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Summary

A system of equations is developed for calculating spatial derivatives of flow quantities behind an expansion fan. For steady two-dimensional inviscid flow, equations in terms of the curvature of the streamline behind the fan are developed. Taylor series expansions of flow quantities within the fan are used and boundary conditions are satisfied to the first and second order so that the curvature of the characteristics in the fan may be determined. The system of linear equations for the spatial derivatives is then developed.

An example of an application of the method is outlined for the problem of shock coalescence in which asymmetric effects are included as derivatives in the circumferential direction. The solution of the coalescence problem may require values for spatial derivatives of the flow variables behind a resulting expansion fan.

Introduction

To solve supersonic-flow problems numerically, it often becomes necessary to know spatial derivatives of the flow quantities. Numerical computation of derivatives by use of difference formulas can sometimes cause difficulties or inaccuracies in the entire numerical scheme. The problem becomes less severe if analytical or semianalytical means of obtaining the derivatives can be found. Included in the first part of this paper is a method for finding spatial derivatives of flow quantities behind two-dimensional, nonuniform flow over a convex corner (i.e., an expansion corner). Taylor series expansions (ref. 1) are made for variables within the expansion fan. Boundary counditions are then used to obtain first- and second-order solutions for the variables and the curvature of the characteristics within the fan. Finally, equations for the flow derivatives behind the fan are obtained in terms of the curvature of the streamline behind the fan.

This method may be applied to the case of shock coalescence including asymmetric effects and can be combined with a sonic boom extrapolation program in reference 2. In that situation, coalescence of shocks often occurs at some distance from the body. This paper contains a description of the entire coalescence system including the resulting shocks, expansions, and contact surface. The asymmetric effects are felt through a second set of governing equations developed by taking the cross derivatives (derivatives in the cross plane) of the original governing equations. It is in solving the asymmetric part of the coalescence problem that it becomes necessary to know derivatives of the flow variables in the axial and radial directions. For calculations away from the body, some of the resulting shocks or expansions are often so weak that flow variables through them may be considered continuous. In reference 2, a "continuous" approximation for flow variables is made at the shock or the expansion of the opposite family which results from the solution of the coalescence problem. This approximation limits the application of that method to cases for which the assumption of continuous flow variables is appropriate. The extension of that method to coalescence problems for which curved shocks of any strength may occur is discussed in reference 3, and the equations developed in the first part of this paper extend the applicability of the method to coalescence problems in which expansion waves of finite strength occur.

The second part of this paper describes the coalescence problem and gives a description of how the equations derived herein are incorporated into that solution.

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Symbols

A	leading characteristic of expansion fan
a	speed of sound
$a_1^{(0)}$	zeroth-order term for speed of sound in region I
B	last characteristic of expansion fan
C^+	any characteristic within expansion fan
D	vertex of convex (expansion) corner
$F_1,F_2,$ F_3,F_4	intersecting surfaces in shock coalescence problem
f_1	wall ahead of convex (expansion) corner
f_2	wall behind convex (expansion) corner
f_2'	slope of wall behind convex (expansion) corner
f_2''	change in slope in wall behind convex (expansion) corner
G	$= \psi - g_1(\sigma)$
g_1	function defining leading characteristic of expansion fan
g_2	function defining trailing characteristic of expansion fan
g_2'	curvature of trailing characteristic of expansion fan
H	enthalpy
h	slipstream surface in coalescence problem
$\mathbf{i}$	unit vector in x direction
$\mathbf{j}$	unit vector in y direction
l	function defining streamline through expansion fan

n	direction normal to streamline
O	order
P	dummy variables
p	pressure
q_m	limit on maximum speed in inviscid, steady flow
R	gas constant
r	radial coordinate
S	entropy
T	temperature
u	velocity in radial direction in polar coordinates
u^*	velocity in x direction in Cartesian coordinates
$\mathbf{V}$	velocity vector
v	velocity in angular direction in polar coordinates
v^*	velocity in y direction in Cartesian coordinates
$X(\Psi)$, $R(\Psi)$	intersection of surfaces in coalescence problem (see fig. 3)
x,y	Cartesian coordinates
β	shock angles; angle of trailing characteristic in fan
γ	ratio of specific heats
θ	flow angularity
μ	Mach angle
ρ	density
σ	unit vector in radial direction
ψ	unit vector in angular direction
σ,ψ	polar coordinates
Ψ	cross-flow direction in coalescence problem

Subscripts:

1	properties in region I; properties in region (1) in coalescence problem
2	properties in region II; properties in region (2) in coalescence problem
3	region ahead of expansion fan in coalescence problem
4,5	regions behind shock and expansion fan in coalescence problem
r	first derivative with respect to r, $\partial/\partial r$
rr	second derivative with respect to r in coalescence problem, $\partial^2/\partial r^2$
x	first derivative with respect to x, $\partial/\partial x$
xx	second derivative with respect to x in coalescence problem, $\partial^2/\partial x^2$
$\Psi\Psi$	second derivative with respect to Ψ in coalescence problem, $\partial^2/\partial\Psi^2$
σ	first derivative with respect to σ, $\partial/\partial\sigma$
ψ	first derivative with respect to ψ, $\partial/\partial\psi$

Superscripts:

0	zeroth order
1	first order, $\partial/\partial\sigma$

A bar over a symbol indicates a quantity in region III (see fig. 1). A prime on a symbol indicates a first derivative. A double prime on a symbol indicates a second derivative.

Part I—Development of Flow Derivatives

Definition of Problem and Assumptions Made

The expansion wave problem is illustrated in figure 1. Assume the flow is around a convex corner with the vertex D at the origin (0,0). Let $y=f_1(x)$ for $x<D$ (D^-) and $y=f_2(x)$ for $x>D$ (D^+) denote the sides of the corner. Flow ahead of the corner in region I is known to the characteristic line C^+ at DA. The expansion fan (region III) is covered by C^+ lines from point D. The last C^+ line, DB, separates the fan from the flow field behind the corner.

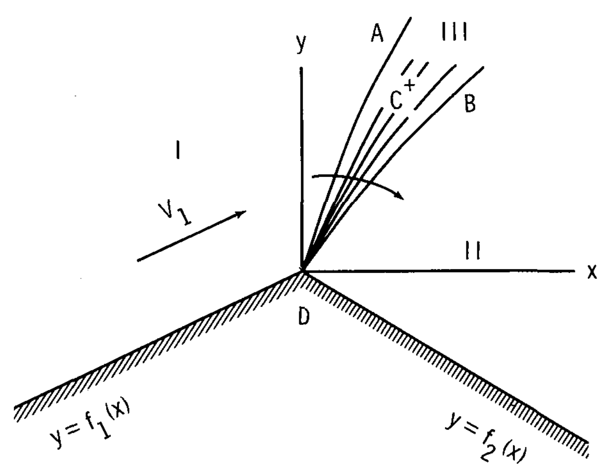

Figure 1. Flow over a convex (expansion) corner.

The problem is to determine the first derivatives of the flow quantities behind the expansion fan, that is, in region II at point D. In doing so, the first two terms in a series expansion in region II shall be defined along with the slope and curvature of the C^+ lines. The following conditions shall be used:

1. The flow quantities and their first derivatives (with respect to x and y) at point D (region I) ahead of the expansion fan are used. These data should be consistent with the slope $f_1'(D^-)$ and the curvature $f_1''(D^-)$ of the surface at point D ahead of the corner.

2. The slope $f_2'(D^+)$ and the curvature $f_2''(D^+)$ of the surface at point D behind the corner (region II) are used.

3. All physical quantities such as velocity **V**, pressure p, and density ρ are continuous across the C^+ lines, but their normal derivatives may not be.

4. In regions I and II, the solutions for the flow properties are regular in x and y near the corner. In particular, their partial derivatives $\partial/\partial x$ and $\partial/\partial y$ exist at point D. If P represents any flow quantity and $j = 1$ and 2 for regions I and II, then the series expansion about (0,0) is

$$P_j(x,y) = P_j(0,0) + x\frac{\partial}{\partial x}P_j(0,0) + y\frac{\partial}{\partial y}P_j(0,0) + \ldots$$

In polar coordinates (σ,ψ) (see fig. 2), this becomes

$$P_j(\sigma,\psi) = P_j(0,0) + \sigma\left(\cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y}\right)P_j(0,0)$$

$$+ O(\sigma^2)$$

where $O(\sigma)$ indicates that the missing terms are of the order of σ, that is,

Expansion (A)

$$P_j(\sigma,\psi) = P_j^{(0)} + \sigma P_j^{(1)}(0,\psi) + O(\sigma^2) \qquad (1)$$

where $P_j^{(0)} = P_j(0,0)$ and $P_j(\sigma,\psi)$ is independent of ψ as $\sigma \rightarrow 0$.

