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EXISTENCE AND NON-UNIQUENESS OF SIMILARITY  
SOLUTIONS OF A BOUNDARY LAYER PROBLEM

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SOLUTIONS OF A BOUNDARY LAYER PROBLEM

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**Abstract**

This work considers a Blasius boundary value problem with inhomogeneous lower boundary conditions  $f(0) = 0$  and  $f'(0) = -\lambda$  with  $\lambda$  strictly positive. The Crocco variable formulation of this problem has a key term which changes sign in the interval of interest. It is shown that solutions of the boundary value problem do not exist for values of  $\lambda$  larger than a positive critical value  $\lambda^*$ . The existence of solutions is proved for  $0 < \lambda < \lambda^*$  by considering an equivalent initial value problem. However, for  $0 < \lambda < \lambda^*$ , solutions of the boundary value problem are found to be nonunique. Physically, this non-uniqueness is related to multiple values of the skin friction.

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## Introduction

The simplest example of Prandtl's boundary layer theory is the flow along a thin, semi-infinite flat plate ( $y = 0, x > 0$ ) immersed in an incompressible liquid of low viscosity flowing with a given constant velocity in the  $x$  direction (Schlichting, 1955). The boundary layer flow in this situation is governed by the Blasius equation

$$f''' + ff'' = 0 \quad (1)$$

subject to the boundary conditions

$$f(0) = 0, f'(0) = 0, \text{ and } f'(\infty) = 1 \quad (2)$$

where the dimensionless streamfunction  $f(\eta)$  is  $\psi(x,y)/(2\nu x)^{1/2}$  with  $\psi$  being the dimensional streamfunction,  $\nu$  the kinematic viscosity, and  $\eta$  the dimensionless similarity variable  $y/(2\nu x)^{1/2}$ . The classical solution of Blasius of (1) and (2) was based on a coordinate perturbation method. It involved a power series expansion in  $\eta$  about the origin and an asymptotic expansion for large  $\eta$ . For the coordinate perturbation method to be consistent, the regions in which the inner and outer expansions are valid must overlap so the expansions can be matched. A rigorous analysis of the Blasius problem (as well as the more general Falkner-Skan problem) was not available until Weyl (1941) used function theoretic techniques to study the existence and uniqueness of solutions of the Blasius equation. Weyl showed that the radius of convergence of the series solution of the Blasius equation lay between 2.08 and 3.11. This approach has been further extended by Hussaini

(1971) to treat the case of the Blasius equation with three-point boundary conditions, a type of situation which arises in connection with the mixing problem behind an infinite step.

A simple proof of Weyl's results for the Falkner-Skan equation has been provided by Coppel (1960) using the theory of differential equations. Coppel approached the Falkner-Skan equation

$$f'''' + ff'' + \beta(1 - f'^2) = 0 \quad (3)$$

by embedding the boundary conditions at  $y = 0$  in (2) in the class of problems with boundary conditions  $f(0) = a$  and  $f'(0) = b$  where  $a$  and  $b$  are arbitrary non-negative constants. He obtained bounds on  $f''(0)$  of the form

$$\frac{4}{3} \beta < [f''(0)]^2 < \frac{4}{3} \beta + \frac{1}{3}. \quad (4)$$

The special cases of the Blasius equation ( $\beta = 0$ ) and the Homann equation ( $\beta = 1/2$ ) were considered separately.

For the Blasius problem, the value of  $f''(0)$  is .469600... . It should be noted that once this value has been determined, the Blasius two-point boundary value problem can be solved in a straightforward manner as an initial value problem. For the Blasius problem,  $f''(0)$  can easily be obtained using homotopy methods (Rosenhead, 1963). The success of the homotopy technique depends critically, however, on the identically zero values of  $f(0)$  and  $f'(0)$ .

