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INFINITE DIMENSIONAL SYSTEMS

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Abstract

In this paper we derive the Chandrasekhar equations for linear time invariant systems defined on Hilbert spaces using a functional analytic technique. An important consequence of this is that the solution to the evolutional Riccati equation is strongly differentiable in time and one can define a ´strong´ solution of the Riccati differential equation. A detailed discussion on the linear quadratic optimal control problem for hereditary differential systems is also included.

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1. Introduction

The Chandrasekhar equations [12] are an alternative form to the Riccati equations from which the optimal feedback gain operator may be calculated directly. If the system has a small number of inputs and outputs, the Chandrasekhar algorithm offers significant reduction in the computational complexity for determining the optimal feedback gain. As observed in [18], this is much more evident in the infinite dimensional case if the optimal feedback gain operator is calculated numerically using some approximation method. In this case, the number of states grows linearly to the order of approximation.

The purpose of this paper is to derive Chandrasekhar equations for systems defined by evolution equations on Hilbert spaces in which the input and output operators are assumed to be bounded. The form of the Chandrasekhar equations derived immediately implies that the solution of the associated Riccati equation is strongly differentiable in time, and it allows us to define a ´strong´ solution of the Riccati equation. Another important consequence of this is that the optimal control for the linear quadratic regulator (LQR) problem is continuously differentiable if the initial datum is sufficiently smooth.

The Chandrasekhar equations for infinite dimensional systems have been discussed in [4] and [6] using a Lions-type framework [15]. However, the equations derived in [4] and [6] are satisfied in the distributional sense. In [19], Sorine derived a set of Chandrasekhar equations satisfied in a strong sense for parabolic systems. Sorine´s derivation relied on the analyticity of the semigroup and thus does not apply to general systems. Our approach differs from those above in that it uses an approximation technique. A

sequence of approximating optimal control problems is chosen for which the Chandrasekhar equations may be derived as in the finite dimensional case (see [5], [12], and [14]). Convergence is then established and the appropriate equations are shown to be satisfied. In this paper, our considerations are restricted to the LQR problem, but the results are also applicable to the Kalman filtering problem [7].

The contents of the paper are as follows. Section 2 briefly recalls the linear quadratic problem and characterizes the optimal control (see [2], [8], and [15] for a survey of the literature). In Section 3 a characterization of the Riccati operator is derived and used to obtain the Chandrasekhar equations. Regularity results for the Riccati operator and optimal control are discussed in Section 4. As a specific example we discuss in Section 5 the linear quadratic optimal control problem for hereditary differential systems in which the input and output spaces are finite dimensional. Because of the smoothing property of the solution semigroup, results stronger than those of the general problem are obtained.

The notation used in this paper is standard. The symbol $\langle \cdot, \cdot \rangle$ stands for the inner product in a Hilbert space where the underlying space will be understood from the context. Also, $\|\cdot\|$ denotes the norm for elements of a Banach space and for operators between Banach spaces, while $|\cdot|$ denotes the Euclidean norm. The adjoint of a densely defined operator A from one Hilbert space to another is denoted by A^*.

2. Riccati Equations

Let Z, U, and Y be Hilbert spaces. We consider the evolution equation on Z

$$\frac{d}{dt} z(t) = A z(t) + \mathcal{B} u(t), \qquad t \geq 0$$

(2.1)

$$z(0) = z \in Z$$

where $u(\cdot)$ is a U-valued, square integrable (control) function and A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on Z. The Y-valued (observation) function y is given by

(2.2) $$y(t) = \mathcal{C} z(t), \qquad t \geq 0.$$

We assume that $\mathcal{B} \in L(U,Z)$ and $\mathcal{C} \in L(Z,Y)$.

For any $T \geq 0$, if u is differentiable almost everywhere on $[0,T]$, $\dot{u} \in L_1(0,T;U)$ and $z \in \mathcal{D}(A)$, then the initial value problem (2.1) has a unique ´strong´ solution [17, Corollary 2.10] in the sense that z is differentiable almost everywhere (a.e.) on $[0,T]$ with $\dot{z} \in L_1(0,T;Z)$ and (2.1) holds a.e. on $[0,T]$. It follows from Corollary 2.2 in [17] that (2.1) has at most one solution and if it has a solution, this solution is given by

(2.3) $$z(t) = S(t)z + \int_0^t S(t - s)\mathcal{B} u(s)ds$$

which we shall call the mild solution of (2.1). Moreover, the mild solution satisfies the ´weak´ differential equation:

$$\frac{d}{dt} \langle z(t),x \rangle = \langle z(t), A^* x \rangle + \langle \mathcal{B}u(t), x \rangle \quad \text{for all} \quad x \in \mathcal{D}(A^*).$$

Consider the linear quadratic optimal control problem on a finite time interval: for given initial data $z \in Z$, choose the control $u \in L_2(0,T; \mathbb{R}^m)$ that minimizes the cost functional

$$(2.4) \qquad J(u,[0,T]) = \int_0^T \left(\|y(t)\|^2 + \|u(t)\|^2 \right) dt + \langle Gz(T), z(T) \rangle_Z$$

where G is a nonnegative (definite), self-adjoint operator on Z and z is the mild solution to (2.1). The next theorem, which characterizes the optimal control, follows from [2], [8] and [20].

Theorem 2.1: <u>The optimal control</u> u^0 <u>of (2.4) is given by</u>

$$(2.5) \qquad u^0(t) = -\mathcal{B}^* \Pi(t) z^0(t), \quad t \geq 0$$

<u>where</u> $\Pi(t)$, $t \leq T$, <u>is strongly continuous on</u> Z. <u>Moreover,</u> $\Pi(t)$ <u>is the unique solution within the class of nonnegative self-adjoint operators for which</u> $\langle \Pi(t)z, z \rangle$ <u>is absolutely continuous for</u> $z \in \mathcal{D}(A)$, <u>and satisfies the</u> <u>'weak differential' Riccati equation</u>

$$(2.6) \qquad \frac{d}{dt} \langle \Pi(t)z, z \rangle + 2\langle Az, \Pi(t)z \rangle - \langle \mathcal{B}^* \Pi(t)z, \mathcal{B}^* \Pi(t)z \rangle + \langle Cz, Cz \rangle = 0$$

<u>for all</u> $z \in \mathcal{D}(A)$

$$\Pi(T) = G.$$

If $U(\cdot,\cdot)$ denotes the perturbed evolution operator of the semigroup $S(t)$ by $-BB^*\Pi$, then for $z \in Z$

$$(2.7) \qquad U(s,t)z = S(s - t)z - \int_t^s S(s - \sigma)BB^* \Pi(\sigma)U(\sigma,t)zd\sigma,$$

$\Pi(t)$ satisfies

$$(2.8) \qquad \Pi(t)z = S^*(T - t)GU(T,t)z + \int_t^T S^*(\sigma - t)C^* CU(\sigma,t)zd\sigma,$$

and

$$z^0(t) = U(t,0)z.$$

3. Chandrasekhar Equations

From here on, we assume that $Gz \in \mathcal{D}(A^*)$ for all $z \in Z$. By the closed graph theorem $A^* G$ is then a bounded operator on Z. Let us define a bounded self-adjoint operator Q on Z by

$$\langle Qx,y \rangle = \langle A^* Gx,y \rangle + \langle x,A^* Gy \rangle - \langle B^* Gx,B^* Gy \rangle$$

$$(3.1)$$

$$+ \langle Cx,Cy \rangle \quad \text{for all } x,y \in Z.$$

The main result of this paper is given in the following theorem.