Stated alternatively,

$$\frac{\partial}{\partial x} = \cos\psi\frac{\partial}{\partial\sigma} - \sin\psi\frac{1}{\sigma}\frac{\partial}{\partial\psi}$$

$$\frac{\partial}{\partial y} = \sin\psi\frac{\partial}{\partial\sigma} + \cos\psi\frac{1}{\sigma}\frac{\partial}{\partial\psi}$$

If $\partial/\partial x$ and $\partial/\partial y$ are finite as $\sigma \rightarrow 0$, it is necessary that $(1/\sigma)(\partial/\partial\psi)$ is finite as $\sigma \rightarrow 0$, that is, $\partial/\partial\psi \rightarrow 0$ as $\sigma \rightarrow 0$.

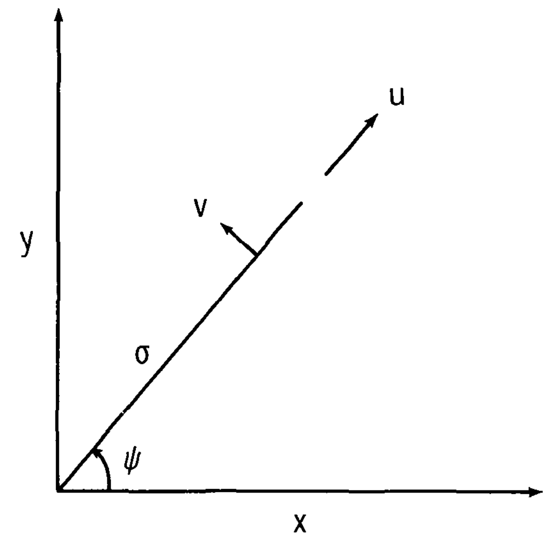

Figure 2. Two-dimensional coordinate system.

5. In region III, the flow quantities must change around point D, that is, they must be functions of ψ as $\sigma \rightarrow 0$. With a bar over a symbol representing quantities in region III,

$$\frac{1}{\sigma}\frac{\partial\bar{P}}{\partial\psi} = O\left(\frac{1}{\sigma}\right)$$

In region III, the power series in terms of σ has the form

Expansion (B)

$$\bar{P}(\sigma,\psi) = \bar{P}^{(0)}(\psi) + \sigma\bar{P}^{(1)}(\psi) + O(\sigma^2) \qquad (2)$$

where $\bar{P}^{(1)}(\psi) = (\partial/\partial\sigma)\bar{P}(0,\psi)$. Expansions (A) and (B) are the two basic ansatzes needed for the analysis. They express the differences in the behavior of the solutions near D in regions I and II from those in region III, the expansion fan.

Outline of solution. Basically, the procedure used in the analysis will be to substitute expansions (A) and (B) (eqs. (1) and (2)) into the governing equations and then to compare the coefficients of like powers of σ. An outline of the solution procedure is the following:

Step (1): Set up the general equations of two-dimensional motion with nonuniform enthalpy and entropy.

Step (2): Determine the slope and curvature of the first characteristic line in region III (DA) and the boundary conditions between regions I and III along DA. This yields the first term in expansion (B).

Step (3): Determine the flow properties at point D in the expansion fan, those behind the fan, and the slope of the last C^+ line (DB) from the slope $f_2'(D^+)$ of the surface f_2.

Step (4): Determine the second term in expansion (B) (eq. (2)) in the expansion fan and the curvature of the characteristic lines.

Step (5): Apply boundary conditions along DB (connection between regions II and III) and determine the first derivatives (with respect to x and y) of the flow properties in region II, the derivative being compatible with the curvature of the surface behind the fan.

Basic definitions and relationships used in solution. Included below are a few definitions and relationships which are used in the subsequent derivations. The components of the velocity along the radial unit vector $\boldsymbol{\sigma}$ and the circumferential unit vector $\boldsymbol{\psi}$ are u and v. Thus,

$$\mathbf{V} = u\boldsymbol{\sigma}(\psi) + v\boldsymbol{\psi}(\psi)$$

The components u and v are in turn expressed as power series in σ as follows:

$$u = \mathbf{V}\cdot\boldsymbol{\sigma}(\psi) = u^{(0)}(\psi) + \sigma u^{(1)}(\psi) + O(\sigma^2)$$

and

$$v = \mathbf{V}\cdot\boldsymbol{\psi}(\psi) = v^{(0)}(\psi) + \sigma v^{(1)}(\psi) + O(\sigma^2)$$

In regions I and II, $\mathbf{V}(\sigma=0)$ exists; hence,

$$u^{(0)}(\psi) = \mathbf{V}(\sigma=0)\cdot\boldsymbol{\sigma}(\psi)$$

$$v^{(0)}(\psi) = \mathbf{V}(\sigma=0)\cdot\boldsymbol{\psi}(\psi)$$

and

$$u^{(1)}(\psi) = \mathbf{V}_\sigma(0,\psi)\cdot\boldsymbol{\sigma}(\psi)$$

$$v^{(1)}(\psi) = \mathbf{V}_\sigma(0,\psi)\cdot\boldsymbol{\sigma}(\psi)$$

Also note that

$$u_\psi^{(0)} = \mathbf{V}(\sigma=0)\cdot\boldsymbol{\psi}(\psi) = v^{(0)}$$

and

$$v_\psi^{(0)} = -\mathbf{V}\cdot\boldsymbol{\sigma}(\psi) = -u^{(0)}$$

Solution Procedure

Step 1. In polar coordinates (σ,ψ), let u be the radial velocity and v be the circumferential velocity. The governing equations in regions I, II, and III are as follows:

Continuity

$$u_\sigma + \frac{u}{\sigma} + \frac{1}{\sigma}v_\psi - \frac{1}{\rho}\left(u\rho_\sigma + \frac{v}{\sigma}\rho_\psi\right) = 0 \tag{3}$$

where ρ is the density.

σ–momentum

$$uu_\sigma + \frac{v}{\sigma}u_\psi - \frac{v^2}{\sigma} = -\frac{1}{\rho}p_\sigma \tag{4a}$$

where p is pressure.

ψ–momentum

$$uv_\sigma + \frac{v}{\sigma}v_\psi + \frac{uv}{\sigma} = -\frac{1}{\rho\sigma}p_\psi \tag{4b}$$

Energy

$$uH_\sigma + \frac{v}{\sigma}H_\psi = 0 \tag{5}$$

where H is the total enthalpy defined by

$$H = \frac{a^2}{\gamma - 1} + \frac{u^2 + v^2}{2}$$

and a is the speed of sound. An additional equation, valid along streamlines, is

$$uS_\sigma + \frac{v}{\sigma}S_\psi = 0 \tag{6}$$

where S is entropy. Eliminating p and ρ from the momentum and continuity equations yields

$$\sigma u_\sigma + u + v_\psi = \frac{1}{a^2}\left(\sigma u\frac{\partial}{\partial\sigma} + v\frac{\partial}{\partial\psi}\right)\left(\frac{u^2 + v^2}{2}\right) \tag{7}$$

and

$$(u^2 + v^2)(u_\psi - \sigma v_\sigma - v) = uH_\psi - \sigma vH_\sigma - T(uS_\psi - v\sigma S_\sigma) \tag{8}$$

where T is the temperature. Equations (5), (6), (7), and (8) are the forms of the governing equations used in this derivation.

