

NASA-CR-172,487

NASA Contractor Report 172487

ICASE REPORT NO. 84-58

NASA-CR-172487
19850006876
7

ICASE

A COMPACT FINITE ELEMENT METHOD FOR ELASTIC BODIES

Milton E. Rose

Contract No. NAS1-17070, NAS1-17130

November 1984

LIBRARY COPY

JAN 15 1985

LANGLEY RESEARCH CENTER
LIBRARY, NASA
HAMPTON, VIRGINIA

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

FOR INFORMATION

NOT FOR CIRCULATION

8

1

1 RN/NASA-CR-172487

DISPLAY 08/2/1

85N15185**# ISSUE 6 PAGE 817 CATEGORY 39 RPT#: NASA-CR-172487

ICASE-84-58 NAS 1.26:172487 CNT#: NAS1-17070 NAS1-17130 84/11/00 40

PAGES UNCLASSIFIED DOCUMENT

UTTL: A compact finite element method for elastic bodies TLSP: Final Report

AUTH: A/ROSE, M. E.

CORP: National Aeronautics and Space Administration. Langley Research Center,
Hampton, Va. AVAIL.NTIS SAP: HC A03/MF A01

MAJS: /*ELASTIC BODIES/*FINITE DIFFERENCE THEORY/*FINITE ELEMENT METHOD

MINS: / LAMINATES/ OPERATORS (MATHEMATICS)/ THEOREMS

ABS: A nonconforming finite method is described for treating linear equilibrium problems, and a convergence proof showing second order accuracy is given. The close relationship to a related compact finite difference scheme due to Phillips and Rose is examined. A condensation technique is shown to preserve the compactness property and suggests an approach to a certain type of homogenization.

ENTER:

A COMPACT FINITE ELEMENT METHOD FOR ELASTIC BODIES

Milton E. Rose

Institute for Computer Applications in Science and Engineering

Abstract

A nonconforming finite element method is described for treating linear equilibrium problems, and a convergence proof showing second order accuracy is given. The close relationship to a related compact finite difference scheme due to Phillips and Rose [2] is examined. A condensation technique is shown to preserve the compactness property and suggests an approach to a certain type of homogenization.

Research was supported by the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-17070 and NAS1-17130 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Introduction

In a recent paper Phillips and Rose [2] described a compact system of finite difference equations to treat the equilibrium of elastic bodies. For any brick cell of volume $O(h^3)$ within the body these equations express, to second order accuracy, the relationship between the traction forces and the displacements on the faces of the cell which results when the cell is in isolated equilibrium. Global equilibrium then occurs when the net traction force across a face common to any two cells vanishes; this condition serves to eliminate the traction forces as variables and results in algebraic conditions for equilibrium which are expressed in terms of the displacement values on the sides of neighboring cells. These algebraic equations seem particularly suited to solution by iterative techniques which can exploit the parallelism of computer architectures.

In this method, since each cell is in isolated equilibrium, the potential energy in the cell is equal to half the work done by the traction forces and displacements on the surface of the cell. The total potential energy within the body can then be calculated, and the principle of minimum potential energy simply results in the condition that the net traction forces vanish across common cell faces.

This paper examines this idea when general volume cells are employed. An approximate equilibrium condition for each cell is obtained by means of the construction of a transmission matrix which, when applied to displacement values on the faces of the cell, yields approximate traction forces on cell faces thus producing equilibrium in the cell. We call these equations compact finite element equations. Equilibrium throughout the body is then obtained by determining displacement values on the faces of cells such that the net traction force vanishes on every face incident to two cells.

The transmission matrix arises as an approximation to a boundary integral operator on a cell. As we shall see, it is also closely related to the ordinary stiffness matrix in finite element methods.

We are content here to describe a method of construction which yields $O(h^2)$ convergence. This is done in Part I, which shows the equivalence to a special nonconforming finite element method. The method can easily be converted to a stress, rather than a displacement, formulation.

Part II discusses results which have a bearing upon the practical implementation of the general construction. Using nonconforming linear elements the compact equations for a tetrahedron are easily obtained. We show that the compactness property is preserved under a condensation technique; as a result, general cells can be handled without constructing an enlarged approximation basis in such cells which the general theory would require.

In the context of the results of [2] the practical implication of these results in three dimensions is: in order to approximate the displacements of a body occupying a volume Ω , we need only cover Ω by regular brick cells and introduce tetrahedral cells near the boundary of Ω . The compact difference equations for brick cells and the compact finite element equations for tetrahedral cells then establish the equilibrium conditions for individual cells in Ω . Global equilibrium in Ω is then obtained by imposing the algebraic balance of traction conditions throughout Ω . The general problem of developing effective iterative methods for this type of construction will be examined elsewhere.

The paper concludes by suggesting an approach to the homogenization problem using the condensation technique mentioned above.

Part I

Equilibrium of Linear Elastic Bodies

Notations:

In the following $x = (x_1, x_2, x_3)$ is a point in \mathbb{R}^3 and Ω is a bounded volume with boundary Γ . $\{\omega\}_M$ indicates a partition of Ω into M subvolumes, or cells; $\gamma = \gamma(\omega)$ is the boundary of ω on which \underline{n} is a unit outward normal. $\hat{\gamma} = \hat{\gamma}(\omega, \omega')$ indicates the surface incident to both ω and ω' and is called an interior face; faces of $\gamma(\omega)$ which lie in Γ are denoted by $\hat{\gamma}_\Gamma$.

$\Delta\omega$ denotes the volume of ω and $\Delta\gamma$ the surface area of γ . We assume positive constants c_0, c_1 which are independent of M such that if $h = O(1/M)$

$$0 < c_0 h^3 \leq \Delta\omega \leq c_1 h^3.$$

Finally, if v is a continuous function on Ω then v_γ denotes the values of v on a cell boundary γ .

I.1. General Results

In the following B is a material body occupying a volume Ω ; \underline{u} = displacement, τ = stress tensor, and ϵ = strain tensor. Linear theory leads to

$$\epsilon(u) = \text{grad}(\underline{u} + \underline{u}^t), \quad (1.1a)$$

$$\tau(u) = \tau(\epsilon(u)); \quad (1.1b)$$

in which (b) represents the constitutive relationship of the material. Both ϵ and τ are symmetric, and the reciprocal principle of Betti-Rayleigh (Timoshenko and Goodier [4]) is expressed by

$$W(\underline{u}, \underline{w}; \Omega) \equiv \frac{1}{2} \int_{\Omega} \epsilon(\underline{u}) \tau(\underline{w}) d\omega = W(\underline{w}, \underline{u}; \Omega). \quad (1.2)$$

Let

$$Lu \equiv \text{div } \tau(\underline{u}) - q^2 \underline{u}, \quad (1.3)$$

where q is a nonzero constant; the case $q = 0$ will be treated later. Then the boundary value problem

$$\begin{aligned} Lu &= 0 & \text{in } \Omega, \\ \underline{u} &= \underline{u}_{\Gamma} & \text{on } \Gamma \end{aligned} \quad (1.4)$$

describes the equilibrium of the body subject to a spring load. The potential energy is given by

$$U(\underline{u}; \Omega) \equiv W(\underline{u}; \Omega) + \frac{1}{2} q^2 \int_{\Omega} \underline{u}^2 d\omega. \quad (1.5)$$

The principle $U(\underline{v}; \Omega) = \min$ then leads to the solution \underline{u} of (1.4) and forms the basis for many finite element methods.

The solution of (1.4) is determined solely by the boundary data and may be represented by a solution operator E as

$$\underline{u}(x) = E(x)u_{\Gamma} \quad (1.6)$$

in which $E(x)$ reduces to the identity operator for $x \in \Gamma$. The traction force on Γ resulting from a displacement \underline{u} is given as

$$p = \tau(\underline{u}) \cdot \underline{n}|_{\Gamma}. \quad (1.7)$$

For the equilibrium solution (1.6) we may write

$$p \equiv p(\underline{u}_{\Gamma}) = \tau(E\underline{u}_{\Gamma}) \cdot \underline{n}|_{\Gamma} \equiv T_{\Gamma} \underline{u}_{\Gamma}; \quad (1.8)$$

we call the boundary operator T_{Γ} a transmission operator.

For arbitrary displacements $\underline{u}, \hat{\underline{u}}$, standard integral theorems are

$$\oint_{\Gamma} \underline{u}_{\Gamma}^t p d\gamma = 2U(\underline{u}; \Omega) + \int_{\Omega} \underline{u}^t L\underline{u} d\omega$$

and, using (1.2),

$$\oint_{\Gamma} (\underline{u}_{\Gamma}^t \hat{p} - \hat{\underline{u}}_{\Gamma}^t p) d\gamma = \int_{\Omega} (\underline{u}^t L\hat{\underline{u}} - \hat{\underline{u}}^t L\underline{u}) d\omega.$$

At equilibrium, $L\underline{u} = 0$; thus

$$\oint_{\Gamma} \underline{u}_{\Gamma}^t p(\underline{u}_{\Gamma}) d\gamma = 2U(\underline{u}; \Omega) \quad (1.9)$$

and

$$\oint_{\Gamma} \underline{u}_{\Gamma}^t p(\hat{\underline{u}}_{\Gamma}) d\gamma = \oint_{\Gamma} \hat{\underline{u}}_{\Gamma}^t p(\underline{u}_{\Gamma}) d\gamma. \quad (1.10)$$

Equation (1.9) simply states that at equilibrium the work due to the traction forces and displacements on Γ equals twice the potential energy in Ω ; equation (1.10) expresses the reciprocal principle (1.2) comparing two equilibrium states.