Theorem 3.1: If $\Pi(t)$, $t \le T$ is the solution to the Riccati equation (2.6), then for $z \in Z$

$$(3.2) \qquad \Pi(t)z = Gz + \int_t^T U^*(T,s)QU(T,s)z\,ds.$$

Proof: If A is a bounded linear operator on Z, then

$$S(t) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

and $t \rightarrow S(t)$ is differentiable in norm. Hence the same arguments as given in [12] for the finite dimensional system allow us to show that the theorem holds for such a case. Consider the Yosida approximation of A given by

$$A_\lambda = \lambda A(\lambda I - A)^{-1} \quad \text{for} \quad \lambda \in \mathbb{R}^+ \cap \rho(A).$$

Then A_λ is a bounded linear operator on Z and from Theorem 5.5 in [17]

$$e^{A_\lambda t} z \rightarrow S(t)z \quad \text{as} \quad \lambda \rightarrow \infty \text{ (strongly)}, \quad z \in Z$$

uniformly on bounded t-intervals. Note that

$$A_\lambda^* = \lambda A^*(\lambda I - A^*)^{-1}.$$

Indeed, for $x \in \mathcal{D}(A)$ and $y \in Z$

$$\langle A_\lambda x, y \rangle = \langle \lambda(\lambda I - A)^{-1} Ax, y \rangle = \langle x, \lambda A^*(\lambda I - A^*)^{-1} y \rangle.$$

But since $\mathcal{D}(A)$ is dense in Z, this shows that $A_\lambda^* = \lambda A^*(\lambda I - A^*)^{-1}$. Thus Theorem 5.5 in [17] again implies that

$$e^{A_\lambda^* t} z \to S^*(t)z, \qquad z \in Z$$

uniformly on bounded t-intervals.

Consider the approximate problem (A_λ, B, C) for which the theorem holds. If $\Pi_\lambda(t)$ and $U_\lambda(\cdot,\cdot)$ denote the solution of the Riccati equation and the perturbated evolution operator corresponding to the perturbation of $e^{A_\lambda t}$ by $-BB^* \Pi_\lambda(t)$, respectively, then

$$\Pi_\lambda(t)z = Gz + \int_t^T U_\lambda^*(T,s)Q_\lambda U_\lambda(T,s)z\,ds \quad \text{for} \quad z \in Z$$

where

$$Q_\lambda = A_\lambda^* G + GA_\lambda - GBB^* G + C^* C.$$

It follows from Theorem 6.1 in Gibson [11] that $\Pi_\lambda(t)$ converges strongly to $\Pi(t)$ for $t \leq T$, and the convergence is uniform on bounded t-intervals. Moreover, statement (6.14) in [11] implies that

$$U_\lambda(t,s)z \to U(t,s)z, \quad z \in Z, \quad 0 \leq s \leq t \leq T$$

where the convergence is uniform in t and s. Hence, for all $x \in Z$

$$\langle \Pi(t)x,x \rangle = \lim_{\lambda \uparrow \infty} \langle \Pi_\lambda(t)x,x \rangle = \langle Gx,x \rangle + \lim_{\lambda \uparrow \infty} \int_t^T \langle Q_\lambda U_\lambda(T,s)x, U_\lambda(T,s)x \rangle ds$$

$$(3.3) \qquad = \langle Gx,x \rangle + \lim_{\lambda \uparrow \infty} \int_t^T \{ 2\langle A^* GU_\lambda(T,s)x, J_\lambda U_\lambda(T,s)x \rangle$$

$$+ \langle (C^* C - GBB^* G)U_\lambda(T,s)x, U_\lambda(T,s)x \rangle \} ds$$

where $J_\lambda = \lambda(\lambda I - A)^{-1}$, $\lambda \in \rho(A)$. Note that

$$J_\lambda \, U_\lambda(T,s)x = (J_\lambda - I)U(T,s)x + J_\lambda(U_\lambda(T,s) - U(T,s))x + U(T,s)x$$

converges strongly to $U(T,s)x$ for $s \leq T$ since J_λ converges strongly to the identity operator I on Z (see, [17]). Since the integrand appearing in equation (3.3) is uniformly bounded in λ and s, the dominated convergence theorem allows us to obtain that for $x \in Z$

$$\langle \Pi(t)x,x \rangle = \langle Gx,x \rangle + \int_t^T \{2\langle A^* GU(T,s)x, U(T,s)x \rangle$$

$$+ \langle (C^* C - GBB^* G)U(T,s)x, U(T,s)x \rangle \} ds$$

$$= \langle (G + \int_t^T U^*(T,s)QU(T,s)ds)x,x \rangle$$

which completes the proof since the operators appearing in both sides of this equation are self-adjoint.

(Q.E.D.)

Remark 3.2: Important in applications is the case $G \equiv 0$. If this occurs, then $Q = C^* C$ and

$$\Pi(t)z = \int_t^T L^*(s)L(s)zds, \qquad z \in Z$$

where $L(s) \equiv CU(T,s)$. Define the gain operator by $K(t) = B^* \Pi(t)$. Then (see, Gibson [10])

$$U(T,t)\tau = S(T - t)z - \int_{t}^{T} U(T,s)BK(s)S(s - t)z\,ds, \quad z \in Z$$

and the operators $K(t)$ and $L(t)$ jointly satisfy

$$K(t)z = \int_{t}^{T} BL^{*}(s)L(s)z\,ds$$

$$L(t)z = CS(T - t)z - \int_{t}^{T} L(s)BK(s)S(s - t)z\,ds,$$

for all $z \in Z$, which are the infinite dimensional Chandrasekhar equations in integral form. Since $K(t)z$ and $L(t)x$ are differentiable for $z \in Z$ and $x \in \mathcal{D}(A)$, $K(t)$ and $L(t)$ also satisfy

$$\frac{d}{dt} K(t)z = -B^{*} L^{*}(t)L(t)z, \quad z \in Z,$$

$$K(T) = 0,$$

$$\frac{d}{dt} L(t)x = -L(t)[A - BK(t)]x, \quad x \in \mathcal{D}(A),$$

$$L(t) = C.$$

Note that these Chandrasekhar differential equations correspond to those derived for finite dimensional systems [12].

4. Strong Differential Riccati Equation

An important consequence of (3.2) is the following theorem.