Step 2. Let $\psi = g_1(\sigma)$, and thus $G = \psi - g_1(\sigma)$, be the equation for the slope of the first characteristic line (DA) in region III. The velocity perpendicular to DA is a, that is,

$$\mathbf{V}\cdot\frac{\nabla G}{|\nabla G|} = \frac{[-u_1 g_1'(\sigma) + (v_1/\sigma)]}{\left[(1/\sigma^2) + g_1'(\sigma)^2\right]^{1/2}} = \pm a \tag{9}$$

where $g_1' = \partial g_1/\partial\sigma$. Since $v_1 < 0$, the negative sign on a is chosen. Expanding both sides of equation (9), one gets

$$\left[-u_1^{(0)}\sigma g_1'(\sigma) + v_1 + \sigma v_1^{(1)} + \ldots\right]\left\{1 - \frac{1}{2}\sigma^2[g_1'(\sigma)]^2 + \ldots\right\} = -\left[a_1^{(0)} + \sigma a_1^{(1)}(\psi) + O(\sigma^2)\right]$$

But

$$v_1^{(1)} = \frac{\partial v_1}{\partial \sigma} = \frac{\partial v_1}{\partial \psi}\frac{\partial \psi}{\partial \sigma} = v_{1\psi}^{(0)} g_1' = -u_1^{(0)} g_1'$$

Therefore,

$$\left[-u_1^{(0)} \sigma g_1'(\sigma) + v_1 - \sigma u_1^{(0)} g_1'\right]\left\{1 - \frac{1}{2}\sigma^2 [g_1'(\sigma)]^2\right\}$$
$$= -\left[a_1^{(0)} + \sigma a_1^{(1)} + O(\sigma^2)\right] \quad (10)$$

Equating σ for the zeroth order, one gets

$$v_1 = a_1^{(0)}$$

implying that $g_1(0) = \tan(\theta_1 + \mu_1)$ at point D (i.e., it is a characteristic line). Equating σ for the first order, one gets

$$-2u_1^{(0)} g_1'(0) = -a_1^{(1)}(\psi = g_1(0))$$

or

$$g_1'(0) = \frac{a_1^{(1)} g_1(0)}{2u_1^{(0)}} \quad (11)$$

Define $\psi_1 = g_1(0)$ to be the inclination of DA at D. Then, the curvature of DA at D is $g_1'(0) = \partial\psi/\partial\sigma\big|_{\sigma=0}$. Also note that

$$a_1^{(1)}(\psi_1) = \frac{\partial a_1(\sigma,\psi_1)}{\partial \sigma}\bigg|_{\sigma=0}$$
$$= \frac{\partial a_1(x,y)}{\partial x}\cos\psi_1 + \frac{\partial a_1(x,y)}{\partial y}\sin\psi_1$$

at $x = 0$ and $y = 0$ in region I.

Along the boundary DA ($\psi = g_1(\sigma) = \psi_1 + \sigma g_1'(0) + \ldots$) the flow quantities are continuous, that is,

$$P_1(\sigma, g_1(\sigma)) = \bar{P}(\sigma, g_1(\sigma))$$

or, from ansatz (A) and ansatz (B) (eqs. (1) and (2)),

$$P_1 + \sigma P_1^{(1)}(g_1(\sigma)) + O(\sigma^2) =$$
$$\bar{P}^{(0)}(g_1(\sigma)) + \sigma \bar{P}^{(1)}(g_1(\sigma)) + O(\sigma^2) \quad (12)$$

where P represents the general flow quantity. Therefore,

$$P_1 = \bar{P}^{(0)}(\psi_1) \quad (13)$$

and

$$P_1^{(1)}(\psi_1) = \bar{P}_\psi^{(0)}(\psi_1)\, g_1'(0) + \bar{P}^{(1)}(\psi_1) \quad (14)$$

since

$$\bar{P}^{(1)}(\psi_1) = \frac{\partial \bar{P}}{\partial \psi}\frac{\partial \psi}{\partial \sigma} + \frac{\partial \bar{P}}{\partial \sigma} = \bar{P}_\psi^{(0)}(\psi_1) g_1'(0) + \bar{P}^{(1)}(\psi_1)$$

at $\sigma = 0$. Recall that in region I, all flow properties are finite and single valued as $\sigma \to 0$; however, in region III, the properties are not single valued but are functions of ψ at $\sigma = 0$. For the velocity $\mathbf{V}$,

$$\mathbf{V}_1(\sigma, g_1(\sigma)) = \bar{\mathbf{V}}(\sigma, g_1(\sigma))$$

The velocity at point D, as indicated in equation (13), is

$$\mathbf{V}_1^{(0)} = \bar{\mathbf{V}}^{(0)}(g_1(0)) = \bar{\mathbf{V}}^{(0)}(\psi_1)$$
$$= \bar{u}^{(0)}(\psi_1)\, \boldsymbol{\sigma}(\psi_1) + \bar{v}^{(0)}(\psi_1)\, \boldsymbol{\psi}(\psi_1)$$

and the velocity to the first order is

$$\mathbf{V}^{(1)}(\psi_1) = \left[\bar{u}_\psi^{(0)} g'(0) + \bar{u}^{(1)}(\psi_1) - \bar{v}^{(0)} g_1'(0)\right]\boldsymbol{\sigma}(\psi_1)$$
$$+ \left[\bar{u}^{(0)} g'(0) + \bar{v}_\psi^{(0)} g'(0) + \bar{v}^{(1)}(\psi_1)\right]\boldsymbol{\psi}(\psi_1)$$

where the properties $\partial\boldsymbol{\sigma}/\partial\psi = \boldsymbol{\psi}$, $\partial\boldsymbol{\psi}/\partial\psi = -\boldsymbol{\sigma}$, $\partial\boldsymbol{\sigma}/\partial\sigma = 0$, and $\partial\boldsymbol{\psi}/\partial\sigma = 0$ are used.

Step 3. From the application of expansion (B) (eq. (2)) to governing equations (3) to (6) and the equating of the coefficients of $\sigma^{(0)}$, the resulting zeroth-order equations are

$$\bar{H}_\psi^{(0)} = 0 \quad (15)$$

$$\bar{S}_\psi^{(0)} = 0 \quad (16)$$

$$\bar{u}^{(0)}\, \bar{a}^{(0)2} - \bar{u}^{(0)}\, \bar{v}^{(0)}\, \bar{u}_\psi^{(0)} + \left(\bar{a}^{(0)2} - \bar{v}^{(0)2}\right)\bar{v}_\psi^{(0)} = 0 \quad (17)$$

$$\left(\bar{u}^{(0)2} + \bar{v}^{(0)2}\right)\left(\frac{\partial \bar{u}^{(0)}}{\partial \psi} - \bar{v}^{(0)}\right) = \bar{u}^{(0)}\, \bar{H}_\psi^{(0)} - \bar{u}^{(0)}\, \bar{T}^{(0)}\, \bar{S}_\psi^{(0)} \quad (18)$$

Equations (15) and (16) imply that H and S are independent of ψ; thus,

$$H_1 = H_2 \quad (19)$$

$$S_1 = S_2 \quad (20)$$

From equations (15) and (16), equation (18) becomes

$$\bar{u}_\psi^{(0)} - \bar{v}^{(0)} = 0 \quad (21)$$

From this, equation (17) may be written as

$$\left(\bar{a}^{(0)2} - \bar{v}^{(0)2}\right)\left(\bar{v}_{\psi}^{(0)} + \bar{u}^{(0)}\right) = 0 \tag{22}$$

The zeroth-order term of equation (4b) gives

$$\bar{v}^{(0)}\left(\bar{v}_{\psi}^{(0)} + \bar{u}^{(0)}\right) = -\frac{1}{\bar{\rho}^{(0)}}\bar{p}_{\psi}^{(0)}$$

Since $\bar{p}_{\psi}^{(0)} \neq 0$ at point D in region III, then

$$\bar{v}_{\psi}^{(0)} + \bar{u}^{(0)} \neq 0$$

and equation (22) becomes

$$\bar{a}^{(0)2} - \bar{v}^{(0)2} = 0$$

From the direction of the flow, the negative sign is chosen; thus,

$$\bar{a}^{(0)} = -\bar{v}^{(0)} \tag{23}$$

which, as expected, defines each local ($\sigma \to 0$) radial line to be a characteristic. Equations (19) and (23) may be used to relate $\bar{u}^{(0)}$ to $\bar{a}^{(0)}$. From the definition of q_m^2 as the limit on maximum speed in inviscid, steady flow, one obtains

$$\frac{\bar{u}^{(0)2} + \bar{v}^{(0)2}}{2} + \frac{\bar{a}^{(0)2}}{\gamma - 1} = \frac{\bar{u}^{(0)2} + a^{(0)2}}{2} + \frac{\bar{a}^{(0)2}}{\gamma - 1}$$

$$= \bar{H}^{(0)} = H_1 = \frac{q_m^2}{2}$$

or

$$\bar{a}^{(0)} = \left(q_m^2 - \bar{u}^{(0)2}\right)^{1/2}\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2} \tag{24}$$

