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A GALERKIN METHOD FOR THE ESTIMATION OF PARAMETERS
IN HYBRID SYSTEMS GOVERNING THE VIBRATION
OF FLEXIBLE BEAMS WITH TIP BODIES

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A GALERKIN METHOD FOR THE ESTIMATION OF PARAMETERS
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ABSTRACT

In this report we develop an approximation scheme for the identification of hybrid systems describing the transverse vibrations of flexible beams with attached tip bodies. In particular, problems involving the estimation of functional parameters (spatially varying stiffness and/or linear mass density, temporally and/or spatially varying loads, etc.) are considered. The identification problem is formulated as a least squares fit to data subject to the coupled system of partial and ordinary differential equations describing the transverse displacement of the beam and the motion of the tip bodies respectively. A cubic spline-based Galerkin method applied to the state equations in weak form and the discretization of the admissible parameter space yield a sequence of approximting finite dimensional identification problems. We demonstrate that each of the approximating problems admits a solution and that from the resulting sequence of optimal solutions a convergent subsequence can be extracted, the limit of which is a solution to the original identification problem. The approximating identification problems can be solved using standard techniques and readily available software. Numerical results for a variety of examples are provided.

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SECTION 1

INTRODUCTION

In this report we develop an approximation scheme for the identifi-
cation of systems describing the transverse vibration of flexible beams
with attached tip bodies. The equations of motion for this type of prob-
lem generally take the form of a hybrid system of coupled partial (gov-
erning the vibration of the beam) and ordinary (describing the motion of
the tip bodies) differential equations with appropriate geometric bound-
ary conditions and initial data. The resulting identification problems,
therefore, tend to have infinite dimensional constraints. Moreover, if
the parameters to be identified are functional (spatially and/or tempor-
arily varying), the admissible parameter space is a function space and as
such is infinite dimensional as well. The solution of the resulting
constrained optimization problem,therefore, necessitates the use of some
form of finite dimensional approximation.

The scheme we develop here is based upon the reformulation of the
equations of motion in weak form. A cubic spline based Rayleigh-Ritz-
Galerkin method is used to define a finite dimensional approximation to
the state equations. Using finite dimensional subspaces to discretize
the admissible parameter space, we obtain a doubly indexed sequence of
approximating identification problems. Using standard variational argu-
ments, we derive a convergence result for the state approximations. We
show next that each of the approximating identification problems admits a

solution and that from the resulting sequence of optimal parameter values a convergent subsequence can be extracted whose limit is a solution to the original infinite dimensional identification problem. In addition to convergence results, we present numerical results which demonstrate the feasibility of our method.

The approach described in this report represents a significant improvement over the method developed in [22]. Indeed, we have developed a scheme which is computationally simpler and, by relaxing the necessary hypotheses on the admissible parameter space, is applicable to a wider class of problems. Our results are similar in spirit to those presented in [7] in the context of parabolic systems, in [8] for hyperbolic systems and in a forthcoming paper by Banks and Crowley [5] for beam equations with standard boundary conditions (e.g., clamped, simply supported, cantilevered, etc.). Other work regarding approximation methods for inverse problems in elasticity can be found in [2], [3], [4], [10] and [14].

We simplify our presentation by only considering a cantilevered beam with an attached tip (point) mass. As is discussed in Section 4, however, our general approach is applicable to a broad class of beam-tip body vibration problems. In Section 2 we derive the weak form of the equations of motion, define weak and strong solutions, and formulate the identification problem. In Section 3 we define the approximation scheme and discuss convergence. Numerical results are presented in Section 4.

We employ standard notation throughout. The Sobolev spaces of real valued functions on the interval [a,b] whose k^{th} derivatives are L_2 are denoted by $H^k(a,b)$, $k = 0,1,2....$ The corresponding Sobolev inner products and their induced norms are denoted by $<\cdot,\cdot>_k$ and $|\cdot|_k$ respectively. For Z a normed linear space with norm $|\cdot|_Z$ and $f : [0,T] \rightarrow Z$, we say that $f \in L_2([0,T],Z)$ if $\int_0^T |f(t)|_Z^2 \, dt < \infty$.

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Similarly, f will be said to be an element in $C^\ell([0,T],Z)$ if the map $t \to f(t)$ from $[0,T]$ into Z is ℓ times continuously differentiable on $(0,T)$. Finally, for a function of one or more real variables the symbol $D_\theta f$ $(D_\theta^k f)$ will be used to denote the 1st (kth) derivative of f with respect to the independent variable θ. If f is a function of a single variable only, the subscript may be deleted. On occasion the short-hand notation $D_\theta f(\theta_0)$ or $Df(\theta_0)$ will be used in place of $D_\theta f|_{\theta_0}$ or $Df|_{\theta_0}$ to denote the derivative of f evaluated at θ_0.

SECTION 2

THE IDENTIFICATION PROBLEM

We consider a long slender beam of length ℓ with spatially varying stiffness EI and linear mass density ρ which is clamped at one end and free at the other with an attached tip mass of magnitude m (see Figure 2.1).
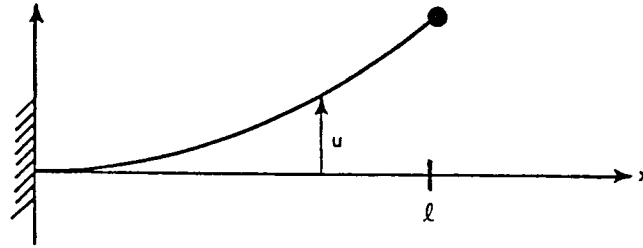

Figure 2.1.

Using the Euler-Bernoulli theory and elementary Newtonian mechanics, we obtain the partial differential equation

$$\rho(x)D_t^2 u(t,x) + D_x^2 EI(x)D_x^2 u(t,x) =$$

$$D_x \sigma(t,x)D_x u(t,x) + f(t,x) \qquad x\varepsilon(0,\ell) \quad t\varepsilon(0,t)$$

(2.1)

and boundary condition at the free end

$$mD_t^2 u(t,\ell) - D_x EI(\ell)D_x^2 u(t,\ell) =$$

$$- \sigma(t,\ell)D_x u(t,\ell) + g(t) \qquad t\epsilon(0,T) \tag{2.2}$$

describing the transverse displacement of the beam and tip mass respectively where σ is the internal tension (as a result of axial loading) f is a distributed lateral load applied to the beam and g is a force directed transversely which acts on the tip mass (see [9] and [26]). Rotational equilibrium at the free end yields

$$D_x^2 u(t,\ell) = 0 \qquad t\epsilon[0,T] , \tag{2.3}$$

while at the clamped end we have the usual geometric boundary conditions

$$u(t,0) = 0 \qquad D_x u(t,0) = 0 \qquad t\epsilon[0,T]. \tag{2.4}$$

The temporal boundary conditions, or initial conditions, are assumed to be of the form

$$u(0,x) = \phi(x) \qquad D_t u(0,x) = \psi(x) \qquad x\epsilon[0,\ell] . \tag{2.5}$$

We make the standing assumptions that m > 0, $EI, \rho \epsilon L_\infty(0,\ell)$ with $EI, \rho > 0$, $\sigma\epsilon L_2([0,T], H^1(0,\ell))$, $g\epsilon L_2(0,T)$, $f\epsilon L_2([0,T], H^0(0,T))$, $\phi\epsilon H^2(0,\ell)$, and $\psi\epsilon H^0(0,\ell)$ with $\psi(\ell)$ specified in R. Define the Hilbert space $H = R \times H^0(0,\ell)$ with inner product

$$< (\eta,\phi), (\zeta,\psi) >_H = \eta\zeta + < \phi,\psi >_0.$$

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We then rewrite Equations (2.1) - (2.5) as

$$M_0 D_t^2 \hat{u}(t) + A_0 \hat{u}(t) = B_0(t)\hat{u}(t) + F_0(t) \quad t\varepsilon(0,T) \quad (2.6)$$

$$\gamma_0 \hat{u}(t)\Big|_{x=0} = 0 \quad \gamma_1 \hat{u}(t)\Big|_{x=0} = 0 \quad \gamma_2 \hat{u}(t)\Big|_{x=\ell} = 0 \quad t\varepsilon[0,T] \quad (2.7)$$

$$\hat{u}(0) = \hat{\phi} \quad D_t \hat{u}(0) = \hat{\psi} \quad (2.8)$$

where $\hat{u}(t) = (u(t,\ell),u(t,\cdot))\varepsilon H$, $F_0(t) = (g(t),f(t,\cdot))$, $\hat{\phi} = (\phi(\ell),\phi)$, $\hat{\psi} = (\psi(\ell),\psi)$ and the operators M_0, A_0, $B_0(t)$ and γ_i, $i=0,1,2$ on H are defined formally by

$$M_0(\eta,\phi) = (m\eta,\rho\phi),$$

$$A_0(\eta,\phi) = (- DEI(\ell)D^2\phi(\ell), D^2EID^2\phi),$$

$$B_0(t)(\eta,\phi) = (- \sigma(t,\ell)D\phi(\ell), D\sigma D\phi),$$

$$\gamma_i(\eta,\phi) = D^i\phi, \quad i = 0,1,2$$

respectively.