Theorem 4.1: <u>If</u> $GZ \subset \mathcal{D}(A^*)$ <u>and</u> $\Pi(t)$, $t \leq T$, <u>is the solution to the</u> <u>Riccati equation (2.6), then for</u> $z \in Z$ $\Pi(t)z$ <u>is continuously differentiable</u> <u>on</u> $[0,T]$ <u>and</u>

$$\frac{d}{dt} \Pi(t)z = L^*(t)L(t)z.$$

The following two lemmas are essential to the derivation of the ´strong differential´ Riccati equation.

Lemma 4.2: <u>Suppose that</u> $\mathcal{B}(t)$ <u>is an operator on</u> Z <u>such that for</u> $z \in Z$, $\mathcal{B}(t)z$ <u>is continuously differentiable on</u> $[0,T]$. <u>Then</u> $A + \mathcal{B}(t)$ <u>generates a perturbed evolution operator</u> $V(t,s)$, <u>of the semigroup</u> $S(t)$ <u>on</u> Z <u>and for</u> $z \in \mathcal{D}(A)$ $V(t,s)z \in \mathcal{D}(A)$, $0 \leq s \leq t \leq T$, $V(t,x)z$ <u>is strongly</u> <u>differentiable in</u> t, <u>and</u>

$$(4.1) \qquad \frac{\partial}{\partial t} V(t,s)z = (A + \mathcal{B}(t))V(t,s)z$$

<u>is satisfied for</u> $0 \leq s \leq t \leq T$. <u>Moreover, the derivative</u> $\frac{\partial}{\partial t} V(t,s)z$ <u>for</u> $z \in \mathcal{D}(A)$ <u>is jointly continuous in</u> t <u>and</u> s.

<u>Proof:</u> Consider a class Ω of evolution operators on Z as follows: Ω consists of bounded linear operators $V(t,s)$, $0 \leq s \leq t \leq T$ on Z such that

(i) $V(s,s) = I$, $V(t,r)V(r,s) = V(t,s)$ for $0 \leq s \leq r \leq t \leq T$

(ii) $(t,s) \to V(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$

(iii) for $z \in \mathcal{D}(A)$, $V(t,s)z \in \mathcal{D}(A)$ is strongly differentiable in t and

the derivative $\frac{\partial}{\partial t} V(t,s)z$ is strongly continuous in t and s for

$0 \leq s \leq t \leq T$.

Note that $V^{(0)}(t,s)z = S(t-s)z$, $z \in Z$ belongs to Ω. Define a sequence of evolution operators $V^{(k)}(t,s)$ by

$$(4.2) \qquad V^{(k+1)}(t,s)z = S(t-s)z + \int_s^t S(t-\sigma)B(\sigma)V^{(k)}(\sigma,s)z\,d\sigma$$

for $z \in Z$ and $0 \leq s \leq t \leq T$.

It then follows from [8], [17] that

$$V^{(k)}(t,s)z \to V(t,s)z \quad \text{for} \quad z \in Z \quad \text{and} \quad 0 \leq s \leq t \leq T$$

where the convergence is uniform in t and s. If $V^{(k)}(t,s)$ belongs to the class Ω, then for $z \in \mathcal{D}(A)$, $B(t)V^{(k)}(t,s)z$ is continuously differentiable in t and

$$\frac{\partial}{\partial t}\left(B(t)V^{(k)}(t,s)z\right) = \dot{B}(t)V^{(k)}(t,s)z + B(t)\frac{\partial}{\partial t}V^{(k)}(t,s)z.$$

It now follows from (4.2) and [13, p. 487] that for $z \in \mathcal{D}(A)$, $V^{(k+1)}(t,s)z$ is continuously differentiable in t and satisfies

$$\frac{\partial}{\partial t} V^{(k+1)}(t,s)z = AV^{(k+1)}(t,s)z + B(t)V^{(k)}(t,s)z$$

or

$$\frac{\partial}{\partial t} V^{(k+1)}(t,s)z = S(t - s)(A + B(t))z + \int_s^t S(t - \sigma)\dot{B}(\sigma)V^{(k)}(\sigma,s)zd\sigma$$

(4.3)
$$+ \int_s^t S(t - \sigma)B(\sigma)\frac{\partial}{\partial t} V^{(k)}(\sigma,s)zd\sigma,$$

for $0 \leq s \leq t \leq T$.

Hence by induction, $V^{(k)}(t,x)z$ belongs to Ω for $k \geq 0$. From (4.3)

$$\frac{\partial}{\partial t} V^{(k+1)}(t,s)z - \frac{\partial}{\partial t} V^{(k)}(t,s)z$$

(4.4)
$$= \int_s^t S(t - \sigma)\dot{B}(\sigma)\big(V^{(k)}(\sigma,s) - V^{(k-1)}(\sigma,s)\big)zd\sigma$$

$$+ \int_s^t S(t - \sigma)B(\sigma)\big(\frac{\partial}{\partial \sigma} V^{(k)}(\sigma,s)z - \frac{\partial}{\partial \sigma} V^{(k-1)}(\sigma,s)z\big)d\sigma.$$

By induction on k one easily verifies the estimate:

$$\|V^{(k)}(t,s) - V^{(k-1)}(t,s)\| \leq C_1 M_1^k \frac{(t - s)^k}{k!}$$

where

$$C_1 = \max_{0 \leq s \leq T} \|S(s)\| \quad \text{and} \quad M_1 = \max_{0 \leq s \leq t \leq T} \|S(t - s)B(s)\|.$$

Since $\dot{B}(t)z$ is continuous for each $z \in Z$, $\|\dot{B}(t)\|$ is uniformly bounded on $[0,T]$. Thus, from (4.4)

$$\|\frac{\partial}{\partial t} V^{(k+1)}(t,s)z - \frac{\partial}{\partial t} V^{(k)}(t,s)z\| \leq C_1 M_2 M_1^k \frac{(t - s)^{k+1}}{(k + 1)!} \|z\|$$

$$+ M_1 \int_s^t \|\frac{\partial}{\partial \sigma} V^{(k)}(\sigma,s)z - \frac{\partial}{\partial \sigma} V^{(k-1)}(\sigma,s)z\|d\sigma$$

where

$$M_2 = \max_{0 \le s \le t \le T} \| S(t - s)\mathring{B}(s) \|.$$

By induction on k one obtains

$$\| \frac{\partial}{\partial t} V^{(k)}(t,s)z - \frac{\partial}{\partial t} V^{(k-1)}(t,s)z \| \le \frac{(t - s)^k}{(k - 1)!} M_2 M_1^{k-1} C_1 \|z\|$$

$$+ \frac{(t - s)^k}{k!} M_1^k C_1 \|Az\|.$$

Hence, $\frac{\partial}{\partial t} V^{(k)}(t,s)z$ converges to a function of $C(s,T;Z)$ for $0 \le s \le t \le T$ and $z \in \mathcal{D}(A)$ where the convergence is uniform in t and s. Note that the differential operator $(\frac{\partial}{\partial t})$ on $C(s,T;Z)$ is closed. These facts, when combined with the convergence of $V^{(k)}(t,s)z$ to $V(t,s)z$ in $C(s,T;Z)$, show that for $z \in \mathcal{D}(A)$ $V(t,s)z$ is continuously differentiable in t, and the derivative is jointly continuous in t and s. Since

$$V(t,s)z = S(t - s)z + \int_s^t S(t - \sigma)B(\sigma)V(\sigma,s)z d\sigma$$

for $z \in Z$ and $0 \le s \le t \le T$,

it now follows from [13, p. 487] that for $z \in \mathcal{D}(A)$, $V(t,s)z \in \mathcal{D}(A)$ and

$$\frac{\partial}{\partial t} V(t,s)z = (A + B(t))V(t,s)z$$

for $0 \le s \le t \le T$.