Equation (21) then becomes

$$\bar{u}_{\psi}^{(0)} = \bar{v}^{(0)} = -\bar{a}^{(0)} = -\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}\left(q_m^2 - \bar{u}^{(0)2}\right)^{1/2}$$

and after integration

$$\bar{u}^{(0)}(\psi) = q_m \sin\phi \tag{25}$$

where $\phi = [(\gamma-1)/(\gamma+1)]^{1/2}(\alpha - \psi)$ and γ is the ratio of specific heats. From boundary condition (6),

$$\bar{u}^{(0)}(\psi_1) = u_1^{(0)} = q_m \sin\left[\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}(\alpha - \psi_1)\right]$$

and

$$\alpha = \left(\sin^{-1}\frac{u_1^{(0)}}{q_m}\right)\left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2} + \psi_1 \tag{26}$$

From equations (24) and (25),

$$\bar{a}^{(0)} = q_m\left(\frac{\gamma - 1}{\gamma + 1}\right)^{1/2}\cos\phi \tag{27}$$

and

$$\frac{\bar{u}^{(0)}}{\bar{v}^{(0)}} = -\frac{\bar{u}^{(0)}}{\bar{a}^{(0)}} = -\left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2}\tan\phi \tag{28}$$

Along a streamline, defined by $\sigma = l(\psi,\sigma_1)$, one obtains (at the boundary DA)

$$\sigma_1 = l(\psi = g_1(\sigma_1),\sigma_1) \tag{29}$$

Expanding from the boundary DA along the streamline

$$\sigma_1 = l(\psi,\sigma_1) = l^{(0)}(\psi) + \sigma l^{(1)}(\psi) + \ldots \tag{30}$$

one gets to the zeroth-order and first-order equations

$$0 = l^{(0)}(\psi_1)$$

and

$$1 = l^{(1)}(\psi_1) \tag{31}$$

Therefore, for a streamline,

$$\sigma = \sigma_1 l^{(1)}(\psi) + O(\sigma_1^2)$$

Recall that

$$\frac{u}{v} = \frac{d\sigma}{\sigma\, d\psi} = \frac{\sigma_1}{\sigma} l_{\psi}^{(1)} = \frac{1}{l^{(1)}} l_{\psi}^{(1)}$$

Thus,

$$\frac{1}{l^{(1)}}\frac{dl^{(1)}}{d\psi} = \frac{\bar{u}^{(0)}}{\bar{v}^{(0)}} = -\left(\frac{\gamma + 1}{\gamma - 1}\right)^{1/2}\tan\phi$$

and

$$\ln l^{(1)} = -\frac{\gamma + 1}{\gamma - 1}\ln\frac{\cos\phi}{\cos\phi_1}$$

From the definition of σ given above,

$$\frac{\sigma(\psi)}{\sigma_1} = l^{(1)}(\psi) = \left(\frac{\cos\phi}{\cos\phi_1}\right)^{-(\gamma+1)/(\gamma-1)} \tag{32}$$

To determine the slope of the last characteristic in region III (DB), write the equation

$$\psi = g_2(\sigma) = g_2(0) + \sigma g_2'(0) + O(\sigma^2)$$

and define $g_2(0) = \psi_2$ as the inclination of the characteristic at D. The boundary conditions along DB between regions III and II are similar to those between regions I and III along DA, that is, the flow quantities along DB are continuous. These conditions imply the following:

$$P_2(\sigma,g_2(\sigma+)) = \bar{P}(\sigma,g_2(\sigma-))$$

$$p_2(\sigma=0) = p_2 = \bar{p}^{(0)}(\psi_2)$$

$$u_2(\sigma=0) = u_2 = \bar{u}^{(0)}(\psi_2)$$

$$v_2(\sigma=0) = v_2 = \bar{v}^{(0)}(\psi_2)$$

An implicit equation for ψ_2 is derived using the slope of the wall behind point D, $y = f_2(x)$. The slope is

$$f_2'(x) = \frac{dy}{dx} = \frac{-u \sin\psi + v\cos\psi}{u\cos\psi + v\sin\psi}$$

$$f_2'(0) = \frac{-u_2\sin\psi_2 + v_2\cos\psi_2}{u_2\cos\psi_2 + v_2\sin\psi_2} \tag{33}$$

The results obtained in this step are identical to those in a centered expansion wave separating two uniform regions (ref. 4). They have been included so that the next step provides a continuation to take care of the nonuniformity of the flow fields ahead of and behind the expansion fan.

Step 4. In this step, values for the next-order terms $\left(\bar{u}^{(1)}(\psi),\ \bar{v}^{(1)}(\psi),\ \bar{H}^{(1)}(\psi),\text{ and } \bar{S}^{(1)}(\psi)\right)$ are determined. Also determined is the curvature of the characteristics at point D by comparing coefficients of σ in equations (7), (8), (5), and (6).

From equation (5), one obtains

$$\bar{u}^{(0)}(\psi)\,\bar{H}^{(1)}(\psi) + \bar{v}^{(0)}\,\bar{H}_{\psi}^{(1)}(\psi) = 0 \tag{34}$$

and therefore

$$\frac{\bar{H}_{\psi}^{(1)}(\psi)}{\bar{H}^{(1)}(\psi)} = -\frac{\bar{u}^{(0)}}{\bar{v}^{(0)}} = \left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan\phi \tag{35}$$

and

$$H^{(1)}(\psi) = H_1^{(1)}\left(\frac{\cos\phi}{\cos\phi_1}\right)^{(\gamma+1)/(\gamma-1)} = \frac{1}{f^{(1)}(\psi)} \tag{36}$$

The boundary condition at DA yields

$$\bar{H}^{(1)}(\psi_1) = \mathrm{H}_1^{(1)} = \lim_{\sigma\to 0}\left.\frac{\partial H_1}{\partial\sigma}\right|_{\psi=\psi_1} = \left.\left(\frac{\partial H_1}{\partial x}\cos\psi_1 + \frac{\partial H_1}{\partial y}\sin\psi_1\right)\right|_{\substack{x=0\\y=0}}$$

Similarly, from equation (6) and the boundary conditions at DA for S,

$$\bar{S}^{(1)}(\psi_1) = S_1^{(1)} = \lim_{\sigma\to 0}\left.\frac{\partial S_1}{\partial\sigma}\right|_{\psi=\psi_1} = \left.\left(\frac{\partial S_1}{\partial x}\cos\psi_1 + \frac{\partial S_1}{\partial y}\sin\psi_1\right)\right|_{\substack{x=0\\y=0}} \tag{37}$$

and

$$\frac{\bar{S}^{(1)}(\psi)}{S_1^{(1)}} = \left(\frac{\cos\phi}{\cos\phi_1}\right)^{(\gamma+1)/(\gamma-1)} = \frac{1}{l^{(1)}(\psi)} \tag{38}$$

The first-order equations for $\bar{u}^{(1)}$ and $\bar{v}^{(1)}$ are derived from the coefficients of σ in equations (7) and (8). The first-order equation from equation (7) is

$$\bar{u}_{\psi}^{(1)}\left(-\bar{v}^{(0)}\bar{u}^{(0)}\right) + \bar{u}^{(1)}\left(2\bar{a}^{(0)2} - \bar{u}^{(0)2} - \bar{v}^{(0)}\bar{u}_{\psi}^{(0)}\right) + \bar{v}_{\psi}^{(1)}\left(\bar{a}^{(0)2} - \bar{v}^{(0)2}\right) + \bar{v}^{(1)}\left(-\bar{u}^{(0)}\bar{v}^{(0)} - \bar{u}^{(0)}\bar{u}_{\psi}^{(0)} - 2\bar{v}^{(0)}\bar{v}_{\psi}^{(0)}\right) + \bar{a}^{(1)2}\left(\bar{u}^{(0)} + \bar{v}_{\psi}^{(0)}\right) = 0 \tag{39}$$