A representation formula for the transmission operator T_Γ can be obtained by setting

$$\hat{u}_\Gamma(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}'), \quad \underline{x}, \underline{x}' \in \Gamma \quad (1.11)$$

in (1.10) where δ is the Dirac function. Defining

$$T_\Gamma(\underline{x}, \underline{x}') \equiv p(\hat{u}_\Gamma(\underline{x}, \underline{x}')) \quad (1.12)$$

(1.10) leads to

$$p(\underline{x}; \underline{u}_\Gamma) = \oint_\Gamma \underline{u}_\Gamma^t(\underline{x}') T_\Gamma(\underline{x}, \underline{x}') d\gamma(\underline{x}') \equiv T_\Gamma(\underline{x}) \cdot (\underline{u}_\Gamma). \quad (1.13)$$

It follows that T_Γ is a symmetric operator. Also, since

$$2U(\underline{u}; \Omega) \geq q^2 \int_\Omega \underline{u}^2 d\omega \geq 0 \quad (1.14)$$

(1.9) results in

$$\oint_\Gamma \underline{u}^t p(\underline{u}_\Gamma) d\gamma \equiv \oint_\Gamma \underline{u}_\Gamma^t T_\Gamma(\underline{u}_\Gamma) d\gamma \geq q^2 \int_\Omega \underline{u}^2 d\omega \geq 0 \quad (1.15)$$

so that T_Γ is positive definite when $q^2 > 0$. We may summarize these results in

Theorem 1: Let B be in equilibrium due to prescribed surface displacements \underline{u}_Γ . Then the resulting traction forces on Γ are determined from \underline{u}_Γ by means of a symmetric positive definite transmission operator T_Γ by $\underline{p}(\underline{u}_\Gamma) = T_\Gamma(\underline{u}_\Gamma)$, and the work done by these traction forces is equal to twice the potential energy, i.e.,

$$\oint_{\Gamma} \underline{u}_\Gamma^t \underline{p}(\underline{u}_\Gamma) d\gamma \equiv \oint_{\Gamma} \underline{u}_\Gamma^t T_\Gamma(\underline{u}_\Gamma) d\gamma = 2U(\underline{u}; \Omega) \quad (1.16)$$

where $\underline{u} = E\underline{u}_\Gamma$.

I.2. The Equilibrium Problem with Constraints

Introduce a partition of Ω onto M volume cells $\{\omega\}_M$. Let V denote the class of continuous functions v on Ω which satisfy the boundary conditions $v = \underline{u}_\Gamma$ on Γ . The values of \underline{v} on the boundary surfaces $\gamma(\omega)$ of cells ω are $v_{\gamma(\omega)}$ or, simply, \underline{v}_γ .

For a given $\underline{v} \in V$ consider the problem of minimizing the potential energy of the system by displacements \underline{w} subject to the constraints $\underline{w}_{\gamma(\omega)} = \underline{v}_{\gamma(\omega)}$, for $\omega \in \Omega$, i.e.,

$$\min_{\underline{w}} U(\underline{w}; \Omega) \geq U(E\underline{u}_\Gamma; \Omega). \quad (1.17)$$

The solution is given by $\underline{w} = E\underline{v}_{\gamma(\omega)}$, $\omega \in \Omega$, and in each cell

$$\min_{\underline{w}} 2U(\underline{w}; \omega) = 2U(E\underline{v}_{\gamma(\omega)}; \omega) = \oint_{\gamma(\omega)} \underline{v}_\gamma^t \underline{p}(\underline{v}_\gamma) d\gamma. \quad (1.18)$$

Denote the total work due to traction forces against the constraints by $Q(\underline{v}; \Omega)$. Then, using (1.13),

$$Q(\underline{v}; \Omega) = \sum_{\omega \in \Omega} \int_{\gamma} \underline{v}_{\gamma}^t p(\underline{v}_{\gamma}) d\gamma = \sum_{\omega \in \Omega} \int_{\gamma} \underline{v}_{\gamma}^t T_{\gamma}(\underline{v}_{\gamma}) d\gamma \geq 2U(\underline{u}; \Omega) \geq 0. \quad (1.19)$$

Any two displacement constraints $\underline{v}, \underline{v}' \in V$ can be considered equivalent if $\underline{v}_{\gamma}(\omega) = \underline{v}'_{\gamma}(\omega)$, $\omega \in \Omega$, since they lead to identically equal equilibrium displacements in each cell, i.e., $\underline{w} - \underline{w}' = E(\underline{v}_{\gamma} - \underline{v}'_{\gamma}) \equiv 0$. The consequences may be stated as

Theorem 2: The total work $Q(\underline{v}; \Omega)$ due to traction forces $p(\underline{v}_{\gamma}) = T_{\gamma}(\underline{v}_{\gamma})$ and the constraints \underline{v}_{γ} is minimized by the class of constraints $\underline{v} \in V$ which satisfy $\underline{v}_{\gamma}(\omega) = \underline{u}_{\gamma}(\omega)$ where \underline{u} is the equilibrium displacement of problem (1.4).

The net traction force acting at a point $\hat{\xi}$ on a face $\hat{\gamma}$ common to neighboring cells ω, ω' will be denoted by

$$\underline{f}_{\hat{\gamma}}(\hat{\xi}, \underline{v}) \equiv p(\hat{\xi}, \underline{v}_{\gamma}) + p(\hat{\xi}, \underline{v}_{\gamma'}) = T_{\gamma}(\hat{\xi}) \underline{v}_{\gamma} + T_{\gamma'}(\hat{\xi}) \underline{v}_{\gamma'}. \quad (1.20)$$

Also we will let $\underline{p}_{\Gamma} = \underline{p}_{\Gamma}(\underline{v}_{\gamma}(\omega))$ indicate the traction force on the face of a cell ω which lies on Γ . Then summation by parts yields

$$Q(\underline{v}, \Omega) = \sum_{\omega \in \Omega} \oint_{\gamma(\Omega)} \underline{v}_{\gamma} p(\underline{v}_{\gamma}) d\gamma = \oint_{\Gamma} \underline{u}_{\Gamma} \underline{p}_{\Gamma} d\gamma + \sum_{\hat{\gamma} \in \Omega} \int_{\hat{\gamma}} \underline{v}_{\hat{\gamma}} \underline{f}_{\hat{\gamma}} d\hat{\gamma}. \quad (1.21)$$

Obviously, the conditions

$$\underline{f}_{\hat{\gamma}}(\hat{\xi}, \underline{v}) = 0, \quad \hat{\gamma} \in \Omega \quad (1.22)$$

are the Euler conditions for $Q(\underline{v}; \Omega) = \min$. The result is

Theorem 3: A necessary and sufficient condition that $\underline{v}_{\gamma(\omega)} = \underline{u}_{\gamma(\omega)}$, $\omega \in \Omega$, is that the net work done by traction forces vanishes on every interior face on which constraints are imposed; at equilibrium, $\underline{p}_{\Gamma} = p(\underline{u}_{\Gamma})$ on Γ in (1.21).

Introduce the norm

$$\|\underline{v}\|_h = \left(h \sum_{\hat{\gamma} \in \Omega} \int_{\hat{\gamma}} \underline{v}^t \underline{v} d\hat{\gamma} \right)^{1/2}. \quad (1.23)$$

If $\underline{w} = E \underline{v}_{\gamma}$,

$$\begin{aligned} \int_{\gamma(\omega)} \underline{v}_{\gamma}^t T_{\gamma}(\underline{v}_{\gamma}) d\gamma &= 2U(\underline{w}; \omega) \geq q^2 \int_{\omega} \underline{w}^t \underline{w} d\omega \\ &= \alpha q^2 h \oint_{\gamma(\omega)} \underline{v}_{\gamma}^t \underline{v}_{\gamma} d\gamma \end{aligned}$$

where $\alpha = 1 + O(h^2)$. Thus a lower bound for the lowest eigenvalue λ_{γ} of the transmission operator T_{γ} is

$$\lambda_{\gamma} = \alpha h q^2. \quad (1.24)$$

An estimate for the difference between the constrained and unconstrained problems can now be obtained: let $\underline{\varepsilon}_{\gamma} = \underline{v}_{\gamma} - \underline{u}_{\gamma}$ so that $\varepsilon_{\Gamma} = 0$. Then (1.21) yields

$$Q(\underline{\varepsilon}, \Omega) = \sum_{\omega \in \Omega} \oint_{\gamma(\omega)} \underline{\varepsilon}_{\gamma}^t T_{\gamma}(\underline{\varepsilon}_{\gamma}) d\gamma = \sum_{\hat{\gamma} \in \Omega} \int_{\hat{\gamma}} \underline{\varepsilon}_{\gamma} \underline{f}_{\gamma} d\hat{\gamma} \quad (1.25)$$

in which $\underline{f}_{\gamma} = \underline{f}_{\gamma}(\underline{\varepsilon}_{\gamma}, \underline{\varepsilon}_{\gamma}^-)$ so that

$$\alpha q^2 \|\varepsilon\|_h^2 \leq h^{-1} \|\varepsilon\|_h \|f(\underline{\varepsilon})\|_h,$$

i.e.,

$$\alpha q^2 \|\varepsilon\|_h \leq h^{-1} \|f(\varepsilon)\|_h.$$

I.3. An Approximation Method

In this section we describe a method for approximating a smooth solution \underline{u} of (1.4) with $O(h^2)$ accuracy.

