TWO-DIMENSIONAL UNSTEADY ANALYSIS OF FLUID FORCES ON A WHIRLING CENTRIFUGAL IMPELLER IN A VOLUTE*

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Destabilizing fluid forces on a whirling centrifugal impeller rotating in a volute have been observed (Ref.1). A quasisteady analysis neglecting shed vorticity (Ref.2) or an unsteady analysis without a volute (Ref.3) does not predict the existence of such destabilizing fluid forces on a whirling impeller.

The present report is intended to take into account the effects of a volute and the shed vorticity. We treat cases when an impeller with an infinite number of vanes rotates with a constant velocity Ω and its center whirls with a constant eccentric radius ε and a constant whirling velocity ω.

Major assumptions are as follows:

(1) The number of the vanes is so large that the impeller can be treated as an actuator impeller in which the flow is perfectly guided.

(2) Flow is inviscid, incompressible and two-dimensional.

(3) The eccentricity ε is so small that unsteady components can be linearized.

(4) Vorticity is transported on a prescribed mean flow, i.e., the operating point is near design flow rate.

(5) The volute can be represented by a curved plate (see Fig.1).

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SYMBOLS

$z = x + iy$: stationary frame with its origin 0 fixed to the center of the whirling motion

$z' = x' + iy' = z - \varepsilon e^{i\omega t}$: translating frame with its origin 0' fixed to the center of the impeller and its axes parallel to those of Z.

r_1 , r_2 : inner and outer radius of the impeller

$\beta(r)$: vane angle measured from circumferential direction

ε : eccentricity

ω : angular velocity of whirling motion

Ω : angular velocity of the impeller

Γ_i : prerotation, Q: flow rate

$\underset{\sim}{V} = u + iv$ (v_r, v_θ) : absolute velocity

$\underset{\sim}{V}' = u' + iv'$ (v_r', v_θ') : velocity relative to x'y' frame

w (w_r, w_θ) : velocity relative to the impeller

t : time, t=0 when 00' is in the direction of x

n : circumferential mode number

Subscripts

$1, 2$: quantities at the inner and outer radius respectively

Superscripts

$-$: steady component ~ ; unsteady component

BASIC EQUATIONS

Euler's equation in the rotating and translating frame fixed to the rotor is,

$$\frac{\partial \underset{\sim}{V}}{\partial t^*} + \nabla\left(\frac{w^2}{2} - \frac{(\underset{\sim}{U} + \Omega \times \underset{\sim}{z}')^2}{2} + \frac{p}{\rho}\right) - \underset{\sim}{w} \times (\nabla \times \underset{\sim}{V}) = \underset{\sim}{f} \tag{1}$$

where $\partial/\partial t^*$ is the time derivative in the rotating frame and $\underset{\sim}{U} = i\omega\varepsilon e^{i\omega t}$ and $\underset{\sim}{\Omega} \times \underset{\sim}{x}' = ir\Omega e^{i\theta}$ are the translational velocity due to whirling and the rotational velocity of the impeller. In an actuator impeller with an infinite number of vanes, the effects of the vorticity distribution on the vanes and the forces exerted by the vanes are represented by the vorticity $\nabla \times \underset{\sim}{v}$ and the external force $\underset{\sim}{f}$ respectively in the above equation. For inviscid flow through actuator impellers, $\underset{\sim}{f}$ and $\underset{\sim}{w} \times (\nabla \times \underset{\sim}{v})$ are normal to the vane surface and the component of Equation (1) parallel to the vane surface is

$$\frac{\partial v_s}{\partial t^*} + \frac{\partial}{\partial s}\left(\frac{w^2}{2} - \frac{V_t^2}{2} + \frac{p}{\rho}\right) = 0 \tag{2}$$

where

$$v_s = w - r\Omega\cos\beta - \omega\varepsilon\cos(\theta+\beta-\omega t)$$

$$w(r,\theta) = w_r(r_2, \theta - \varphi(r))\, r_2 / (r(s)\sin\beta(r))$$

$$V_t^2 = r^2\Omega^2 + \varepsilon^2\omega^2 + 2r\omega\varepsilon\Omega\cos(\theta-\omega t)$$

$$\partial/\partial t^* = \partial/\partial t + \Omega\,\partial/\partial\theta$$

Integrating Equation (2) along the vane surface we get the following total pressure increase.