The following relations, derived in step 3, are used to simplify equation (39) and a similar equation for equation (8). From equation (21),

$$\bar{u}_{\psi}^{(0)} = \bar{v}^{(0)}$$

and from equation (22),

$$\bar{a}^{(0)2} - \bar{v}^{(0)2} = 0$$

Because of flow direction, choose (eq. (23))

$$\bar{a}^{(0)} = -\bar{v}^{(0)}$$

Likewise, equations (23) and (27) give

$$\bar{a}^{(0)} = q_m\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\cos\phi = -\bar{v}^{(0)}$$

As given in equation (25),

$$\bar{u}^{(0)} = q_m\sin\phi$$

where $\phi = [(\gamma-1)/(\gamma+1)]^{1/2}\,(\alpha-\psi)$ and $d\phi = -[(\gamma-1)/(\gamma+1)]^{1/2}\,d\psi$. Differentiating $\bar{a}^{(0)}$, one obtains

$$\bar{a}_{\psi}^{(0)} = q_m\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}(\sin\phi)\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2} = \frac{\gamma-1}{\gamma+1}\,\bar{u}^{(0)}$$

and differentiating equation (23) gives

$$\bar{v}^{(0)}_{\psi} = -\bar{a}^{(0)}_{\psi} = -\left(\frac{\gamma-1}{\gamma+1}\right)\bar{u}^{(0)}$$

Substituting for $\bar{v}^{(0)}_{\psi}$, one then arrives at the following expression for $\bar{u}^{(0)} + \bar{v}^{(0)}_{\psi}$ in equation (39):

$$\bar{u}^{(0)} + \bar{v}^{(0)}_{\psi} = \frac{2}{\gamma+1}\bar{u}^{(0)}$$

From the definition of enthalpy and $\partial H/\partial\sigma$, one obtains

$$\bar{a}^{(1)2} = (\gamma-1)(\bar{H}^{(1)} - \bar{u}^{(0)}\,\bar{u}^{(1)} - \bar{v}^{(0)}\,\bar{v}^{(1)}) \tag{40}$$

Solving equation (39) for $\bar{v}^{(1)}$ yields

$$\bar{v}^{(1)} = -\frac{\bar{u}^{(1)}_{\psi}}{2} + \frac{\bar{u}^{(1)}\{\bar{a}^{(0)2} - \bar{u}^{(0)2}[(3\gamma-1)/(\gamma+1)]\}}{2\bar{u}^{(0)}\,\bar{v}^{(0)}} + \frac{\gamma-1}{\gamma+1}\frac{\bar{H}^{(1)}}{\bar{v}^{(0)}} \tag{41}$$

A similar procedure for equation (8) yields

$$\bar{v}^{(1)} = \frac{\bar{u}^{(1)}_{\psi}}{2} - \frac{1}{2\bar{a}^{(0)}}(H^{(1)} - \bar{T}^{(0)}\bar{S}^{(1)}) \tag{42}$$

By equating equations (41) and (42), the following equation for $\bar{u}^{(1)}$ is obtained:

$$\bar{u}^{(1)}_{\psi} + \bar{u}^{(1)}\left[-\frac{3\gamma-1}{2(\gamma+1)}\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\tan\phi + \frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\cot\phi\right] = \frac{\gamma-3}{\gamma+1}\frac{\bar{H}^{(1)}}{2\bar{v}^{(0)}} + \frac{\bar{T}^{(0)}\bar{S}^{(1)}}{2\bar{v}^{(0)}} \tag{43}$$

The integrating factor for this equation is

$$\mathscr{F} = \exp\left[-\frac{3\gamma-1}{2(\gamma+1)}\left(\frac{\gamma+1}{\gamma-1}\right)^{1/2}\int\tan\phi\,d\psi + \frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\int\cot\phi\,d\psi\right]$$

$$= \exp\left[\frac{3\gamma-1}{2(\gamma-1)}\int\tan\phi\,d\phi - \frac{1}{2}\int\cot\phi\,d\phi\right]$$

$$\mathscr{F} = (\cos\phi)^{-(3\gamma-1)/[2(\gamma-1)]}\,(\sin\phi)^{-1/2} \tag{44}$$

Thus,

$$\bar{u}^{(1)} = (\cos\phi)^{(3\gamma-1)/[2(\gamma-1)]}\,(\sin\phi)^{1/2}\left(\int_{\psi_1}^{\psi}\mathscr{A}_1\,d\psi + \int_{\psi_1}^{\psi}\mathscr{A}_2\,d\psi\right) + \frac{\bar{u}^{(1)}_1(\psi_1)}{\mathscr{F}(\psi_1)} \tag{45}$$

where

$$\bar{u}^{(1)}_1(\psi_1) = u^{(1)}_1(\psi_1) - \bar{u}^{(0)}_{\psi}(\psi_1)\,g_1'(0) + \bar{v}^{(0)}(\psi_1)\,g_1'(0) = u^{(1)}(\psi_1) \tag{46}$$

and

$$\int\mathscr{A}_1\,d\psi = \int\frac{\gamma-3}{\gamma+1}(\cos\phi)^{-(3\gamma-1)/[2(\gamma-1)]}(\sin\phi)^{-1/2}\frac{\bar{H}^{(1)}}{2\bar{v}^{(0)}}\,d\psi$$

$$= \frac{H_1}{q_m}\left(\frac{1}{\cos\phi_1}\right)^{(\gamma+1)/(\gamma-1)}\int\frac{(\cos\phi)^{(3\gamma+5)/[2(\gamma-1)]}}{(\sin\phi)^{1/2}}\,d\phi$$

$$\int\mathscr{A}_2\,d\psi = \int(\cos\phi)^{-(3\gamma-1)/[2(\gamma-1)]}(\sin\phi)^{-1/2}\frac{\bar{T}^{(0)}\,\bar{S}^{(1)}}{2\bar{v}^{(0)}}\,d\psi$$

$$= \frac{S_1 q_m}{2\gamma R}\left(\frac{1}{\cos\phi_1}\right)^{(\gamma+1)/(\gamma-1)}\int\frac{(\cos\phi)^{\gamma+1/[2(\gamma-1)]}}{(\sin\phi)^{1/2}}\,d\phi$$

If $\gamma = 7/5$, then

$$\int_{\psi_1}^{\psi}\mathscr{A}_1\,d\psi = 2\left(\frac{H_1}{q_m}\right)\left(\frac{1}{\cos\phi_1}\right)^6(\sin^{1/2}\phi - \sin^{1/2}\phi_1) \tag{47}$$

and

$$\int_{\psi_1}^{\psi}\mathscr{A}_2\,d\psi = \frac{5S_1q_m}{14R}\left(\frac{1}{\cos\phi_1}\right)^6\left(\frac{2}{7}\cos^2\phi\,\sin^{-1/2}\phi + \frac{8}{35}\sin^{1/2}\phi - \frac{2}{7}\cos^2\phi_1\,\sin^{-1/2}\phi_1 - \frac{8}{35}\sin^{1/2}\phi_1\right) \tag{48}$$

Knowing $\bar{u}^{(1)}$, one can evaluate equation (43) for $\bar{u}^{(1)}_{\psi}$, equation (42) for $\bar{v}^{(1)}$, and equation (40) for $\bar{a}^{(1)2}$.