Callegari and Friedman (1968) have noted that while Weyl's iterative technique for solution of the Blasius problem is convergent, in practice it

cannot be carried beyond two iterations. As an alternative, they considered the problem in Crocco variables. In this approach, the tangential velocity  $u = f'(\eta)$  becomes the new independent variable while the new dependent variable is the shear stress  $g(u) = f''(\eta)$ . The Blasius problem now belongs to a class of nonlinear boundary value problems of the form

$$g(u)g'(u) + 2uk(u) = 0, \quad 0 < u < 1 \quad (5)$$

$$g'(0) = 0 \quad (6a)$$

$$g(1) = 0 \quad (6b)$$

where  $k(u)$  is a non-negative continuous function for  $u$  in the closed interval  $[0,1]$ . For the Blasius problem,  $k(u) = 1$ . Using techniques of analytic function theory, Callegari and Friedman prove the existence, uniqueness, and analyticity of solutions of the Blasius problem. Their proofs are critically dependent, however, on the fact that the function  $uk(u)$  in their singular differential equation does not change sign on the interval  $0 < u < 1$ . Thus, if  $k(u)$  is non-negative and the solution exists at a point  $u_0$ , then  $g'(u_0) < 0$  so  $g(u)$  is a decreasing function.

The analyticity of solutions of equation (5) with  $k(u) = 1$  and the two point boundary conditions

$$g(u_1) = 0, \quad g'(u_2) = 0 \quad (7)$$

has also been discussed by Callegari and Nachman (1978). This problem has applications to the flow behind weak expansion and shock waves and the boundary layer flow on a conveyor belt.

The present work considers a two-point boundary value problem consisting of the Blasius equation (1) and the nonhomogeneous boundary conditions

$$f(0) = 0, \quad (8)$$

$$f'(0) = -\lambda,$$

together with the outer boundary condition

$$f'(\infty) = 1 \quad (9)$$

where  $\lambda > 0$ . The Crocco variable representation of this boundary value problem is

$$gg'' + (x - \lambda) = 0, \quad 0 < x < 1 + \lambda \quad (10)$$

$$g'(0) = 0 \quad \text{and} \quad g(1 + \lambda) = 0 \quad (11)$$

where  $x = \{f'(\eta) + \lambda\}$  and  $g(x) = f''(\eta)$ . Unlike the usual Blasius problem with homogeneous conditions at  $\eta = 0$  which leads to (5), the second term in equation (10) changes sign in the interval of interest. Hence,  $g(x)$  will not be monotone on the interval. A critical condition present in previous work on the Blasius problem is thus violated here.

In the next section, we show that if a solution to the boundary value problem (10) and (11) exists, then there is an upper bound  $\lambda^*$  such that  $\lambda < \lambda^*$ . Section 3 proves the existence of a continuous solution of (10) and (11) when  $0 < \lambda < \lambda^*$ . However, as shown in section 4, a solution for fixed  $\lambda$



in this range is not unique. The existence of  $\lambda^*$  and the non-uniqueness of solutions of the boundary value problem are previously unsuspected properties of (10) and (11) directly traceable to the fact that  $x - \lambda$  is not purely non-negative on the entire interval of interest.

## 2. Dependence on the Parameter $\lambda$

Suppose that a solution  $g(x)$  exists for the boundary value problem (10) and (11). Then, the identity

$$(gg')' = gg'' + (g')^2$$

can be used to write equation (10) in the form

$$(gg')' + (x - \lambda) = (g')^2. \quad (12)$$

Let  $g(0) = \alpha$ . Then, using the fact that  $g'(0) = 0$ , integrating equation (12) from 0 to  $x$  and integrating the result a second time from 0 to  $X = 1 + \lambda$  gives that

$$x^2(x - 3\lambda)/6 = \alpha^2/2 + \int_0^x dx \int_0^x g'^2(s)ds. \quad (13)$$