Consider a typical cell ω whose boundary $\gamma(\omega)$ consists of k faces $\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_k$. Recall the representation (1.13) of the transmission operator T_Y as the integral operator

$$T_Y(\underline{x}; \underline{v}_Y) = \oint_Y T_Y(\underline{x}, \underline{\xi}) \underline{v}(\underline{\xi}) d\gamma(\underline{\xi})$$

in which $T_Y(\underline{x}, \underline{\xi})$ is the traction force due to a displacement $\delta(|\underline{x} - \underline{\xi}|)$ on γ . Let $\hat{\underline{\xi}}$ indicate the centroid of a face $\hat{\gamma}$ whose area is $\Delta\hat{\gamma}$. Define

$$[\underline{v}_Y]^t \equiv [v(\hat{\underline{\xi}}_1) \Delta\hat{\gamma}_1, v(\hat{\underline{\xi}}_2) \Delta\hat{\gamma}_2, \dots, v(\hat{\underline{\xi}}_k) \Delta\hat{\gamma}_k], \quad (1.27a)$$

$$T_Y^h(\underline{x})[\underline{v}_Y] \equiv \sum_{i=1}^k T_Y(\underline{x}, \hat{\underline{\xi}}_i) v(\hat{\underline{\xi}}_i) \Delta\hat{\gamma}_i, \quad (1.27b)$$

$$[T_Y^h] \equiv (T_Y(\hat{\underline{\xi}}_i; \hat{\underline{\xi}}_j)). \quad (1.27c)$$

The $k \times k$ matrix $[T_Y^h]$ is symmetric and positive definite and

$$p^h(\hat{\underline{\xi}}, \underline{v}_Y) \equiv T_Y^h(\hat{\underline{\xi}})[\underline{v}_Y] \quad (1.28)$$

approximates the traction force on a side $\hat{\gamma}$.

$$Q^h(\underline{v}; \omega) = [\underline{v}_{\gamma(\omega)}]^t [T_{\gamma(\omega)}^h] [\underline{v}_{\gamma(\omega)}] \quad (1.29)$$

approximates the work done by traction forces and displacements on $\gamma(\omega)$. The net traction force on a face $\hat{\gamma}$ common to adjacent cells ω and ω' is then approximated by

$$\underline{f}_{\hat{\gamma}}^h(\hat{\xi}, [\underline{v}]) \equiv T_{\hat{\gamma}}^h(\hat{\xi})[\underline{v}_{\hat{\gamma}}] + T_{\hat{\gamma}'}^h(\hat{\xi})[\underline{v}_{\hat{\gamma}'}] = \underline{p}^h(\hat{\xi}, \underline{v}_{\hat{\gamma}}) + \underline{p}^h(\hat{\xi}, \underline{v}_{\hat{\gamma}'}). \quad (1.30)$$

Let us agree to approximate the boundary condition $\underline{v}_{\Gamma} = \underline{u}_{\Gamma}$ for $v \in V$ by

$$\Delta \gamma_{\Gamma} \cdot \underline{v}(\hat{\xi}_{\Gamma}) = \int_{\gamma_{\Gamma}} \underline{u}_{\Gamma} d\gamma_{\Gamma}, \quad \gamma_{\Gamma} \in \Gamma \quad (1.31)$$

in which $\hat{\xi}_{\Gamma}$ indicates the centerpoint of a face γ_{Γ} of a cell which lies on Γ .

We propose to show that the solution \underline{u}^h of the following algebraic problem p^h provides an $O(h^2)$ approximation to the solution \underline{u} of (1.4) at points $\hat{\xi}$ of $\hat{\gamma} \in \Omega$.

Problem p^h : Solve

$$\underline{f}_{\hat{\gamma}}^h(\hat{\xi}; [\underline{v}]) = 0, \quad \hat{\gamma} \in \Omega \quad (1.32)$$

for $[\underline{v}_{\gamma}]$ subject to the boundary conditions (1.31).

First, using (1.29), define

$$Q^h(\underline{v}; \Omega) = \sum_{\omega \in \Omega} Q^h(\underline{v}; \omega). \quad (1.33)$$

Then summation by parts leads to

$$\begin{aligned} Q^h(\underline{v}; \Omega) &= \sum_{\omega \in \Omega} [\underline{v}_\gamma]^t [T_\gamma^h] [\underline{v}_\gamma] = \sum_{\hat{\xi}_\Gamma \in \Gamma} \underline{v}^t(\hat{\xi}_\Gamma) \tilde{p}^h(\hat{\xi}_\Gamma) \Delta \gamma_\Gamma \\ &\quad + \sum_{\hat{\gamma} \in \Omega} \underline{v}_\gamma(\hat{\xi}) f_\gamma^h(\hat{\xi}, [\underline{v}]) \Delta \hat{\gamma} \end{aligned} \quad (1.34)$$

in which $\tilde{p}^h(\hat{\xi}_\Gamma)$ represents an approximate traction force arising from a cell which has a face lying on Γ .

The value $1/2 Q^h(\underline{v}; \Omega)$ given by (1.33) is an approximation to the potential energy of the constrained system. The problem of minimizing $Q^h(\underline{v}; \Omega)$ among the class of algebraic constraints satisfying (1.31) leads, using (1.34), to (1.32) as Euler conditions.

Next, introduce the norm

$$[\underline{v}]_h \equiv \left(h \sum_{\hat{\xi} \in \Omega} \underline{v}^t(\hat{\xi}) \underline{v}(\hat{\xi}) \Delta \hat{\gamma}(\hat{\xi}) \right)^{1/2}. \quad (1.35)$$

From earlier results

$$Q^h(\underline{v}; \Omega) \geq \alpha q^2 [\underline{v}]_h^2 \quad (1.36)$$

where $\alpha = (1 + O(h^2))$.

With \underline{u} indicating the solution of (1.4), define

$$\varepsilon \equiv \underline{u} - \underline{v}. \quad (1.37)$$

Then ε satisfies the boundary conditions $\varepsilon(\underline{x}_\Gamma) = 0$. Also, from (1.32),

$$\underline{f}_{\hat{\gamma}}^h(\underline{\xi}; [\underline{\varepsilon}]) = \underline{f}_{\hat{\gamma}}^h(\underline{\xi}; [\underline{u}]). \quad \text{Hence (1.34) and (1.36) lead to the estimate}$$

$$\alpha q^2 \|\underline{\varepsilon}\|_h \leq h^{-1} \|\underline{f}_{\hat{\gamma}}^h[\underline{u}]\|_h. \quad (1.38)$$

Using (1.27) and (1.13), define

$$(\underline{p}^h - \underline{p})(\underline{u}_\gamma) \equiv T_\gamma^h[\underline{u}_\gamma] = T_\gamma(\underline{u}_\gamma). \quad (1.39)$$

Then, since $\underline{f}_{\hat{\gamma}}(\underline{u}) = 0$

$$\underline{f}_{\hat{\gamma}}^h[\underline{u}] = (\underline{f}_{\hat{\gamma}}^h - \underline{f}_{\hat{\gamma}})(\underline{u}) = (\underline{p}^h - \underline{p})(\underline{u}_\gamma + \underline{u}_{\gamma'}) \quad (1.40)$$

where $\hat{\gamma}$ is the face common to γ and γ' .

For illustration, consider a boundary value problem for $\nabla^2 u - q^2 u = 0$. In this case both the solution u and the kernel of the transmission operator can be represented as a superposition of exponential solutions of the form $\exp q(\underline{r} \cdot \underline{x})$. Then

$$|(\underline{p}^h - \underline{p})(\underline{u}_\gamma)| = q^2 O(h^3).$$

A similar superposition principle applies to the problem (1.4) when, as we may assume, the material properties do not vary in a cell. Using (1.40) we then obtain the estimate

$$[\underline{f}_{\gamma}^h(\underline{u})]_h = q^2 O(h^3) \quad (1.41)$$

and (1.38) yields

$$[\epsilon]_h = O(h^2). \quad (1.42)$$

Thus

Convergence of p^h : The solution \underline{u}^h of problem p^h converges to the solution \underline{u} of (1.4) with an accuracy $O(h^2)$ when \underline{u} is sufficiently smooth. By continuity this remains true when $q = 0$.