To evaluate the curvature of the characteristic C^+ in region III at point D with inclination ψ_0, define the curvature with the equation $\psi = g(\sigma,\psi_0)$, that is, $\psi_0 = g(0,\psi_0)$. For the first and last C^+ lines in region III,

$$g_1(\sigma) = g(\sigma,\psi_1)$$

$$g_2(\sigma) = g(\sigma,\psi_2)$$

Given this definition for $g(\sigma,\psi_0)$, the dependence of g on ψ_0 shall not be shown. The normal component of velocity is $\bar{a}(\sigma,g(\sigma))$, or

$$\frac{(\bar{u}\sigma + \bar{v}\psi) \bullet \nabla[-\psi + g(\sigma)]}{|\nabla[\psi - g(\sigma)]|} = \bar{a}$$

$$\frac{\bar{u}g'(\sigma) - (\bar{v}/\sigma)}{[g'(\sigma)^2 + (1/\sigma^2)]^{1/2}} = \bar{a}$$

Recall that $\psi = g(\sigma) = g(0) + \sigma g^{(1)}(0) + \ldots$, where $g(0) = \psi_0$ is the angle at $\sigma = 0$, or

$$[\bar{u}g'(\sigma) - (\bar{v}/\sigma)]\sigma[1 + \sigma^2 g^2(\sigma)]^{1/2}$$

$$= \bar{a}^{(0)}(\psi_0) + \sigma[g'(0)\bar{a}^{(0)}_{\psi} + \bar{a}^{(1)}(\psi_0)]$$

Expanding the left side and equating the coefficients of σ, one obtains

$$\bar{u}^{(0)}(\psi_0)\, g'(0) = \bar{a}^{(1)}(\psi_0) + \bar{v}^{(1)}(\psi_0)$$

$$\text{Curvature} = g'(0) = \frac{\bar{a}^{(1)}(\psi_0) + \bar{v}^{(1)}(\psi_0)}{\bar{u}^{(0)}(\psi_0)} \tag{49}$$

Also note that

$$\bar{a}^{(1)^2} = \lim_{\sigma \to 0} \frac{\partial a^2}{\partial \sigma} = \lim_{\sigma \to 0} 2\bar{a}^{(0)} \frac{\partial \bar{a}}{\partial \sigma} = 2\bar{a}^{(0)}\, \bar{a}^{(1)}(\psi)$$

Therefore,

$$\bar{a}^{(1)}(\psi) = \frac{\bar{a}^{(1)^2}(\psi)}{2\bar{a}^{(0)}(\psi)}$$

Step 5. The boundary conditions along DB, the last C^+ line in region III, are developed in the same manner as those along DA. Along DB, $\psi = g_2(\sigma)$ and $g_2(0) = \psi_2$. The term $g_2'(0)$ is given by equation (49) with ψ_0 replaced by ψ_2. Applying the boundary conditions yields

$$H_2^{(1)}(\psi_2) = \bar{H}_2^{(1)}(\psi_2) \tag{50}$$

$$S_2^{(1)}(\psi_2) = \bar{S}_2^{(1)}(\psi_2) \tag{51}$$

$$u^{(1)}\sigma + v^{(1)}\psi = \left(\bar{u}^{(0)}_{\psi}\sigma + \bar{u}^{(0)}\psi + \bar{v}^{(0)}_{\psi}\psi - \bar{v}^{(0)}\sigma\right)g'$$
$$+ \bar{u}^{(1)}\sigma + \bar{v}^{(1)}\psi$$

or, equating σ and ψ components

$$u^{(1)} = \left(\bar{u}^{(0)}_{\psi} - \bar{v}^{(0)}\right)g' + \bar{u}^{(1)}$$

$$v^{(1)} = \left(\bar{u}^{(0)} + \bar{v}^{(0)}_{\psi}\right)g' + \bar{v}^{(1)}$$

yields

$$u_2^{(1)}(\psi_2) = \bar{u}^{(1)}(\psi_2) \tag{52}$$

$$v_2^{(1)}(\psi_2) = \bar{v}^{(1)}(\psi_2) + \left(\bar{v}^{(0)}_{\psi} + u^{(0)}\right)\Big|_{\psi_2} g_2'(\psi_2) \tag{53}$$

Note that

$$H_2^{(1)}(\psi_2) = \lim_{\sigma \to 0} \frac{\partial H_2}{\partial \sigma}\bigg|_{\psi = \psi_2} = \frac{\partial H_2}{\partial x}\cos\psi_2 + \frac{\partial H_2}{\partial y}\sin\psi_2$$

Since we know the slope θ_2 of the streamline, we obtain

$$\frac{dH_2}{dn}\sin(\theta_2 - \mu_2) = \frac{\partial H}{\partial \sigma}$$

$$\frac{dH_2}{dn}\bigg|_D = \frac{H_2^{(1)}(\psi_2)}{\sin(\theta_2 + \mu_2)} \tag{54}$$

Similarly,

$$\frac{dS_2}{dn}\bigg|_D = \frac{S_2^{(1)}(\psi_2)}{\sin(\theta_2 + \mu_2)}. \tag{55}$$

Denote the velocity components in the x and y directions in Cartesian coordinates by u^* and v^*; then,

$$u_2^{(1)}(\psi_2) = \mathbf{V}_{2,\sigma}(\sigma,\psi_2) \bullet \sigma(\psi_2)$$

$$= \left(u^*_{2x}\cos\psi_2 + u^*_{2y}\sin\psi_2\right)\Big|_{\substack{x=0\\y=0}} \cos\psi_2$$

$$+ \left(v^*_{2x}\cos\psi_2 + v^*_{2y}\sin\psi_2\right)\Big|_{\substack{x=0\\y=0}} \sin\psi_2$$

or

$$u_2^{(1)}(\psi_2) = u^*_{2x}(0,0)\cos^2\psi_2$$

$$+ \cos\psi_2\sin\psi_2\left[u^*_{2y}(0,0) + v^*_{2x}(0,0)\right]$$

$$+ v^*_{2y}(0,0)\sin^2\psi_2 \tag{56}$$

Similarly,

$$v_2^{(1)}(\psi_2) = \sin\psi_2\cos\psi_2\left[-u_{2x}^*(0,0) + v_{2y}^*(0,0)\right]$$

$$- u_{2y}^*(0,0)\sin^2\psi_2 + v_{2x}^*(0,0)\cos^2\psi_2 \qquad (57)$$

where

$$\mathbf{V}_2(x,y) = \mathbf{V}_2(0,0) + \sigma\left\{\left[xu_2^*(0,0) + yu_y^*(0,0)\right]\mathbf{i}\right.$$

$$\left. + \left[v_x^*(0,0)\,x + v_y^*(0,0)y\right]\mathbf{j}\right\} + O(\sigma^2)$$

Equations (7) and (8) or (3) and (4) in Cartesian coordinates are equivalent to characteristic equations. Since the boundary conditions along DB are used, the equations along the C^+ lines are fulfilled to the σ order. Only one independent equation is available from equations (7) and (8) or (3) and (4). Choosing equation (4) we get

$$-q^2(v_x - u_y) = q\frac{dH}{dn} - Tq\frac{dS}{dn}$$

where $q(\partial/\partial n) = u(\partial/\partial y) - v(\partial/\partial x)$ and $\partial/\partial n$ is the derivative normal to the streamline. The leading term of this equation (σ^0) yields

$$q_2(0,0)\left[u_y^*(0,0) - v_x^*(0,0)\right]$$

$$= \frac{dH_2(0,0)}{dn} - \frac{a_2^2(0,0)}{\gamma R}\frac{dS_2(0,0)}{dn} \qquad (58)$$

where R is the gas constant.

One more equation, $y=f_2(x)$, is available from the boundary condition along the wall. Thus,

$$u_2^* f_2'(x) - v_2^* = 0$$

$$\left.\frac{d}{dx}\right|_{(0,0)}\left[u_2^* f_2'(x) - v_2^*\right]$$

along $y = f_2(x)$ yields

$$\left[u_{2x}^*(0,0) + u_{2y}^*(0,0)f_2'(0)\right]f_2'(0)$$

$$- \left[v_{2x}^*(0,0) + v_{2y}^*(0,0)\,f_2'(0)\right] = -u_2^*(0,0)f_2''(0) \qquad (59)$$

Equations (56), (57), (58), and (59) are four equations for $u_{2x}^*(0,0)$, $v_{2x}^*(0,0)$, $u_{2y}^*(0,0)$, and $v_{2y}^*(0,0)$ in region II. In coefficient matrix form, these become

$$\begin{bmatrix} \cos^2\psi_2 & \sin\psi_2\cos\psi_2 & \sin\psi_2\cos\psi_2 & \sin^2\psi_2 \\ -\sin\psi_2\cos\psi_2 & -\sin^2\psi_2 & \cos^2\psi_2 & \sin\psi_2\cos\psi_2 \\ 0 & 1 & -1 & 0 \\ f_2' & (f_2')^2 & -1 & -f_2' \end{bmatrix}\begin{bmatrix} u_{2x}^* \\ u_{2y}^* \\ v_{2x}^* \\ v_{2y}^* \end{bmatrix}$$

$$= \begin{bmatrix} u_2^{(1)}(\psi_2) \\ v_2^{(1)}(\psi_2) \\ \dfrac{1}{q_2}\dfrac{dH_2}{dn} - \dfrac{a_2^2}{\gamma R}\dfrac{dS_2}{dn} \\ -u_2^* f_2''(0) \end{bmatrix}$$

The determinate of the coefficient matrix yields

$$\det = \begin{bmatrix} \cos^2\psi & \sin 2\psi & \sin^2\psi \\ -\sin\psi\cos\psi & \cos 2\psi & \sin\psi\cos\psi \\ f_2' & (f_2')^2 - 1 & -f_2' \end{bmatrix}$$

$$= \frac{-\sin 2\mu_2}{2\cos^2\theta_2} \neq 0 \qquad (0 < u_2 < 90°)$$

Part II—Application of the Method to the Case of Shock Coalescence Including Asymmetric Effects

The coalescence system is briefly described herein. For a detailed derivation of the governing equations of the system, see reference 2.