The right-hand side of equation (13) is intrinsically positive and thus

$$x^2(x - 3\lambda) > 0.$$

Setting  $X = 1 + \lambda$  now gives that

$$\lambda < 1/2. \quad (14)$$

Thus, if a solution to the boundary value problem (10) and (11) exists,  $\lambda$  must be bounded above by a finite number  $\lambda^*$ , and an upper bound on  $\lambda^*$  is given by (14). In fact, while (14) shows that a bound on  $\lambda$  is required, it is a rather crude upper bound as it is obtained by simply noting that the right-hand side of (13) is positive. A sharper upper bound can be obtained by an analysis which obtains a sharper positive lower bound for the right-hand side of (13). This analysis will be done in a subsequent paper. In section 4 of the present paper, numerical results are given which show that  $\lambda^* = 0.3541078\dots$

### 3. Existence of Solutions

Consider now the existence of solutions of the boundary value problem (10) and (11) for  $0 < \lambda < \lambda^*$ . The required result will be shown by proving the following theorem for an equivalent initial value problem.

Theorem: There exists at least one positive constant  $\alpha$  such that the initial value problem consisting of equation (10) and the initial conditions

$$g(0) = \alpha, \quad g'(0) = 0 \quad (15)$$

has a continuous solution which is positive for  $0 \leq x < 1 + \lambda$  and  
 $g(1 + \lambda) = 0.$

Proof: As a preliminary step, the initial value on  $g$  in (15) will be normalized to unity and dependence on  $\alpha$  transferred into the governing differential equation. This is accomplished by the transformation

$$g(x) = \alpha h(t) \quad \text{with} \quad x = \alpha^{2/3} t \quad (16)$$

which leads to the initial value problem

$$hh'' + t - L = 0 \quad (17)$$

$$h(0) = 1, \quad h'(0) = 0 \quad (18)$$

with

$$L(\alpha, \lambda) = \alpha^{-2/3} \lambda. \quad (19)$$

Let  $y(t)$  be the vector  $(y_1(t), y_2(t))^t$  with  $y_1 = h(t)$  and  $y_2 = h'(t)$ . Then, (17) and (18) can be written as the first order system

$$dy/dt = F(t, y)^t \quad (20)$$

$$y(0) = (1, 0)^t \quad (21)$$

with  $F = (F_1, F_2)^t$  and

$$F_1(t, y) = y_2(t) \quad (23)$$

$$F_2(t, y) = (L - t)/y_1(t). \quad (24)$$

Thus,  $F_1$  and  $F_2$  are continuous and bounded functions of  $t$  and  $y$  provided  $|t| < \infty$ ,  $|y_1| > 0$ , and  $|y_2| < \infty$ . Further, with these restrictions on  $t$  and  $y$ , partial derivatives of  $F$  with respect to the components of  $y$  are continuous and bounded. Thus,  $F$  satisfies a Lipschitz condition on the set in  $(t, y)$ -space with  $|t|$  and  $|y_2|$  bounded and  $|y_1 - 1| < 1 - \epsilon$  with  $0 < \epsilon < 1$ .

The Picard-Lindelof Theorem (Coddington and Levinson, 1955) now gives the existence of a unique continuous solution of (20) and (21), and hence (17) and (18) on the interval  $|t| < t_0$  with  $t_0 > 0$ . Further, for  $t$  in this interval, if  $h(t)$  is non-zero, then

$$h'(t) = \int_0^t (L - s)/h(s) ds. \quad (25)$$

In particular, if  $h(t_0^-)$  is non-zero, then  $h(t_0)$  is non-zero and  $|h'(t_0)|$  is bounded. The solution of (17) and (18) can thus be continuously extended past  $t_0$ . Two cases must now be considered.

1. A continuous solution exists for all bounded  $t$ : Consider the range  $0 < t \leq L$ . Then, by (25), as  $h(0)$  is positive, both  $h(t)$  and  $h'(t)$  are positive, monotone increasing functions. Further,  $h'(t)$  is bounded above by  $L^2/2$  and has a local maximum at  $t = L$ . Beyond  $t = L$ , the factor  $L - s$  in (25) changes sign and becomes negative, and  $h'(t)$  becomes a monotone decreasing function which passes through zero at the point  $t^*$  defined by

$$\int_0^L (L - s)/h(s) ds = \int_L^{t^*} (s - L)/h(s) ds. \quad (26)$$

For  $L < t < t^*$ ,  $h'(t)$  is positive and remains bounded above by  $L^2/2$ . Consequently,  $h(t)$  is monotone increasing on this range and has a finite positive maximum value  $h^*$  at  $t^*$ .