(Q.E.D)

Lemma 4.3: <u>If</u> $GZ \subset \mathcal{D}(A^*)$, <u>then for</u> $z \in \mathcal{D}(A)$ <u>and</u> $t \leq T$, $\Pi(t)z \in \mathcal{D}(A^*)$ <u>and</u> $t \to \Pi(t)z$ <u>is strongly continuous in the Hilbert space</u> $\mathcal{D}(A^*)$ <u>equipped with the graph norm.</u>

<u>Proof</u>: As a result of Theorem 2.1 it is only necessary to show that $A^* \Pi(t)z$ is continuous for $z \in \mathcal{D}(A)$. Recall that for $z \in Z$

$$(4.5) \qquad \Pi(t)z = S^*(T - t)GU(T,t)z + \int_t^T S^*(\sigma - t)C^* CU(\sigma,t)zd\sigma.$$

From Theorem 4.1 and Lemma 4.2, for $z \in \mathcal{D}(A)$ $U(\sigma,t)z$ is continuously differentiable in σ. Thus it again follows from [13, p. 487] that

$$\Pi(t)z \in \mathcal{D}(A^*), \qquad t \leq T$$

and

$$A^* \Pi(t)z = S^*(T - t)(A^* G + C^* C)U(T,t)z$$

$$- C^* Cz - \int_t^T S^*(\sigma - t)C^* C \frac{\partial}{\partial \sigma} U(\sigma,t)zd\sigma.$$

From Lemma 4.2, $\sigma \to \frac{\partial}{\partial \sigma} U(\sigma,t)z$ is strongly continuous for $0 \leq t \leq \sigma \leq T$ and $z \in \mathcal{D}(A)$. Hence $t \to A^* \Pi(t)z$ is strongly continuous for $z \in \mathcal{D}(A)$.

(Q.E.D)

We are now ready to state the main result of this section.

Theorem 4.4: <u>For</u> $z \in \mathcal{D}(A)$, $\Pi(t)z$ <u>is a unique strong solution to the Riccati equation in the sense that</u> $\Pi(t)z$ <u>is continuously differentiable on</u> $[0,T]$, $\Pi(t)z \in \mathcal{D}(A^*)$ <u>for</u> $0 \leq t \leq T$ <u>and the strong differential Riccati equation:</u>

(4.6) $(\frac{d}{dt} \Pi(t) + A^* \Pi(t) + \Pi(t)A - \Pi(t)BB^* \Pi(t) + C^* C)z = 0$

for all $z \in \mathcal{D}(A)$

$\Pi(t) = G$

<u>is satisfied on</u> $[0,T]$.

Proof: We only need to prove that (4.6) holds for $0 \leq t \leq T$. From (2.6), we have for all $x, y \in \mathcal{D}(A)$

$\frac{d}{dt} \langle \Pi(t)x, y \rangle + \langle Ax, \Pi(t)y \rangle + \langle \Pi(t)x, Ay \rangle - \langle B^* \Pi(t)x, B^* \Pi(t)y \rangle + \langle x, y \rangle = 0.$

It then follows from Theorem 4.1 and Lemma 4.3 that

(4.7) $\langle (\frac{d}{dt} \Pi(t) + A^* \Pi(t) + \Pi(t)A - \Pi(t)BB^* \Pi(t) + C^* C)x, y \rangle = 0$

for all $x, y \in \mathcal{D}(A)$.

Since $\mathcal{D}(A)$ is dense in Z, (4.7) holds for all $y \in Z$, which completes the proof.

(Q.E.D)

Corollary 4.5: <u>If $GZ \subset \mathcal{D}(A^*)$ and the initial data $z \in \mathcal{D}(A)$, then the optimal control u^0 to (2.4) is continuously differentiable on</u> $[0,T]$.

<u>Proof:</u> From Theorem 4.1 and Lemma 4.2, $z^0(t) = \mathcal{U}(t,0)z$ is continuously differentiable for $z \in \mathcal{D}(A)$. Therefore, the continuous differentiability of u^0 follows from (2.5) and Theorem 4.1.

<u>Remark:</u> From Lemma 4.2 and Theorem 4.1, if $z \in \mathcal{D}(A)$, then $z(t) \in \mathcal{D}(A)$, $t \geq 0$ and

$$(4.8) \quad \frac{d}{dt} u^0(t) = -B^* \left(\frac{d}{dt} \Pi(t)z(t) + \Pi(t)\dot{z}(t) \right) = B^*(A^* \Pi(t) + C^* C)z(t).$$

5. Hereditary Differential System

In this section we discuss the hereditary differential system:

$$\frac{d}{dt} x(t) = \int_{-r}^{0} d\mu(\theta)x(t + \theta) + Bu(t), \quad t \geq t_0$$

(5.1)

$$x(t_0) = \eta \quad \text{and} \quad x(t_0 + \theta) = \phi(\theta), \quad -r \leq \theta < 0,$$

where $\mu(\cdot)$ is an $n \times n$ matrix valued function of bounded variation which vanishes at $\theta = 0$ and is left continuous on $(-r,0)$. Without loss of generality we can assume that for $\theta \geq 0$, $\mu(\theta) = 0$ and for $\theta \leq -r$, $\mu(\theta) = \mu(-r)$. B is an $n \times m$ matrix. The observation y is given by

$$(5.2) \quad y(t) = Cx(t)$$

where C is a p×n matrix. We will denote by Z, the product space

$\mathbb{R}^n$ x $L_2(-r,0;\mathbb{R}^n)$ in this section. Given an element $z \in Z$, $\eta \in \mathbb{R}^n$ and

$\phi \in L_2$ denote the two coordinates of $z : z = (\eta,\phi)$. It is well known [3],

[9] that for $(\eta,\phi) \in Z$ and u locally square integrable, (5.1) admits a

unique solution $x \in L_2(t_0 - r,T;\mathbb{R}^n) \cap H^1(t_0,T;\mathbb{R}^n)$ for any $T \geq t_0$. If

$t_0 = 0$, then (5.1) can be formulated as an evolution equation on Z.