We may summarize matters as follows: for each cell which is in equilibrium assume that an approximate algebraic relationship can be established between the traction forces $[\underline{p}_{\gamma}^h]$ and displacements $[\underline{v}_{\gamma}]$ on the faces of the cell by means of a transmission matrix $[\underline{T}_{\gamma}^h]$, viz.,

$$[\underline{p}_{\gamma}^h] = [\underline{T}_{\gamma}^h][\underline{v}_{\gamma}]. \quad (1.43)$$

We call (1.43) a compact finite volume scheme. Next, establish global

equilibrium in Ω by imposing the balance of traction conditions

$\underline{f}_{\gamma}^h(\underline{v}) = 0$; this corresponds, at the algebraic level, to the elimination of stresses and strains in (1.4) so as to obtain a second order system of equilibrium of partial differential equations for the displacements. When the boundary conditions (1.31) are imposed, the resulting algebraic equations have a unique solution \underline{u}^h and $[\underline{u} - \underline{u}^h]_h = O(h^2)$ when \underline{u} is smooth.

I.4. An Abstract Formulation

Consider again the equilibrium equations $\underline{L}u = \text{div } \tau(\underline{u}) - q^2 u = 0$. We may, in principle, determine a family of solutions $\theta_i(\underline{x}, \underline{\xi})$, $i=1,2,\dots,k$ in any cell which has k faces such that

$$\theta_i(\underline{\xi}_j, \underline{\xi}_i) = \delta_{ij}.$$

Then

$$\underline{w}(\underline{x}) = \sum_{i=1}^k \theta_i(\underline{x}, \underline{\xi}_i) \underline{v}(\underline{\xi}_i) \quad (1.44)$$

provides a solution such that $\underline{w}(\underline{\xi}_i) = \underline{v}(\underline{\xi}_i)$ for $\underline{\xi}_i \in \hat{\gamma}_i$. In any cell we may write

$$\underline{w}(\underline{x}) = E(\underline{x}) P^h[\underline{v}_\gamma] \quad (1.45)$$

where $E(\underline{x})$ is the solution operator for the boundary value problem (1.4) and P^h is a projection operator. The traction forces which result are

$$\underline{p}^h(\underline{x})[\underline{v}_\gamma] = \tau(E P^h[\underline{v}_\gamma]) \cdot \underline{n}|_{\underline{x}} \equiv T_\gamma^h(\underline{x})[\underline{v}_\gamma]. \quad (1.46)$$

Thus

$$[\underline{p}_\gamma^h] = [T_\gamma^h][\underline{v}_\gamma] \quad (1.47)$$

with $[T_\gamma^h] = P_\gamma T_\gamma^h$ yields the required compact equations on the cell.

This abstract formulation indicates how higher order approximations can, at least in principle, be obtained. We shall not pursue this matter here, however.

The construction (1.44) was proposed in [3] as a general method for obtaining the solution of elliptic partial differential equations. The present paper clarifies the physical ideas involved and presents the estimates required to establish a general convergence proof.

I.5. Further Considerations

The previous discussion was focused upon the Dirichlet problem for (1.4) in which displacement values \underline{u}_Γ were prescribed on Γ . The numerical treatment of this type of boundary condition is described by (1.31). Problems in which the traction force is prescribed on a part of the boundary Γ^0 , say $\underline{p}(\underline{\xi}) = \underline{p}^0(\underline{\xi})$, $\underline{\xi} \in \Gamma^0$, can be treated by using (1.28) and imposing the traction condition in the form

$$\underline{T}_{\gamma^0}^h(\hat{\underline{\xi}})[\underline{v}_\gamma] = \underline{p}^0(\hat{\underline{\xi}}), \quad (1.48)$$

where $\hat{\underline{\xi}}$ indicates the centroid of the face of a cell boundary γ^0 which lies on Γ^0 .

Finally, instead of formulating the algebraic equilibrium equations in terms of the displacements by means of the balance of traction conditions $\underline{f}_\gamma^h(\underline{v}) = 0$, the tractions may be considered as the primary variables and the problem solved in terms of them as follows: First note that the tractions $[\underline{p}_\gamma^h]$ as given by (1.43) are necessarily compatible with the displacements $[\underline{v}_\gamma]$. From (1.24), when $q \neq 0$, the minimum eigenvalue of the transmission matrix $[\underline{T}_\gamma^h]$ was shown to approximate $\alpha h q^2 > 0$. We may thus set

$$[\underline{S}_\gamma^h] \equiv [\underline{T}_\gamma^h]^{-1} \quad (1.49)$$

and write

$$[\underline{v}_\gamma^h] = [S_\gamma^h][\underline{p}_\gamma]. \quad (1.50)$$

Consider

$$R^h(\underline{p}) \equiv \sum_{\omega \in \Omega} [\underline{p}_{\gamma(\omega)}]^t [S_{\gamma(\omega)}^h] [\underline{p}_{\gamma(\omega)}]. \quad (1.51)$$

Let us agree to use (1.50) to express a boundary condition of the form (1.31) in terms of $[\underline{p}_\gamma]$. Consider the variational problem

$$\max_{\underline{p}_\gamma} R^h(\underline{p}_\gamma) \quad (1.52)$$

where \underline{p}_γ satisfies the balance of traction conditions $f_{\hat{\gamma}}^h = 0$ across any interior face $\hat{\gamma}$ and also satisfies boundary conditions on Γ in the form just indicated. A summation by parts procedure similar to that indicated by (1.34) now leads to Euler equations which express conditions that the jump in displacement values of \underline{v}_γ^h across any interior face $\hat{\gamma}$ must vanish. Using (1.50), these conditions are expressible in terms of the traction variables on the faces of neighboring cells. The result (c.f. [4]) is simply a restatement of Castigliano's principle in discrete form.

Part II

Practical Developments

We now turn our attention to placing the previously described theoretical development into a practical framework. Our preliminary discussion will concern the Laplace equation in two-dimensions; this will enable us to illustrate several key features in a more familiar context. A later section

describes the construction details required to treat the equilibrium problem for the elastic body described in Part I.

Suppose Ω is a domain in \mathbb{R}^2 . Ω can be conveniently covered by rectangular cells as shown in Figure 1.

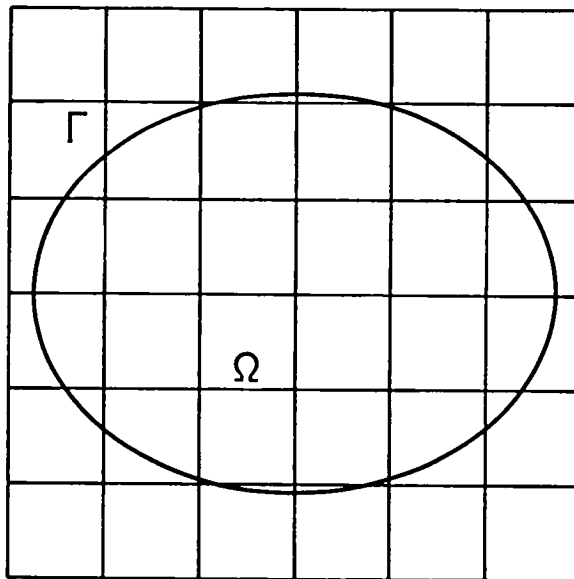


Figure 1. A covering of a domain Ω by rectangular cells.

Interior cells can be treated by compact schemes for rectangles; cells at the boundary can be reduced to the treatment of compact schemes for triangles.

Compact schemes for rectangles have been developed using the finite difference calculus in a manner described by Phillips and Rose [1], [2]. We shall indicate below how these can be obtained by a simple condensation process applied to more elementary results for triangular cells which we first discuss.

II.1. A Model Problem

In this section we treat the Laplace equation $\nabla^2 u = 0$ as a model problem.

a) Triangular Cells

Consider a triangular cell ω whose area is $\Delta\omega$. Choose a local coordinate system having the centroid of the vertices $\underline{x}_i, i=1,2,3$, as origin 0. The centerpoints of the sides are $\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3$, as shown in Figure 2.

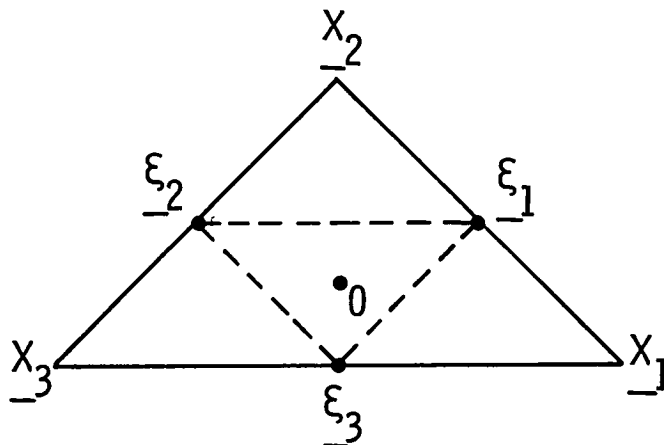


Figure 2. A triangular cell.