The first two shock surfaces, $F_1(r,\Psi)$ and $F_2(r,\Psi)$, coalesce to form a resultant shock surface $F_3(r,\Psi)$, a contact surface $h(r,\Psi)$, and a weak (isentropic) shock or expansion of the opposite family $F_4(r,\Psi)$. The solution developed in part I is applicable to the case for which F_4 is an expansion wave, as shown in figure 3.

The problem is first solved axisymmetrically, and at some distance away from the body, it may be treated as a two-dimensional problem, as shown in figure 4. All conditions in regions (1), (2), and (3) are assumed to be known, as is the point of coalescence. The unknowns for

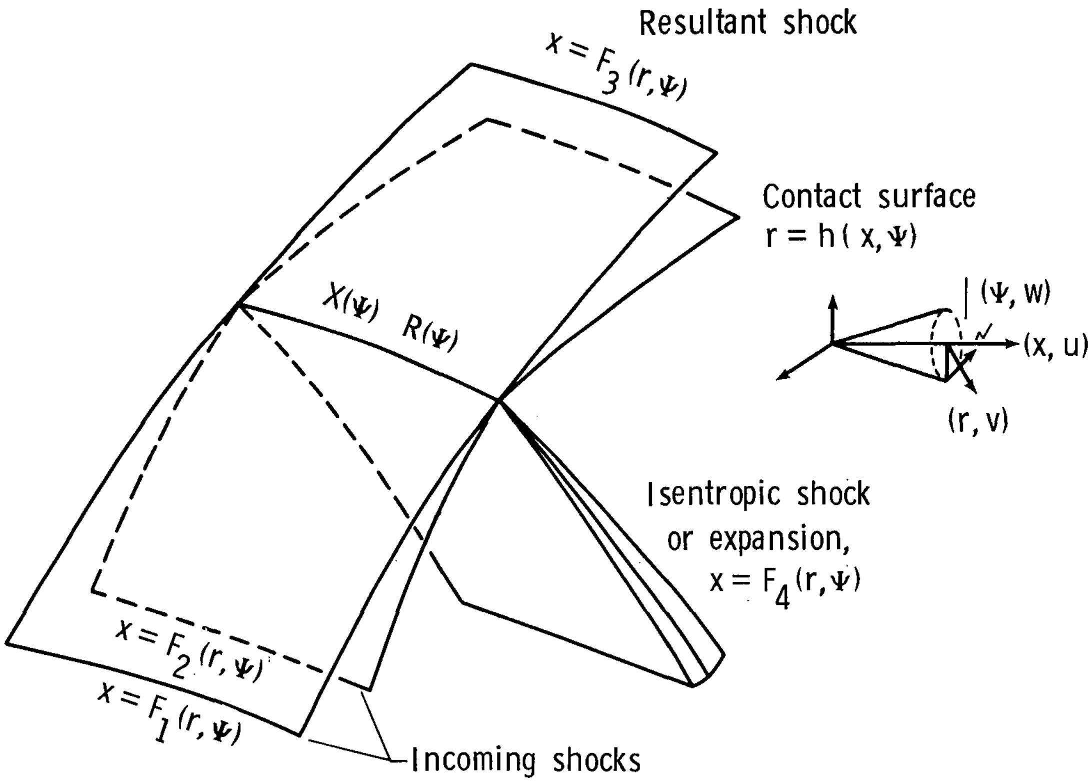

Figure 3. Three-dimensional shock coalescence.

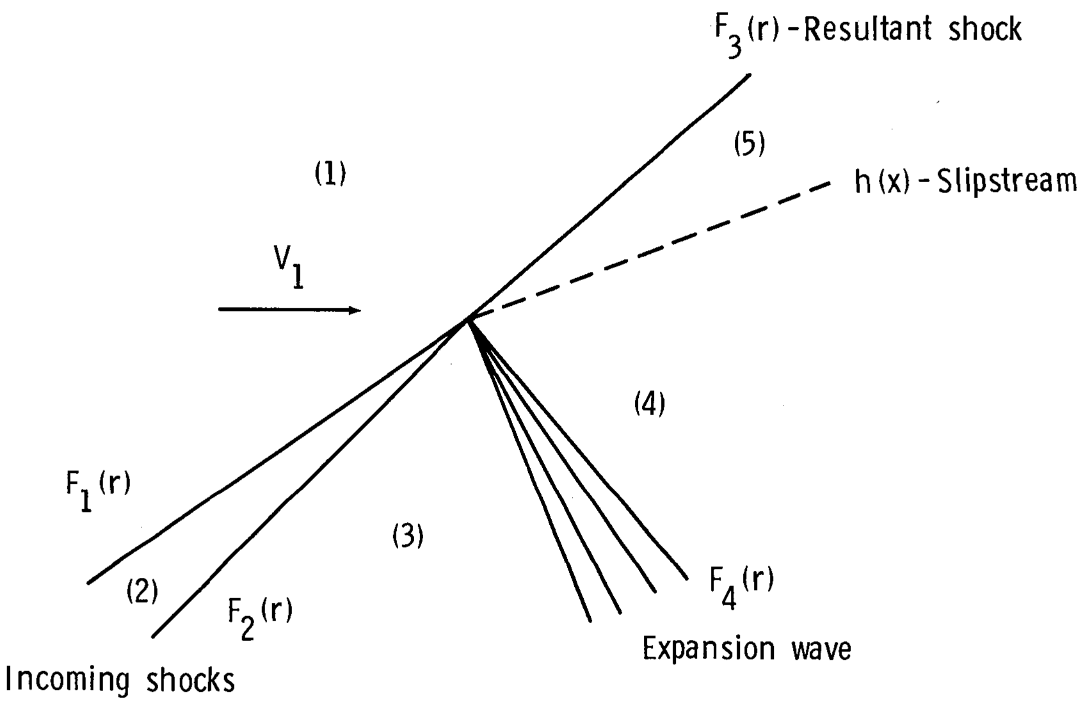

Figure 4. Two-dimensional representation of shock coalescence.

this problem are the following: velocities u_4^*, v_4^*, u_5^*, and v_5^*; pressures p_4 and p_5; temperatures T_4 and T_5; the angle of the resultant shock with respect to the upstream flow β_3; and the angle of the last characteristic of the expansion fan β_4. The system of equations consists of the four shock equations across F_3, the four isentropic conditions across the expansion wave F_4, and the matching pressure and flow direction at the slipstream surface h.

The system is closed, with 10 unknowns and 10 equations. To derive the asymmetric system of governing equations, the system is treated as five intersecting surfaces along $X(\Psi)$, $R(\Psi)$. (See fig. 3.) The governing set of asymmetric equations are the 5 intersection equations, the 5 shock equations at F_3 (1 continuity, 1 energy, and 3 momentum), 5 equations across the expansion wave F_4, and 3 slipstream equations at h for a total of 18 equations. The second derivative with respect to Ψ (the circumferential direction) is taken for each of the governing equations along the corresponding surface. The equations are then reduced to the $\Psi=0$ plane, resulting in 18 equations and 18 unknowns of the form $u_{\Psi\Psi}|_{\Psi=0}$, $P_{\Psi\Psi}|_{\Psi=0}$, and so forth. The asymmetric system would be closed except that in the asymmetric equations there appear spatial derivatives of u^*, v^*, p, and T in both regions (4) and (5). The method derived in part I is now applied to provide a method for obtaining the spatial derivatives. Equations (56), (57), (58), and (59) give u_{4x}^*, v_{4x}^*, u_{4r}^*, and v_{4r}^* in terms of the curvature of the streamline behind the expansion fan (h_{xx} for this problem).