There exist several ways in which the notion of a solution to the system (2.6) - (2.8) can be made precise. Of particular interest to us here are the ideas of a weak or variational solution and a strong solution.

Define the Hilbert space $\{V, < \cdot, \cdot >_V\}$ by

$$V = \{(\eta, \phi) \epsilon H : \phi \epsilon H^2(0, \ell), \phi(0) = D\phi(0) = 0, \eta = \phi(\ell)\} ,$$

$$< (\phi(\ell), \phi), (\psi(\ell), \psi) >_V = < D^2\phi, D^2\psi >_0.$$

It is not difficult to show that V can be densely embedded in H. Choosing H as our pivot space, we have therefore that $V \subset H \subset V'$ where V' denotes the space of continuous linear functionals on V. Consider the second order initial value problem

$$< M_0 D_t^2 \hat{u}(t), \hat{\theta} >_H + a(\hat{u}(t), \hat{\theta}) =$$

$$b(t)(\hat{u}(t), \hat{\theta}) + < F_0(t), \hat{\theta} >_H \quad t\epsilon(0,T), \hat{\theta}\epsilon V \tag{2.9}$$

$$\hat{u}(0) = \hat{\phi} \qquad D_t \hat{u}(0) = \hat{\psi} \tag{2.10}$$

where $\hat{\theta} = (\theta(\ell), \theta)$ and the bilinear forms a: $V \times V \to R$ and b(t): $V \times V \to R$ are given by

$$a(\hat{\phi}, \hat{\psi}) = < EID^2\phi, D^2\psi >_0$$

and

$$b(t)(\hat{\phi}, \hat{\psi}) = < -\sigma D\phi, D\psi >_0$$

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respectively. A solution $\hat{u}$ to (2.9) and (2.10) with $\hat{u}(t)\epsilon V$ is known as weak or variational solution to (2.6) - (2.8). Indeed, if the derivatives in (2.6) and (2.7) are taken in the distributional sense, A_0 and $B_0(t)$ become bounded linear operators from V into V' with

$$< A_0\hat{\phi},\hat{\psi} >_H = a(\hat{\phi},\hat{\psi})$$

and

$$< B_0(t)\hat{\phi},\hat{\psi} >_H = b(t)(\hat{\phi},\hat{\psi})$$

where the H inner product is interpreted as its natural extension to the duality pairing between V and V' (see [1], [19], [24]). Since $F_0(t)\epsilon H \subset V'$ we have therefore that the systems (2.6) - (2.8) and (2.9) and (2.10) are two representations for the same initial value problem in V'.

Under the assumptions which we have made above, standard arguments (see [16], [17]) can be used to demonstrate the existence of a unique solution $\hat{u}$ to (2.9) and (2.10) with $\hat{u}\epsilon C([0,T],V)$, $D_t\hat{u}\epsilon C([0,T],H)$ and $D_t^2\hat{u}\epsilon L_2([0,T],V')$.

In order to characterize strong solutions we rewrite (2.6) - (2.8) as an equivalent abstract first order system and then rely upon the theory of semigroups and evolution operators. Let Z = V × H with inner product

$$< (\hat{v}_1,\hat{h}_1), (\hat{v}_2,\hat{h}_2) >_Z = a(\hat{v}_1,\hat{v}_2) + < M_0\hat{h}_1,\hat{h}_2 >_H .$$

We assume that $EI \epsilon H^2(0,\ell)$ and $\sigma \epsilon C^1([0,T], H^1(0,\ell))$ and define the opera-
$A: \text{Dom}(A) \subset Z \to Z$ by

$$\text{Dom}(A) = \text{Dom}(A_0) \times V$$

$$A = \begin{bmatrix} 0 & I \\ -M_0^{-1}A_0 & 0 \end{bmatrix}$$

where I is the identity on V, M_0 and A_0 are as they were defined
above, and

$$\text{Dom}(A_0) = \{\hat{\phi} = (\phi(\ell), \phi) \epsilon V: \phi \epsilon H^4(0,\ell), D^2\phi(\ell) = 0\}.$$

Similarly, define the operators $B(t): Z \to Z$ by

$$B(t) = \begin{bmatrix} 0 & 0 \\ M_0^{-1}B_0(t) & 0 \end{bmatrix}$$

and let $A(t): \text{Dom}(A) \subset Z \to Z$ be given by $A(t) = A + B(t)$. Let $F(t) \epsilon Z$ be
defined by $F(t) = (0, M_0^{-1}F_0(t))$, $z_0 \epsilon Z$ by $z_0 = (\hat{\phi}, \hat{\psi})$ and consider the
initial value problem

$$D_t z(t) = A(t)z(t) + F(t) \qquad t\epsilon(0,T) \qquad (2.11)$$

$$z(0) = z_0. \qquad (2.12)$$

It is not difficult to argue that the operator A is densely defined and conservative. That is

$$< Az,z >_Z = 0 \qquad z \epsilon Dom(A). \qquad (2.13)$$

Moreover, we have

Theorem 2.1: The operator $A:Dom(A) \subset Z \to Z$ is skew self adjoint.

Proof

We first argue that $-A \subset A^*$. That is $Dom(A) \subset Dom(A^*)$ and for $z \epsilon Dom(A)$, $A^*z = -Az$. Let $z_1, z_2 \epsilon Dom(A)$ with $z_i = (\hat{\phi}_i, \hat{\psi}_i)$. Then

$$< Az_1, z_2 >_Z = a(\hat{\psi}_1, \hat{\phi}_2) + < - A_0 \hat{\phi}_1, \hat{\psi}_2 >_H$$

$$= < EID^2 \psi_1, D^2 \phi_2 >_0 - < EID^2 \phi_1, D^2 \psi_2 >_0$$

$$= - < \hat{\psi}_1, -A_0 \hat{\phi}_2 >_H - a(\hat{\phi}_1, \hat{\psi}_2)$$

$$= - <z_1, Az_2 >_Z$$

where we have used integration by parts, the definition of $Dom(A_0)$ and the definition of V in performing the above computation. This, of course, implies that $z_2 \epsilon Dom(A^*)$ and $A^*z_2 = -Az_2$.

We next argue that $Dom(A^*) \subset Dom(A)$. Let $w \epsilon Dom(A^*)$ and $y = A^*w$. Then for $z \epsilon Dom(A)$

$$< z, y >_Z = < z, A^*w >_Z = < Az, w >_Z.$$

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Recalling that $z, w, y \epsilon Z$, let $z = (\hat{z}_1, \hat{z}_2)$, $w = (\hat{w}_1, \hat{w}_2)$ and $y = (\hat{y}_1, \hat{y}_2)$. Then

$$0 = \langle z, y \rangle_Z - \langle Az, w \rangle_Z$$

$$= a(\hat{z}_1, \hat{y}_1) + \langle M_0 \hat{z}_2, \hat{y}_2 \rangle_H - a(\hat{z}_2, \hat{w}_1) + \langle A_0 \hat{z}_1, \hat{w}_2 \rangle_H$$

$$= \langle EID^2 z_1, D^2 y_1 \rangle_0 + m z_2(\ell) \hat{y}_2^1 + \langle \rho z_2, \hat{y}_2^2 \rangle_0 -$$

$$\langle EID^2 z_2, D^2 w_1 \rangle_0 - DEI(\ell) D^2 z_1(\ell) \hat{w}_2^1 + \langle D^2 EID^2 z_1, \hat{w}_2^2 \rangle_0 \qquad (2.14)$$

where $\hat{y}_2 = (\hat{y}_2^1, \hat{y}_2^2) \epsilon H$ and $\hat{w}_2 = (\hat{w}_2^2, \hat{w}_2^2) \epsilon H$. Let $\theta \epsilon H^2(0, \ell)$ be defined by

$$D^2 \theta = \rho \hat{y}_2^2, \quad \theta(\ell) = 0 \quad \text{and} \quad D\theta(\ell) = - m\hat{y}_2^1 .$$

Then substituting into (2.14) and integrating by parts, we obtain

$$0 = - DEI(\ell) D^2 z_1(\ell)(y_1(\ell) + \hat{w}_2^1) +$$

$$\langle D^2 EID^2 z_1, y_1 + \hat{w}_2^2 \rangle_0 + \langle D^2 z_2, \theta - EID^2 w_1 \rangle_0$$

which implies

(i) $-DEI(\ell) D^2 z_1(\ell)(y_1(\ell) + \hat{w}_2^1) + \langle D^2 EID^2 z_1, y_1 + \hat{w}_2^2 \rangle_0 = 0$

and

(ii) $\langle D^2 z_2, \theta - EID^2 w_1 \rangle_0 = 0 .$

Let (η,ϕ) be an arbitrary element in H. Choose $z_1 \epsilon H^4(0,\ell)$ which satisfies

$$D^2 EID^2 z_1 = \phi, \quad DEI(\ell)D^2 z_1(\ell) = -\eta, \quad D^2 z_1(\ell) = 0, \quad Dz_1(0) = 0$$

and

$$z_1(0) = 0.$$

Then $\hat{z}_1 = (z_1(\ell),z_1)\epsilon Dom(A_0)$ and (i) therefore implies that

$$< (y_1(\ell) + \hat{w}_2^1, y_1 + \hat{w}_2^2), (\eta,\phi) >_H = 0 \qquad (\eta,\phi)\epsilon H$$

from which it follows that $y_1(\ell) = -\hat{w}_2^1$ and $y_1 = -\hat{w}_2^2$ in $H^0(0,\ell)$. We have, therefore, that $\hat{w}_2 = -\hat{y}_1 \epsilon V$.

