NASA Contractor Report 172563

FRACTURE ANALYSIS OF NOTCHED COMPOSITES

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Grant NAG1-480
February 1985

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

Fracture Analysis of Notched Composites(*)

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Abstract

In this study the problem of part-through cracks emanating from a central notch in a composite laminate is considered. The composite laminate is modeled as an orthotropic plate and along the branches, it is assumed that the stresses are carried by the fibers only. The stresses along the uncracked portions are assumed to be proportional to the crack opening displacement at that point. After formulating the branched crack problem, the model is applied to a $(\pm 45°)_{2s}$ composite laminate. The stress concentration factors for the fibers are computed for various values of the spring constant K. Sample results showing the crack opening displacement are also displayed in the figures.

1. INTRODUCTION

The load carrying capacity of a composite laminate may be reduced significantly due to notches or crack-like defects. Thus, notched composites have been studied extensively. In certain composites, cracks making an angle with the pre-existing notch may emanate from its tip. These cracks are part-through, meaning that only some of the laminae are broken and they tend to meander along the fracture path [1,2]. As the geometry suggests, an exact stress analysis of the problem is extremely complicated. In this study, the following approximate model is used in the analysis. The laminate is assumed to be an orthotropic plate. The stresses in the unbroken laminae of the

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* This work was supported by NASA-Langley under the grant NAG-1-480

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part-through cracks (along the fracture path) are represented by springs and at each point are assumed to be proportional to the crack opening displacement, i.e., $\sigma = K\Delta v$.

Starting with the formulation of the crack geometry shown in figure 2, the through crack problem is formulated in terms of the displacement derivatives of the crack surfaces. This formulation yields a system of singular integral equations which can be solved numerically. Next, the part-through crack problem is formulated, by relating the crack surface tension to the crack opening displacement. A new set of singular integral equations is obtained. At the crack tips the stress intensity factors are computed. Also the stress concentration factors at the tip of the original slit are computed for various values of K. These factors can be used in studying fiber damage. Also some examples showing the crack opening along the fracture path are displayed in the figures.

2. FORMULATION OF THE PROBLEM

To formulate the part-through crack problem, we start with the geometry shown in figure 2. In this symmetric configuration, all the cracks are assumed to be through cracks. Since the problem is linear, the solution will be constructed by the superposition of several cracked plates.

2.a. The Centrally Cracked Plate

Consider an infinite orthotropic plate with a central crack extending from -a to +a. The governing equation of the problem can easily be written in terms of the stress function F(x,y) as follows [3,4]:

$$a_{22} \frac{\partial^4 F}{\partial x^4} + (2a_{12}+a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} + a_{11} \frac{\partial^4 F}{\partial y^4} = 0 \qquad (1)$$

or

$$\frac{\partial^4 F}{\partial x^2} + \beta_2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \beta_1 \frac{\partial^4 F}{\partial y^4} = 0 \qquad (2)$$

where

$$\beta_1 = \frac{a_{11}}{a_{22}} \quad ; \quad \beta_2 = \frac{2a_{12}+a_{66}}{a_{22}} \tag{3}$$

and

$$a_{11} = \frac{1}{E_x} \quad , \quad a_{12} = -\frac{\nu_{xy}}{E_x} \quad , \quad a_{22} = \frac{1}{E_y} \quad , \quad a_{66} = \frac{1}{G_{xy}}$$

E_x, E_y, G_{xy} and ν_{xy} being the engineering elastic constants for the orthotropic material.

Define:

$$F(x,y) = \int_{-\infty}^{\infty} g(\alpha,y)e^{-i\alpha x}\, d\alpha \tag{4}$$

Substituting in (2) we obtain:

$$\beta_1 \frac{d^4 g}{dy^4} - \beta_2 \alpha^2 \frac{d^2 g}{dy^2} + \alpha^4 g = 0 \tag{5}$$

The solution of (5) may be written as:

$$g(\alpha,y) = A(\alpha)e^{-\omega_1 \alpha y} + B(\alpha)\, e^{-\omega_2 \alpha y} + C(\alpha)e^{\omega_1 \alpha y} + D(\alpha)e^{\omega_2 \alpha y} \tag{6}$$

where ω_1 and ω_2 are the roots of

$$\beta_1 \omega^4 - \beta_2 \omega^2 + 1 = 0 \tag{7}$$

such that $Re(\omega_1) > 0$ and $Re(\omega_2) > 0$.

Taking the symmetry of the problem into account, one may consider one half of the plane only and equation (6) may then be rewritten as:

$$g(\alpha,y) = A(\alpha)e^{-\omega_1 |\alpha| y} + B(\alpha)e^{\omega_2 |\alpha| y} \quad \text{for } y>0 \tag{8}$$

Thus, the stress function F becomes:

$$F(x,y) = \int_{-\infty}^{\infty} [Ae^{-\omega_1 |\alpha| y} + Be^{-\omega_2 |\alpha| y}]e^{-i\alpha x}\, d\alpha \quad y>0 \tag{9}$$

Taking derivatives of the stress function, we obtain the following expressions for the stresses:

$$\sigma_x = \int_{-\infty}^{\infty} A[\omega_1^2\, e^{-\omega_1|\alpha|y} - \omega_1\omega_2\, e^{-\omega_2|\alpha|y}]\alpha^2 e^{-i\alpha x} d\alpha$$

$$\sigma_y = -\int_{-\infty}^{\infty} A[e^{-\omega_1|\alpha|y} - \frac{\omega_1}{\omega_2}\, e^{-\omega_2|\alpha|y}]\alpha^2\, e^{-i\alpha x}\, d\alpha \qquad (10a\text{-}c)$$

$$\tau_{xy} = -i\int_{-\infty}^{\infty} A\omega_1[e^{-\omega_1|\alpha|y} - e^{-\omega_2|\alpha|y}]\alpha|\alpha|e^{-i\alpha x} d\alpha$$

To formulate the crack problem, the crack surface derivative is taken as the unknown function.

Defining,

$$\lim_{y\to 0} + \frac{\partial v}{\partial x} = f(x) \qquad (11)$$

and using the following stress-strain relationship,

$$\varepsilon_y = a_{12}\sigma_x + a_{22}\sigma_y \qquad (12)$$

with (10), we obtain:

$$\frac{\partial v}{\partial y} = \varepsilon_y = \int_{-\infty}^{\infty} A[(a_{12}\omega_1^2 - a_{22})e^{-\omega_1|\alpha|y} - (a_{12}\omega_1\omega_2 - a_{22}\frac{\omega_1}{\omega_2})e^{-\omega_2|\alpha|y}]\alpha^2 e^{-i\alpha x} d\alpha \qquad (13)$$

Integrating (13) with respect to y, and then differentiating the resulting expression of $v(x,y)$ with respect to x, we have:

$$\frac{\partial v}{\partial x} = i\int_{-\infty}^{\infty} A|\alpha|\alpha[(a_{12}\omega_1 - \frac{a_{22}}{\omega_1})e^{-\omega_1|\alpha|y} - (a_{12}\omega_1 - a_{22}\frac{\omega_1}{\omega_2^2})e^{-\omega_2|\alpha|y}]e^{-i\alpha x} d\alpha \qquad (14)$$

Noting that:

$$\lim_{y\to 0} \frac{\partial v}{\partial x} = f(x) \tag{15}$$

equation (14) yields:

$$f(x) = i \int_{-\infty}^{\infty} A|\alpha|\alpha \; [(a_{12}\omega_1 - \frac{a_{22}}{\omega_1}) - (a_{12}\omega_1 - a_{22}\frac{\omega_1}{\omega_2^2})] e^{-i\alpha x} d\alpha \tag{16}$$