Let $\ell(\underline{x}, \underline{\xi}_i), i=1,2,3$, denote area coordinates for the triangle $(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3)$, i.e.,

$$\underline{x} = \sum_{i=1}^3 \ell(\underline{x}, \underline{\xi}_i) \underline{\xi}_i, \quad \sum_{i=1}^3 \ell(\underline{x}, \underline{\xi}_i) = 1. \quad (2.1)$$

The vertices $[\underline{\xi}]$ and $[\underline{x}]$ are thus related by

$$\begin{aligned}
 [\underline{\xi}] &= \Omega [\underline{x}], & \Omega &= 1/2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \\
 [\underline{x}] &= \Omega^{-1} [\underline{\xi}], & \Omega^{-1} &= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.
 \end{aligned} \tag{2.2}$$

Also, $\underline{n}(\underline{\xi}_i) \equiv \underline{n}_i$, $i=1,2,3$, indicates the outward normal vectors on the sides. Finally, $\Delta\gamma_i \equiv |\underline{x}_{i+1} - \underline{x}_i|$ is the length of a side containing $\underline{\xi}_i$ and

$$\underline{m}(\underline{\xi}_i) \equiv \Delta\gamma_i \cdot \underline{n}(\underline{\xi}_i) / \Delta\omega. \tag{2.3}$$

Noting our choice of origin, $\ell(\underline{x}, \underline{\xi})$ then has the form

$$\ell(\underline{x}, \underline{\xi}_i) = \left(\frac{1}{3} + \underline{x} \cdot \underline{m}(\underline{\xi}_i) \right). \tag{2.4}$$

The function

$$w(\underline{x}) = \sum_{i=1}^3 \ell(\underline{x}, \underline{\xi}_i) v(\underline{\xi}_i) \tag{2.5}$$

is a solution of $\nabla^2 w = 0$ for which $w(\underline{\xi}_i) = v(\underline{\xi}_i)$. The 'traction' defined by

$$p^h(\underline{\xi}_i, v) = \underline{n}^t(\underline{\xi}_i) \cdot \nabla w(\underline{\xi}_i) \tag{2.6}$$

is thus given by

$$p^h(\underline{\xi}_i, v) = \sum_{j=1}^3 \underline{n}^t(\underline{\xi}_i) \underline{n}(\underline{\xi}_j) (\Delta\omega)^{-1} \cdot v(\underline{\xi}_j) \Delta\gamma_j$$

so that

$$[p_Y^h] = [T_Y^h][v_Y] \quad (2.7)$$

is a compact scheme on ω with the transmission matrix

$$[T_Y^h] \equiv (n^t(\underline{\xi}_i) \underline{n}(\underline{\xi}_j)) \Delta\omega^{-1}. \quad (2.8)$$

Next consider two adjacent triangular cells ω, ω' having the side $\hat{\gamma}$ in common (Figure 3).

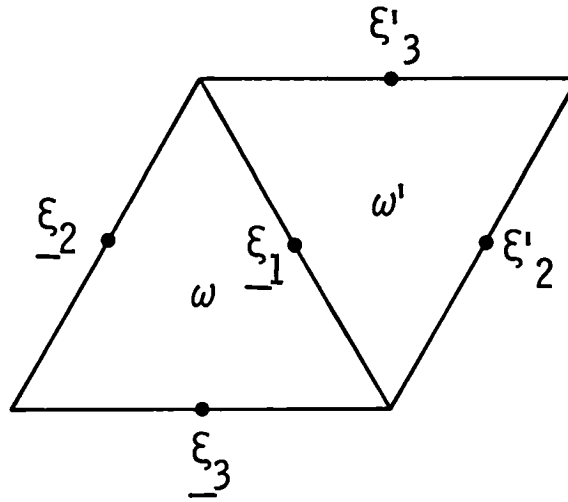


Figure 3. Two neighboring triangular cells ω, ω' .

The balance of traction condition at $\underline{\xi}_1$ is

$$f_Y^h(\underline{\xi}_1, v) \equiv p_Y^h(\underline{\xi}_1, v) + p_Y^h(\underline{\xi}_1, v) = 0 \quad (2.9)$$

which reduces, using (2.8) to

$$\begin{aligned} \Delta\gamma_1 \left(\frac{1}{\Delta\omega} + \frac{1}{\Delta\omega'} \right) v(\underline{\xi}_1) &= - \frac{1}{\Delta\omega} \sum_{i=2}^3 \underline{n}_1^t \underline{n}_i v(\underline{\xi}_i) \Delta\gamma_i \\ &\quad - \frac{1}{\Delta\omega'} \sum_{i=2}^3 (\underline{n}_1')^t \underline{n}_i' v(\underline{\xi}_i') \Delta\gamma_i'. \end{aligned} \quad (2.10)$$

Suppose ω, ω' are acute triangles; then

$$\underline{n}_1^t \underline{n}_i < 0, \quad i=2,3. \quad (2.11)$$

Define

$$v_{ij} \equiv - \underline{n}_i^t \underline{n}_j \Delta\gamma_j / \Delta\gamma_i,$$

$$\rho \equiv \Delta\omega' / (\Delta\omega + \Delta\omega'),$$

$$\rho' \equiv \Delta\omega / (\Delta\omega + \Delta\omega'). \quad (2.12)$$

Then $v_{1j} > 0$ and (2.10) leads to

$$v(\underline{\xi}_1) = \sum_{i=2}^3 \{ \rho v_{1i} v(\underline{\xi}_i) + \rho' v_{1i}' v(\underline{\xi}_i') \}. \quad (2.13)$$

However,

$$\oint_{\gamma(\omega)} \underline{n} d\gamma = 0 \quad (2.14)$$

so that

$$v_{12} + v_{13} = - \underline{n}_1^t (\underline{n}_2 \Delta\gamma_2 + \underline{n}_3 \Delta\gamma_3) / \Delta\gamma_1 = 1. \quad (2.15)$$

Thus

$$\sum_{i=2}^3 (\rho v_{1i} + \rho' v_{1i}') = \rho + \rho' = 1. \quad (2.16)$$

Hence $v(\underline{\xi}_1)$ as given by (2.13) is an average with positive weights of the neighboring values of v in the triangles.

Figure 4 illustrates two situations.

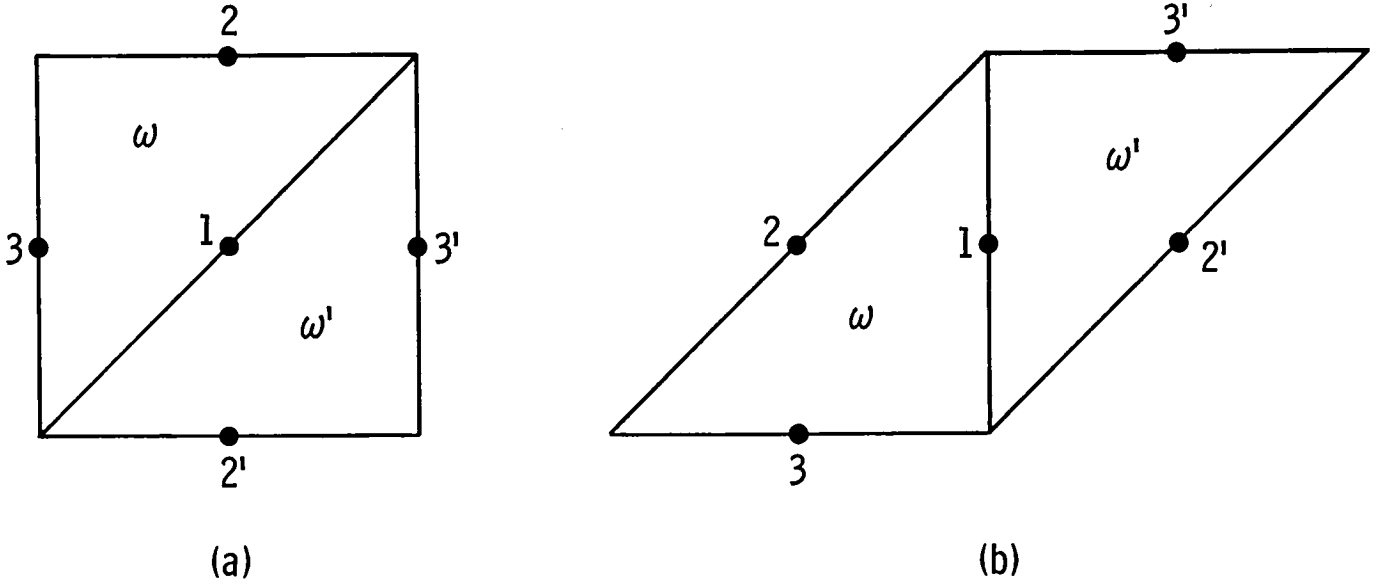


Figure 4. Cases of two neighboring right triangular cells.

