0 3.0 5.0 5.0 0.0 5.0 1 2 2 1
Card 21: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 22: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

As described above, cards 2, 3, and 4 are repeated for each cutout size. The results from BUCKO are summarized for ratios of the hole diameter to plate width d/W below.

d/W	Buckling load, lb	Buckling coefficient	Critical end shortening, in.
0	2037	4.50	0.00110
.1	1967	4.37	.00108
.2	1777	3.93	.00106
.3	1641	3.63	.00114
.4	1560	3.45	.00126
.5	1512	3.34	.00145
.6	1564	3.46	.00186

Sample Problem 2

The buckling load, buckling coefficient, and critical end shortening for a rectangular orthotropic plate with a centrally located rectangular cutout are calculated. Buckling results are calculated for plates having a width of 10 inches and lengths of 20 inches, 25 inches, and 30 inches. The rectangular cutout is 2 inches wide and 4 inches long with the 2-inch-wide side perpendicular to the loading. The stiffness coefficients used in sample problem 1 are also used for this sample problem. The loaded edges are clamped, the unloaded edges are simply supported, and the loading is a uniform edge stress. The corresponding input data are

Card 1: 1.0 6.2E+5 42
Card 2: 2.0 1.0 5.0 10.0 2.5 15.0 1 1 1 2
Card 3: 18.698E+5 0.566E+5 1.617E+5 0.832E+5
Card 4: 15.582E+2 0.472E+2 1.348E+2 0.693E+2

The results obtained from BUCKO are summarized as follows:

Aspect ratio	Buckling load, lb	Buckling coefficient	Critical end shortening, in.
2.0	1997	4.37	0.00283
2.5	1729	3.82	.00280
3.0	1511	3.34	.00283

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