The inverse transform of (16) gives:

$$A(\alpha) = \frac{1}{2\pi_i} \frac{1}{\alpha|\alpha|} \frac{\omega_1\omega_2^2}{a_{22}(\omega_1^2-\omega_2^2)} \int_{-a}^{a} f(t) e^{i\alpha t} dt \tag{17}$$

Finally, substituting the expression of $A(\alpha)$ into expressions (10) and integrating over α the stresses can be written in terms of the unknown function $f(t)$ as follows:

$$\sigma_x^{(1)} = \frac{1}{\pi} \frac{\omega_1^2\omega_2^2}{a_{22}(\omega_1^2-\omega_2^2)} \int_{-a}^{a} [\frac{\omega_1(t-x)}{\omega_1^2 y^2+(t-x)^2} - \frac{\omega_2(t-x)}{\omega_2^2 y^2+(t-x)^2}] f(t) dt \tag{18}$$

$$\sigma_y^{(1)} = -\frac{1}{\pi} \frac{\omega_1\omega_2}{a_{22}(\omega_1^2-\omega_2^2)} \int_{-a}^{a} [\frac{\omega_2(t-x)}{\omega_1^2 y^2+(t-x)^2} - \frac{\omega_1(t-x)}{\omega_2^2 y^2+(t-x)^2}] f(t) dt \tag{19}$$

$$\tau_{xy}^{(1)} = -\frac{1}{\pi} \frac{\omega_1^2\omega_2^2}{a_{22}(\omega_1^2-\omega_2^2)} \int_{-a}^{a} [\frac{\omega_1 y}{\omega_1^2 y^2+(t-x)^2} - \frac{\omega_2 y}{\omega_2^2 y^2+(t-x)^2} \tag{20}$$

The expressions given in (18-20) may be used to obtain the stresses at any point in the plane in terms of the unknown function $f(t)$.

2.b. Formulation of the inclined crack

Consider now a crack making an angle θ with the x-axis. For this case the problem will be formulated in the s-n coordinate system. Again referring to [3,4] the governing equation for an anisotropic material in terms of the stress function $F_2(s,n)$ can be written as:

$$b_{22}\frac{\partial^4 F_2}{\partial s^4} - 2b_{26}\frac{\partial^4 F_2}{\partial s^3 \partial n} + (2b_{12} + b_{66})\frac{\partial^4 F_2}{\partial s^2 \partial n^2}$$

$$- 2b_{16}\frac{\partial^4 F_2}{\partial s \partial n^3} + b_{11}\frac{\partial^4 F_2}{\partial n^4} = 0 \tag{21}$$

where

$$\begin{aligned}
b_{11} &= a_{11}\cos^4\theta + (2a_{12} + a_{66})\sin^2\theta\cos^2\theta + a_{22}\sin^4\theta \\
b_{22} &= a_{11}\sin^4\theta + (2a_{12} + a_{66})\sin^2\theta\cos^2\theta + a_{22}\cos^4\theta \\
b_{12} &= a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\theta\cos^2\theta \\
b_{66} &= a_{66} + 4(a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\theta\cos^2\theta \\
b_{16} &= [a_{22}\sin^2\theta - a_{11}\cos^2\theta + \tfrac{1}{2}(2a_{12} + a_{66})\cos 2\theta] - \sin 2\theta \\
b_{26} &= [a_{22}\cos^2\theta - a_{11}\sin^2\theta - \tfrac{1}{2}(2a_{12} + a_{66})\cos 2\theta]\sin 2\theta
\end{aligned} \tag{22}$$

To simplify the formulation, we note that for (±45°) composites the angle θ defining the path of fracture is close to 45° and $E_x = E_y$ (see for example [1]). Thus, for $\theta = 45°$ and $E_x = E_y$ we have:

$$b_{16} = b_{26} = 0 \tag{23}$$

and equation (21) reduces to:

$$\frac{\partial^4 F_2}{\partial s^4} + \gamma_2 \frac{\partial^4 F_2}{\partial s^2 \partial n^2} + \gamma_4 \frac{\partial^4 F_2}{\partial n^4} = 0 \tag{24}$$

with

$$\gamma_2 = \frac{2b_{12}+b_{66}}{b_{22}} \; ; \quad \gamma_4 = \frac{b_{11}}{b_{22}} \tag{25}$$

Again we use the Fourier transform to formulate the problem. Defining

$$F_2(s,n) = \int_{-\infty}^{\infty} h(n,\alpha)\, e^{-i\alpha s} d\alpha \tag{26}$$

substituting into (24) and after some algebra we obtain:

$$\begin{aligned} F_2(s,n) &= \int_{-\infty}^{\infty} [C_1(\alpha) e^{-r_1|\alpha|n} + C_2(\alpha) e^{-r_2|\alpha|n}] e^{-i\alpha s} d\alpha. \text{ for } n>0 \\ F_2(s,n) &= \int_{-\infty}^{\infty} [C_3(\alpha) e^{r_1|\alpha|n} + C_4(\alpha) e^{r_2|\alpha|n}] e^{-i\alpha s} d\alpha, \text{ for } n<0 \end{aligned} \tag{27}$$

In these expressions r_1 and r_2 are the roots of:

$$\gamma_4 r^4 - \gamma_2 r^2 + 1 = 0 \tag{28}$$

such that $Re(r_1) > 0$, $Re(r_2) > 0$.

Using the continuity conditions along the axis $n = 0$:

$$F_2(s,+0) = F_2(s,-0) \tag{29}$$

and

$$\frac{\partial F_2}{\partial n}(s,+0) = \frac{\partial F_2}{\partial n}(s,-0) \tag{30}$$

we obtain:

$$C_3 = -\frac{r_1+r_2}{r_1-r_2} C_1 - \frac{2r_2}{r_1-r_2} C_2 \tag{31}$$

$$C_4 = \frac{2r_1}{r_1-r_2} C_1 + \frac{r_1+r_2}{r_1-r_2} C_2 \tag{32}$$

Then, the expressions given in (27) become:

$$F_2(s,n) = \int_{-\infty}^{\infty}[C_1 e^{-r_1|\alpha|n} + C_2 e^{-r_2|\alpha|n}]e^{-i\alpha s} d\alpha, \quad n>0$$

$$F_2(s,n) = \int_{-\infty}^{\infty}[(-\lambda_2 C_1+\lambda_3 C_2)e^{r_1|\alpha|n} + (\lambda_1 C_1+\lambda_2 C_2)e^{r_2|\alpha|n}]e^{-i\alpha s}d\alpha, \quad n<0 \tag{33}$$

with

$$\lambda_1 = \frac{2r_1}{r_1-r_2} \; ; \; \lambda_2 = \frac{r_1+r_2}{r_1-r_2} \; ; \; \lambda_3 = -\frac{2r_2}{r_1-r_2} \tag{34}$$

Since the inclined crack under consideration is denoted by A in figure 2, from here on, for the quantities related to this crack, a superscript A will be used.