Next, let ϕ be an arbitrary element in $H^0(0,\ell)$. Choosing $z_2 = \int_0 \int_0 \phi$, (ii) implies that

$$< \theta - EID^2 w_1, \phi >_0 = 0 \qquad \phi\epsilon H^0(0,\ell)$$

and hence that $\theta = EID^2 w_1$. This in turn implies that $w_1 \epsilon H^4(0,\ell)$ and $D^2 w_1(\ell) = \frac{1}{EI(\ell)} \theta(\ell) = 0$. Since $w\epsilon Z$, $\hat{w}_1 \epsilon V$ and therefore $\hat{w}_1 \epsilon Dom(A_0)$, we conclude that $w = (\hat{w}_1, \hat{w}_2)\epsilon Dom(A_0) \times V = Dom(A)$ and the theorem is proven.

Since A is densely defined [12, Theorem 3, page 142] implies that A*
is closed and, by Theorem 2.1 above, that A is closed as well. This fact
together with (2.13) and Theorem 2.1 yield that A is maximal dissipative,
and hence by [13, Theorem 4.2, page 84] that it is the infinitesimal
generator of a C_0 semigroup of contractions $\{S(t): t \geq 0\}$ on Z. It is
in fact the case that (see [13], [28]) S(t) is defined for $t < 0$ and that
$\{S(t): -\infty < t < \infty\}$ forms a C_0 group of unitary operators on Z. Since
the operators B(t), $0 \leq t \leq T$ are bounded [20, Theorem 2.3, page 132]
implies that $\{A(t)\}_{t\epsilon[0,T]}$ is a stable family of infinitesimal
generators. Since Dom(A) is independent of t, [20, Theorem 4.8, page
145] yields that the family $\{A(t)\}_{t\epsilon[0,T]}$ generates an evolution system
$\{U(t,s): 0 \leq s \leq t \leq T\}$ on Z.

Define

$$z(t) = (\hat{u}(t),\hat{v}(t)) \qquad (2.15)$$

in Z by

$$z(t) = U(t,0)z_0 + \int_0^t U(t,s)F(s)ds. \qquad (2.16)$$

The continuous function z given by (2.16) above is the unique mild
solution to the initial valve problem (2.11), (2.12). If, in addition,
$z_0\epsilon Dom(A)$ (that is $\hat{\phi}\epsilon Dom(A_0)$ and $\psi\hat{\epsilon}V$) then z is a strong solution to
(2.11), (2.12). Indeed z is differentiable almost everywhere on [0,T]
with $D_t z\epsilon L_2([0,T],Z)$, satisfies (2.12) and satisfies (2.11) for almost
every $t\epsilon[0,T]$ and is such that $z(t)\epsilon Dom(A)$ a.e. on [0,T].

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We shall call $\hat{u}$ given by (2.15) and (2.16) a strong solution to the initial value problem (2.6) - (2.8). The following result is easily obtained.

Theorem 2.2: Suppose $EI \varepsilon H^2(0,\ell)$, $\sigma \varepsilon C^1([0,T], H^1(0,\ell))$, $\hat{\phi} \varepsilon \text{Dom}(A_0)$, and $\hat{\psi} \varepsilon V$. Then $\hat{u}$ given by (2.15) and (2.16) is the unique strong solution to the initial value problem (2.6) - (2.8). We have that $\hat{u}$ satisfies (2.8) and (2.6) and (2.7) a.e. on [0,T]. Moreover, $\hat{u}$ is twice differentiable in H and differentiable in V almost everywhere on [0,T] with $D_t^2 \hat{u} \varepsilon L_2([0,T],H)$ and $D_t \hat{u} \varepsilon L_2([0,T],V)$.

It is also not difficult to show that if a strong solution $\hat{u}$ to (2.6) - (2.8) exists it coincides with the weak solution, and in which case, it is given either by (2.15) and (2.16) or as the solution to the initial value problem (2.9), (2.10).

In formulating the identification problem, for ease of exposition, we consider a (reasonably broad) class of inverse problems which are of particular interest in the structural modeling of large flexible spacecraft and shuttle attached payloads (see [25], [27]). We assume that we are only interested in estimating the parameters $m \varepsilon R$, EI, $\rho \varepsilon L_\infty(0,\ell)$, and $\sigma \varepsilon L_2([0,T],H^1(0,\ell))$ where $\sigma = \sigma(a_0,m,\rho)$ with $a_0 \varepsilon L_2(0,T)$. It is not difficult to further generalize the results which follow to allow for the identification of initial data, the external loads f and g, and more general forms of the internal tension σ (see [7]). The motivation for choosing σ to be a function of a time varying function a_0, the magnitude of the tip mass m, and the linear mass density of the beam ρ will be made clear below.

Let $\mathscr{Q} = R \times L_\infty(0,\ell) \times L_\infty(0,\ell) \times L_2(0,T)$ with the usual product topology. Let Q be a subset of $\mathscr{Q}$ which satisfies

(H1) Q is compact.

(H2) There exist constants m_i, M_i, $i=1,2,3$ such that

$$0 < m_1 \leq m \leq M_1$$

$$0 < m_2 < EI < M_2$$

$$0 < m_3 < \rho < M_3$$

for all $q = (m, EI, \rho, a_0) \epsilon Q$.

(H3) For all $q \epsilon Q$, $\sigma(q) \epsilon L_2([0,T]), H^1(0,\ell))$ with the mapping $q \to \sigma(q)$ continuous from Q into $L_2([0,T], H^1(0,\ell))$.

We assume that we have been provided with displacement measurements $\{\overline{u}(t_i, x_j)\}_{\substack{i=1,\ldots\mu \\ j=1,\ldots\nu}}$ for the beam at positions $x_j \epsilon [0,\ell]$, $j = 1,2,\ldots\nu$, at times $t_i \epsilon [0,T]$, $i = 1,2,\ldots\mu$, which result from a known input disturbance applied to the system in a known initial state and formulate the identification problem as a least squares fit to data:

(ID)

Find $q = (m, EI, \rho, a_0) \epsilon Q$ which minimizes

$$J(q; \hat{u}(q)) = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} \left| \overline{u}(t_i, x_j) - u(t_i, x_j, q) \right|^2 \qquad (2.17)$$

where $\hat{u}(t; q) = (u(t,\ell; q), u(t,\cdot; q))$ is the solution to (2.9), (2.10) corresponding to $q \epsilon Q$.

The infinite dimensionality of both the state, which is governed by the system (2.9), (2.10), and the admissible parameter space Q (being a

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function space) necessitates the use of some form of approximation in solving Problem (ID). We develop and analyze one particular scheme in the next section. We note that the approximation theory to be developed below will also permit the formulation of the identification problem based upon criteria other than displacement; for example, velocity (see [22]).

SECTION 3

AN APPROXIMATION SCHEME

Our scheme is based upon the construction of a sequence of approximating identification problems, in each of which both the state constraint and the admissible parameter space are finite dimensional. We argue that each of the approximating problems admits a solution. The resulting sequence is shown to have a sub-sequential limit which is a solution to the original identification problem. The state approximation is constructed using a spline based Galerkin method. The admissible parameter space is discretized using splines as well. We begin by discussing the state approximation.

Working abstractly at first, let $V^N \subset V$ be a finite dimensional subspace of H and let P^N denote the orthogonal projection of H onto V^N with respect to the H inner product. The Galerkin equations corresponding to (2.9), (2.10) are given by

$$< M_0 D_t^2 \hat{u}^N(t), \hat{\theta}^N >_H + a(\hat{u}^N(t), \hat{\theta}^N) =$$

$$b(t)(\hat{u}^N(t), \hat{\theta}^N) + < F_0(t), \hat{\theta}^N >_H \quad t\epsilon(0,T), \quad \hat{\theta}^N \epsilon V^N \quad (3.1)$$

$$\hat{u}^N(0) = P^N \hat{\phi} \qquad D_t \hat{u}^N(0) = P^N \hat{\psi} \qquad (3.2)$$

where $\hat{u}^N(t) = (u^N(t,\ell), u^N(t,\cdot)) \epsilon V^N$. We define the following sequence of approximating identification problems.