$$\frac{p_{t2}-p_{t1}}{\rho} = \left[r\Omega v_\theta'\right]_1^2 - M\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)v_{r2}' - \varepsilon\omega(\Omega-\omega)\int_1^2 \sin(\theta+\beta-\omega t)\,ds$$
$$+\varepsilon\omega\left[(r\Omega + v_\theta')\cos(\theta-\omega t) + v_r'\sin(\theta-\omega t)\right]_1^2 \tag{3}$$

where

$$M = \int_1^2 \frac{r_2}{r\sin\beta(r)}\,dr$$

Downstream of the impeller, where $\underset{\sim}{f}=0$, Equation (1) can be expressed as

$$\underset{\sim}{w} \times (\nabla \times \underset{\sim}{v}) = \frac{\partial \underset{\sim}{v}}{\partial t^*} + \frac{1}{2}\nabla H', \quad H' = \frac{p}{\rho} + \frac{w^2}{2} - \frac{V_t^2}{2} \tag{4}$$

By multiplying $\underset{\sim}{e}_x dx + \underset{\sim}{e}_y dy = r_2(\underset{\sim}{e}_y\cos\theta - \underset{\sim}{e}_x\sin\theta)d\theta$ on both sides of the above equation we get the following vorticity distribution at the outer radius of the impeller;

$$\zeta(r_2,\theta) = -\frac{1}{r_2 w_{r2}}\frac{\partial H_2'}{\partial\theta} - \frac{1}{w_{r2}}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)v_{\theta 2}'$$
$$= \frac{1}{r_2 w_{r2}}\Big[r_1\Omega\frac{\partial v_{\theta 1}'}{\partial\theta} + \omega\varepsilon\Big\{v_{r1}'\cos(\theta_1-\omega t) - v_{\theta 1}'\sin(\theta_1-\omega t)$$
$$+\frac{\partial v_{\theta 1}'}{\partial\theta}\cos(\theta_1-\omega t) + \frac{\partial v_{r1}'}{\partial\theta}\sin(\theta_1-\omega t) - r_1\Omega\sin(\theta_1-\omega t)\Big\}$$
$$+\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)\frac{\partial v_{r2}'}{\partial\theta}\cdot M + \varepsilon\omega(\Omega-\omega)\int_1^2\cos(\theta+\beta-\omega t)\,ds$$
$$-\frac{1}{w_r}\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)v_{\theta 2}' + \frac{1}{w_r}\varepsilon\omega(\Omega-\omega)\sin(\theta-\omega t) \tag{5}$$

ELEMENTARY FLOW COMPONENTS

The flow tangency condition at the impeller outlet is,

$$v_{\theta 2}' = r_2 \Omega - v_{r2}' \cot \beta_2 \tag{6}$$

The first term on the right hand side of the above equation is cancelled by the flow component

$$v_r' - i v_\theta' = \frac{1}{2\pi r} \left\{ Q - 2\pi i r_2 \left(r_2 \Omega - \frac{Q}{2\pi r_2} \cot \beta_2 \right) \right\} \tag{7}$$

and other disturbance components should satisfy the following equation

$$v_{\theta 2}' = - v_{r2}' \cot \beta_2 \ . \tag{8}$$

For the region outside the impeller we consider two types of elementary velocity disturbances satisfying Equation (8):

First, consider the following velocity field:

$$u' - i v' = \frac{i\Gamma}{2\pi} \left\{ \frac{1}{z' - z_0'} + e^{-2i\beta_2} \left(\frac{1}{z'} - \frac{1}{z' - r_2^2 / \bar{z}_0'} \right) \right\} \tag{9}$$

This velocity field has a vortex Γ at $z'=z'_0$ and satisfies Equation (8) with no circulation/net flow around/from the impeller. Consider a vortex distribution $\Gamma(s) = \Gamma s(s) + \varepsilon \Gamma_d^s(s) \sin \omega t + \varepsilon \Gamma_d^c(s) \cos \omega t$ on the volute surface $z_0(s)=z'_0+\varepsilon e^{i\omega t}$. Assuming $\varepsilon \ll r_2$, we get the following steady and unsteady velocity components

$$\bar{u}' - i \bar{v}' |_{z'} = \frac{i}{2\pi} \int_0^{S_e} \Gamma_s(s) \left[\frac{1}{z' - z_0(s)} - \frac{e^{-2i\beta_2}}{z'} \frac{r_2^2}{(z' \bar{z}_0(s) - r_2^2)} \right] ds \tag{10}$$

$$\begin{aligned} \tilde{u}' - i \tilde{v}' |_{z'} = & -\frac{i\varepsilon}{2\pi} \int_0^{S_e} \Gamma_s(s) \left[\frac{e^{i\omega t}}{(z' - z_0(s))^2} + e^{-2i\beta_2} \frac{r_2^2 e^{-i\omega t}}{(z' \bar{z}_0(s) - r_2^2)^2} \right] ds \\ & + \frac{i\varepsilon}{2\pi} \int_0^{S_e} (\Gamma_d^s(s) \sin \omega t + \Gamma_d^c(s) \cos \omega t) \left[\frac{1}{z' - z_0(s)} - \frac{r_2^2 e^{-2i\beta_2}}{z'(z' \bar{z}_0(s) - r_2^2)} \right] ds \end{aligned} \tag{11}$$