Differentiating (33), the stress expressions are found to be for $n>0$,

$$\sigma_s^A = \int_{-\infty}^{\infty} [C_1 r_1^2 e^{-r_1|\alpha|n} + C_2 r_2^2 e^{-r_2|\alpha|n}]\cdot\alpha^2 e^{-i\alpha s} d\alpha \tag{35}$$

$$\sigma_n^A = -\int_{-\infty}^{\infty} [C_1 e^{-r_1|\alpha|n} + C_2 e^{-r_2|\alpha|n}]\cdot\alpha^2 e^{-i\alpha s} d\alpha \tag{36}$$

$$\tau_{ns}^A = -i\int_{-\infty}^{\infty} [C_1 r_1 e^{-r_1|\alpha|n} + C_2 r_2 e^{-r_2|\alpha|n}]\cdot\alpha|\alpha| e^{-i\alpha s} d\alpha \tag{37}$$

for $n<0$,

$$\sigma_s^A = \int_{-\infty}^{\infty}[r_1^2(-\lambda_2 C_1 + \lambda_3 C_2)e^{r_1|\alpha|n} + r_2^2(\lambda_1 C_1 + \lambda_2 C_2)e^{r_2|\alpha|n}]\alpha^2 e^{-i\alpha s} d\alpha \tag{38}$$

$$\sigma_n^A = - \int_{-\infty}^{\infty} [(-\lambda_2 C_1 + \lambda_3 C_2) e^{r_1|\alpha|n}$$
$$+ (\lambda_1 C_1 + \lambda_2 C_2) e^{r_2|\alpha|n}] \alpha^2 e^{-i\alpha s} d\alpha \tag{39}$$

$$\tau_{ns}^A = +i \int_{-\infty}^{\infty} [(-\lambda_2 C_1 + \lambda_3 C_2) r_1 e^{r_1|\alpha|n}$$
$$+ (\lambda_1 C_1 + \lambda_2 C_2) r_2 e^{r_2|\alpha|n}] \alpha|\alpha| e^{-i\ s} d\alpha \tag{40}$$

Again, the problem will be formulated in terms of the crack surface derivatives. Thus, defining:

$$g_1(s) = \frac{\partial}{\partial s} [u(s,0^+) - u(s,0^-)] \tag{41}$$

$$g_2(s) = \frac{\partial}{\partial s} [v(s,0^+) - v(s,0^-)] \tag{42}$$

and after some lengthy algebra one may show that:

$$C_1 = - \frac{1}{2\pi(\delta_2 - \delta_1)} [\frac{1}{\alpha^2} q_1(\alpha) - \frac{\delta_2}{i\delta_3 \alpha|\alpha|} q_2(\alpha)] \tag{43}$$

$$C_2 = \frac{1}{2\pi(\delta_2 - \delta_1)} [\frac{1}{\alpha^2} q_1(\alpha) - \frac{\delta_1}{i\delta_3 \alpha|\alpha|} q_2(\alpha)] \tag{44}$$

where

$$q_1(\alpha) = \int_b^c g_1(t) e^{i\alpha t} dt \tag{45}$$

$$q_2(\alpha) = \int_b^c g_2(t) e^{i\alpha t} dt \tag{46}$$

and

$$\delta_1 = 2b_{11} r_1 (r_1 + r_2)$$
$$\delta_2 = 2b_{11} r_2 (r_1 + r_2) \tag{47}$$
$$\delta_3 = -2b_{22} \frac{(r_1 + r_2)}{r_1 r_2}$$

Substituting (43) and (44) into expressions (35-40), and performing the infinite integrals over α, we obtain:

$$\sigma_s^A = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_2^3 n}{(r_2 n)^2+(t-s)^2} - \frac{r_1^3 n}{(r_1 n)^2+(t-s)^2}\right] g_1(t)dt$$

$$- \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_1(t-s)}{(r_1 n)^2+(t-s)^2} - \frac{r_2(t-s)}{(r_2 n)^2+(t-s)^2}\right] g_2(t)dt \qquad (48)$$

$$\sigma_n^A = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_1 n}{(r_1 n)^2+(t-s)^2} - \frac{r_2 n}{(r_2 n)^2+(t-s)^2}\right] g_1(t)dt$$

$$+ \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{(t-s)/r_1}{(r_1 n)^2+(t-s)^2} - \frac{(t-s)/r_2}{(r_2 n)^2+(t-s)^2}\right] g_2(t)dt \qquad (49)$$

$$\tau_{ns}^{A} = - \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c [\frac{r_1(t-s)}{(r_1 n)^2+(t-s)^2}$$

$$- \frac{r_2(t-s)}{(r_2 n)^2+(t-s)^2}]g_1(t)dt$$

$$\tag{50}$$

$$+ \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c [\frac{r_1 n}{(r_1 n)^2+(t-s)^2}$$

$$- \frac{r_2 n}{(r_2 n)^2+(t-s)^2}]g_2(t)dt$$

For cracks B, C, D shown in figure 2, similar expressions may be obtained. Taking into account the symmetry conditions, for each crack the stresses may be written as:

$$\sigma_n^{B} = - \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c [\frac{r_2^3 s}{(r_2 s)^2+(t+n)^2}$$

$$- \frac{r_1^3 s}{(r_1 s)^2+(t+n)^2}]g_1(t)dt$$

$$\tag{51}$$

$$- \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c [\frac{r_1(t+n)}{(r_1 s)^2+(t+n)^2}$$

$$- \frac{r_2(t+n)}{(r_2 s)^2+(t+n)^2}]g_2(t)dt$$

$$\sigma_s^{\,B} = -\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1 s}{(r_1 s)^2+(t+n)^2}$$

$$-\frac{r_2 s}{(r_2 s)^2+(t+n)^2}\Big] g_1(t)dt$$

$$+\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{(t+n)/r_1}{(r_1 s)^2+(t+n)^2} \tag{52}$$

$$-\frac{(t+n)/r_2}{(r_2 s)^2+(t+n)^2}\Big] g_1(t)dt$$

$$\tau_{ns}^{B} = -\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1(t+n)}{(r_1 s)^2+(t+n)^2}$$

$$-\frac{r_2(t+n)}{(r_2(s)^2+(t+n)^2}\Big] g(t)dt$$

$$-\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1 s}{(r_1 s)^2+(t+n)^2} \tag{53}$$

$$-\frac{r_2 s}{(r_2 s)^2+(t+n)^2}\Big] g_2(t)dt$$

$$\sigma_n{}^c = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \Big[\frac{r_2^3(s+\sqrt{2})}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}$$

$$- \frac{r_1^3(s+a\sqrt{2})}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big] g_1(t)dt$$

$$- \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \Big[\frac{r_1(t-n+a\sqrt{2})}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2} \tag{54}$$

$$- \frac{r_2(t-n+a\sqrt{2})}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big] g_2(t)dt$$

$$\sigma_s{}^c = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \Big[\frac{r_1(s+a\sqrt{2})}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}$$

$$- \frac{r_2(s+a\sqrt{2})}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big]\ g_1(t)dt$$

$$+ \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \Big[\frac{(t-n+a\sqrt{2})/r_1}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2} \tag{55}$$

$$- \frac{(t-n+a\sqrt{2})/r_2}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big]\ g_2(t)dt$$

$$\tau_{ns}^{c} = -\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1(t-n+a\sqrt{2})}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}$$

$$-\frac{r_2(t-n+a\sqrt{2})}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big]\; g_1(t)dt$$

$$+\frac{1}{2\pi_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1(s+a\sqrt{2})}{(r_1(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2} \tag{56}$$

$$-\frac{r_2(s+a\sqrt{2})}{(r_2(s+a\sqrt{2}))^2+(t-n+a\sqrt{2})^2}\Big]\; g_2(t)dt$$

$$\sigma_s{}^{D} = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_2^3(-n+a\sqrt{2})}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}$$

$$-\frac{r_1^3(-n+a\sqrt{2})}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\Big]\; g_1(t)dt$$

$$-\frac{1}{2\pi b_{11}(r_2^2-r_1^2)}\int_b^c \Big[\frac{r_1(t+s+a\sqrt{2})}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2} \tag{57}$$

$$-\frac{r_2(t+s+a\sqrt{2})}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\Big]\; g_2(t)dt$$

$$\sigma_n^D = \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_1(-n+a\sqrt{2})}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2} - \frac{r_2(-n+a\sqrt{2})}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\right] g_1(t)dt$$

$$+ \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{(t+s+a\sqrt{2})/r_1}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2} - \frac{(t+s+a\sqrt{2})/r_2}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\right] g_2(t)dt \tag{58}$$

$$\tau_{ns}^D = - \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_1(t+s+a\sqrt{2})}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2} - \frac{r_2(t+s+a\sqrt{2})}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\right] g_1(t)dt$$

$$+ \frac{1}{2\pi b_{11}(r_2^2-r_1^2)} \int_b^c \left[\frac{r_1(-n+a\sqrt{2})}{(r_1(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2} - \frac{r_2(-n+a\sqrt{2})}{(r_2(-n+a\sqrt{2}))^2+(t+s+a\sqrt{2})^2}\right] g_2(t)dt \tag{59}$$