In the situation in Figure 4a, (2.13) leads to

$$v(\underline{\xi}_1) = 1/4 (v(\underline{\xi}_2) + v(\underline{\xi}_3) + v(\underline{\xi}_2') + v(\underline{\xi}_3')) \quad (2.17)$$

which is a familiar 5-point approximation to $\nabla^2 u = 0$. In the situation in Figure 4b, the coefficients of the terms $v(\underline{\xi}_3)$, $v(\underline{\xi}_3')$ in (2.13) vanish so that

$$v(\underline{\xi}_1) = 1/2 (v(\underline{\xi}_2) + v(\underline{\xi}_2')). \quad (2.18)$$

Here we observe a difference between the result in this paper and that of the typical finite element treatment of $\nabla^2 u = 0$ (Zienkiewicz [5]). In the latter a linear approximation to $\nabla^2 u = 0$ in ω is conventionally described in terms of area functions $L(\underline{x}, \underline{x}_i)$ for the triangle using the vertex points $\{\underline{x}_i\}$, viz.,

$$w(\underline{x}) = \sum_{i=1}^3 L(\underline{x}, \underline{x}_i) v(\underline{x}_i). \quad (2.19)$$

This construction leads to conforming approximations, i.e., approximations which are continuous across the sides of neighboring triangles. In contrast, the approximation (2.5) is discontinuous across cell sides except at the center points (\underline{x}_i) , i.e., (2.5) is nonconforming. For conforming approximations the standard 5-point finite difference formula expresses the relationship between a vertex value and its four neighboring vertex values, shown in Figure 5, viz.,

$$v_1 = 1/4 (v_2 + v_{2'} + v_3 + v_{3'}) \quad (2.20)$$

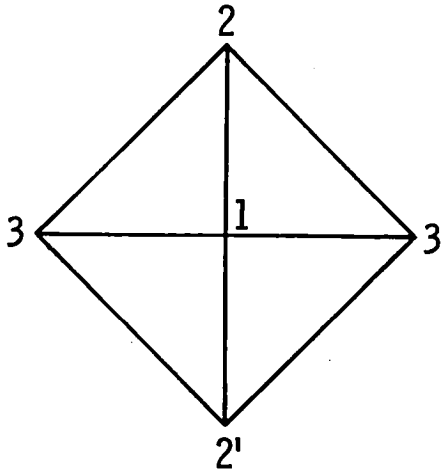


Figure 5. The relationship of vertices involved in the 5-point formula (2.20) using conforming approximations in cells.

This example is often used to point to the close relationship between finite element and finite difference formulations of this problem. However, the theory given in Part I shows that the scheme represented by (2.17) and (2.18) also converges with $O(h^2)$ accuracy and thus represents a valid, if less conventional, finite difference approximation to $\nabla^2 u = 0$.

The next example will help illustrate an important property of approximations based on the nonconforming approximation (2.5) which is not shared by the conforming approximation (2.19).

b) Rectangular Cells

We propose to show that, for $\nabla^2 u = 0$, compact schemes for rectangular cells can be developed by one of two equivalent means. One uses the general approximation method described in Part I. The other uses a condensation process based upon the example just given for triangular cells. Both lead to the same result. For reasons to be described, the condensation technique fails when the conforming approximation (2.19) is employed.

1) An application of the previous theory:

The functions $(1, x, y, x^2 - y^2)$ are solutions of $\nabla^2 u = 0$. From these we may construct solutions $\theta(\underline{x}, \underline{\xi}_i)$ such that

$$w(\underline{x}) = \sum_{i=1}^3 \theta(\underline{x}; \underline{\xi}_i) v(\underline{\xi}_i) \quad (2.21)$$

is a solution of $\nabla^2 u = 0$ in ω such that $w(\underline{\xi}_i) = v(\underline{\xi}_i)$.

Write $\nabla^2 u = 0$ in system form as

$$r_x + s_y = 0$$

$$r = u_x, \quad s = u_y \quad (2.22)$$

Figure 6 illustrates a square cell on whose sides the indices $(i \pm 1, j \pm 1)$ indicate points corresponding to the centerpoints of sides of length $2h$.

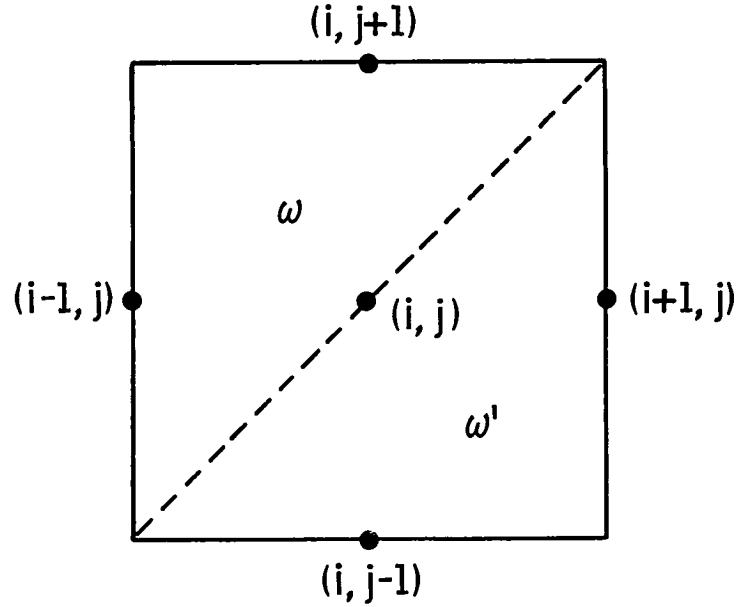


Figure 6. A square cell having indicial labeled sides of length $2h$ based upon a triangular decomposition.

The 'traction' values at the centerpoints of the sides, labeled counterclockwise, are

$$[p_Y^h] \equiv [s_{i,j+1}, -r_{i-1,j}, -s_{i,j-1}, r_{i+1,j}]^t. \quad (2.23)$$

The transmission matrix $[T_Y^h]$ provides the relationship between $[p_Y^h]$ and $[u_Y]$.

Introduce the standard finite difference notations

$$\delta_x r_{i,j} = (r_{i+1,j} - r_{i-1,j})/2h,$$

$$\mu_x r_{i,j} = (r_{i+1,j} + r_{i-1,j})/2$$

with corresponding definitions for $\delta_y s_{i,j}, \mu_y s_{i,j}$. In [1] the following compact scheme was described:

$$\delta_x r_{i,j} + \delta_y s_{i,j} = 0$$

$$\mu_x r_{i,j} = \delta_x u_{i,j}, \quad \mu_y s_{i,j} = \delta_y u_{i,j},$$

$$\mu_x r_{i,j} - \frac{h^2}{2} \delta_x r_{i,j} = \mu_y s_{i,j} - \frac{h^2}{2} \delta_y s_{i,j}. \quad (2.24)$$

The fact that the approximation (2.21) leads to (2.24) may be verified by noting that each of the functions $[1, x, y, x^2 - y^2]$ in the approximation basis satisfies these equations. We may write (2.24) as

$$r_{i\pm 1,j} = [\delta_x \pm h^{-1}(\mu_x - \mu_y)]u_{i,j}$$

$$s_{i,j\pm 1} = [\delta_y \pm h^{-1}(\mu_y - \mu_x)]u_{i,j}.$$

The definition (2.23) for $[p_Y^h]$ then allows (2.24) to be expressed in the form $[p_Y^h] = [T_Y^h][u_Y]$.

The algebraic equations which result from the balance of traction force conditions between cell sides have been studied in [1]. Standard iterative methods appear to work quite successfully.

ii) A condensation method:

The representation (2.8) for the transmission matrix for a triangular cell ω relates the 'traction' values $[p_Y^h]$ on the sides of a triangular cell ω to the corresponding displacement values $[v_Y]$ on the sides. As a result, with reference to Figure 6, $p_{i,j+1}^h, p_{i-1,j}^h, p_{i+1,j}^h, p_{i,j-1}^h$ can be expressed in terms of the values $v_{i,j+1}, v_{i-1,j}, v_{i,j-1}, v_{i+1,j}$ as well as $v_{i,j}$. Using (2.13), we can also solve for $v_{i,j}$ in terms of $v_{i\pm 1,j}$ and $v_{i,j\pm 1}$. The result of this condensation process, the details of which we shall not present here, is again the compact scheme (2.24).

More generally, this process leads to a compact scheme for any cell based upon a condensation procedure for triangular cells, i.e., for the nonconforming approximation (2.5) condensation preserves compactness.

An explanation of this is: Recall that for a triangle, the components of $[v_Y]$ in (2.7) are the values

$$[v_Y] \equiv [v(\underline{x}_1)\Delta\gamma_1, v(\underline{x}_2)\Delta\gamma_2, v(\underline{x}_3)\Delta\gamma_3]. \quad (2.25)$$

Because a linear nonconforming element is employed, each component is the value of the integral

$$\int_{\gamma_i} v d\gamma_i. \quad (2.26)$$

In contrast, when using the conforming approximation (2.19) a trapezoidal approximation to (2.26) using vertex values is natural. In the latter case, however, the imposition of the balance of 'traction' face condition does not reduce the number of displacement unknowns; this is in contrast to the nonconforming approximation (2.5) and highlights a basic difference between these two approaches.