For  $t > t^*$ ,  $h'(t)$  is strictly negative and both  $h(t)$  and  $h'(t)$  are monotone decreasing. Two possibilities now exist. Either  $h(t)$  remains positive for all  $t > t^*$ , or there is a finite  $T$  at which  $h(T) = 0$ .

The first possibility leads to a contradiction. Suppose there is a  $\delta$  such that  $0 < \delta < h^* < \infty$  and  $h(t) > \delta$  for all  $t > t^*$ . Then,

$$h'(t) = - \int_{t^*}^t (s - L)/h(s) ds < -(1/\delta) \int_{t^*}^t (s - L) ds. \quad (27)$$

This implies that as  $t$  tends to  $\infty$ ,  $h(t)$  remains positive but  $h'(t)$  tends to  $-\infty$ , which is a contradiction.

If the solution  $h(t)$  of the initial value problem can be continuously extended for all bounded  $t$ , there must thus be a finite  $T$  at which  $h(T) = 0$ . This, in turn, requires that if  $g(x)$  is the corresponding solution of the original initial value problem (10) and (15), then there is a finite  $X$  given by

$$X = \alpha^{2/3} T \quad (28)$$

at which  $g(X) = 0$ . As by (15)  $g'(0) = 0$ ,  $g(x)$  will be a non-negative continuous solution of the boundary value problem (10) and (11) if  $X = 1 + \lambda$ . Equation (28) now shows that one appropriate choice for the initial value  $\alpha$  is

$$\alpha(\lambda) = [(1 + \lambda)/T]^{3/2}. \quad (29)$$

2. The solution  $h(t)$  cannot be continuously extended to all bounded values of  $t$ : In this case, there is a finite value  $\tau$  such that a continuous solution exists for  $0 \leq t < \tau$ , but no continuous solution exists for  $t > \tau$ .

To treat this case, we first note that  $\tau$  must be greater than  $t^*$ . For if  $h(t)$  can be continued past  $t_0$  to some point  $t_1$  less than  $t^*$ , then  $h(t)$  is monotone increasing,  $h(t_1)$  is positive, and  $|h'(t_1)| < L^2/2$ . Consequently,  $h(t)$  can be continued past  $t_1$  and in particular can be continued to  $t^*$ . As  $h(t^*)$  is positive and  $h'(t^*) = 0$ ,  $h(t)$  can now be continuously extended beyond  $t^*$ . For  $t > t^*$ , because  $t^* > L$ , the term  $t - L$  in (17) is now strictly positive, so  $h'(t)$  will be strictly negative and both  $h(t)$  and  $h'(t)$  will be monotone decreasing.

Two possibilities must now be considered. Either there is a  $T$  in the range  $t^* < T < \tau$  such that  $h(T) = 0$ , or  $h(t) > 0$  for  $t^* < t < \tau$ . In the former case, we can proceed as before and define an appropriate  $\alpha$  in terms of  $\lambda$  and  $T$  by (29). Therefore, it is only necessary to consider the second alternative.

Suppose now that  $h(t)$  is positive for  $t < \tau$ . Then, it is clear that the limit of  $h(t)$  as  $t$  tends to  $\tau$  exists and hence  $h(\tau)$  is well defined. Further, if  $h(\tau) > 0$ , then  $|h'(\tau)|$  is bounded. Under these circumstances, a continuous solution  $h(t)$  would exist in a neighborhood of  $\tau$ , and in particular, for some  $t > \tau$ . This would contradict the definition of  $\tau$ . Consequently, we must have  $h(\tau) = 0$ . An appropriate initial value  $\alpha$  in terms of  $\lambda$  and  $\tau$  is now defined by (29) with  $T$  replaced by  $\tau$ .

The theorem is now established.