(5.3) $$\frac{d}{dt} z(t) = Az(t) + \mathcal{B}u(t), \quad t \geq 0$$

where $z(t) = (x(t),x(t + \cdot)) \in Z$, $t \geq 0$ and $\mathcal{B}u = (Bu,0) \in Z$ for $u \in \mathbb{R}^m$.
The infinitesimal generator A is then defined by

$$\mathcal{D}(A) = \{(\eta,\phi) \in Z \mid \eta = \phi(0) \quad \text{and} \quad \dot{\phi} \in L_2\}$$

and for $(\phi(0),\phi) \in \mathcal{D}(A)$

$$A(\phi(0),\phi) = (\int_{-r}^{0} d\mu(\theta)\phi(\theta),\dot{\phi}),$$

and generates the strong continuous semigroup S(t):

$S(t)(\eta,\phi) = (x(t),x(t + \cdot))$, $t \geq 0$ where x is the solution of (5.1) with

$t_0 = 0$ and $u \equiv 0$. Within this framework, the observation equation (5.2) is

written as

(5.4) $$y(t) = \mathcal{C}z(t), \quad t \geq 0$$

where $\mathcal{C}(\eta,\phi) = C\eta \in \mathbb{R}^p$ for $(\eta,\phi) \in Z$. Thus the system (5.1)-(5.2) is

formulated as the model system (2.1)-(2.2) in which $Z = \mathbb{R}^n$ x L_2, $U = \mathbb{R}^m$

and $Y = \mathbb{R}^p$.

The following lemma gives two important properties of the hereditary differential system which shall be used extensively in the subsequent development.

Lemma 5.1:

(i) <u>If</u> X <u>denotes the Hilbert space</u> $\mathcal{D}(A)$ <u>equipped with the graph norm,</u> <u>then</u> $\int_0^t S(t-s)\mathcal{B}u(s)ds$ <u>is an X-valued function continuous in</u> t <u>for each</u> $u \in L_2(0;T;\mathbb{R}^m)$ <u>and continuous in</u> u <u>for each</u> $t \in [0,T]$.

(ii) <u>If</u> γ <u>is a p×n matrix-valued function of bounded variation on</u> $[-r,0]$ <u>and</u> H <u>denotes an operator defined by</u> $H(\eta,\phi) = \int_{-r}^0 d\gamma(\theta)\phi(\theta)$ <u>for</u> $(\eta,\phi) \in Z,$ <u>then there exists a nondecreasing function</u> $M(\cdot) : [0,\infty] \longrightarrow \mathbb{R}^+$ <u>such that for</u> $z \in Z$

$$\int_0^T |HS(t)z|^2 \, dt \le M(T)\|z\|^2.$$

<u>Remark</u>: The proof of (i) makes explicit use of the hereditary structure and is straightforward (though tedious). In (5.5) the expression $HS(t)z$ only makes sense when $z \in \mathcal{D}(A)$. However, because of (ii), we will use the expression $HS(t)z$, $0 \le t \le T$ to denote the function in $L_2(0,T;\mathbb{R}^p)$ which is obtained by a continuous extension of the operator:

$$z \in \mathcal{D}(A) \longrightarrow HS(t)z \in L_2(0,T;\ \mathbb{R}^p).$$

Let us consider the linear quadratic optimal control problem: for given $(\eta,\phi) \in Z$ choose the control $u \in L_2(t_0,T;\mathbb{R}^m)$ that minimizes the cost functional

$$(5.6) \qquad J(u,[t_0,T]) = \int_{t_0}^{T} \left(|Cx(t)|^2 + |u(t)|^2\right)dt + \langle G_0\, x(T), x(T)\rangle_{\mathbb{R}^n}$$

where G_0 is a nonnegative, symmetric matrix on $\mathbb{R}^n$ and $x(\cdot)$ is the solution to (5.1). Note that (5.6) can be equivalently written as

$$J(u,[t_0,T]) = \int_{t_0}^{T} \left(|Cz(t)|^2 + |u(t)|^2\right)dt + \langle Gz(T), z(T)\rangle_Z$$

where G is a nonnegative, self-adjoint operator on Z defined by $G(\eta,\phi) = (G_0\,\eta, 0) \in Z$ for $(\eta,\phi) \in Z$ and $z(\cdot)$ is given by

$$z(t) = S(t - t_0)(\eta,\phi) + \int_{t_0}^{t} S(t - s)\mathcal{B}u(s)ds, \qquad t \geq t_0.$$

Hence Theorem 2.1 applies to the minimization problem (5.6).

It follows from [11], [21] that if $(y,\psi) \in \mathcal{D}(A^*)$ then

$$\psi(\theta) - (\mu(\theta) - \mu(-r))^T\, y \in H^1(-r,0)$$

and

$$\psi(-r) = (\mu((-r)^+) - \mu(-r))^T\, y.$$

Obviously $Gz \notin \mathcal{D}(A^*)$ in general. So, Theorem 3.1 does not apply for (5.6) unless $G_0 = 0$. However, as a result of Lemma 5.1 one can extend the results in Sections 3 and 4 to this case. We will discuss such an extension later and for the present consider the case $G_0 = 0$.

If $G_0 = 0$, then the solution $\Pi(t)$ to the Riccati equation (2.6) is given by

$$(5.7) \qquad \Pi(t)z = \int_{t}^{T} S^*(\sigma - t)C^*\, \mathcal{C}U(\sigma,t)zd\sigma.$$

Let A_T be the infinitesimal generator on Z defined by $\mathcal{D}(A_T) = \mathcal{D}(A)$ and for $\phi \in H^1$,

$$A_T(\phi(0),\phi) = \left(\int_{-r}^{0} d\mu^T(\theta)\phi(\theta), \dot\phi \right)$$

and let $S_T(t)$ denote the C_0-semigroup generated by A_T. Define the structural operator F on Z by

$$F(\eta,\phi) = \left(\eta, \int_{-r}^{\theta-} d\mu(\xi)\phi(\xi - \theta) \right) \quad \text{for} \quad (\eta,\phi) \in Z.$$

Then, the following result has been proven by Manitius [16].

Theorem 5.2:

(i) $FS(t) = S_T^*(t)F$, $F^* S_T(t) = S^*(t)F^*$, $t \geq 0$.

(ii) <u>If</u> $z \in \mathcal{D}(A)$, <u>then</u> $Fz \in \mathcal{D}(A_T^*)$ <u>and</u> $A_T^* Fz = FAz$.

(iii) <u>If</u> $z \in \mathcal{D}(A_T)$, <u>then</u> $F^* z \in \mathcal{D}(A^*)$ <u>and</u> $A^* F^* z = F^* A_T z$.

Since $C^* = F^* C^*$, it follows from (5.7) and Theorem 5.2 that

$$\Pi(t)z = F^* \int_{t}^{T} S_T(\sigma - t)C^* CU(\sigma,t)z d\sigma.$$

Note that $C^* y = (C^T y, 0) \in Z$ for $y \in \mathbb{R}^p$. Thus from (i) of Lemma 5.1 and (iii) of Theorem (5.2), $\Pi(t)z \in \mathcal{D}(A^*)$ for $z \in Z$. Moreover, since the evolution operator $U(\sigma,t)$ is jointly continuous for $0 \leq t \leq \sigma \leq T$, $A^* \Pi(t)z$ is strongly continuous in Z for $z \in Z$, and hence $\Pi(t)A$ has a bounded

extension to all of Z. The next result now follows from Theorem 4.4 and Corollary 4.5.