where z' is fixed to $x'y'$-plane. At $z=z_s=z'+\varepsilon e^{i\omega t}$, which is fixed to the volute, we get the following expressions:

$$\bar{u}' - i \bar{v}' |_{z_s} = \frac{i}{2\pi} \int_0^{S_e} \Gamma_s(s) \left[\frac{1}{z_s - z_0(s)} - \frac{r_2^2 e^{-2i\beta_2}}{z_s (z_s \bar{z}_0(s) - r_2^2)} \right] ds \tag{12}$$

$$\tilde{u}' - i\tilde{v}'|_{z_s} = -\frac{i\varepsilon}{2\pi}\int_0^{S_e}\Gamma_s(s)\frac{r_2^2 e^{-2i\beta_2}}{z_s(z_s\bar{z}_o(s) - r_2^2)}\left(\frac{e^{i\omega t}}{z_s} + \frac{\bar{z}_o(s)e^{i\omega t} + z_s e^{-i\omega t}}{z_s\bar{z}_o(s) + r_2^2}\right)ds$$
$$+\frac{i\varepsilon}{2\pi}\int_0^{S_e}(\Gamma_d^c(s)\cos\omega t + \Gamma_d^s(s)\sin\omega t)\left[\frac{1}{z_s - z_o(s)} - \frac{r_2^2 e^{-2i\beta_2}}{z_s(z_s\bar{z}_o(s) - r_2^2)}\right]ds \tag{13}$$

In the same way the velocity component (7) has the following steady and unsteady components at the volute surface $z_s = r_s e^{i\theta_s} = z' + \varepsilon e^{i\omega t}$.

$$\bar{v}_r' - i\bar{v}_\theta'|_{z_s} = \frac{1}{2\pi r_s}\left\{Q - 2\pi i r_2\left(r_2\Omega - \frac{Q}{2\pi r_2}\cot\beta_2\right)\right\} \tag{14}$$

$$\tilde{v}_r' - i\tilde{v}_\theta'|_{z_s} = \frac{1}{2\pi r_s}\left\{Q - 2\pi i r_2\left(r_2\Omega - \frac{Q}{2\pi r_2}\cot\beta_2\right)\right\}\frac{\varepsilon}{r_s}e^{i(\omega t - \theta_s)} \tag{15}$$

Next consider the velocity field due to shed vorticity. If we assume that the vorticity is transported on a streamline

$$v_r' - iv_\theta' = (Q - i\Gamma)/(2\pi r)$$

we have the following elementary vorticity field

$$\zeta_n = \zeta_{cn}\cos\left\{\pm\omega\left(t - \frac{\pi}{Q}(r^2 - r_2^2)\right) + n\left(\theta - \frac{\Gamma}{Q}\log(r/r_2)\right)\right\}$$
$$+ \zeta_{sn}\sin\left\{\pm\omega\left(t - \frac{\pi}{Q}(r^2 - r_2^2)\right) + n\left(\theta - \frac{\Gamma}{Q}\log(r/r_2)\right)\right\} \tag{16}$$

and the corresponding velocity field

$$v_{rn}^\zeta - iv_{\theta n}^\zeta = (\zeta_{cn}Z_{Rn} + \zeta_{sn}Z_{In})\cos(n\theta \pm \omega t) + (\zeta_{sn}Z_{Rn} - \zeta_{cn}Z_{In})\sin(n\theta \pm \omega t) \tag{17}$$

where

$$Z_{Rn} = R_{Rn} - i\Theta_{Rn}, \qquad Z_{In} = R_{In} - i\Theta_{In}$$

$$R_{Rn} + jR_{In} = -j[F(r) + G(r)], \qquad \Theta_{Rn} + j\Theta_{In} = -F(r) + G(r)$$

$$F(r) = \frac{1}{2}\int_{r_2}^{r}\exp\left\{j\left(\pm\omega\frac{\pi}{Q}(r_0^2 - r_2^2) - n\frac{\Gamma}{Q}\log(r_0/r_2)\right)\right\}(r_0/r)^{n+1}dr_0$$

$$G(r) = \frac{1}{2}\int_{r_2}^{r}\exp\left\{j\left(\pm\omega\frac{\pi}{Q}(r_0^2 - r_2^2) - n\frac{\Gamma}{Q}\log(r_0/r_2)\right)\right\}(r/r_0)^{n-1}dr_0$$