(IDN)

Find $q = (m, EI, \rho, a_0) \epsilon Q$ which minimizes $J(q; \hat{u}^N(q))$ where J is given by (2.17) and $\hat{u}^N(t; q) = (u^N(t, \ell; q), u^N(t, \cdot\ ; q))$ is the solution to (3.1), (3.2) corresponding to $q \epsilon Q$.

Let $\{\bar{B}_j^N\}_{j=-1}^{N+1}$ denote the standard cubic B-splines on the interval $[0, \ell]$ corresponding to the uniform partition $\Delta^N = \{0, \frac{\ell}{N}, \frac{2\ell}{N}, \ldots \ell\}$ (see [21]). Let $\{B_j^N\}_{j=1}^{N+1}$ denote the modified cubic B-splines which satisfy $B_j^N(0) = DB_j^N(0) = 0$, $j = 1, 2, \ldots N+1$. That is

$$B_1^N = \bar{B}_0^N - 2\bar{B}_1^N - 2\bar{B}_{-1}^N$$

$$B_j^N = \bar{B}_j^N \qquad j = 2, 3 \ldots N+1.$$

Let $\hat{B}_j^N = (B_j^N(\ell), B_j^N)$ and let $V^N \subset V$ be defined by

$$V^N = \text{SPAN } \{\hat{B}_j^N\}_{j=1}^{N+1} .$$

The Galerkin equations (3.1), (3.1) take the form

$$M_0^N \ddot{w}^N(t) + A_0^N w^N(t) = B_0^N(t) w^N(t) + F_0^N(t) \tag{3.3}$$

$$w^N(0) = (W^N)^{-1} \hat{\phi}^N \qquad \dot{w}^N(0) = (W^N)^{-1} \hat{\psi}^N \tag{3.4}$$

where

$$[M_0^N]_{ij} \quad = \quad mB_i^N(\ell)B_j^N(\ell) + \int_0^\ell \rho B_i^N B_j^N$$

$$[A_0^N]_{ij} \quad = \quad \int_0^\ell EID^2B_i^N D^2B_j^N$$

$$[B_0^N(t)]_{ij} \quad = \quad - \int_0^\ell \sigma(t,\cdot)DB_i^N DB_j^N$$

$$[F_0^N(t)]_i \quad = \quad g(t)B_i^N(\ell) + \int_0^\ell f(t,\cdot)B_i^N$$

$$[\hat{\phi}^N]_i \quad = \quad \phi(\ell)B_i^N(\ell) + \int_0^\ell \phi B_i^N$$

$$[\hat{\psi}^N]_i \quad = \quad \psi(\ell)B_i^N(\ell) + \int_0^\ell \psi B_i^N$$

$$[W^N]_{ij} \quad = \quad B_i^N(\ell)B_j^N(\ell) + \int_0^\ell B_i^N B_j^N$$

$i,j = 1,2..N+1$ and $\hat{u}^N(t) = \sum_{j=1}^{N+1} w_j^N(t)\hat{B}_j^N$.

Our convergence arguments are based upon the approximation properties of spline functions. Let $\overline{S}_3(\Delta^N) = \text{SPAN}\{\overline{B}_j^N\}_{j=-1}^{N+1}$ and let $S^3(\Delta^N) = \text{SPAN}\{B_j^N\}_{j=1}^{N+1}$. For ϕ a function defined on the interval $[0,\ell]$ let $\overline{I}^N\phi$ denote that element in $\overline{S}^3(\Delta^N)$ which satisfies the interpolatory constraints $(\overline{I}^N\phi)(\frac{j\ell}{N}) = \phi(\frac{j\ell}{N})$, $j = 0,1,2...N$, $D(\overline{I}^N\phi)(\frac{j\ell}{N}) = D\phi(\frac{j\ell}{N})$ $j = 0$

and N and let $I^N\phi$ denote that element in $S^3(\Delta^N)$ which satisfies the interpolatory constraints $(I^N\phi)(\frac{j\ell}{N}) = \phi(\frac{j\ell}{N})$ $j = 1,2...N$, $D(I^N\phi)(\ell) = D\phi(\ell)$. The interpolatory spline $\bar{I}^N\phi$ will be well defined whenever ϕ is well defined at the node points and $D\phi$ at the end points. A similar statement can be made for $I^N\phi$.

We require the following two standard results concerning the approximation properties of interpolatory splines (see [23]).

<u>Proposition 3.1</u>: For $\phi \in H^2(0,\ell)$

$$\left| D^k (\bar{I}^N\phi - \phi) \right|_0 \leq c_k^1 N^{-2+k} \left| D^2\phi \right|_0 \qquad k = 0,1$$

where c_k^1 is independent of ϕ and N.

<u>Proposition 3.2</u>: For $\phi \in H^4(0,\ell)$

$$\left| D^k (\bar{I}^N\phi - \phi) \right|_0 \leq c_k^2 N^{-4+k} \left| D^4\phi \right|_0 \qquad k = 0,1,2$$

where c_k^2 is independent of ϕ and N.

<u>Lemma 3.1</u>

(1) Let $\hat{\phi}^N = (\phi(\ell),\phi)\epsilon V$ and let $\hat{\phi}^N = P^N\hat{\phi} = (\phi^N(\ell),\phi^N)$. Then $\phi^N \to \phi$ in $H^2(0,\ell)$ and consequently $\hat{\phi}^N \to \hat{\phi}$ in V.

(2) $P^N \to I$ strongly in H.

Proof

(1)

$$\left|\phi^N - \phi\right|_0 \le \left|\hat{\phi}^N - \hat{\phi}\right|_H \le \left|(I^{\hat{N}}\phi) - \hat{\phi}\right|_H$$

$$= \left|I^N\phi - \phi\right|_0 = \left|\overline{I}^N\phi - \phi\right|_0$$

$$\le C_0^1 N^{-2}\left|D^2\phi\right|_0 \to 0 \quad \text{as} \quad N \to \infty$$

where $(I^{\hat{N}}\phi) = ((I^N\phi)(\ell),\ I^N\phi)$.

$$\left|D(\phi^N - \phi)\right|_0 \le \left|D(\phi^N - \overline{I}^N\phi)\right|_0 + \left|D(\overline{I}^N\phi - \phi)\right|_0$$

$$\le k_1 N\left|\phi^N - \overline{I}^N\phi\right|_0 + C_1^1 N^{-1}\left|D^2\phi\right|_0$$

$$\le k_1 N\left|\phi^N - \phi\right|_0 + k_1 N\left|\overline{I}^N\phi - \phi\right|_0 + C_1^1 N^{-1}\left|D^2\phi\right|_0$$

$$\le k_1 C_0^1 N^{-1}\left|D^2\phi\right|_0 + k_1 C_0^1 N^{-1}\left|D^2\phi\right|_0 + C^1 N^{-1}\left|D^2\phi\right|_0$$

$$= (2k_1 C_0^1 + C_1^1)N^{-1}\left|D^2\phi\right|_0 \to 0 \quad \text{as} \quad N \to \infty$$

where we have made use of the Schmidt inequality (see [23]) in making the estimates above.

Using the Schmidt inequality together with the first integral relation (see [23]), we obtain

$$\left|D^2\phi^N\right|_0^2 \;\leq\; 2\left|D^2(\phi^N - \bar{I}^N\phi)\right|_0^2 + 2\left|D^2\bar{I}^N\phi\right|_0^2$$

$$\leq\; 2k_2N^4\left|\phi^N - \bar{I}^N\phi\right|_0^2 + 2\left|D^2\phi\right|_0^2 - 2\left|D^2(\phi - \bar{I}^N\phi)\right|_0^2$$

$$\leq\; 2k_2N^4\left|\phi^N - \phi\right|_0^2 + 2k_2N^4\left|\bar{I}^N\phi - \phi\right|_0^2 + 2\left|D^2\phi\right|_0^2$$

$$\leq\; \alpha^2\left|D^2\phi\right|_0^2$$

where α is independent of ϕ and N. Let $\psi\epsilon\{\theta\colon \theta\epsilon H^4(0,\ell), \theta(0) = D\theta(0) = 0\}$. Using arguments similar to those used above together with Proposition 3.2, it can be shown that

$$\left|D^2(\psi^N - \psi)\right|_0 \;\leq\; \tilde{k}N^{-2}\left|D^4\psi\right|_0$$

where $\tilde{k}$ is independent of ψ and N. Then

$$\left|D^2(\phi^N - \phi)\right|_0 \;\leq\; \left|D^2(\phi^N - \psi^N)\right|_0 + \left|D^2(\psi^N - \psi)\right|_0 + \left|D^2(\psi - \phi)\right|_0$$

$$\leq\; (\alpha + 1)\left|D^2(\psi - \phi)\right|_0 + \tilde{k}N^{-2}\left|D^4\psi\right|_0.$$

Standard density arguments guarantee that ψ can be chosen so as to make the first term arbitrarily small and therefore that

$$\left|D^2(\phi^N - \phi)\right|_0 \to 0 \qquad \text{as} \quad N \to \infty$$

which proves statement (1) of the theorem.