3. THE INTEGRAL EQUATIONS AND THE RESULTS

3.a. The case of through cracks

For the through crack configuration shown in figure 2, the integral equations can be obtained by superimposing the solutions for the central crack and the inclined cracks A, B, C, D. The infinite plate is under a uniform tension σ parallel to the y-direction. For the uncracked plate, the stresses along the crack lines are:

$$\text{along } y = 0\ ,\ \sigma_y = \sigma_0\ ,\quad \tau_{xy} = 0$$
$$\text{along } n = 0\ ,\ \sigma_n = \sigma_0 \cos^2\theta = \sigma_0/2 \tag{60}$$

$$\tau_{ns} = \sigma_0 \sin\theta\cos\theta = \sigma_0/2 \tag{61}$$

Thus, the crack surface tractions may be written as:

$$\sigma_y(x,0) = -\sigma_0 \tag{62}$$

$$\sigma_n(s,0) = -\sigma_0/2 \tag{63}$$

$$\tau_{ns}(s,0) = -\sigma_0/2 \tag{64}$$

With the use of equations (18-20) and (48-59), the boundary conditions given in (62-64) yield the following system of singular integral equations:

$$\frac{1}{\pi}\ \frac{\omega_1\omega_2}{a_{22}(\omega_1+\omega_2)} \int_{-a}^{a} \frac{f(t)}{t-x}\,dt$$
$$+ \frac{1}{2}(\sigma_n^A + \sigma_s^A + \sigma_n^B + \sigma_s^B + \sigma_n^C + \sigma_s^C + \sigma_n^D + \sigma_s^D\big|_{y=0} \tag{65}$$
$$+ (\tau_{ns}^A + \tau_{ns}^B + \tau_{ns}^C + \tau_{ns}^D\big|_{y=0} = -\sigma_0$$

$$\text{for } y = 0\ ,\quad s = \frac{1}{\sqrt{2}}(-x+a)\ ;\quad n = \frac{1}{\sqrt{2}}(-x+a)$$
$$\tag{66}$$
$$\frac{1}{2\pi b_{11}(r_2+r_1)r_1r_2} \int_{b}^{c} \frac{g_2(t)}{t-s}\,dt$$

$$+ (\sigma_n^B + \sigma_n^C + \sigma_n^D\big| + \frac{1}{2}(\sigma_x^{(1)} + \sigma_y^{(1)}\big|_{n=0} - \tau_{xy}^{(1)}\big|_{n=0} = -\sigma_0/2$$

$$\text{for } n = 0\,, \quad x = a + \frac{\sqrt{2}}{2}s$$

$$y = \frac{\sqrt{2}}{2}s \tag{67}$$

$$\frac{1}{2\pi b_{11}(r_2+r_1)} \int_b^c \frac{g_1(t)}{t-s}\,dt$$

$$+ (\tau_{ns}^B + \tau_{ns}^C + \tau_{ns}^D|_{n=0} + \frac{1}{2}(\sigma_y^{(1)} - \sigma_x^{(1)}|_{n=0} = -\sigma_0/2$$

$$\text{for } n = 0\,, \quad x = a + \frac{\sqrt{2}}{2}s$$

$$y = \frac{\sqrt{2}}{2}s$$

To complete the formulation of the problem, one has also to write the following single-valuedness conditions for the displacements:

$$\int_{-a}^{a} f(t)dt = 0 \tag{68}$$

$$\int_b^c g_1(t)dt = 0 \tag{69}$$

$$\int_b^c g_2(t)dt = 0 \tag{70}$$

The singular integral equations (65-67) may be solved by one of the collocation methods given in [5-7]. However, to apply the methods, first the system must be normalized. Thus, we make use of the following transformations:

for equation (65),

$$\begin{aligned} x &= az \qquad -a<x,\ t<a \\ t &= a\rho \qquad -1<z,\ \rho<1 \\ f(t) &= f^*(\rho) \end{aligned} \tag{71}$$

for equations (66,67),

$$s = \frac{c-b}{2} s_1 + \frac{c+b}{2} \qquad b<s\,,\ t<c$$

$$t = \frac{c-b}{2} t_1 + \frac{c+b}{2} \qquad -1<s_1,\ t_1<1 \tag{72}$$

$$g_1(t) = g_1^*(t_1)$$

$$g_2(t) = g_2^*(t_1)$$

By substituting (71) and (72) into (65-70), the system will be normalized. In this case, since only through cracks are considered, the important fracture parameters, i.e., the stress intensity factors are computed. The stress intensity factors are defined as:

$$k_1(a) = \lim_{x \to a} \sqrt{2(x-a)}\ \sigma_y(x,0) \tag{73}$$

$$k_1(c) = \lim_{s \to c} \sqrt{2(s-c)}\ \sigma_n(s,0) \tag{74}$$

$$k_1(b) = \lim_{s \to b} \sqrt{2(b-s)}\ \sigma_n(s,0) \tag{75}$$

$$k_2(c) = \lim_{s \to c} \sqrt{2(s-c)}\ \tau_{ns}(s,0) \tag{76}$$

$$k_2(b) = \lim_{s \to b} \sqrt{2(b-s)}\ \tau_{ns}(s,0) \tag{77}$$

Noting that the normalized unknown functions will have the usual square-root singularity at both ends, one may write:

$$f^*(\rho) = \frac{F(\rho)}{\sqrt{1-\rho^2}} \tag{78}$$

$$g_1^*(t_1) = \frac{G_1(t_1)}{\sqrt{1-t_1^2}} \tag{79}$$

$$g_2^*(t_1) = \frac{G_2(t_1)}{\sqrt{1-t_1^2}} \tag{80}$$

where the functions $F(\rho)$, $G_1(t_1)$ and $G_2(t_1)$ are bounded in the interval [-1,1]. Thus, with the definitions given in (73-77), we obtain:

$$k_1(a) = -\sqrt{a}\,\frac{\omega_1\omega_2}{a_{22}(\omega_1+\omega_2)}\,F(1) \tag{81}$$

$$k_1(c) = -\sqrt{\ell}\,\frac{1}{2b_{11}(r_2+r_1)r_1r_2}\,G_2(1) \tag{82}$$

$$k_1(b) = \sqrt{\ell}\,\frac{1}{2b_{11}(r_2+r_1)r_1r_2}\,G_2(-1) \tag{83}$$

$$k_2(c) = -\sqrt{\ell}\,\frac{1}{2b_{11}(r_2+r_1)}\,G_1(1) \tag{84}$$

$$k_2(b) = \sqrt{\ell}\,\frac{1}{2b_{11}(r_2+r_1)}\,G_1(-1) \tag{85}$$

with $\ell = (c-b)/2$

Results have been obtained for a $(\pm 45)_{2s}$ composite. The material properties used in the computations and given below were taken from [8].