II.2. Compact Schemes for Elastic Bodies

In \mathbb{R}^3 write $\underline{\epsilon}$ and $\underline{\tau}$ in vector form as

$$\begin{aligned}\underline{\epsilon} &= (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12})^t, \\ \underline{\tau} &= (\tau_{11}, \tau_{22}, \tau_{33}, 2\tau_{23}, 2\tau_{13}, 2\tau_{12})^t.\end{aligned}\tag{2.27}$$

Let $\partial_i \equiv \frac{\partial}{\partial x_i}$ and $\underline{\partial} \equiv (\partial_1, \partial_2, \partial_3)$. Define symmetric matrix operators $D_1(\underline{\partial}), D_2(\underline{\partial})$ by

$$D_1(\underline{\partial}) \equiv \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \end{pmatrix}; \quad D_2(\underline{\partial}) \equiv \begin{pmatrix} 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}, \tag{2.28}$$

and let

$$D(\underline{\partial}) \equiv \begin{pmatrix} D_1(\underline{\partial}) \\ D_2(\underline{\partial}) \end{pmatrix} \tag{2.29}$$

with $\underline{u} = (u_1, u_2, u_3)^t$. Then (1.1) can be written

$$\begin{aligned}\underline{\epsilon}(\underline{u}) &= D(\underline{\partial})\underline{u} \\ \underline{\tau}(\underline{u}) &= C\underline{\epsilon}(\underline{u})\end{aligned}\tag{2.30}$$

where C is a symmetric 6×6 positive definite matrix. We shall restrict our discussion to the equilibrium problem

$$\begin{aligned} \operatorname{div} \tau(\underline{u}) &\equiv D^t(\underline{\partial})\tau(\underline{u}) = 0 & \text{in } \Omega, \\ \underline{u} &= \underline{u}_\Gamma & \text{on } \Gamma, \end{aligned} \quad (2.31)$$

(i.e., $q = 0$ in (2.4)).

Let $\underline{n}(\underline{\xi}) = (n_1(\underline{\xi}), n_2(\underline{\xi}), n_3(\underline{\xi}))$ denote the outward unit normal at a point $\underline{\xi}$ which lies at the center of a face $\gamma(\underline{\xi})$ on the boundary $\gamma(\omega)$ of a cell. If \underline{u} is a solution of (2.31) the traction force $\underline{p}(\underline{\xi})$ can be calculated as

$$\underline{p}(\underline{\xi}) = D^t(\underline{n}(\underline{\xi})) \cdot \operatorname{CD}(\underline{\partial})\underline{u} \Big|_{\underline{\xi}}. \quad (2.32)$$

On a cell ω having k faces whose centerpoints are $\underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_k$ suppose $\phi(\underline{x}; \underline{\xi}_i), i=1, 2, \dots, k$ are a basis set of solutions of (2.31) satisfying $\phi(\underline{\xi}_i, \underline{\xi}_j) = \delta_{ij}$. Then

$$\underline{w}(\underline{x}) = \sum_{i=1}^k \phi(\underline{x}, \underline{\xi}_i) \underline{v}(\underline{\xi}_i) \quad (2.33)$$

provides a solution of (2.31) which interpolates the constraint values $\underline{v}(\underline{\xi}_i)$ on $\gamma(\omega)$. Recalling the notation

$$[\underline{v}_\gamma] \equiv [\underline{v}(\underline{\xi}_1)\Delta\gamma_1, \underline{v}(\underline{\xi}_2)\Delta\gamma_2, \dots, \underline{v}(\underline{\xi}_k)\Delta\gamma_k] \quad (2.34)$$

let

$$\Psi(\underline{x}; \underline{\xi}_i) = \phi(\underline{x}; \underline{\xi}_i) / \Delta\gamma_i. \quad (2.35)$$

Then, from (2.32)

$$\underline{p}_\gamma^h(\underline{\xi}) = \sum_{i=1}^k \left(D^t(\underline{n}(\underline{\xi})) \operatorname{CD}(\underline{\partial}) \Psi(\underline{x}, \underline{\xi}_i) \right)_{\underline{x}=\underline{\xi}} \cdot \underline{v}(\underline{\xi}_i) \Delta\gamma_i \quad (2.36)$$

so that if

$$T_Y^h(\underline{\xi}, \underline{\xi}_1) \equiv D^t(\underline{n}(\underline{\xi})) CD(\underline{\partial}) \Psi(\underline{x}, \underline{\xi}_1) \Big|_{\underline{x}=\underline{\xi}}$$

$$T_Y^h(\underline{\xi}) \equiv [T_Y(\underline{\xi}, \underline{\xi}_1), T_Y(\underline{\xi}, \underline{\xi}_2), \dots, T_Y(\underline{\xi}, \underline{\xi}_\ell)] \quad (2.37)$$

then

$$[T_Y^h] = (T_Y^h(\underline{\xi}_1, \underline{\xi}_j)) \quad (2.38)$$

is the transmission matrix and

$$p_Y^h(\underline{\xi}) = T_Y^h(\underline{\xi}) [\underline{v}_Y]. \quad (2.39)$$

When $k = 4$, i.e., the case of a tetrahedron, the linear basis $(1, x_1, x_2, x_3)$ allows a simple construction. When $k \neq 4$, the practical development of a basis set of equilibrium solutions in (2.33) can present a formidable problem when the constitutive matrix C in (2.30) is general. For $k = 6$ the result given in [2] can be applied directly when the cell is a brick; we shall describe this below. Otherwise the condensation technique indicated earlier can be used. Hence we can confine ourselves to examining the development for a tetrahedron.

The Tetrahedral Cells

Let $(\underline{\xi}_1)$ indicate the centerpoints of the faces (γ_1) of a tetrahedral cell ω ; $\Delta\omega$ is the volume and $(\Delta\gamma_1)$ the areas of its faces. Similar to the example given earlier, use the centroid of the cell as origin and introduce volume coordinates $\mathfrak{L}(\underline{x}, \underline{\xi}_1)$ for the tetrahedron $(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4)$ where

$$\ell(\underline{x}, \underline{\xi}_i) = \frac{1}{4} + \Delta\gamma_i (\underline{n}^t(\underline{\xi}_i) \cdot \underline{x}) / \Delta\omega. \quad (2.40)$$

Then

$$w(\underline{x}) = \sum_{i=1}^4 \ell(\underline{x}, \underline{\xi}_i) v(\underline{\xi}_i) \quad (2.41)$$

provides for the construction (2.33).

Define (compare to (2.8))

$$T_Y^h(\underline{x}, \underline{\xi}) \equiv (\Delta\omega)^{-1} D^t(\underline{n}(\underline{x})) CD(\underline{n}(\underline{\xi})). \quad (2.42)$$

Using (2.36)-(2.39) we obtain

$$p_Y^h(\underline{\xi}_i) = \sum_{j=1}^4 T_Y^h(\underline{\xi}_i, \underline{\xi}_j) v(\underline{\xi}_j) \Delta\gamma_j \equiv T_Y^h(\underline{\xi}_i) \cdot [\underline{v}_Y]. \quad (2.43)$$

This provides a compact scheme on ω .

Consider two tetrahedra ω, ω' having the face γ_1 in common. The balance of traction condition at $\underline{\xi}_1$ is

$$p_Y^h(\underline{\xi}_1) + p_{Y'}^h(\underline{\xi}_1) = 0. \quad (2.44)$$

This leads, using (2.43), to

$$\begin{aligned} (T_Y^h(\underline{\xi}_1, \underline{\xi}_1) + T_{Y'}^h(\underline{\xi}_1, \underline{\xi}_1)) v(\underline{\xi}_1) \Delta\gamma_1 &= - \sum_{j=2}^4 T_Y^h(\underline{\xi}_1, \underline{\xi}_j) v(\underline{\xi}_j) \Delta\gamma_j \\ &\quad - \sum_{j=2}^4 T_{Y'}^h(\underline{\xi}_1, \underline{\xi}_j') v(\underline{\xi}_j') \Delta\gamma_j' \end{aligned}$$

which may be solved for $\underline{v}(\underline{\xi}_1)$ in the form

$$\underline{v}(\underline{\xi}_1)\Delta\gamma_1 = \sum_{j=2}^4 [S_{\gamma}(\underline{\xi}_1, \underline{\xi}_j)\underline{v}(\underline{\xi}_j)\Delta\gamma_j + S_{\gamma'}(\underline{\xi}_1, \underline{\xi}_j')\underline{v}(\underline{\xi}_j')\Delta\gamma_j']. \quad (2.45)$$

The equations should be compared to a similar result for $\nabla^2 u = 0$ given by (2.13). We believe, but have not been able to show, that the positive definiteness of the constitutive matrix C as well as geometrical properties arising from ω and ω' induce properties in the coefficient matrices $S_{\gamma}, S_{\gamma'}$ in (2.45) which can allow standard iterative methods to be applied effectively.