The above proof shows that for  $0 < \lambda < \lambda^*$  there is at least one positive initial value  $\alpha$  which leads to a solution of the boundary value problem. In the case when  $\lambda = 0$ , Callegari and Friedman showed that there was in fact exactly one positive  $\alpha$ ; i.e., the solution of their boundary value problem is unique. Their uniqueness proof is critically dependent, however, on the fact that the term  $2uk(u)$  in their equation, which plays the role of the  $x - \lambda$  term in equation (10), is non-negative over the full range of  $x$  values. By contrast,  $x - \lambda$  in (10) clearly violates this property. Indeed, as will be seen in the next section, for  $0 < \lambda < \lambda^*$ , there are two positive values of  $\lambda$  which lead to solutions of the boundary value problem. Hence, for this range of  $\lambda$ , solutions of the boundary value problem (10) and (11) will exist but will not be unique.

#### 4. Non-uniqueness of Solutions

To examine the dependence of solutions on  $\lambda$  and  $\alpha$ , the initial value problem

$$f'''' + ff'' = 0 \tag{30}$$

$$f(0) = 0, \quad f'(0) = -\lambda, \quad f''(0) = \alpha$$

was solved numerically. This problem in the original similarity variables corresponds to the Crocco variable initial value problem (10) and (11). In this formulation, it is clear that the initial value  $\alpha$ , physically, is related to the skin friction.

The initial value problem (30) was first solved for fixed  $\lambda$  to determine the limiting value  $f'(\infty)$  as a function of  $\alpha$ . Results of the calculations are shown in Figure 1. For values of  $\alpha$  at which the curves cross the horizontal line  $f'(\infty) = 1$ , a solution of the boundary value problem (1), (8), and (9) is obtained.

Consider first the curve for  $\lambda = 0$ , which is the usual Blasius problem considered by Calligari and Friedman. In this case,  $f'(\infty)$  is a monotone increasing function of  $\alpha$  which crosses the line  $f'(\infty) = 1$  at  $\alpha_0 = .46960\dots$ . This value of  $\alpha_0$  is well known from the homotopy technique.

For  $\lambda > 0$ , the curves in Figure 1 have a local minimum value of  $f'(\infty)$  which will be denoted by  $M(\lambda)$ . Calculations show that  $M(\lambda)$  is an increasing function of  $\lambda$ . When  $\lambda = \lambda^*$ ,  $M(\lambda^*) = 1$ , and there will thus be exactly one value of  $\alpha$ ,  $\alpha^*$  say, such that a solution of the initial value problem (30) is also a solution of the boundary value problem (1), (8), and (9). For  $\lambda > \lambda^*$ ,  $M(\lambda) > 1$  and there will thus be no solutions of the boundary value problem. When  $0 < \lambda < \lambda^*$ ,  $M(\lambda) < 1$ , and hence there will be two values of  $\alpha$  which give solutions of the boundary value problem, i.e., non-uniqueness. These values lie in the ranges  $0 < \alpha_1 < \alpha^*$  and  $\alpha^* < \alpha_2 < \alpha_0$ .

To accurately determine  $\lambda^*$  and  $\alpha^*$  and refine the points at which the curves in Figure 1 cross the line  $f'(\infty) = 1$ , the basic initial value solver for fixed  $\lambda$  was augmented by an iteration procedure on  $\alpha$  which terminated when  $|f'(\infty) - 1|$  was less than  $10^{-9}$ . Figure 2 shows the resulting  $\alpha$ 's as a function of  $\lambda$ . Critical values were found to be



$$\lambda^* = 0.3541078\dots \quad \text{and} \quad \alpha^* = 0.2180238\dots . \quad (31)$$

The non-uniqueness of solutions of the boundary value problem (1), (8), and (9) for  $0 < \lambda < \lambda^*$  is clearly evident in Figure 2.