E_x = 19.6 GPa
E_y = 19.6 GPa
G_{xy} = 33.7 GPa
ν_{yx} = 0.735

Some sample results are presented in tables (1,2) and figure 3. First, the results were checked for the case when the inclined cracks were located far away from the central crack. For the normalized stress intensity factor at the tip of the central crack, the expected value of 1 was recovered. Also the normalized stress intensity factors at the tips of the inclined cracks reached the expected limiting value of 0.5. Table 1 shows the results for an inclined crack whose center is kept fixed, but the length of which varies. As expected the stress intensity factors at the tips $x = +a$ and $s = b$, assume higher values as the two cracks approach each other. Table 2 and figure 3 show the results for a cracked plate, where the length of the inclined cracks is kept constant. Again, as expected, the stress intensity factors at tips $x = +a$, and $s = b$ decrease as the cracks are further apart from each other.

3.b. The branched crack case

In this case we still have through cracks, but we assume that all cracks are joined. Thus, we obtain a centrally cracked plate with branches running at ±45° from the tips of the central crack. The formulation of the problem is the same except that one has to put $b = 0$ for the limits of the integrals in equations (65-67) and the single valuedness conditions (68-70) are not necessary anymore. However, by putting $b = 0$, the nature of the singularity at $x = a$ and $s = 0$ will change since the integral equations will now have generalized Cauchy kernels. A brief inspection of the crack geometry shows that, the crack displacement derivatives must be bounded at $x = a$, and $s = 0$, since all wedge angles are less than π. (For stress singularities in isotropic wedges see [9] and for stress singularities in orthotropic wedges see [10,11].) The solution of singular integral equations with generalized Cauchy kernels may be obtained by the methods suggested in [12-13]. In this study the system is solved by assuming that the crack surface displacements for the central crack are symmetric and that the crack surface derivatives are bounded at $x=a$ and $s=0$. Thus, the conditions (68-70) are replaced by:

$$\int_{-1}^{1} f^*(\rho)\,d\rho = 0 \tag{86}$$

$$G_1(-1) = 0 \tag{87}$$

and

$$G_2(-1) = 0 \tag{88}$$

The numerical computations show that, the convergence of the system is excellent and as expected $F(1) \equiv 0$ since the crack surface displacement derivative is known to be bounded at $x = a$ (or $z = 1$). For this case the only relevant fracture parameters are the stress intensity factors at the tip of the branched crack at $s = +c$. The normalized stress intensity factors $k_1(c)$ and $k_2(c)$ are given in table 3 and figure 4. The results indicate that the normalized stress intensity factors decrease with increasing crack length.

In literature, only for the isotropic case one may find some results for the branched crack. Therefore, it is very difficult to compare the results directly. As a check, the anisotropic results found in this study were compared with the isotropic results found in [14]. Although the material and geometry were close but not exactly the same, the results were found to be of the same order.

3.c. A central crack with part-through branches

The foregoing formulation was performed mainly to study this case which appears in (±45) and other composites. In a centrally cracked plate under uniaxial tension, some inclined cracks which make up an angle with the initial central crack may develop. However, some of the laminae are not broken, i.e., the cracks are not through cracks. In this work, the following model is adopted: It is assumed that the unbroken laminae are weakened and they carry only tension or compression. Furthermore, it is assumed that the unknown normal stress in the unbroken laminae is proportional to the crack opening displacement at that point, i.e.:

$$\sigma = K\Delta v \tag{89}$$

where K is called the spring coefficient and Δv is the opening of the crack. Thus at each point of the fracture path one may compute a bounded stress and thus a stress concentration factor. It is expected that generally the stress concentration factor will be larger near the original crack tip, namely, near $s = 0$. The spring coefficient in relation (89) may be a constant giving a linear stress-crack opening relationship or may vary from point to point making relation (89) non-linear. This relationship which must be determined experimentally, in general is believed to be non-linear [15]. However, in this study, for simplicity it is assumed to be linear, thus K is taken as a constant.

To incorporate this relationship in the formulation, we may replace boundary condition (63) by:

$$\sigma_n(s,0) = -\sigma_0/2 + K\Delta v \tag{90}$$

or

$$\sigma_n(s,0) = -\sigma_0/2 - K \int_s^c g_2(t)dt \tag{91}$$

The problem may then be solved as in case (b) with the difference that, in the singular equations system (65-67), equation (66) must be replaced with equation (91). After solving the system, one may compute the stress intensity factors $k_1(c)$ and $k_2(c)$ at the tip of the branched crack, the normal stress distribution σ along the fracture path, the stress concentration factor for the fibers at $s = 0$ (and $x = a$). As stated earlier, the spring coefficient K is determined experimentally. In [15] the value of K is given for a certain composite. In this study, results are obtained for four different values of K, but chosen of the same order of K given in [15]. The stress intensity factors are given in table 4, and the stress concentration factors are presented in table 5 and figure 5. The results indicate that the stress concentration factors which may be used to study the breaking of fibers near the tip of the original crack, vary significantly with the spring coefficient K and the ratio c/a. As K increases, i.e., for stiffer materials, the stress concentration will be higher, meaning that for such materials it is more likely that the fibers ahead of the slit will break. Here it must be noted that, the stress concentration factor (SCF) is not always maximum at $s = 0$. In some instances, the maximum stress concentration factor was found to be maximum at a small distance from the tip of the original slit. Two examples showing the crack opening along the fracture path are displayed in figures 6 and 7. Figure 6 shows the crack opening for a short crack ($c/a = 1$) and figure 7 shows the crack opening for a larger crack ($c/a = 5$). It is observed that the crack profiles may be significantly different and for the long crack the maximum stress concentration occurs not at $s = 0$ but at $s/c \cong 0.2$.

REFERENCES

[1] C.C. Poe, Jr. and J.A. Sova, "Fracture Toughness of Boron/Aluminum Laminates with Various Proportions of 0° and ±45° Plies", NASA Technical Paper 1707, 1980.

[2] C.C. Poe, Jr., " A Unifying Strain Criterion for Fracture of Fibrous Composite Laminates", Journal of Engineering Fracture Mechanics, Vol. 17, No. 2, pp. 153-171, 1983.

[3] S.G. Lekhnitskii, "Theory of Elasticity of an Anisotropic Elastic Body", Holden-Day, Inc. 1963.

[4] S.G. Lekhnitskii, "Anisotropic Plates", Gordon and Breach Sci. Pub., 1968.

[5] F. Erdogan, "Mixed Boundary-Value Problems in Mechanics", Mechanics Today, (Edited by S. Nemat-Nasser), Vol. 4, p. 1, Pergamon Press, Oxford, 1978.

[6] P.S. Theocaris and N.I. Ioakimidis, "Numerical Integration Methods for the Solution of Singular Integral Equations", Quarterly of Applied Mathematics, pp. 173-183, 1977.

[7] S. Krenk, "Quadrature Formulae of Closed Type for Solution of Singular Integral Equations", J. Inst. Maths. Appl., Vol. 22, pp. 99-107, 1978.

[8] C.C. Poe, Jr., "Fracture Toughness of Fibrous Composite Materials", Private Communication.

[9] M.L. Williams, "Stress Singularities Resulting From Various Boundary Conditions in Angular Corners of Plates in Extension", J. of Appl. Mech., Vol. 19, pp. 526-528, 1952.

[10] M.C. Kuo and D.B. Bogy "Plane Solutions for the Displacement and Traction-Displacement Problems for Anisotropic Elastic Wedges", ASME J. of App. Mech., pp. 197-202, 1974.

[11] M.C. Kuo and D.B. Bogy, "Plane Solution for Traction Problems on Orthotropic Unsymmetrical Wedges and Symmmetrically Twinned Wedges", J. of Appl. Mech., Vol. 41, pp. 203-208, 1974.