Equation (17) includes the effects of the vorticity in $r_2 < r < R$. Since $G(r)$ is convergent only for $n \geq 3$ as $R \to \infty$, we should artificially prescribe some appropriate finite value for R. Equation (17) does not satisfy Equation (8) and we should add a potential component

$$v_{rn}^p - iv_{\theta n}^p = -e^{-i\beta_2}(\zeta_{cn} + i\zeta_{sn})\left[(\Theta_R - i\Theta_I)\sin\beta_2 + (R_R - iR_I)\cos\beta_2\right]_{r=r_2}\frac{e^{i\theta \mp i\omega t}}{z'^{n+1}} \tag{18}$$

such that

$$v'_{rn} - i\,v'_{\theta n} = (\,v^{\zeta}_{rn} - i\,v^{\zeta}_{\theta n}\,) + (\,v^{P}_{rn} - i\,v^{P}_{\theta n}\,) \tag{19}$$

satisfies the boundary condition of Equation (8). We may now note that there are steady and unsteady components as follows:

(i) Steady component; in Equations (16)-(19) we put ω=0 and represent the related quantities with superscript $\bar{}$ and suffix n (n=1,2...) for each mode, $\bar{\zeta}_{cn}$ etc.

(ii) Unsteady component; correspondingly to the sign on ω and the mode n, we represent the related quantities such that $\hat{\tilde{\zeta}}^{\pm}_{cn}$ etc.

On the volute $z=z_s=z'+\varepsilon e^{i\omega t}$, the steady and unsteady velocity fields are expressed as follows:

$$\begin{aligned}\bar{u}'-i\bar{v}'|_{z_s} = \sum_{n=1}^{\infty} \Big[\!\Big[&\big[(\bar{\zeta}_{cn}\bar{z}_{Rn}+\bar{\zeta}_{sn}\bar{z}_{In})\cos n\theta_s + (\bar{\zeta}_{sn}\bar{z}_{Rn}-\bar{\zeta}_{cn}\bar{z}_{In})\sin n\theta_s\big]e^{-i\theta_s} \\ &- e^{-i\beta_2}(\bar{\zeta}_{cn}+i\bar{\zeta}_{sn})\big[(\bar{\Theta}_{Rn}-i\bar{\Theta}_{In})\sin\beta_2+(\bar{R}_{Rn}-i\bar{R}_{In})\cos\beta_2\big]_{r=r_2}/z_s^{n+1}\Big]\!\Big]\end{aligned} \tag{20}$$

$$\begin{aligned}\hat{\tilde{u}}'-i\hat{\tilde{v}}'|_{z_s}/\varepsilon = \sum_{n=1}^{\infty} \Big[\!\Big[&\big[(\bar{\zeta}_{cn}\bar{z}_{Rn}+\bar{\zeta}_{sn}\bar{z}_{In})(-n\sin n\theta_s - i\cos n\theta_s) \\ &+ (\bar{\zeta}_{sn}\bar{z}_{Rn}-\bar{\zeta}_{cn}\bar{z}_{In})(n\cos n\theta_s - i\sin n\theta_s)\big]\frac{1}{r_s}\sin(\theta_s-\omega t)\,e^{-i\theta_s} \\ &-\big[(\bar{\zeta}_{cn}\bar{z}'_{Rn}+\bar{\zeta}_{sn}\bar{z}'_{In})\cos n\theta_s + (\bar{\zeta}_{sn}\bar{z}'_{Rn}-\bar{\zeta}_{cn}\bar{z}'_{In})\sin n\theta_s\big]\cos(\theta_s-\omega t)\,e^{-i\theta_s} \\ &-(n+1)e^{-i\beta_2}(\bar{\zeta}_{cn}+i\bar{\zeta}_{sn})\big[(\bar{\Theta}_{Rn}-i\bar{\Theta}_{In})\sin\beta_2+(\bar{R}_{Rn}-i\bar{R}_{In})\cos\beta_2\big]_{r=r_2}\frac{e^{i\omega t}}{z_s^{n+2}}\Big]\!\Big] \\ +\sum^{\pm}\sum_{n=1}^{\infty}\Big[\!\Big[&\big[(\hat{\tilde{\zeta}}^{\pm}_{cn}\hat{\tilde{z}}^{\pm}_{Rn}+\hat{\tilde{\zeta}}^{\pm}_{sn}\hat{\tilde{z}}^{\pm}_{In})\cos(n\theta_s\pm\omega t)+(\hat{\tilde{\zeta}}^{\pm}_{sn}\hat{\tilde{z}}^{\pm}_{Rn}-\hat{\tilde{\zeta}}^{\pm}_{cn}\hat{\tilde{z}}^{\pm}_{In})\sin(n\theta_s\pm\omega t)\big]e^{-i\theta_s} \\ &-e^{i\beta_2}(\hat{\tilde{\zeta}}^{\pm}_{cn}+i\hat{\tilde{\zeta}}^{\pm}_{sn})\big[(\hat{\tilde{\Theta}}^{\pm}_{Rn}-i\hat{\tilde{\Theta}}^{\pm}_{In})\sin\beta_2+(\hat{\tilde{R}}^{\pm}_{Rn}-i\hat{\tilde{R}}^{\pm}_{In})\cos\beta_2\big]_{r=r_2}\frac{e^{\mp i\omega t}}{z_s^{n+1}}\Big]\!\Big]\end{aligned} \tag{21}$$