Statement (2) follows from Statement (1) and the fact that V is dense in H and the P^N being orthogonal projections are uniformly bounded.

<u>Theorem 3.1</u>: Let $\{q^N\} \subset Q$ with $q^N \to q$ as $N \to \infty$. Let $\hat{u}$ denote the solution to (2.9), (2.10) corresponding to q and let $\hat{u}^N$ denote the solution to (3.1), (3.2) corresponding to q^N. Suppose further that $\hat{u}$ is a strong solution (see Theorem 2.2). Then $\hat{u}^N \to \hat{u}$ in V and $D_t \hat{u}^N \to D_t \hat{u}$ in H as $N \to \infty$ for each $t \in [0,T]$.

<u>Proof</u>

Our argument is similar in spirit to the one used in [11] to demonstrate the convergence of a Galerkin method for the integration of a class of hyperbolic systems. We adopt the convention that the superscript N on a form or operator indicates that it be computed with respect to $q^N = (m^N, EI^N, \rho^N, a_0^N)$ while no superscript indicates that it be computed with respect to $q = (m, EI, \rho, a_0)$.

Applying the triangle inequality, we have

$$\left|\hat{u}^N - \hat{u}\right|_V \leq \left|\hat{u}^N - P^N\hat{u}\right|_V + \left|P^N\hat{u} - \hat{u}\right|_V \qquad (3.5)$$

and

$$\left|D_t\hat{u}^N - D_t\hat{u}\right|_H \leq \left|D_t\hat{u}^N - D_t P^N\hat{u}\right|_H + \left|P^N D_t\hat{u} - D_t\hat{u}\right|_H . \qquad (3.6)$$

Lemma 3.1 and the fact that $\hat{u}$ has been assumed to be a strong solution imply that $\left|P^N\hat{u} - \hat{u}\right|_V \to 0$ and $\left|P^N D_t\hat{u} - D_t\hat{u}\right|_H \to 0$ as $N \to \infty$ for each $t \in [0, T]$.

Since $\hat{u}^N$ satisfies (3.1) and (3.2) and since $\hat{u}$ satisfies (2.9) and (2.10) then for $\hat{\theta}^N \epsilon V^N$ we have

$$\langle M_0^N D_t^2(\hat{u}^N - P^N\hat{u}), \hat{\theta}^N \rangle_H + a^N(\hat{u}^N - P^N\hat{u}, \hat{\theta}^N)$$

$$= \langle M_0^N D_t^2(\hat{u} - P^N\hat{u}), \hat{\theta}^N \rangle_H + \langle (M_0 - M_0^N) D_t^2\hat{u}, \hat{\theta}^N \rangle_H$$

$$+ a^N(\hat{u} - P^N\hat{u}, \hat{\theta}^N) + a(\hat{u}, \hat{\theta}^N) - a^N(\hat{u}, \hat{\theta}^N)$$

$$+ b^N(t)(\hat{u}^N - P^N\hat{u}, \hat{\theta}^N) + b^N(t)(P^N\hat{u} - \hat{u}, \hat{\theta}^N)$$

$$+ b^N(t)(\hat{u}, \hat{\theta}^N) - b(t)(\hat{u}, \hat{\theta}^N).$$

Let $\hat{u}_N = (u_N(\ell), u_N) = P^N\hat{u}$, let $\hat{v}^N = (v^N(\ell), v^N) = \hat{u}^N - \hat{u}_N$ and choose $\hat{\theta}^N = D_t \hat{v}^N \epsilon V^N$. Then

$$\langle M_0^N D_t^2\hat{v}^N, D_t\hat{v}^N \rangle_H + a^N(\hat{v}^N, D_t\hat{v}^N)$$

$$= \langle M_0^N D_t^2(\hat{u} - P^N\hat{u}), D_t\hat{v}^N \rangle_H + \langle (M_0 - M_0^N) D_t^2\hat{u}, D_t\hat{v}^N \rangle_H$$

$$+ a^N(\hat{u} - P^N\hat{u}, D_t\hat{v}^N) + a(\hat{u}, D_t\hat{v}^N) - a^N(\hat{u}, D_t\hat{v}^N)$$

$$+ b^N(t)(\hat{v}^N, D_t\hat{v}^N) + b^N(t)(P^N\hat{u} - \hat{u}, D_t\hat{v}^N)$$

$$+ b^N(t)(\hat{u}, D_t\hat{v}^N) - b(t)(\hat{u}, D_t\hat{v}^N)$$

or

$$\frac{1}{2} D_t(\langle M_0^N D_t \hat{v}^N, \ D_t \hat{v}^N \rangle_H + a^N(\hat{v}^N, \ \hat{v}^N))$$

$$= \langle M_0^N (I - P^N) D_t^2 \hat{u}, \ D_t \hat{v}^N \rangle_H + \langle (M_0 - M_0^N) D_t^2 \hat{u}, \ D_t \hat{v}^N \rangle_H$$

$$+ D_t a^N((I - P^N)\hat{u}, \ \hat{v}^N) - a^N((I - P^N) D_t \hat{u}, \ \hat{v}^N)$$

$$+ D_t(a(\hat{u}, \ \hat{v}^N) - a^N(\hat{u}, \ \hat{v}^N)) - (a(D_t \hat{u}, \ \hat{v}^N) - a^N(D_t \hat{u}, \ \hat{v}^N))$$

$$+ b^N(t)(\hat{v}^N, \ D_t \hat{v}^N) + b^N(t)(P^N \hat{u} - \hat{u}, \ D_t \hat{v}^N)$$

$$+ b^N(t)(\hat{u}, \ D_t \hat{v}^N) - b(t)(\hat{u}, \ D_t \hat{v}^N).$$

Integrating both sides of the above expression from 0 to t, invoking hypotheses (H1) - (H3) on Q and using standard estimates we obtain

$$\min \ (m_1, \ m_2, \ m_3)(|D_t \hat{v}^N|_H^2 + |\hat{v}^N|_V^2)$$

$$\leq \int_0^t \ \{ \max \ (M_1^2, \ M_3^2) \big| (I - P^N) D_s^2 \hat{u} \big|_H^2 + \big| D_s \hat{v}^N \big|_H^2$$

$$+ \max(\big| m - m^N \big|^2, \ \big| \rho - \rho^N \big|_\infty^2) \big| D_s^2 \hat{u} \big|_H^2 + \big| D_s \hat{v}^N \big|_H^2$$

$$+ M_2^2 \big| (I - P^N) D_s \hat{u} \big|_V^2 + \big| \hat{v}^N \big|_V^2 + \big| EI - EI^N \big|_\infty^2 \big| D_s \hat{u} \big|_V^2$$

$$+ \big| \hat{v}^N \big|_V^2 + C_0 \big| \sigma(q^N) \big|_1 \big| \hat{v}^N \big|_V^2 + C_0 \big| \sigma(q^N) \big|_1 \big| D_s \hat{v}^N \big|_H^2$$

$$+ C_0^2 \big| \sigma(q^N) \big|_1^2 \big| (I - P^N) \hat{u} \big|_V^2 + \big| D_s \hat{v}^N \big|_H^2$$

$$+ C_1 \big| \sigma(q) - \sigma(q^N) \big|_1^2 \big| \hat{u} \big|_V^2 + \big| D_s \hat{v}^N \big|_H^2 \} \ ds$$

$$+ \frac{1}{C_2} \ \big| (I - P^N) \hat{u} \big|_V^2 + C_2 \big| \hat{v}^N \big|_V^2 + \big| (I - P^N) \hat{\phi} \big|_V^2$$

$$+ \frac{1}{C_3} \ \big| EI - EI^N \big|_\infty^2 \big| \hat{u} \big|_V^2 + C_3 \big| \hat{v}^N \big|_V^2 + \big| EI - EI^N \big|_\infty^2 \big| \hat{\phi} \big|_V^2$$

where C_0 and C_1 are constants which are independent of N and C_2 and C_3 may be chosen arbitrarily. Choosing C_2 and C_3 sufficiently small $(C_2 + C_3 < \min(m_1, m_2, m_3))$, we obtain

$$\left|D_t \hat{v}^N\right|^2_H + \left|\hat{v}^N\right|^2_V \leq \Delta^N(t) + \int_0^t K_0^N(s)\{\left|D_s \hat{v}^N\right|^2_H + \left|\hat{v}^N\right|^2_V\} \, ds$$