Theorem 5.3: <u>If</u> $G_0 = 0$, <u>then for</u> $z \in Z$, $\Pi(t)z$ <u>is a unique strong solution to the Riccati equation in the sense that</u> $\Pi(t)z$ <u>is continuously differentiable on</u> $[0,T)$, $\Pi(t)z \in \mathcal{D}(A^*)$ <u>for</u> $0 \leq t \leq T$, <u>and the strong differential Riccati equation:</u>

$$\left(\frac{d}{dt} \Pi(t) + A^* \Pi(t) + \Pi(t)A - \Pi(t)BB^* \Pi(t) + C^* C\right) z = 0$$

<u>for all</u> $z \in Z$

<u>is satisfied on</u> $[0,T)$. <u>Moreover, the optimal control</u> $u^0(\cdot)$ <u>to (5.6) is continuously differentiable on</u> $(0,T]$ <u>for</u> $(\eta,\phi) \in Z$.

<u>Proof:</u> From (4.8), if $z = (\eta,\phi) \in \mathcal{D}(A)$, then u^0 is continuously differentiable and

$$\dot{u}^0(t) = B^*(A^* \Pi(t) + C^* C)z^0(t)$$

$$z^0(t) = U(t,0)(\eta,\phi).$$

It has been proven that $A^* \Pi(t)z$ is strongly continuous in Z for $z \in Z$. So, the theorem follows since $\mathcal{D}(A)$ is dense in Z and $U(t,0)$ is continuous on Z for $t \geq 0$.

(Q.E.D.)

Let us turn to the case $G_0 \neq 0$. Consider the λth approximate problem to (5.6) in which the cost functional is given by

$$(5.8) \qquad J^\lambda(u,[t_0,T]) = \int_{t_0}^{T} (|Cz(t)|^2 + |u(t)|^2)dt + \langle G_\lambda z(T), z(T)\rangle_Z$$

where $G_\lambda = J_\lambda^* GJ_\lambda$ and $J_\lambda = \lambda(\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. Note that $G_\lambda Z \subset \mathcal{D}(A^*)$ and $G_\lambda \to G$ in trace norm since G has a finite rank. If $\Pi_\lambda(t)$, $t \leq T$ denotes the solution of the Riccati equation associated with the problem (5.8), then it follows from Theorem 3.1 that

$$\Pi_\lambda(t)z = G_\lambda z + \int_t^t \mathcal{U}_\lambda^*(T,s)Q_\lambda \, \mathcal{U}_\lambda(T,s)zds, \qquad z \in Z$$

where Q_λ is a self-adjoint operator on Z defined by

$$Q_\lambda = A^* G_\lambda + G_\lambda A - G_\lambda BB^* G_\lambda + C^* C.$$

Such a representation for Q_λ exists since $G_\lambda A$ can be extended to all Z via (3.1). If we denote the optimal control for the original problem (5.6) by u^0 and the optimal control for the λth approximate problem (5.8) by u_λ, it follows from [11, pp. 114-115] that u_λ converges strongly to u^0 in $L_2(t_0,T;\mathbb{R}^m)$, and the convergence is uniform in t_0 for $0 \leq t_0 \leq T$.

The following three results are essential to discuss the extension of Theorem 3.1 and Corollary 4.5 to the case when $G_0 \neq 0$.

Lemma 5.4: GA has a bounded extension to all elements $z \in Z$ of the form $z = (\phi(0),\phi)$ with $\phi \in C(-r,0;\mathbb{R}^n)$ and there exists a nondecreasing function $M(\cdot) : [0,\infty) \longrightarrow \mathbb{R}^+$ such that for $z \in Z$

$$(5.9) \qquad \int_0^T |GAS(t)z|^2 \, dt \le M(T) \|z\|^2.$$

Proof: For $z = (\phi(0),\phi) \in \mathcal{D}(A)$

$$|GAz| = |G(\int_{-r}^0 d\mu(\theta)\phi(\theta),\dot{\phi})| = |G_0 \int_{-r}^0 d\mu(\theta)\phi(\theta)|$$

$$\le |G_0| \int_{-r}^0 |d\mu| \, \|\phi\|_{C(-r,0;\mathbb{R}^n)}.$$

Since $H^1(-r,0;\mathbb{R}^n)$ is dense in $C(-r,0;\mathbb{R}^n)$, GA has the prescribed extension. Upon identifying GA with H of Lemma 5.1, (5.9) follows.

(Q.E.D)

Lemma 5.5: For $x,y \in Z$

$$\langle G_\lambda \, AU_\lambda(T,t)x, U_\lambda(T,t)y \rangle \to \langle GAU(T,t)x, U(T,t)y \rangle \quad \underline{\text{in}} \quad L_2(t_0,T).$$

Remark: To be precise, Lemma 5.4 only extends $G_\lambda A$ and GA to $x = (\phi(0),\phi)$ such that $\phi \in C(-r,0;\mathbb{R}^n)$. However, as functions in $L_2(t_0,T)$, the inner products may be extended to all Z.

Proof: First note that $U_\lambda(T,t)z$ converges strongly to $U(T,t)z$ for $z \in Z$ and the convergence is uniform in t, and that

$$U_\lambda(T,t)z = S(T-t)z + \int_t^T S(T-s)Bu_\lambda(s)ds$$

where $u_\lambda(\cdot)$ is the optimal control for the λth approximate (5.8) on the time interval $[t,T]$ with given initial condition $z \in Z$. Since J_λ converges strongly to I as $\lambda \to \infty$ on $X = \mathcal{D}(A)$, it follows from (i) of Lemma 5.1 and the fact that $u_\lambda \to u^0$ in $L_2(t,T:\mathbb{R}^m)$ that for $t \leq T$

$$f_\lambda(t) = J_\lambda \int_t^T S(T - s)\mathcal{B}u_\lambda(s)ds$$

converges strongly to

$$f^0(t) = \int_t^T S(T - s)\mathcal{B}u_0(s)ds$$

in X. Since $\|f_\lambda(t)\|_X$ is uniformly bounded in λ and $t \in [t_0,T]$, by the dominated convergence theorem, $f_\lambda(t)$ converges strongly to $f^0(t)$ in $L_2(t_0,T;X)$. Hence $\langle GAf_\lambda(t), J_\lambda U_\lambda(T,t)y \rangle$ converges strongly to $\langle GAf^0(t), U(T,t)y \rangle$ in $L_2(t_0,T)$ for $y \in Z$. The remainder of the proof is to show that for $z,y \in Z$

(5.10) $\langle GAJ_\lambda S(T - t)z, J_\lambda U_\lambda(T,t)y \rangle \to \langle GAS(T - t)z, U(T,t)y \rangle$ in $L_2(t_0,T)$.