In the region upstream of the impeller ($r \leq r_1$), we consider a source Q and prerotation of strength Γ_i at the center of the volute. Then the velocity can be expressed in the series

$$v'_r - i\,v'_\theta = \Big[\frac{Q+i\Gamma_i}{2\pi}\,\frac{1}{z'+\varepsilon e^{i\omega t}} + \sum_{n=1}^{\infty}\big\{\bar{A}_n + \varepsilon(A^s_n \sin\omega t + A^c_n\cos\omega t)\big\}z'^{\,n-1}\Big]e^{i\theta} \tag{22}$$

where $\bar{A}_n=\bar{A}_{Rn}+i\bar{A}_{In}$, $A^s_n=A^s_{Rn}+iA^s_{In}$, $A^c_n=A^c_{Rn}+iA^c_{In}$ are complex constants.

Now we have given all of the elementary flow components necessary for the construction of the entire flow field. Each of them contains several unknowns that are determined in the following sections.

BOUNDARY CONDITIONS ON THE VOLUTE

The flow tangency condition on the volute surface is,

$$\varepsilon\omega\cos(\alpha-\omega t) + v'\cos\alpha - u'\sin\alpha = v_n = 0 \tag{23}$$

where α is the angle between volute and x-axis. If we put

$$u' = \bar{u}' + \tilde{u}_c'\cos\omega t + \tilde{u}_s'\sin\omega t$$

$$v' = \bar{v}' + \tilde{v}_c'\cos\omega t + \tilde{v}_s'\sin\omega t$$

we get the following conditions,

$$\bar{v}'\cos\alpha - \bar{u}'\sin\alpha = 0, \tag{24}$$

$$\tilde{v}_c'\cos\alpha - \tilde{u}_c'\sin\alpha = -\varepsilon\omega\cos\alpha, \tag{25}$$

$$\tilde{v}_s'\cos\alpha - \tilde{u}_s'\sin\alpha = -\varepsilon\omega\sin\alpha. \tag{26}$$

The steady velocity $\bar{u}', \bar{v}'$ is given as a sum of the velocity components in Equations (12), (14) and (20). The unsteady components $\tilde{u}'_c$, $\tilde{v}'_c$ and $\tilde{u}'_s$, $\tilde{u}'_s$ are the cosine and sine components of the velocity of Equations (13), (15) and (21). Equations (24)-(26) constitute integral equations for $\Gamma_s(s)$, $\Gamma_d^s(s)$ and $\Gamma_d^c(s)$.

CONTINUITY EQUATION

The continuity equation across the impeller is

$$v_r'(r_1, \theta_1 = \theta + \vartheta_0) = \frac{r_2}{r_1}\, v_r'(r_2, \theta) \tag{27}$$

where ϑ_0 is the angle between corresponding leading and trailing edges of a vane. At the outer radius the total of the steady velocity components given by Equations (7), (10) and (19) can be expressed as a Fourier series; namely,

$$\left.\begin{aligned} \bar{v}_r'(r_2,\theta) &= \frac{Q}{2\pi r_2} + \sum_{n=1}^{\infty}\left(\bar{v}_{rn}^{s}\sin n\theta + \bar{v}_{rn}^{c}\cos n\theta\right) \\ \bar{v}_\theta'(r_2,\theta) &= r_2\Omega - \frac{Q}{2\pi r_2}\cot\beta_2 + \sum_{n=1}^{\infty}\left(\bar{v}_{\theta n}^{s}\sin n\theta + \bar{v}_{\theta n}^{c}\cos n\theta\right) \end{aligned}\right\} \tag{28}$$

In the same way, the unsteady components of Equations (11), and (19) can be expressed as

$$\left.\begin{aligned} \tilde{v}_r'(r_2,\theta) &= \varepsilon \sum_{n=1}^{\infty} [\![(\tilde{v}_{rn}^{cc}\cos n\theta + \tilde{v}_{rn}^{cs}\sin n\theta)\cos\omega t + (\tilde{v}_{rn}^{sc}\cos n\theta + \tilde{v}_{rn}^{ss}\sin n\theta)\sin\omega t]\!] \\ \tilde{v}_\theta'(r_2,\theta) &= \varepsilon \sum_{n=1}^{\infty} [\![(\tilde{v}_{\theta n}^{cc}\cos n\theta + \tilde{v}_{\theta n}^{cs}\sin n\theta)\cos\omega t + (\tilde{v}_{\theta n}^{sc}\cos n\theta + \tilde{v}_{\theta n}^{ss}\sin n\theta)\sin\omega t]\!] \end{aligned}\right\} \quad (29)$$