[12] F. Erdogan, G.D. Gupta and T.S. Cook, "Numerical Solution of Singular Integral Equations", Mechanics of Fracture, Vol. 1: Methods of Analysis and Solutions of Crack Problems, (Edited by G.C. Sih), Chap. 7, pp. 368-425, Noordhoff, Leyden, The Netherlands, 1973.

[13] N.I. Ioakimidis and P.S. Theocaris, "On the Solution of Collocation Points for the Numerical Solution of Singular Integral Equations with Generalized Kernels Appearing in Elasticity Problems", Computers & Structures, Vol. 11, pp. 289-295, 1980.

[14] P.S. Theocaris and N. Ioakimidis, "The Symmetrically Branched Crack in an Infinite Elastic Medium", Zeit. ang. Math. Phys. (ZAMP), Vol. 27, pp. 801-814, 1976.

[15] J.R. Walton and Y. Weitsman, "Deformations and Stress Intensities Due to a Craze in an Extended Elastic Material", Journal of Applied Mechanics, Vol. 51, pp. 84-92, 1984.

Table 1. Stress Intensity factors for the cracked plate (through cracks)

$(\frac{c+b}{2a} = 2)$

b/a	$\frac{k_1(a)}{\sigma_0\sqrt{a}}$	$\frac{k_1(c)}{\sigma_0\sqrt{\ell}}$	$\frac{k_1(b)}{\sigma_0\sqrt{\ell}}$	$\frac{k_2(c)}{\sigma_0\sqrt{\ell}}$	$\frac{k_2(b)}{\sigma_0\sqrt{\ell}}$
1.9	1.003	0.535	0.537	0.577	0.580
1.5	1.075	0.532	0.544	0.570	0.592
1.25	1.175	0.531	0.548	0.565	0.602
1.0	1.332	0.531	0.550	0.560	0.617
0.75	1.571	0.532	0.549	0.554	0.641
0.50	1.955	0.536	0.545	0.548	0.691
0.25	2.717	0.545	0.563	0.548	0.823
0.10	4.003	0.560	0.677	0.559	1.098

Table 2. Stress Intensity factors for the cracked plate (through cracks)

$(\ell = \frac{c-b}{2} = 1)$

b/a	$\frac{k_1(a)}{\sigma_0\sqrt{a}}$	$\frac{k_1(c)}{\sigma_0\sqrt{\ell}}$	$\frac{k_1(b)}{\sigma_0\sqrt{\ell}}$	$\frac{k_2(c)}{\sigma_0\sqrt{\ell}}$	$\frac{k_2(b)}{\sigma_0\sqrt{\ell}}$
0.1	2.745	0.593	0.728	0.597	1.019
0.5	1.607	0.548	0.582	0.574	0.689
1.0	1.332	0.531	0.550	0.560	0.617
1.5	1.221	0.523	0.534	0.548	0.584
2.0	1.161	0.518	0.526	0.540	0.564
2.5	1.125	0.514	0.520	0.533	0.550
3.0	1.100	0.512	0.516	0.527	0.541
5.0	1.050	0.507	0.509	0.515	0.520
10.0	1.017	0.503	0.503	0.505	0.506

Table 3. The stress intensity factors for a completely cracked laminate
(Branched crack case)

c/a	$\frac{k_1(c)}{\sigma_0\sqrt{c}}$	$\frac{k_2(c)}{\sigma_0\sqrt{c}}$
0.1	2.098	0.829
0.25	1.305	0.705
0.5	0.943	0.638
1.0	0.722	0.584
2.0	0.547	0.538
5.0	0.513	0.492
10.0	0.483	0.473
100.0	0.455	0.454
1000.0	0.452	0.452

Table 4. The stress intensity factors at the tip of the branched crack. (Part-through cracks case)

	$K = 0.5 \times 10^7$ 1/m		$K = 10^7$ 1/m		$K = 5 \times 10^7$ 1/m		$K = 10^8$ 1/m	
c/a	$k_1(c)/\sigma_0\sqrt{c}$	$k_2(c)/\sigma_0\sqrt{c}$	$k_1(c)/\sigma_0\sqrt{c}$	$k_2(c)/\sigma_0\sqrt{c}$	$k_1(c)/\sigma_0\sqrt{c}$	$k_2(c)/\sigma_0\sqrt{c}$	$k_1(c)/\sigma_0\sqrt{c}$	$k_2(c)/\sigma_0\sqrt{c}$
0.25	1.211	0.725	1.132	0.741	0.765	0.820	0.567	0.863
0.50	0.820	0.662	0.729	0.679	0.417	0.739	0.295	0.762
1.0	0.563	0.610	0.469	0.624	0.236	0.660	0.165	0.669
2.0	0.394	0.561	0.308	0.571	0.144	0.587	0.100	0.590
5.0	0.248	0.509	0.182	0.512	0.081	0.516	0.055	0.516

Table 5. The stress concentration factors for the stressed fibers at the tip of the notch

c/a	$K=0.5x10^7$ 1/m SCF	$K=10^7$ 1/m SCF	$K=5x10^7$ 1/m SCF	$K=10^8$ 1/m SCF
0.25	0.19	0.35	1.19	1.75
0.50	0.24	0.42	1.12	1.50
1.0	1.28	0.45	0.93	1.12
2.0	0.32	0.45	0.68	0.75
5.0	0.36	0.43	0.48	0.49

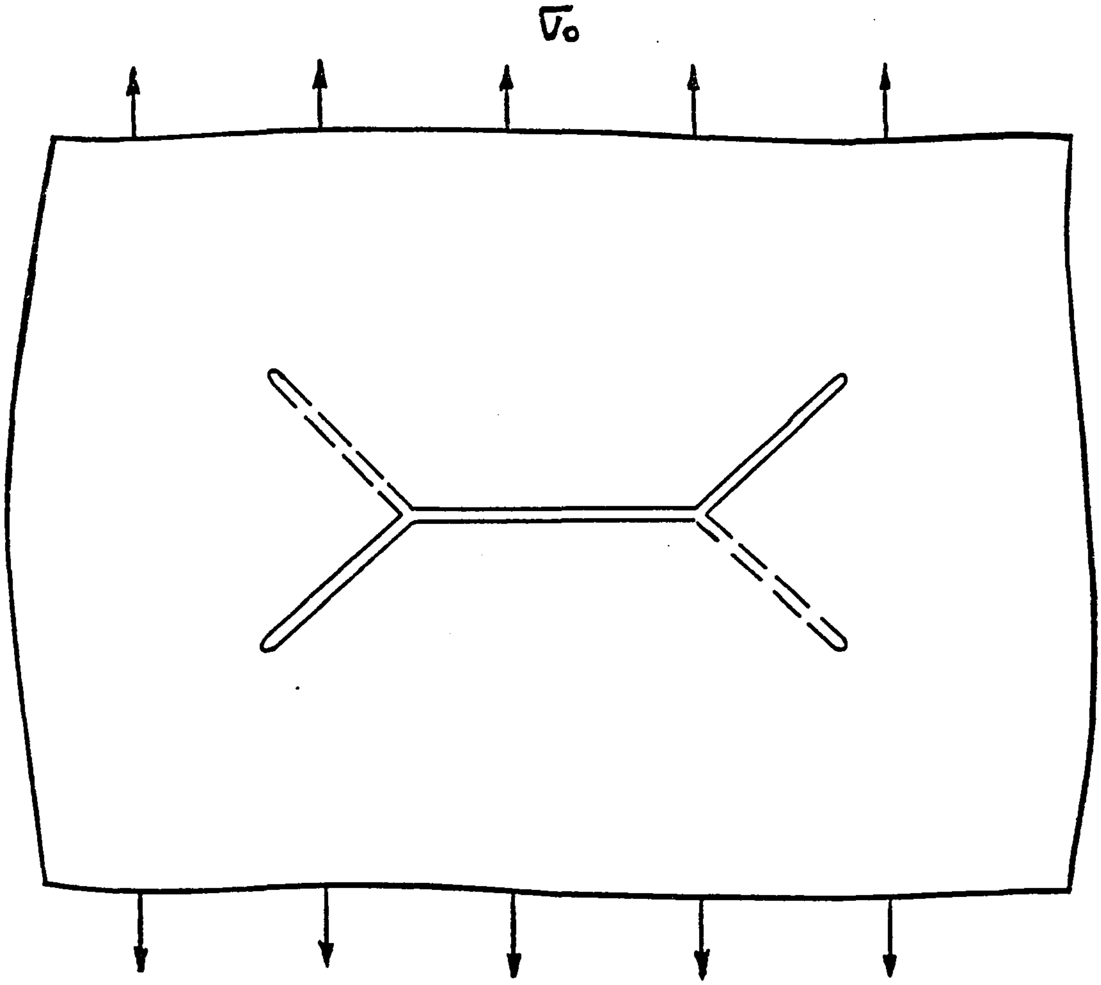

Figure 1. Part-through cracks emanating from the tip of an existing notch.