As in Lemma 5.4, it can be shown that

$$\int_{t_0}^T |GAJ_\lambda S(T - t)z|^2 dt \leq M\|z\|^2 \quad z \in Z,$$

since $\|J_\lambda\|$ is bounded uniformly in λ. The desired result follows from direct applications of the triangle inequality and the dominated convergence theorem.

Lemma 5.6: <u>There exist a finite rank</u> $(\tilde{p})$ <u>operator</u> H <u>on</u> Z <u>and a</u> <u>nonsingular diagonal matrix</u> Λ <u>on</u> $\mathbb{R}^{\tilde{p}}$ <u>such that</u>

$$2\langle GAz,z\rangle + \langle(C^* C - G\mathcal{B}\mathcal{B}^* G)z,z\rangle = \langle\Lambda Hz,Hz\rangle_{\mathbb{R}^{\tilde{p}}} \quad \text{for} \quad z \in \mathcal{D}(A)$$

and H can be continuously extended to all elements $z \in Z$ of the form $z = (\phi(0),\phi)$ with $\phi \in C(-r,0;\mathbb{R}^n)$.

Proof: Let X' denote the strong dual space of X. We identify Z with its dual, so that $X \subset Z \subset X'$. If j is the canonical injection from X into Z: $j\phi = (\phi(0),\phi) \in Z$, $\phi \in X$, then j is an embedding from X into Z; i.e., j is injective and $j(X)$ is dense in Z; thus it follows from Proposition 4 in [1, p. 65] that j' from Z to V' and $j'j$ from X to X' are embeddings:

$$X \xrightarrow{\ j\ } Z \xrightarrow{\ j'\ } X'$$

and the bilinear form $(x,y)_{X',X}$ on $X' \times X$ is the unique extension by continuity of the scalar product (x,y) of Z restricted to $Z \times X$. Here $(')$ stands for dual operators. Let us define an operator $Q \in L(X,X')$ by

$$Q = A'G + j'GA - j'G\mathcal{B}\mathcal{B}^* Gj + j'C^* Cj.$$

If i is the norm-preserving canonical map from X' into X, then iQ is a self-adjoint operator on X. Indeed,

$$\langle iQx,y\rangle_X = \langle Qx,y\rangle_{X',X} = \langle x,Qy\rangle_{X,X'} = \langle x,iQy\rangle_X.$$

Since G and C have finite rank, Q has a finite rank, and so iQ does also. Suppose rank $(iQ) = \tilde{p}$. Then there exist an operator H on X and a nonsingular diagonal matrix Λ on $\mathbb{R}^{\tilde{p}}$ such that

$$iQz = H^* \Lambda Hz \quad \text{for all} \quad z \in X = \mathcal{D}(A).$$

It now follows that for $z \in X$

$$\langle Qz,z \rangle_{X^-,X} = \langle iQz,z \rangle_X = \langle H^* \Lambda Hz,z \rangle_X = \langle \Lambda Hz, Hz \rangle_{\mathbb{R}^{\tilde{p}}} .$$

The proof is completed if we note that

$$\langle Qz,z \rangle_{X^-X} = 2\langle GAz,z \rangle_Z + \langle (C^* C - GBB^* G)z,z \rangle_Z$$

and that from Lemma 5.4 the right-hand-side of this equality is continuous on $\phi \in C(-r,0;\mathbb{R}^n)$.

(Q.E.D.)

The next theorem gives the extension of Remark 3.2 and Theorem 3.1 to the case $G_0 \neq 0$.

Theorem 5.7: If $\Pi(t)$, $t \leq T$ is the solution of the Riccati equation (2.6) with $G(\eta,\phi) = (G_0 \, \eta, 0)$ for $(\eta,\phi) \in Z$, then for $z \in Z$

$$\Pi(t)z = Gz + \int_t^T (HU(T,s))^* \Lambda HU(T,s)zds$$

where H and Λ are defined in Lemma 5.6.

<u>Proof</u>: Recall that for $t \leq T$ and $z \in Z$

$$\Pi_\lambda(t)z = G_\lambda z + \int_t^T U_\lambda^*(T,s)Q_\lambda U_\lambda(T,s)z\,ds.$$

Since $\Pi_\lambda(t)$, $t \leq T$ converges strongly to $\Pi(t)$, uniformly on bounded t-intervals, for $t \leq T$ and $x,y \in Z$

$$\langle \Pi(t)x,y \rangle = \lim_{\lambda \uparrow \infty} \langle \Pi_\lambda(t)x,y \rangle$$

$$= \lim_{\lambda \uparrow \infty} \left(\langle G_\lambda x,y \rangle + \int_t^T \left\{ \langle A^* G_\lambda U_\lambda(T,s)x, U_\lambda(T,s)y \rangle \right. \right.$$

$$+ \langle U_\lambda(T,s)x, A^* G_\lambda U_\lambda(T,s)y \rangle + \langle (C^* C - G_\lambda BB^* G_\lambda)$$

$$\left. \left. U_\lambda(T,s)x, U_\lambda(T,s)y \rangle \right\} ds \right).$$

Hence from Lemma 5.5 and the fact that $U_\lambda(T,s)z$ converges strongly to $U(T,s)z$ for $z \in Z$ and the convergence is uniform in s, the dominated convergence theorem allows us to show that for $t \leq T$ and $x,y \in Z$

$$\langle \Pi(t)x,y \rangle = \langle Gx,y \rangle + \int_t^T \left\{ \langle GAU(T,s)x, U(T,s)y \rangle + \langle U(T,x)x, GAU(T,s)y \rangle \right.$$

$$\left. + \langle (C^* C - GBB^* G)U(T,s)x, U(T,s)y \rangle \right\} ds.$$

Since $U(T,t)z \in \mathcal{D}(A)$ for $z \in \mathcal{D}(A)$, it follows from Lemma 5.6 that for $x,y \in \mathcal{D}(A)$

$$\langle \Pi(t)x,y \rangle = \langle Gx,y \rangle + \int_t^T \langle \Lambda HU(T,s)x, HU(T,s)y \rangle ds.$$

But since H can be continuously extended to all elements z of the form $z = (\phi(0),\phi)$ with $\phi \in C(-r,0;\mathbb{R}^n)$, it follows from (ii) of Lemma 5.1 and the arguments in the proof of Lemma 5.5 that (5.11) holds for all $x,y \in Z$.

(Q.E.D)

The following results are concerned with the differentiability of the optimal control $u^0(\cdot)$ of (5.8).

Theorem 5.8: <u>For</u> $z \in Z$ <u>the optimal control</u> $u^0(\cdot)$ <u>of (5.8) is differentiable a.e. on</u> $[t_0,T]$ <u>with</u> $\dot{u}^0 \in L_2[t_0,T;\mathbb{R}^m]$.