At the inner radius $r=r_1$ we expand Equation (22) in $\theta=\theta_1-\varphi_0$ rather than in θ_1; then,

$$\left.\begin{aligned} \bar{v}_r'(r_1,\theta_1=\theta+\varphi_0) &= \frac{Q}{2\pi r_1} + \sum_{n=1}^{\infty}\{\bar{\bar{v}}_{rn}^{s}\sin n\theta + \bar{\bar{v}}_{rn}^{c}\cos n\theta\} \\ \bar{v}_\theta'(r_1,\theta_1=\theta+\varphi_0) &= -\frac{\Gamma_2}{2\pi r_1} + \sum_{n=1}^{\infty}\{\bar{\bar{v}}_{\theta n}^{s}\sin n\theta + \bar{\bar{v}}_{\theta n}^{c}\cos n\theta\} \end{aligned}\right\} \quad (30)$$

$$\left.\begin{aligned} \tilde{v}_r'(r_1,\theta_1=\theta+\varphi_0) &= \varepsilon \sum_{n=1}^{\infty} [\![(\tilde{\tilde{v}}_{rn}^{cc}\cos n\theta + \tilde{\tilde{v}}_{rn}^{cs}\sin n\theta)\cos\omega t + (\tilde{\tilde{v}}_{rn}^{sc}\cos n\theta + \tilde{\tilde{v}}_{rn}^{ss}\sin n\theta)\sin\omega t]\!] \\ \tilde{v}_\theta'(r_1,\theta_1=\theta+\varphi_0) &= \varepsilon \sum_{n=1}^{\infty} [\![(\tilde{\tilde{v}}_{\theta n}^{cc}\cos n\theta + \tilde{\tilde{v}}_{\theta n}^{cs}\sin n\theta)\cos\omega t + (\tilde{\tilde{v}}_{\theta n}^{sc}\cos n\theta + \tilde{\tilde{v}}_{\theta n}^{ss}\sin n\theta)\sin\omega t]\!] \end{aligned}\right\} \quad (31)$$

From Equation (27) we get the following relations

$$\bar{\bar{v}}_{rn}^{s} = (r_2/r_1)\,\bar{v}_{rn}^{s} \quad (32) \qquad \bar{\bar{v}}_{rn}^{c} = (r_2/r_1)\,\bar{v}_{rn}^{c} \quad (33)$$

$$\tilde{\tilde{v}}_{rn}^{cc} = (r_2/r_1)\,\tilde{v}_{rn}^{cc} \quad (34) \qquad \tilde{\tilde{v}}_{rn}^{cs} = (r_2/r_1)\,\tilde{v}_{rn}^{cs} \quad (35)$$

$$\tilde{\tilde{v}}_{rn}^{sc} = (r_2/r_1)\,\tilde{v}_{rn}^{sc} \quad (36) \qquad \tilde{\tilde{v}}_{rn}^{ss} = (r_2/r_1)\,\tilde{v}_{rn}^{ss} \quad (37)$$

Equations (32) and (33) give the relations to determine $\bar{A}_{Rn}$ and $\bar{A}_{In}$, and Equations (34)-(37) determine A_{Rn}^{c}, A_{Rn}^{s}, A_{In}^{c} and A_{In}^{s}.

STRENGTH OF SHED VORTICITY

If we use the expressions (28-31) in Equation (5), the steady vorticity components can be expressed as

$$\bar{\zeta}_{cn} = -\frac{2\pi r_2}{Q}\Omega n\left(\frac{Mn}{r_2}\bar{v}_{rn}^{c} + \bar{v}_{\theta n}^{s} - \frac{r_1}{r_2}\bar{\bar{v}}_{\theta n}^{s}\right) \quad (38)$$

$$\bar{\zeta}_{sn} = -\frac{2\pi r_2}{Q}\Omega n\left(\frac{Mn}{r_2}\bar{v}_{rn}^{s} - \bar{v}_{\theta n}^{c} + \frac{r_1}{r_2}\bar{\bar{v}}_{\theta n}^{c}\right) \quad (39)$$

where we have used $w_r=\frac{Q}{2\pi r_2}$ because of the assumption on the transport of the vorticity. In the same way, if we express the vorticity by

$$\tilde{\zeta}(r_2,\theta) = \frac{2\pi r_2}{Q}\varepsilon\sum_{n=1}^{\infty}[\![(\tilde{\zeta}_n^{sc}\cos n\theta + \tilde{\zeta}_n^{ss}\sin n\theta)\sin\omega t + (\tilde{\zeta}_n^{cc}\cos n\theta + \tilde{\zeta}_n^{cs}\sin n\theta)\cos\omega t]\!] \quad (40)$$