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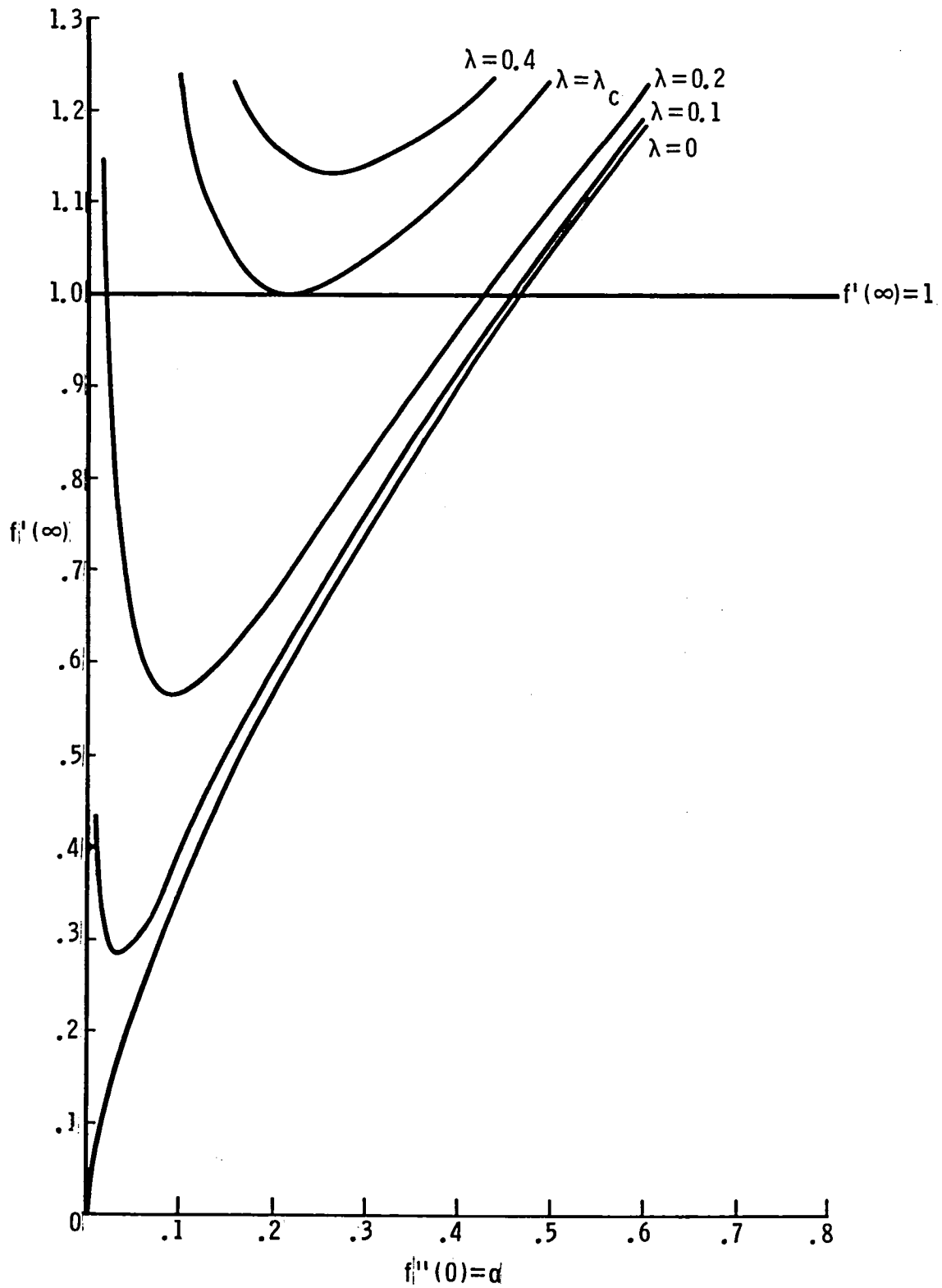


Figure 1. The limiting value  $f'(\infty)$  as a function of the initial value  $f''(0) = \alpha$  for  $\lambda = 0, 0.1, 0.2, \lambda_c$ , and  $0.4$ . At the critical value  $\lambda_c = \lambda^*$ , the minimum value  $M(\lambda^*) = 1$ .