Brick Cells

In order to develop a compact scheme for a brick cell, we could employ a condensation technique using the result just given for a tetrahedron. However, as indicated earlier, in this case a compact scheme has been derived from a slightly different viewpoint in Phillips and Rose [2]. Here we will be satisfied to present their scheme in a form closer to the notation presented earlier in this paper.

Similar to the finite difference notations used in (2.24), let

$$\delta_1 \theta_{i,j,k} = (\theta_{i+1/2,j,k} - \theta_{i-1/2,j,k})/2h_1$$

$$\mu_1 \theta_{i,j,k} = (\theta_{i+1/2,j,k} + \theta_{i-1/2,j,k})/2$$

using similar definitions for $\mu_2, \delta_2, \mu_3, \delta_3$. Let the faces of ω be defined by the planes $x_1 = (i \pm 1/2)\Delta x_1$, $x_2 = (j \pm 1/2)\Delta x_2$, $x_3 = (k \pm 1/2)\Delta x_3$, where

$\Delta x_v = 2h_v, v = 1, 2, 3$. If $\underline{n}_1 = (1, 0, 0)^t, \underline{n}_2 = (0, 1, 0)^t, \underline{n}_3 = (0, 0, 1)^t$, then the outward normal vectors on the faces are $\pm \underline{n}_v, v = 1, 2, 3$. We let $\underline{p}^h(\pm \underline{n}_v)$ denote the traction force on the face for which $\pm \underline{n}_v$ is the outward normal. Finally, again write $\underline{\delta} = (\delta_1, \delta_2, \delta_3)$.

The compact scheme given in [2] can, with a suitable choice of certain nonessential parameters, be written as

$$\underline{p}^h(\pm \underline{n}_v) = \pm D^t(\underline{n}_v) CD(\underline{\partial}) \underline{u} + h_v^{-1} (\mu_v \underline{u} - \underline{\lambda}) \quad v = 1, 2, 3 \quad (2.46)$$

where

$$\underline{\lambda} = \left(\sum_{v=1}^3 h_v^{-1} \mu_v \underline{u} \right) / \left(\sum_{v=1}^3 h_v^{-1} \right)$$

using the notation indicated in (2.28)-(2.29).

The balance of flux condition across the face $x_v = \text{const.}$ common to neighboring cells $\omega, \tilde{\omega}$ is, then, simply

$$\begin{aligned} 0 = \underline{f}_v^h = \underline{p}^h(\underline{n}_v) + \tilde{\underline{p}}^h(-\underline{n}_v) &= D^t(\underline{n}_v) CD(\underline{\partial}) \underline{u} - D^t(\underline{n}_v) \tilde{CD}(\underline{\partial}) \tilde{\underline{u}} \\ &+ h_v^{-1} (\mu_v (\underline{u} + \tilde{\underline{u}}) - (\underline{\lambda} + \tilde{\underline{\lambda}})) \end{aligned} \quad (2.47)$$

In [2] a plane stress calculation involving an isotropic material was performed satisfactorily using several standard iterative methods. We refer the reader to that paper for details.

II.3. Homogenization

In the treatment of bonded materials, it is often useful to replace the laminate by an approximately equivalent homogeneous material. The methods considered in this paper offer an algebraic approach to this problem which we shall briefly describe.

With reference to Figure 7, consider two bonded thin materials L and L' within which ω and ω' are typical small rectangular cells. We wish to treat ω and ω' as a single cell $\tilde{\omega}$ with its own characteristic properties. Specifically, we wish to describe a compact scheme for $\tilde{\omega}$ on the basis of the material properties reflected in the transmission matrices for ω and ω' . In order to accomplish this we exploit the principle that condensation preserves compactness.

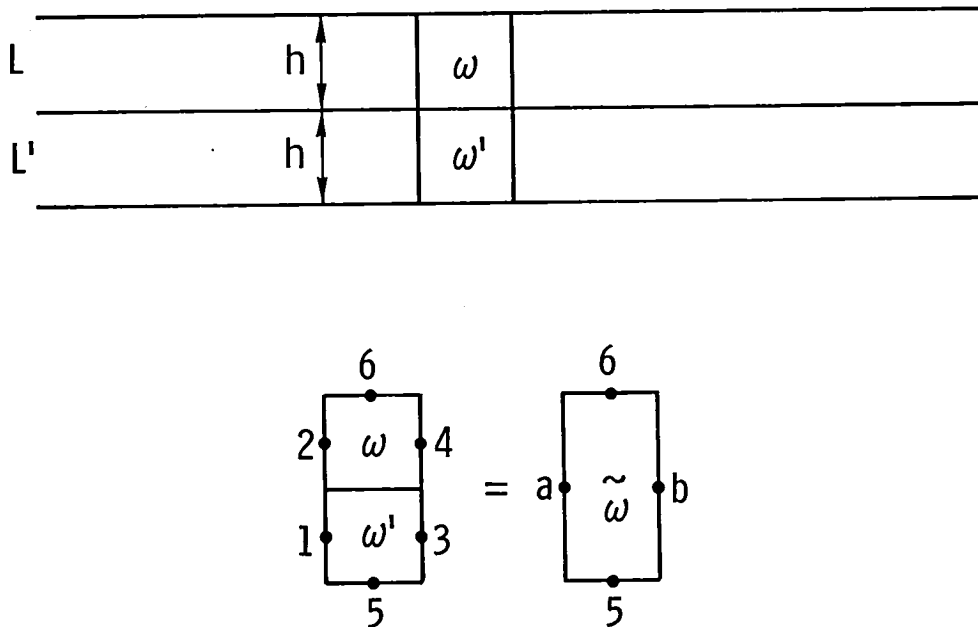


Figure 7. Homogenization of a laminate.

Assume transmission matrices T_{ω} and $T_{\omega'}$ are known. As described earlier, we may use a condensation technique to express the tractions on those sides of ω and ω' which lie on the boundary of $\tilde{\omega}$ in terms of the displacements which correspond to these sides. With reference to the figure, set $\underline{v}_1 = \underline{v}_2 = \underline{v}_a/2$, $\underline{v}_3 = \underline{v}_4 = \underline{v}_b/2$ and, in terms of \underline{v}_a , \underline{v}_b and \underline{v}_5 , \underline{v}_6 , calculate $\underline{p}_a = (\underline{p}_1 + \underline{p}_2)/2$ and $\underline{p}_b = (\underline{p}_3 + \underline{p}_4)/2$. We thus obtain a compact scheme for $\tilde{\omega}$ which relates the tractions $(\underline{p}_a, \underline{p}_b, \underline{p}_5, \underline{p}_6)$ to the displacement values $(\underline{v}_a, \underline{v}_b, \underline{v}_5, \underline{v}_6)$ and for which the transmission matrix $T_{\tilde{\omega}}$ incorporates the material properties of both L and L' . By this means, we suggest, L and L' can be treated as if it were a single material \tilde{L} .

References

1. T. N. PHILLIPS AND M. E. ROSE, "A finite difference scheme for a class of first-order elliptic partial differential equations," Comput. Math. Appl., to appear, 1984.
2. T. N. PHILLIPS AND M. E. ROSE, "A finite difference scheme for the equilibrium equations of elastic bodies," SIAM J. Sci. Statist. Comput., to appear.
3. M. E. ROSE, "Weak-element approximations to elliptic differential equations," Numer. Math., Vol. 24, 1975, pp. 185-204.
4. S. P. TIMOSHENKO AND J. N. GOODIER, Theory of Elasticity, McGraw-Hill, New York, 1970.
5. O. C. ZIENKIEWICZ, The Finite Element Method in Engineering Science, McGraw-Hill, London, 1974.

1. Report No. NASA CR-172487 ICASE Report No. 84-58		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle A COMPACT FINITE ELEMENT METHOD FOR ELASTIC BODIES				5. Report Date November 1984	
				6. Performing Organization Code	
7. Author(s) Milton E. Rose				8. Performing Organization Report No. 84-58	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665				10. Work Unit No.	
				11. Contract or Grant No. NAS1-17070, NAS1-17130	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546				13. Type of Report and Period Covered Contractor Report	
				14. Sponsoring Agency Code 505-31-83-01	
15. Supplementary Notes Langley Technical Monitor: J. C. South, Jr. Final Report					
16. Abstract A nonconforming finite element method is described for treating linear equilibrium problems, and a convergence proof showing second order accuracy is given. The close relationship to a related compact finite difference scheme due to Phillips and Rose [2] is examined. A condensation technique is shown to preserve the compactness property and suggests an approach to a certain type of homogenization.					
17. Key Words (Suggested by Author(s)) finite element finite difference transmission operator elasticity homogenization			18. Distribution Statement 39 - Structural Mechanics 64 - Numerical Analysis Unclassified - Unlimited		
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 39	22. Price A03		