and use the expressions (28-31) in Equation (5), we can express $\tilde{\zeta}$ in terms of the Fourier coefficients in Equations (28)-(31). Comparing Equation (40) and Equation (16), we get the following relations,

$$\tilde{\zeta}^{+}_{sm} = \frac{\pi r_2}{Q}(\tilde{\zeta}^{cs}_{m} + \tilde{\zeta}^{sc}_{m}) \quad (41) \qquad \tilde{\zeta}^{-}_{sm} = \frac{\pi r_2}{Q}(\tilde{\zeta}^{cs}_{m} - \tilde{\zeta}^{sc}_{m}) \quad (42)$$

$$\tilde{\zeta}^{+}_{cm} = \frac{\pi r_2}{Q}(\tilde{\zeta}^{cc}_{m} - \tilde{\zeta}^{ss}_{m}) \quad (43) \qquad \tilde{\zeta}^{-}_{cm} = \frac{\pi r_2}{Q}(\tilde{\zeta}^{cc}_{m} + \tilde{\zeta}^{ss}_{m}) \quad (44)$$

METHOD OF SOLUTION

We have used the following unknowns for the expression of the flow field

	steady component	unsteady component
$r \geqq r_2$	$\Gamma_s(s)$	$\Gamma^c_d(s)$, $\Gamma^s_d(s)$
	$\bar{\zeta}_{cm}$, $\bar{\zeta}_{sm}$	$\tilde{\zeta}^{+}_{cm}$, $\tilde{\zeta}^{+}_{sm}$, $\tilde{\zeta}^{-}_{cm}$, $\tilde{\zeta}^{-}_{sm}$
$r \leqq r_2$	$\bar{A}_{Rm}$, $\bar{A}_{Im}$	A^s_{Rm}, A^s_{Im} , A^c_{Rm}, A^c_{Im}

These unknowns are determined by the following relations:

	Steady Component	Unsteady Component
B.C. on the volute:	Eq.(24)	Eqs.(25),(26)
Continuity:	Eqs.(32),(33)	Eqs.(34),(35),(36),(37)
Vorticity:	Eqs.(38),(39)	Eqs.(41),(42),(43),(44)

These equations include integrals related to the vortex distribution on the volute surface, which should be evaluated by some appropriate method. Equations (24)-(26) are integral equations for the vortex distributions on the volute surface and could be reduced to simultaneous linear equations by a singularity method. In the solution of the vortex distributions the "Kutta condition" at the trailing edge should be applied. Strictly speaking the circulation around the volute fluctuates and a free vortex sheet is shed from the trailing edge of the volute. Since we are mainly interested in the forces on the impeller, we will neglect the effect of the free vortex sheet but apply the following conditions at the trailing edge.

Steady part:

$$\Gamma_s(S_e) = 0 \tag{45}$$

Unsteady parts:

$$\omega \int_0^{S_e} \Gamma^c_d(s)\,ds = \Gamma^s_d(S_e)\,\overline{W}(S_e) \tag{46}$$

$$-\,\omega \int_0^{S_e} \Gamma^s_d(s)\,ds = \Gamma^c_d(S_e)\,\overline{W}(S_e) \tag{47}$$

Now we can express all the relations as a set of simultaneous linear equations which can be solved numerically. The steady component may be solved independently of unsteady component, and the result used in the analysis of the unsteady components.

UNSTEADY FORCES ON THE IMPELLER

By considering the balance of the momentum of the fluid in the impeller, we can express the forces on the impeller as follows;

Steady component

$$\begin{aligned}\bar{X} - i\bar{Y} &= -i\left[\oint_{C_2} \bar{p}_t\, d\bar{z}' - \oint_{C_1} \bar{p}_t\, d\bar{z}'\right] \\ &\quad + \frac{i\rho}{2}\left[\oint_{C_2} (\bar{u}' - i\bar{v}')^2 dz' - \oint_{C_1} (\bar{u}' - i\bar{v}')^2 dz'\right]\end{aligned} \tag{48}$$