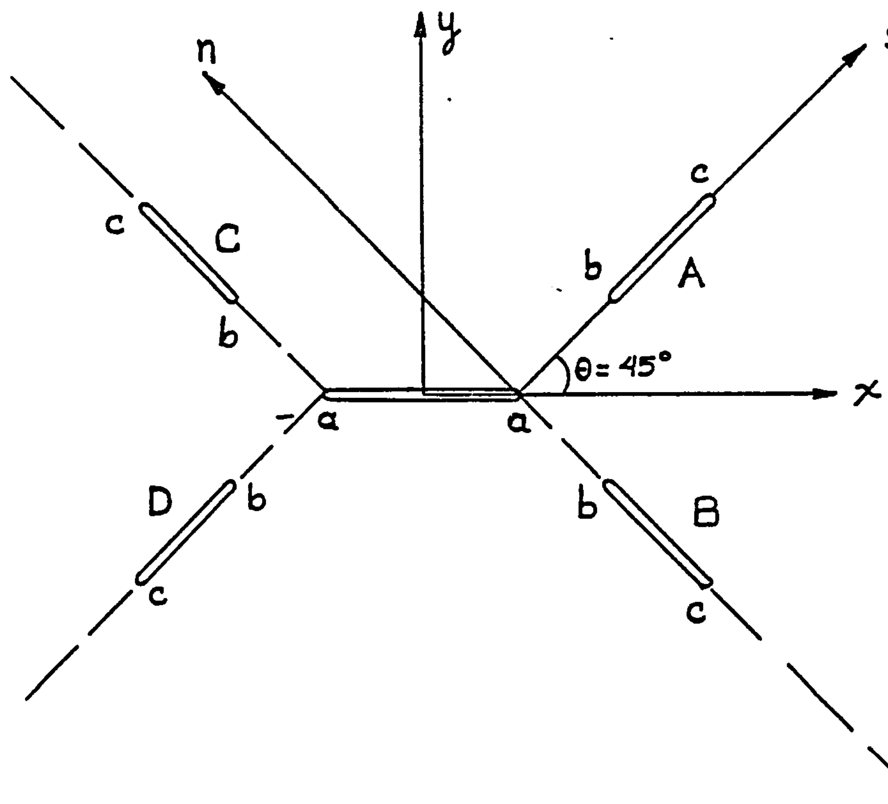

Figure 2. The crack geometry used in the formulation.

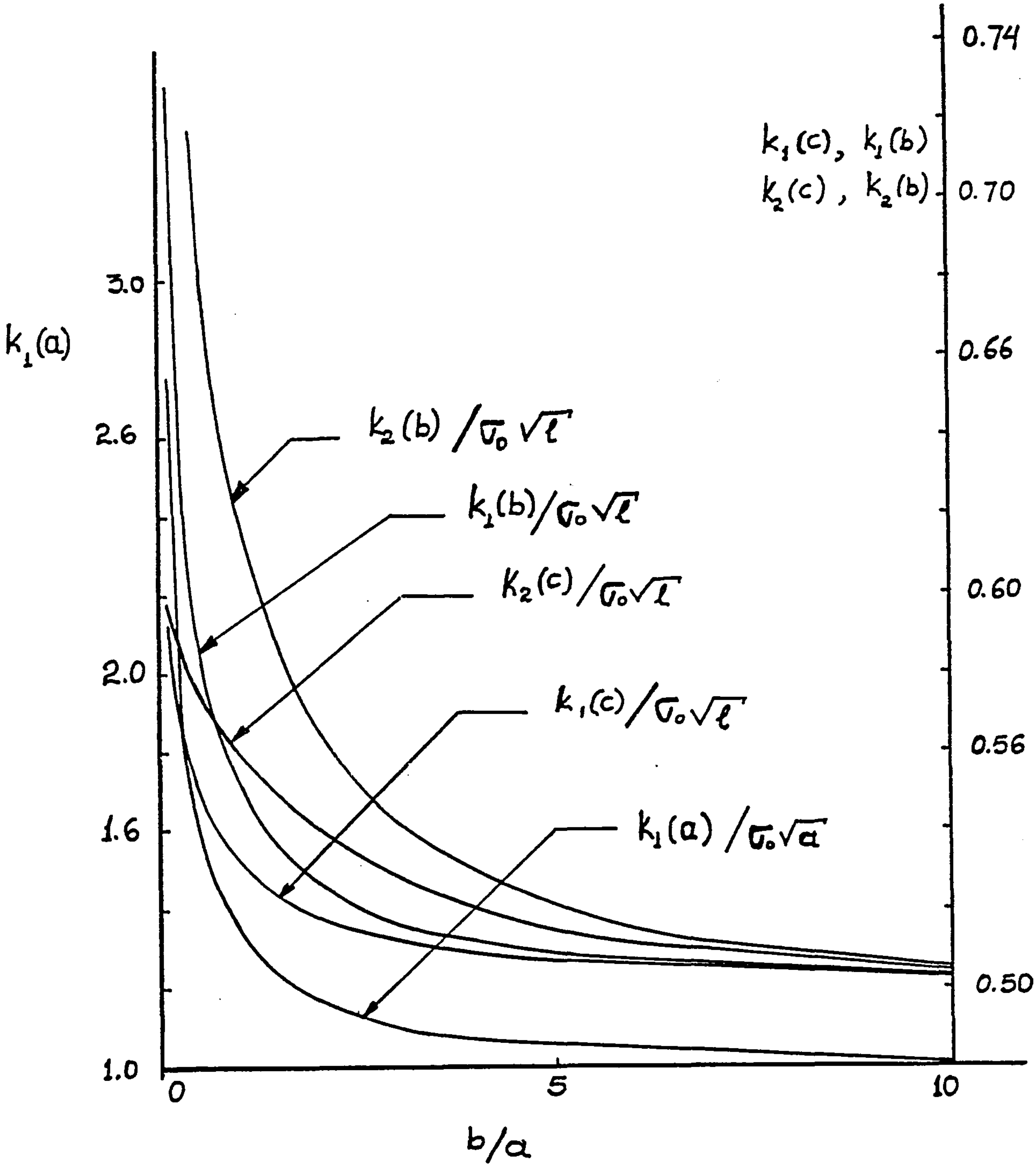

Figure 3. Variation of the normalized stress intensity factors with b/a for a cracked plate (ℓ = 1).