where

$$\Delta^N(t) = K_1\{\left|(I - P^N)\hat{u}(t)\right|^2_V + \left|(I - P^N)\hat{\phi}\right|^2_V$$

$$+ \left|EI - EI^N\right|^2_\infty \left|\hat{u}(t)\right|^2_V + \left|EI - EI^N\right|^2_\infty \left|\hat{\phi}\right|^2_V$$

$$+ \int_0^t \{\left|(I - P^N)D_s^2 \hat{u}(s)\right|^2_H + \max(\left|m - m^N\right|^2, \left|\rho - \rho^N\right|^2_\infty)$$

$$\cdot \left|D_s^2 \hat{u}(s)\right|^2_H + \left|(I - P^N)D_s \hat{u}(s)\right|^2_V + \left|EI - EI^N\right|^2_\infty \left|D_s \hat{u}(s)\right|^2_V$$

$$+ \left|\sigma(s, \cdot \,; q^N)\right|^2_1 \left|(I - P^N)\hat{u}(s)\right|^2_V + \left|\sigma(s, \cdot \,; q)\right.$$

$$\left. - \sigma(s, \cdot \,; q^N)\right|^2_1 \left|\hat{u}(s)\right|^2_V\} \, ds$$

and

$$K_0^N(s) = K_2\{K_3 + \left|\sigma(s, \cdot \,; q^N)\right|_1\}$$

where K_i, $i = 1, 2, 3$ are constants independent of N. Using the fact that $q^N \to q$ in Q as $N \to \infty$ and $u(\hat{q})$ is a strong solution together with Lemma 3.1, the compactness of Q and Hypothesis (H3) we have that $\Delta^N \to 0$ and K_0^N is uniformly bounded in $L_2(0, T)$ as $N \to \infty$. An application of the Gronwall inequality therefore yields

$$\left|D_t \hat{v}^N\right|^2_H + \left|\hat{v}^N\right|^2_V \to 0 \qquad \text{as } N \to \infty$$

for each $t \in [0, T]$. Consequently $\left|D_t \hat{u}^N - D_t P^N \hat{u}\right|_H \to 0$ and $\left|\hat{u}^N - P^N \hat{u}\right|_V \to 0$ as $N \to \infty$ which together with (3.5) and (3.6) proves the theorem.

The continuous dependence results given in Theorems 3.2 and 3.3 below can be verified using arguments similar to those used in the proof of Theorem 3.1.

Theorem 3.2: Let $\{q_k\} \subset \mathcal{Q}$ with $q_k \to q^*$ as $k \to \infty$. If for each N fixed, $\hat{u}^N(q)$ denotes the solution to (3.1), (3.2) correspondingly to q then $\hat{u}^N(q_k) \to \hat{u}^N(q^*)$ in V and $D_t\hat{u}^N(q_k) \to D_t\hat{u}^N(q^*)$ in H for each $t\epsilon[0, T]$ as $k \to \infty$. That is, the mapping $q \to (\hat{u}^N(q), D_t u^N(q))$ from Q into $V \times H$ is continuous for each N = 1, 2, ...

Theorem 3.3: Let $\{q_k\} \subset \mathcal{Q}$ with $q_k \to q^*$ as $k \to \infty$ and let $\hat{u}(q)$ denote the solution to (2.9), (2.10) corresponding to q. If $\hat{u}(q^*)$ is a strong solution then $\hat{u}(q_k) \to \hat{u}(q^*)$ in V and $D_t\hat{u}(q_k) \to D_t u(q^*)$ in H for each $t\epsilon[0, T]$ as $k \to \infty$. The mapping $q \to (\hat{u}(q), D_t\hat{u}(q))$ is continuous from Q into $V \times H$ in neighborhoods of those $q\epsilon Q$ for which $\hat{u}(q)$ is a strong solution.

We are now prepared to prove our first major convergence result.

Theorem 3.4: For each N = 1, 2, ... fixed, problem (IDN) has a solution $\bar{q}^N$. The sequence $\{\bar{q}^N\}$ admits a convergent subsequence $\{\bar{q}^{N_k}\}$ with $\bar{q}^{N_k} \to \bar{q} \epsilon Q$ as $k \to \infty$. If $\hat{u}(\bar{q})$ is a strong solution to (2.9), (2.10) then $\bar{q}$ is a solution to Problem (ID).

Proof

Theorem 3.2 implies that the mapping $q \to J(q; \hat{u}^N(q))$ from Q into R is continuous. This together with the fact that Q is compact yields the existence of a solution $\bar{q}^N$ to problem (IDN). The existence of a convergent subsequence $\{\bar{q}^{N_k}\} \subset \{\bar{q}^N\}$ also follows from the compactness of Q. If

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$\overline{q}^{N_k} \to \overline{q}$ as $k \to \infty$ and $u(\overline{q})$ is a strong solution to (2.9), (2.10) then Theorem 3.1 implies

$$J(\overline{q}, \hat{u}(\overline{q})) = \lim_{k \to \infty} J(\overline{q}^{N_k}; \hat{u}^{N_k}(\overline{q}^{N_k}))$$

$$\leq \lim_{k \to \infty} J(q; \hat{u}^{N_k}(q)) = J(q; \hat{u}(q))$$

for all $q \in Q$ and the theorem is proven.

Turning our attention next to the other infinite dimensional aspect of Problem (ID), we introduce a second level of finite dimensional approximation to effect a discretization of the admissible parameter space.

For each $M = 1, 2, \ldots$ let the sets $Q^M \subset \mathscr{Q}$ be given by $\mathscr{I}^M(Q)$ where the mappings $\mathscr{I}^M$ satisfy

(P1) $\mathscr{I}^M : Q \to \mathscr{Q}$ is continuous.

(P2) $\mathscr{I}^M(q) \to q$ as $M \to \infty$ uniformly in q for all $q \in Q$.

We assume further that the sets Q^M have the property

(P3) For each $q \in Q^M$, $u(\hat{q})$ is a strong solution.

We define a doubly indexed sequence of approximating identification problems by
(IDNM)

Find $q = (m, EI, \rho, a_0) \in Q^M$ which minimizes $J(q; \hat{u}^N)$ subject to $J(q; \hat{u}^N)$ subject to $\hat{u}^N$ being the solution to (3.1), (3.2) corresponding to q.

Note that if the sets Q^M are of dimension $K_M < \infty$ then the optimization in Problem (IDNM) is simply over a compact subset of the space R^{k_M} and is subject to finite dimensional constraints; a computationally tractable problem.

The convergence arguments now go as follows. Property (P1) and Q compact imply that the sets Q^M are compact as well. For each $N = 1, 2, \ldots$ and each $M = 1, 2, \ldots$ Problem (IDNM) therefore has a solution $\bar{q}_M^N$. For each $M = 1, 2, \ldots$ fixed, the sequence $\{\bar{q}_M^N\}_{N=1}^\infty$ admits a convergent subsequence $\{\bar{q}_M^{N_k}\}$ with $\bar{q}_M^{N_k} \to \bar{q}_M$ as $k \to \infty$. Recalling property (P3) and arguing as we have in the proof of Theorem 3.4, we conclude that $\bar{q}_M$ is a solution to the problem of minimizing $J(q; \hat{u})$ over Q^M. Since $Q^M = \mathscr{S}^M(Q)$ there exists $\tilde{q}_M \in Q$ such that $\bar{q}_M = \mathscr{S}^M(\tilde{q}_M)$. Now $\{\tilde{q}_M\} \subset Q$ and Q compact imply the existence of a convergent subsequence $\{\tilde{q}_{M_j}\}$ with $\tilde{q}_{M_j} \to \tilde{q}$ as $j \to \infty$. Property (P2) then implies that $\mathscr{S}^{M_j}(\tilde{q}_{M_j}) \to \tilde{q}$ as $j \to \infty$ and hence that $\bar{q}_{M_j} \to \tilde{q}$ as $j \to \infty$ as well. Then

$$J(\bar{q}_{M_j}; \hat{u}) \leq J(q; \hat{u}) \qquad q \in Q^{M_j}$$

and $Q^{M_j} = \mathscr{S}^{M_j}(Q)$ yield

$$J(\bar{q}_{M_j}; \hat{u}(\bar{q}_{M_j})) \leq J(\mathscr{S}^{M_j}(q); \hat{u}(\mathscr{S}^{M_j}(q))) \qquad q \in Q. \qquad (3.7)$$

Taking the limit as $j \to \infty$ on both sides of (3.7) and recalling Theorem 3.3 and Property (P3), we obtain

$$J(\bar{q}; \hat{u}(\bar{q})) \leq J(q; \hat{u}(q)) \qquad q \in Q$$

from which we conclude that $\bar{q}$ is a solution to Problem (ID). In summary, we have

Theorem 3.5: Each of the approximating identification problems (IDNM) has a solution $\bar{q}_M^N$. From the tableau $\{\bar{q}_M^N\}$ a sequence $\{\bar{q}(k)\}$ can be extracted with $\bar{q}(k) \to \bar{q}$, a solution to Problem (ID), as $k \to \infty$.