The Euler equations in the plane of symmetry ($\Psi=0$) are valid in regions (4) and (5). The following four equations are applied in region (5). Because the curvature of the characteristics in the expansion fan have already been determined by the method developed in part I, only equations (60) to (62) need to be applied in region (4).

x-momentum

$$p_5 u_5^* u_{5x}^* + p_5 v_5^* u_{5r}^* + RT_5 p_{5x} = 0 \tag{60}$$

r-momentum

$$p_5 u_5^* v_{5x}^* + p_5 v_5^* v_{5r}^* = -g p_5 - RT_5 P_{5r} \tag{61}$$

Continuity

$$p_5 T_5 u_{5x}^* + p_5 T_5 v_{5r}^* - u_5^* p_5 T_{5x} - v_5^* p_5 T_{5r} + u_5^* T_5 p_{5x} + v_5^* T_5 p_{5r} = -\frac{p_5 T_5}{r} v_5^* \tag{62}$$

Energy

$$(u_5^*)^2 T_5 u_{5x}^* + u_5^* v_5^* v_{5x}^* + u_5^* v_5^* u_{5r}^* + (v_5^*)^2 v_{5r}^* + \frac{\gamma}{\gamma-1} u_5^* T_{5x} + \frac{\gamma}{\gamma-1} v_5^* T_{5r} = g v_5^* \tag{63}$$

The tangential derivative of the three properties along the slipstream h yield

$$p_{4x} + p_{4r} h_x = p_{5x} + p_{5r} h_x \tag{64}$$

$$u_{4x}^* h_x + u_4^* h_{xx} - v_{4x}^* + u_{4r}^* h_x^2 - v_{4r}^* h_x = 0 \tag{65}$$

$$u_{5x}^* h_x + u_5^* h_{xx} - v_{5x}^* + u_{5r}^* h_x^2 - v_{5r}^* h_x = 0 \tag{66}$$

The remaining equations at f_3 are derived from the following four shock equations:

Continuity of mass

$$p_1 T_5 Q_1 = p_5 T_1 Q_5$$

r-momentum

$$(u_5^* - u_1^*) F_{3r} + (v_5^* - v_1^*) = 0$$

Energy

$$(u_5^* + u_1^*)(u_5^* - u_1^*) + (v_5^* + v_1^*)(v_5^* - v_1^*) + \frac{2\gamma R(T_5 - T_1)}{\gamma - 1} = 0$$

Combination of all shock equations

$$2\gamma T_1\left(1 + F_r^2 + \frac{F_\Psi^2}{r^2}\right) = (\gamma+1)Q_1 Q_5 - (\gamma-1)Q_1^2$$

where $Q = u^* - v^* F_r$. Taking the tangential derivative of each of these equations along the resultant shock F_3 yields the following four equations:

$$\begin{aligned}(p_1 T_{5x} Q_1 &- p_{5x} T_1 Q_5 - p_5 T_1 u_{5x}^* + p_5 T_1 v_{5x}^* F_{3r}) F_{3r} \\ &+ p_1 T_{5r} Q_1 - p_{5r} T_1 Q_5 - p_5 T_1 u_{5r}^* \\ &+ p_5 T_1 v_{5r}^* F_{3r} + p_5 T_1 F_{3rr} + p_1 T_5 v_1^* F_{3rr} \\ &= (p_5 T_{1x} Q_5 - p_{1x} T_5 Q_1 - p_1 T_5 Q_{1x}) F_{3r} \\ &+ p_5 T_{1r} Q_5 - p_{1r} T_5 Q_1 - p_1 T_5 u_{1r} \\ &+ p_1 T_5 v_{1r}^* F_{3r} + p_1 T_5 v_1^* F_{3rr}\end{aligned} \tag{67}$$

$$\begin{aligned}(u_{5x}^* F_{3r} &+ v_{5x}^*) F_{3r} + u_{5r}^* F_{3r} + u_5^* F_{3rr} + v_{5r}^* \\ &= \left(u_{1x}^* F_{3r}^2 + v_{1x}^* F_{3r} + u_{1r}^* F_{3r} + u_1^* F_{3rr} + v_{1r}^*\right)\end{aligned} \tag{68}$$

$$\left(u_5^* u_{5x}^* - u_1^* u_{1x}^* + v_5^* v_{5x}^* - v_1^* v_{1x}^* + \frac{\gamma}{\gamma-1} T_{5x} - \frac{\gamma}{\gamma-1} T_{1x}\right) F_{3r}$$

$$+ u_5^* u_{5r}^* - u_1^* u_{1r}^* + v_5^* v_{5r}^* - v_1^* v_{1r}^*$$

$$+ \frac{\gamma}{\gamma-1} T_{5r} - \frac{\gamma}{\gamma-1} T_{1r} = 0 \qquad (69)$$

and

$$2\gamma[(T_{1x}F_{3r} + T_{1r})(1 + F_{3r}^2) + 2T_1 F_{3r} F_{3rr}]$$

$$= [(\gamma-1)(Q_{1x}Q_5 + Q_1 u_{5x}^* - Q_1 v_{5x}^* F_{3r})$$

$$- (\gamma-1)2Q_1 Q_{1x}]F_{3r} + [(\gamma+1)(u_{1r}^* Q_5$$

$$- v_{1r}^* Q_5 F_{3r} - Q_5 v_1^* F_{3rr} + Q_1 u_{5r}^* - Q_1 v_{5r}^* F_{3r}$$

$$- Q_1 v_5^* F_{3rr}) - 2(\gamma-1)Q_1(u_{1r} - v_{1r}^* F_{3r} - v_1^* F_{3rr})] \qquad (70)$$

The unknowns of the system are u_{4x}^*, v_{4x}^*, u_{4r}^*, v_{4r}^*, u_{5x}^*, u_{5r}^*, v_{5x}^*, v_{5r}^*, p_{4x}, p_{5r}, p_{5x}, p_{4r}, T_{4x}, T_{4r}, T_{5x}, T_{5r}, and the change in slopes of the slipstream h_{xx} and the resultant shock F_{3rr}. There are four shock equations ((67) to (70)) and four flow equations ((60) to (63)) in region 5, four expansion equations ((56) to (59)) and three flow equations ((60) to 63)) in region (4), and three equations ((64) to (66)) valid at the slipstream h. The resulting system of linear equations is closed, with 18 unknowns and 18 equations. A matrix solution of the system yields the needed spatial derivatives to complete the solution for shock coalescence resulting in a shock, a contact surface, and an expansion wave.

Concluding Remarks

A method for finding spatial derivatives of flow quantities behind two-dimensional, nonuniform-expansion flow over a convex corner has been developed. Equations for the axial and radial derivatives of the two velocity components in the region behind the corner have been derived in terms of the curvature of the streamline behind the expansion fan. Taylor series expansions of flow quantities within the fan are used and boundary conditions are satisfied at the leading and terminal characteristics of the fan to obtain the first- and second-order solutions and the curvature of the characteristics of the fan.

The method is applied to the case of sonic-boom extrapolation including the solution of shock coalescence with asymmetric effects. The calculation of the asymmetric effects requires finding the spatial derivatives behind the coalesced system. An outline is given for incorporating the results derived in this paper into the system for those spatial derivatives. Application of this method will remove the restriction of continuous properties through the expansion wave and extend the applicability of the sonic-boom extrapolation technique to include expansion waves of finite strength.

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16. Abstract

A method of determining spatial derivatives of flow quantities behind an expansion fan as a function of the curvature of the streamline behind the fan is developed. Taylor series expansions of flow quantities within the fan are used and boundary conditions satisfied to the first and second order so that the curvature of the characteristics in the fan may be determined. A system of linear equations for the spatial derivatives is then developed. An application of the method to shock coalescence including asymmetric effects is described.

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