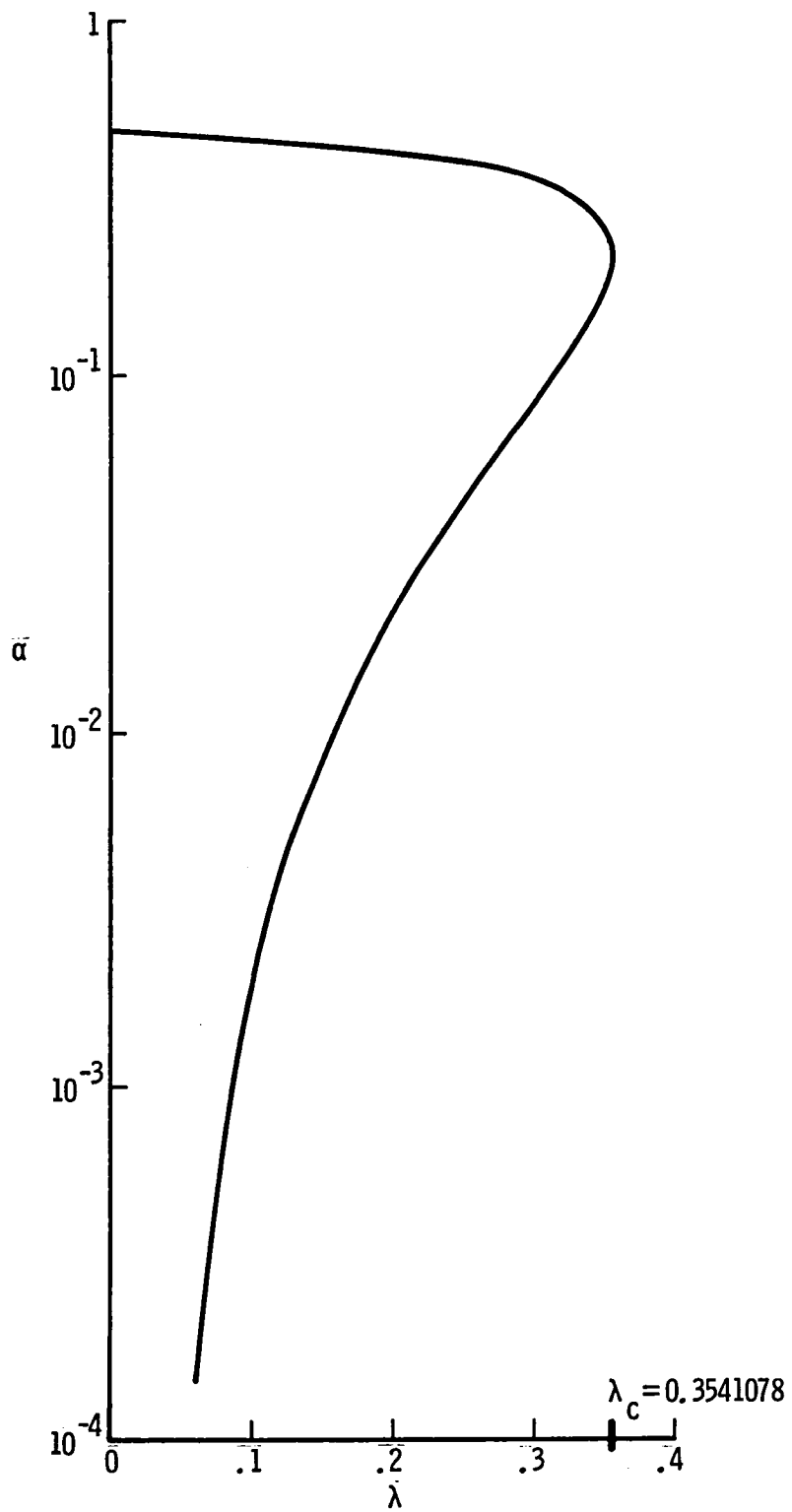


Figure 2. Values of the parameter  $\alpha = f''(0)$  for which  $f'(\infty) = 1$  as a function of  $\lambda$ .



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