Typically an appropriate choice for the sets Q^M are the spaces of linear or cubic interpolatory spline functions with the mappings $\mathscr{J}^M$ being constructed from the usual interpolation operators. If the elements in Q are sufficiently regular, it is not difficult to verify that Properties (P1) - (P3) are satisfied for this approximation. A detailed discussion and several examples of this particular choice for the Q^M in the context of inverse problems for parabolic systems can be found in [6]. Similar results for identification problems involving the estimation of functional parameters in beam equations with simple boundary conditions (clamped, simply supported, cantilevered, etc.) are presented in [5] and [14].

SECTION 4

NUMERICAL RESULTS

In this section we discuss numerical results for a variety of ex-
amples. Although the analysis presented in the previous sections was
based primarily on a simple example involving a cantilevered beam with
tip mass, only minor modifications would be required so as to make our
general approach applicable to a broad class of inverse problems for beam
vibration. Some of these will be considered and outlined below.

Although one of the major features of our scheme is its ability to
identify or estimate functional (spatially and/or temporally varying)
parameters, our numerical findings for this important class of problems
are, at present, incomplete. For this reason, this report includes exam-
ples involving the identification of constant parameters only. Our
results for functional parameters will appear in a forthcoming paper.

In all of the examples below, the fits were based upon artificially
generated observations. By this we mean that so-called "true" values for
the parameters were selected and a modal based Galerkin method was used
to generate the solution to the resulting system of equations from which
the sampled displacement measurements $\{\overline{u}(t_i, x_j)\}_{\substack{i=1,\ldots,\mu \\ j=1,\ldots,\nu}}$ were obtained.
Results for examples involving fits based upon actual experimental data
will be discussed elsewhere.

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The finite dimensional optimization problems were solved using an iterative steepest descent routine, ZXSSQ, from the IMSL Library. The algorithm used is the one suggested in the papers by Levenberg [15] and Marquardt [18]. The finite dimensional second order initial value problem given by (3.3), (3.4) which has to be solved at each iteration to compute the value of the least squares payoff functional J and approximations to its gradient and the corresponding Jacobian matrix is integrated using a variable order Adams predictor corrector method (IMSL routine DGEAR). The system (3.3), (3.4) did not in general demonstrate stiff behavior.

The integral inner products in the generalized mass and stiffness matrices and the generalized Fourier coefficient vectors for the external loads and initial data were computed using a composite two point Gauss-Legendre quadrature rule. The four subinterval support of the cubic B-splines and their derivatives leads to 7-banded matrices and consequently contributes to the overall computational efficiency of the method.

All examples were run on the IBM 3081 processor at Draper Laboratory.

Example 4.1

In this example we consider a cantilevered beam of length 1.0 with an attached tip mass. We seek to identify the spatially invariant stiffness EI and linear mass density ρ of the beam and the magnitude m of the tip mass. We assume that the system is initially at rest ($\phi = 0$, $\psi = 0$) and then excited by the distirbuted lateral load along the beam given by

$$f(t, x) = e^x \sin 2\pi t$$

and the point force applied to the tip mass

$$g(t) = 2e^{-2t}.$$

We assume that there is no axial loading, or, $\sigma = 0$.

Displacement observations were generated by taking $EI = 1.0$, $\rho = 3.0$ and $m = 1.5$ to be the true values of the parameters. Measurements were taken at positions $x_j = 0.25j$, $j = 2, 3, 4$, at times $t_i = 0.5i$, $i = 1, 2, \ldots 10$, from a solution to the system (2.1) - (2.5) generated using a Galerkin method and the first two natural modes of the unforced system (see [25]). The "start up" values for the steepest descent routine were $EI_0 = 0.7$, $\rho_0 = 2.7$ and $m_0 = 0.7$. The final converged values for $\overline{EI}^N$, $\overline{\rho}^N$, and $\overline{m}^N$ together with the residual sum of squares $\overline{J}^N$ and required CPU times for various values of N are given in Table 4.1 below.

Table 4.1.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{m}^N$	$\overline{J}^N$	CPU (m:s)
2	1.00057	3.04455	1.48957	0.12×10^{-3}	0:9.19
3	1.00067	3.01256	1.49707	0.11×10^{-3}	0:22.10
4	1.00027	3.00922	1.49721	0.11×10^{-3}	0:57.60
5	1.00016	2.98936	1.50262	0.11×10^{-3}	1:22.52
6	0.99912	2.99720	1.49952	0.11×10^{-3}	2:52.76

Example 4.2

We consider the system described in the previous example. We assume that it is initially at rest and then excited by the distributed load

$$f(t, x) = e^x \sin 2\pi t$$

along the beam and the point force

$$g(t) = 2e^{-t}$$

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acting on the tip mass. We also assume that the entire system is sub-
jected to a base acceleration which is given by

$$a_0(t) = \begin{cases} 1 & 0 \le t \le 1.5 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

The internal tension resulting from the axial load (see [26], [27]) is
given by

$$\sigma(t, x) = -a_0(t)(\rho(\ell - x) + m).$$

We are interested once again in estimating the stiffness EI, linear mass
density ρ, and the magnitude of the tip mass m. The true values of the
parameters were taken to be EI = 1.0, ρ = 3.0, and m = 1.5 with the
reference solution being generated using the first two natural modes of
vibration for the unforced, unaccelerated system. Displacement measure-
ments were taken at positions x_j = 0.75, 0.875, 1.0, j = 1, 2, 3, at
times t_i = 0.5i, i = 1, 2, ..., 10. The start up values for the itera-
tive search routine were taken as EI_0 = 0.7, ρ_0, and m_0 = 1.7. Our
results are summarized in Table 4.2.

Table 4.2.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{m}^N$	$\overline{J}^N$	CPU (m:s)
2	1.00057	3.09966	1.47928	0.17×10^{-4}	0:18.01
3	1.00121	3.06360	1.48727	0.18×10^{-4}	0:35.93
4	1.00092	3.04144	1.49207	0.19×10^{-4}	1:26.19
5	1.00057	3.03063	1.49413	0.19×10^{-4}	4:15.87
6	1.00117	3.03186	1.49436	0.35×10^{-4}	5:21.34

We note that although strictly speaking the convergence theory developed in Section 3 requires that $a_0 \epsilon C^1$, the scheme performed satisfactorily with a_0 given by (4.1) above.

Example 4.3

In this example we consider a free-free beam of length 1.0 with an attached tip body at each end (see Figure 4.1).
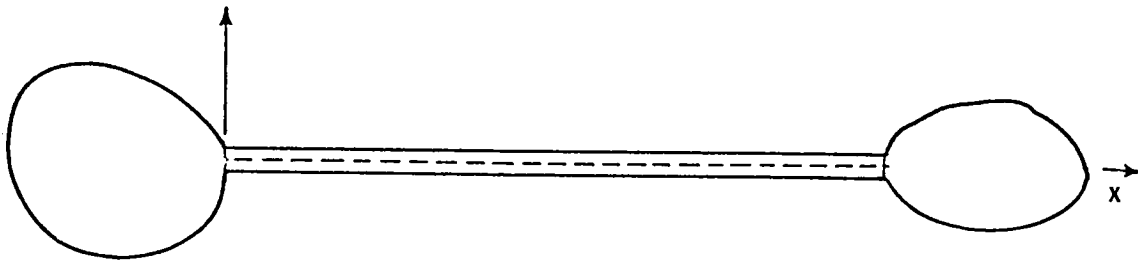

Figure 4.1.

The tip bodies are assumed to have known mass properties which are given by:

Tip Body 0 (at $x = 0$):

$$m_0 = 0.75, \quad c_0 = 0.1, \quad \delta_0 = \pi/6, \quad J_0 = 0.6$$

Tip Body 1 (at $x = 1$):

$$m_1 = 1.5, \quad c_1 = 0.2, \quad \delta_1 = \pi/3, \quad J_1 = 0.4$$

where for tip body i, $i = 0, 1$, m_i, c_i, δ_i, and J_i are respectively its mass, the distance from its center of mass to the tip of the beam, its mass center offset as measured from the extension of the longitudinal axis of the beam, and its moment of inertia about its center of mass.