<u>Proof:</u> It follows from Corollary 4.5 and (4.8) that if $z = (\eta,\phi) \in \mathcal{D}(A)$, then $u_\lambda(\cdot)$ is continuously differentiable on $[t_0,T]$ and is given by

$$\frac{d}{dt} u_\lambda(t) = B^*(A^* \Pi_\lambda(t) + C^* C)U_\lambda(t,t_0)z.$$

From (2.8)

$$\Pi_\lambda(t)z = S^*(T - t)G_\lambda U_\lambda(T,t)z + \int_t^T S^*(\sigma - t)C^* CU_\lambda(\sigma,t)zd\sigma \quad \text{for} \quad z \in Z.$$

Note that $G_\lambda Z \subset \mathcal{D}(A^*)$. Hence using the same arguments as those in the proof of Lemma 4.3, one can show that $\Pi_\lambda(t)z \in \mathcal{D}(A^*)$ for $z \in Z$ and $t \leq T$, and moreover, $A^* \Pi_\lambda(t)$ is strongly continuous on $[t_0,T]$. Since $U_\lambda(t,\cdot)$ is strongly continuous on Z, this fact along with the closedness of the differential operator $(\frac{d}{dt})$ on $C(t_0,T;\mathbb{R}^m)$, shows that for $z \in Z$ $u_\lambda(t)$ is continuously differentiable on $[t_0,T]$. We now note that for $t \leq T$

$$B^* A^* S^*(T - t)G_\lambda = B^* A^* S^*(T - t)J_\lambda^* GJ_\lambda$$

$$= B^* A^* S^*(T - t)J_\lambda^* F^* GJ_\lambda \quad (\text{using } F^* G = G)$$

$$= B^* A^* S^*(T - t)F^* I_\lambda GJ_\lambda$$

$$= B^* F^* A_T S_T(T - t)I_\lambda GJ_\lambda$$

$$= B^* A_T S_T(T - t)I_\lambda GJ_\lambda \quad (\text{using } B^* F^* = B^*)$$

where $I_\lambda = \lambda(\lambda I - A_T)^{-1}$, $\lambda \in \rho(A)$ and we have used Theorem 5.2 successively. Since $B^*(\eta,\phi) = B^T \eta$, the arguments, as in the proof of Lemma 5.4, yield that $B^* A_T$ has a bounded extension to all elements $z \in Z$ of the form $z = (\phi(0),\phi)$ with $\phi \in C(-r,0;\mathbb{R}^n)$ and

$$(5.12) \qquad \int_0^T |B^* A_T S_T(T - t)z|^2 \, dt \leq M(T) \|z\|^2$$

for $M(\cdot) : [0,\infty) \longrightarrow \mathbb{R}^+$ nondecreasing. Hence one obtains

$$\frac{d}{dt} u_\lambda(t) = B^* A^* \{ S^*(T - t)G_\lambda \, U_\lambda(T,t)$$

$$+ \int_t^T S^*(\sigma - t)C^* CU_\lambda(\sigma,t)d\sigma \} U_\lambda(t,t_0)z + B^* C^* CU_\lambda(t,t_0)z$$

$$= B^* A_T S_T(T - t)I_\lambda GJ_\lambda \, U_\lambda(T,t_0)z$$

$$+ B^* A_T \int_t^T S_T(\sigma - t)C^* CU_\lambda(\sigma,t_0)zd\sigma$$

$$+ B^* C^* C \mathcal{U}_\lambda(t,t_0)z.$$

Note that $\mathcal{U}_\lambda(t,t_0)z$ converges strongly to $\mathcal{U}(t,t_0)z$ for $z \in Z$, and the convergence is uniform on $[t_0,T]$. Since I_λ and J_λ converge strongly to the identity operator on Z, $I_\lambda G J_\lambda \mathcal{U}_\lambda(T,t_0)z$ converges strongly to $G\mathcal{U}(T,t_0)z$ for $z \in Z$. It now follows from (i) of Lemma 5.1 and (5.12) that $\{\frac{d}{dt} u_\lambda(\cdot)\}$ is a convergence sequence in $L_2(t_0,T;\mathbb{R}^m)$, which completes the proof when combined with the closedness of the differential operator $(\frac{d}{dt})$ on $L_2(t_0,T;\mathbb{R}^m)$.

(Q.E.D.)

This last corollary establishes the Chandrasekhar equations for hereditary differential systems.

Corollary 5.9 <u>Define the operator</u> $L(\cdot)$ <u>on</u> Z <u>by</u> $L(t)z = H\mathcal{U}(T,t)z$, $0 \le t \le T$ <u>for all</u> $z \in Z$, <u>and the gain operator</u> $K(t) = B^* \Pi(t)$, $t \le T$. <u>Then</u> $K(t)z$ <u>and</u> $L(t)x$ <u>are differentiable on</u> $[0,T]$ <u>for</u> $z \in Z$ <u>and</u> $x \in \mathcal{D}(A)$, <u>and they also satisfy</u>

$$\frac{d}{dt} K(t)z = -B^* L^*(t) \Lambda L(t)z, \qquad z \in Z$$

$$K(t) = B^* G$$

<u>and</u>

$$\frac{d}{dt} L(t)x = -L(t)(A - BK(t))x, \quad x \in \mathcal{D}(A)$$

$$L(T)x = Hx.$$

<u>Proof</u>: From (2.7),

$$L(t)Bv = HS(T - t)Bv - H \int_t^T S(T - s)BK(s)U(s,t)Bv ds \quad \text{for } v \in \mathbb{R}^m.$$

Here note that $HS(\tau)Bv = H(x(\tau), x(\tau + \cdot))$, $\tau \geq 0$ where x is the homogeneous solution of (5.1) with initial condition Bv and $x \in BV(-r, T; \mathbb{R}^n)$ for any $T \geq 0$. This means that $L(t)B \in \mathbb{R}^{\tilde{p} \times m}$ exists for each t, and it is not difficult to show that $L(t)B$ is of bounded variation on $[0,T]$. So $B^* L^*(\cdot) = (L(\cdot)B)^* \in \mathbb{R}^{m \times \tilde{p}}$. It then follows from Theorem 5.7 that for $z \in Z$

$$K(t)z = B^* Gz + \int_t^T B^* L^*(s)\Lambda L(s)z ds;$$

and hence $K(t)z$ is differentiable on $[0,T]$.

Note that for $x \in \mathcal{D}(A)$, $U(T,t)x$ is continuously differentiable with respect to t and

$$U(T,t)x - x = \int_t^T U(T,s)(A - BK(s))x ds.$$

Since $HU(T,t)B = L(t)B$ is integrable,

$$L(t)x - Hx = HU(T,t)x - Hx = \int_t^T HU(T,s)(A - BK(s))x ds$$

$$= \int_t^T L(s)(A - BK(s))x \quad \text{for } x \in \mathcal{D}(A).$$

(Q.E.D)

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16. Abstract

In this paper we derive the Chandrasekhar equations for linear time invariant systems defined on Hilbert spaces using a functional analytic technique. An important consequence of this is that the solution to the evolutional Riccati equation is strongly differentiable in time and one can define a ´strong´ solution of the Riccati differential equation. A detailed discussion on the linear quadratic optimal control problem for hereditary differential systems is also included.

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