and unsteady component

$$\begin{aligned}&(\tilde{X}_c - i\tilde{Y}_c)\cos\omega t + (\tilde{X}_s - i\tilde{Y}_s)\sin\omega t \\ &= -i\left[\oint_{C_2} \tilde{p}_t\, d\bar{z}' - \oint_{C_1} \tilde{p}_t\, d\bar{z}'\right] + \rho\omega^2\varepsilon\pi(r_2^2 - r_1^2)e^{-i\omega t} \\ &\quad + i\rho\left[\oint_{C_2} (\bar{u}' - i\bar{v}')(\tilde{u}' - i\tilde{v}')dz' - \oint_{C_1} (\bar{u}' - i\bar{v}')(\tilde{u}' - i\tilde{v}')dz'\right] \\ &\quad + i\rho\omega\varepsilon\left[\oint_{C_2} (\bar{u}' - i\bar{v}')\cos(\theta - \omega t)\, r_2 d\theta - \oint_{C_1} (\bar{u}' - i\bar{v}')\cos(\theta - \omega t)\, r_1 d\theta\right] \\ &\quad - \rho\frac{d}{dt}\iint v_r'(r_2, \theta - \varphi(r))\frac{r_2}{r\sin\beta(r)} e^{-i(\theta - \frac{\pi}{2} + \beta)}\, r\, dr\, d\theta\end{aligned} \tag{49}$$

The total pressure is given by Equation (3) and the integrals can be evaluated analytically by using the expressions (28–31).

CONCLUDING REMARKS

The unsteady forces can eventually be expressed in the form of stiffness matrix,

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} \tilde{X}_c/\varepsilon\ , & \tilde{X}_s/\varepsilon \\ \tilde{Y}_c/\varepsilon\ , & \tilde{Y}_s/\varepsilon \end{pmatrix} \begin{pmatrix} x = \varepsilon\cos\omega t \\ y = \varepsilon\sin\omega t \end{pmatrix} \tag{50}$$

The time average of the force component in the direction of whirling motion is given by $\frac{1}{2}(\tilde{Y}_c - \tilde{X}_s)$ and the sign of this quantity determines whether or not the fluid forces have destabilizing effects on the whirling motion. The sum $\frac{1}{2}(\tilde{X}_c + \tilde{Y}_s)$ gives the time average of the force component in the radial direction and thus the hydrodynamic stiffness. The ultimate goal of the present study is to examine these factors for realistic impeller-volute combinations.

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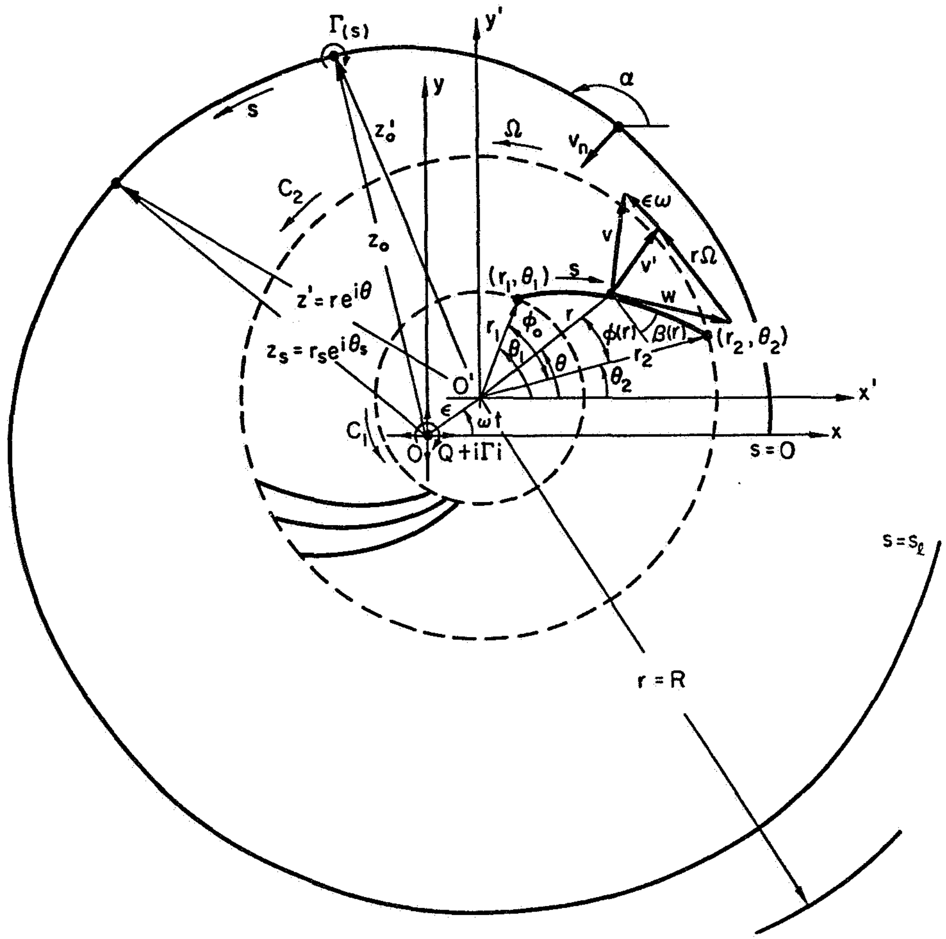

Figure 1. - Impeller and volute configuration.