The equations describing the transverse displacement of the beam and the translational and rotational equilibrium of the tip bodies are given by (see [25], [27])

$$\rho D_t^2 u + D_x^2 EI D_x^2 u = f, \qquad x\epsilon\ (0,\ 1),\ t\epsilon\ (0,T) \qquad (4.2)$$

$$m_0 D_t^2 u - m_0 c_0 \cos \delta_0 D_t^2 D_x u + D_x EI D_x^2 u = 0,\ x = 0,\ t\epsilon\ (0,\ t)$$

$$-m_0 c_0 \cos \delta_0 D_t^2 u + (J_0 + m_0 c_0^2) D_t^2 D_x u - EI D_x^2 u = 0,\ x = 0,\ t\epsilon\ (0,\ T)$$

$$m_1 D_t^2 u + m_1 c_1 \cos \delta_1 D_t^2 D_x u - D_x EI D_x^2 u = 0,\ x = 1,\ t\epsilon\ (0,\ T)$$

$$+m_1 c_1 \cos \delta_1 D_t^2 u + (J_1 + m_1 c_1^2) D_t^2 D_x u + EI D_x^2 u = 0,\ x = 1,\ t\epsilon\ (0,\ T).$$

The initial conditions are of the form

$$u\ =\ \phi \qquad\qquad D_t u = \psi \qquad\qquad x\epsilon\ (0,\ 1),\ t = 0\ . \qquad (4.3)$$

Letting $H = R^4 \times H^0(0,\ 1)$ with inner product

$$\langle (\eta,\ \phi),\ (\zeta,\ \psi) \rangle_H\ =\ \eta^T \zeta + \langle \phi,\ \psi \rangle_0$$

and

$$V\ =\ \{\hat{\phi}\epsilon H : \hat{\phi}\ =\ (\eta,\ \phi),\ \phi\epsilon H^2(0,\ 1),\ \eta = (\phi(0),\ D\phi(0),\ \phi(1),\ D\phi(1))^T\}$$

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with inner product

$$\langle \hat{\phi}, \hat{\psi} \rangle_V = \langle \phi, \psi \rangle_2.$$

The weak form of the system given by (4.2), (4.3) above becomes

$$\langle M_0 D_t^2 \hat{u}(t), \hat{\theta} \rangle_H + a(\hat{u}(t), \hat{\theta}) = \langle F_0(t), \hat{\theta} \rangle_H \qquad t \in (0, T), \ \hat{\theta} \in V \qquad (4.4)$$

$$\hat{u}(0) = \hat{\phi} \qquad D_t \hat{u}(0) = \hat{\psi} \qquad\qquad (4.5)$$

where

$$\hat{u}(t) = ((u(t, 0), D_x u(t, 0), u(t, 1), D_x u(t, 1))^T, u(t, \cdot)) \in V,$$

$$M_0(\eta, \phi) = (\overline{M}_0 \eta, \rho \phi)$$

with

$$\overline{M}_0 = \begin{bmatrix} \begin{matrix} m_0 & -m_0 c_0 \cos \delta_0 \\ -m_0 c_0 \cos \delta_0 & J_0 + m_0 c_0^2 \end{matrix} & 0 \\ 0 & \begin{matrix} m_1 & m_1 c_1 \cos \delta_1 \\ m_1 c_1 \cos \delta_1 & J_1 + m_1 c_1^2 \end{matrix} \end{bmatrix}$$

a: $V \times V \to R$ given by

$$a(\hat{\phi}, \hat{\psi}) = \langle EID^2 \phi, D^2 \psi \rangle_0,$$

$$F_0(t) = (0, f(t, \cdot))$$

$$\hat{\phi} = ((\phi(0), D\phi(0), \phi(1), D\phi(1))^T, \phi)$$

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and

$$\hat{\psi} = ((\psi(0), D\psi(0), \psi(1), D\psi(1))^T, \psi).$$

The appproximation scheme for the system (4.4), (4.5) is then constructed in essentially the same manner as it was in Section 3.

In this example we seek to identify EI and ρ. The system was assumed to be initially at rest and then excited via the input disturbance

$$f(t, x) = 10e^x \sin 2\pi t.$$

Observations at positions $x_j = 0.25j$, $j = 0, 1, \ldots, 4$, at times $t_i = 0.2i$, $i = 1, 2, \ldots, 10$, were generated using the first six natural modes of the system; four flexible modes plus rigid body rotation and translation. The true values of the parameters were assumed to be EI = 1.0 and $\rho = 3.0$ with start up values taken as $EI_0 = 0.7$ and $\rho_0 = 2.7$. Our results are given in Table 4.3.

Table 4.3.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	0.99567	3.00092	0.20×10^{-4}
3	0.99374	2.99900	0.16×10^{-4}
4	0.99849	2.99798	0.78×10^{-5}
5	0.99888	2.99910	0.28×10^{-5}

Example 4.4

In this example we estimate the flexural stiffness EI and linear mass density ρ for a cantilevered beam of length 1.0 with an attached tip body (see Figure 4.2).
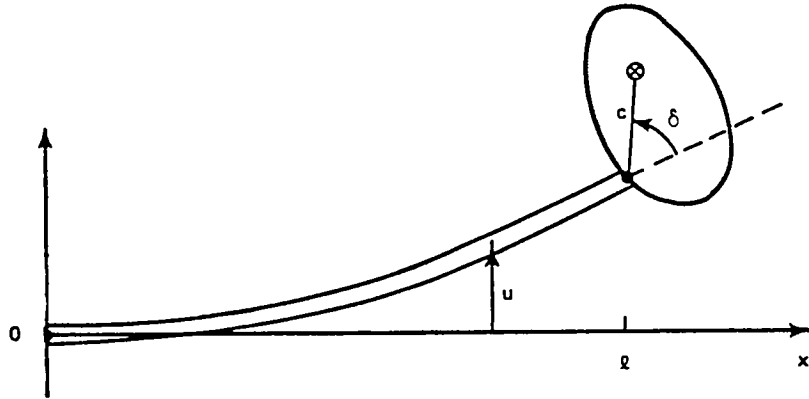

Figure 4.2.

We assume that the system is initially at rest and then acted upon by the distributed load

$$f(t, x) = 20e^{-2t} e^{-20(1-x)}$$

and base acceleration

$$a_0(t) = \begin{cases} 1.0 & 0 \le t \le 1.5 \\ 0 & 1.5 < t < 3.0 \\ 1.0 & 3.0 \le t < 4.0 \\ 0 & 4.0 \le t. \end{cases}$$

The equations of motion are given by (see [26], [27])

$$\rho D_t^2 u + EI D_x^4 u = -a_0 D_x(\rho(1-x) + m)D_x u + f \qquad x\epsilon(0,1), \; t\epsilon(0,T)$$

$$m D_t^2 u + mc \cos \delta D_t^2 D_x u - EI D_x^3 u = ma_0 D_x u, \quad x = 1, \; t\epsilon(0,T) \tag{4.6}$$

$$mc \cos \delta D_t^2 u + (J + mc^2)D_t^2 D_x u + EI D_x^2 u = mc \cos \delta a_0 D_x u + mc \sin \delta a_0,$$
$$x = 1, \; t\epsilon(0,T)$$

with boundary and initial conditions

$$u = 0 \qquad D_x u = 0 \qquad x = 0, \; t\epsilon[0, T] \tag{4.7}$$

and

$$u = \phi \qquad D_t u = \psi \qquad x\epsilon[0, 1], \; t = 0 \tag{4.8}$$

respectively where the quantities m, c, δ, and J are as they were defined in Example 4.3. Once again only minor modifications (similar to those outlined in the previous Example) are necessary so as to make the theory presented in Sections 2 and 3 applicable to the system given by (4.6) - (4.8) above. Taking the true values of EI and ρ to be 1.0 and 2.0 respectively and setting m = 4.0, c = 0.2, δ = $\pi/3$, and J = 0.4, displacement observations at positions x_j = 0.75, 0.875, 1.0, j = 1, 2, 3, at times t_i = 4.0, 4.5, 5.0, i = 1, 2, 3, were generated using the first three natural modes of the unforced, unaccelerated system. The start up values were taken as EI_0 = 0.7 and ρ_0 = 2.5. The final converged values for $\overline{EI}^N$ and $\overline{\rho}^N$ together with the residual sum of squares $\overline{J}^N$ for several values of N are given in Table 4.4 below.

Table 4.4.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	1.00046	2.07731	0.4×10^{-3}
3	1.00135	2.06652	0.22×10^{-3}
4	1.00117	2.05180	0.23×10^{-3}
5	1.00086	2.04776	0.26×10^{-3}

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15. Supplementary Notes

16. Abstract

In this report we develop an approximation scheme for the identification of hybrid systems describing the transverse vibrations of flexible beams with attached tip bodies. In particular, problems involving the estimation of functional parameters (spatially varying stiffness and/or linear mass density, temporally and/or spatially varying loads, etc.) are considered. The identification problem is formulated as a least squares fit to data subject to the coupled system of partial and ordinary differential equations describing the transverse displacement of the beam and the motion of the tip bodies respectively. A cubic spline-based Galerkin method applied to the state equations in weak form and the discretization of the admissible parameter space yield a sequence of approximting finite dimensional identification problems. We demonstrate that each of the approximating problems admits a solution and that from the resulting sequence of optimal solutions a convergent subsequence can be extracted, the limit of which is a solution to the original identification problem. The approximating identification problems can be solved using standard techniques and readily available software. Numerical results for a variety of examples are provided.

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