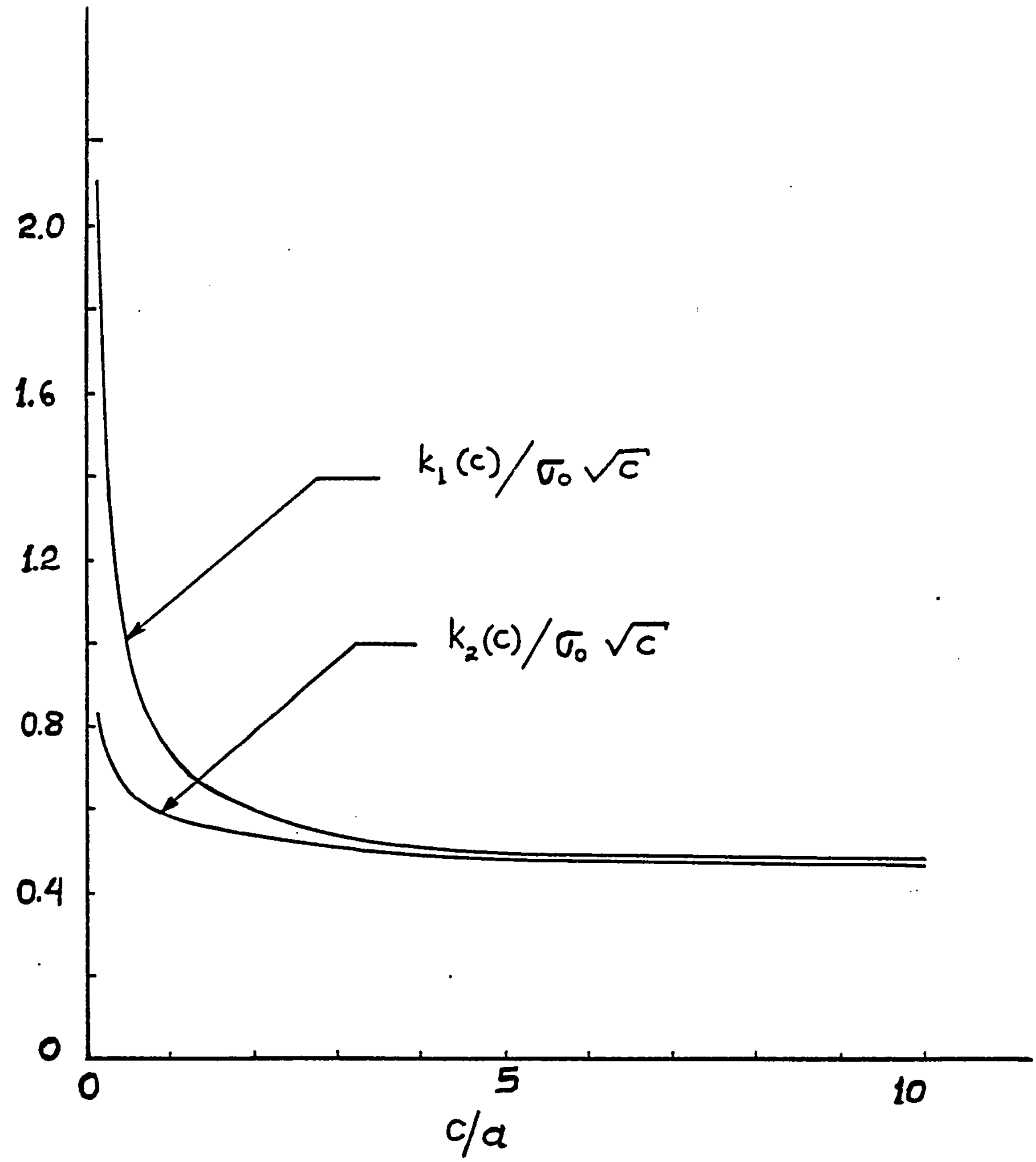

Figure 4. Variation of the normalized stress intensity factors with c/a for a completely cracked laminate (branched crack case).

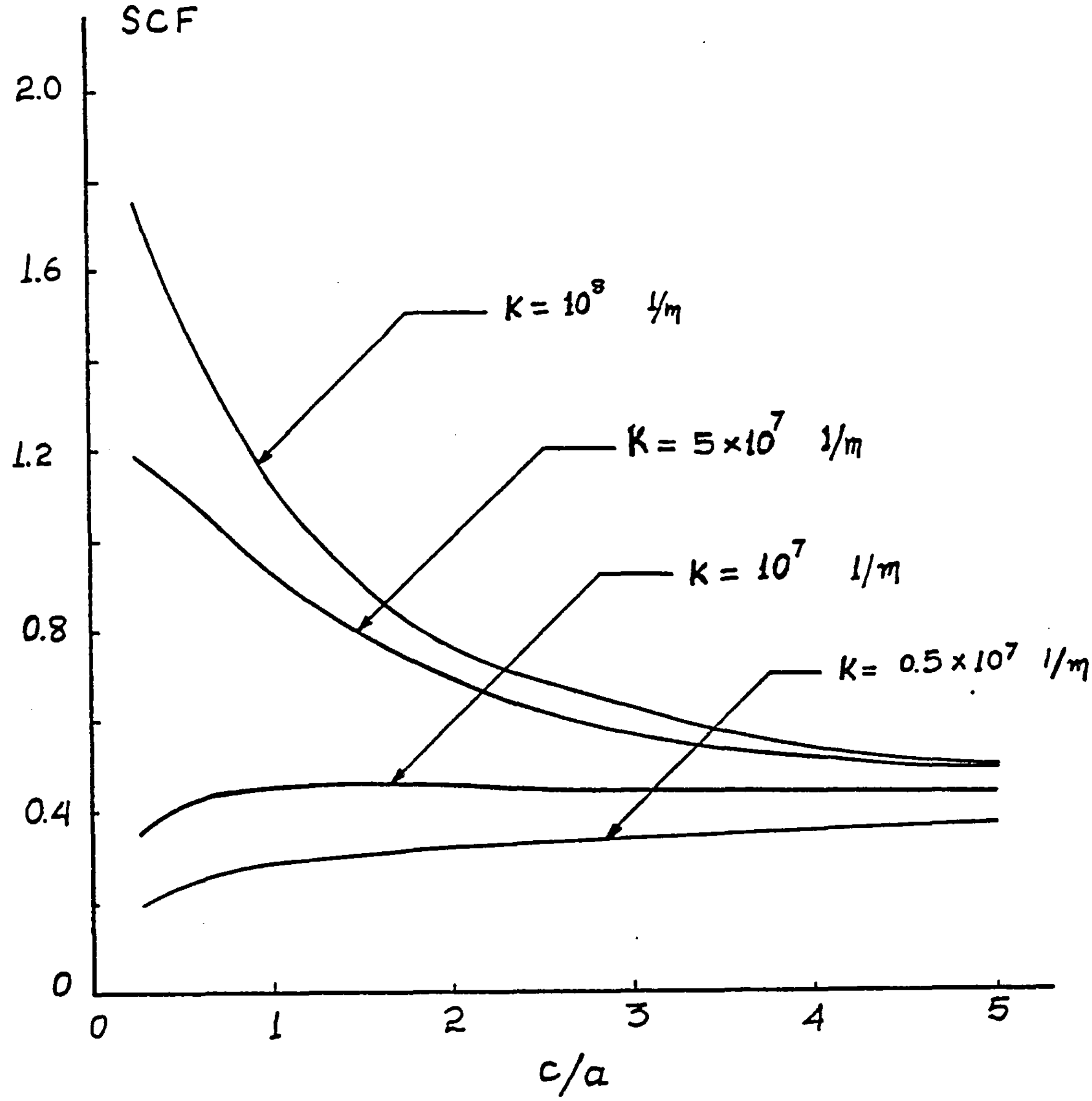

Figure 5. The stress concentration factors for various values of the spring constant K.

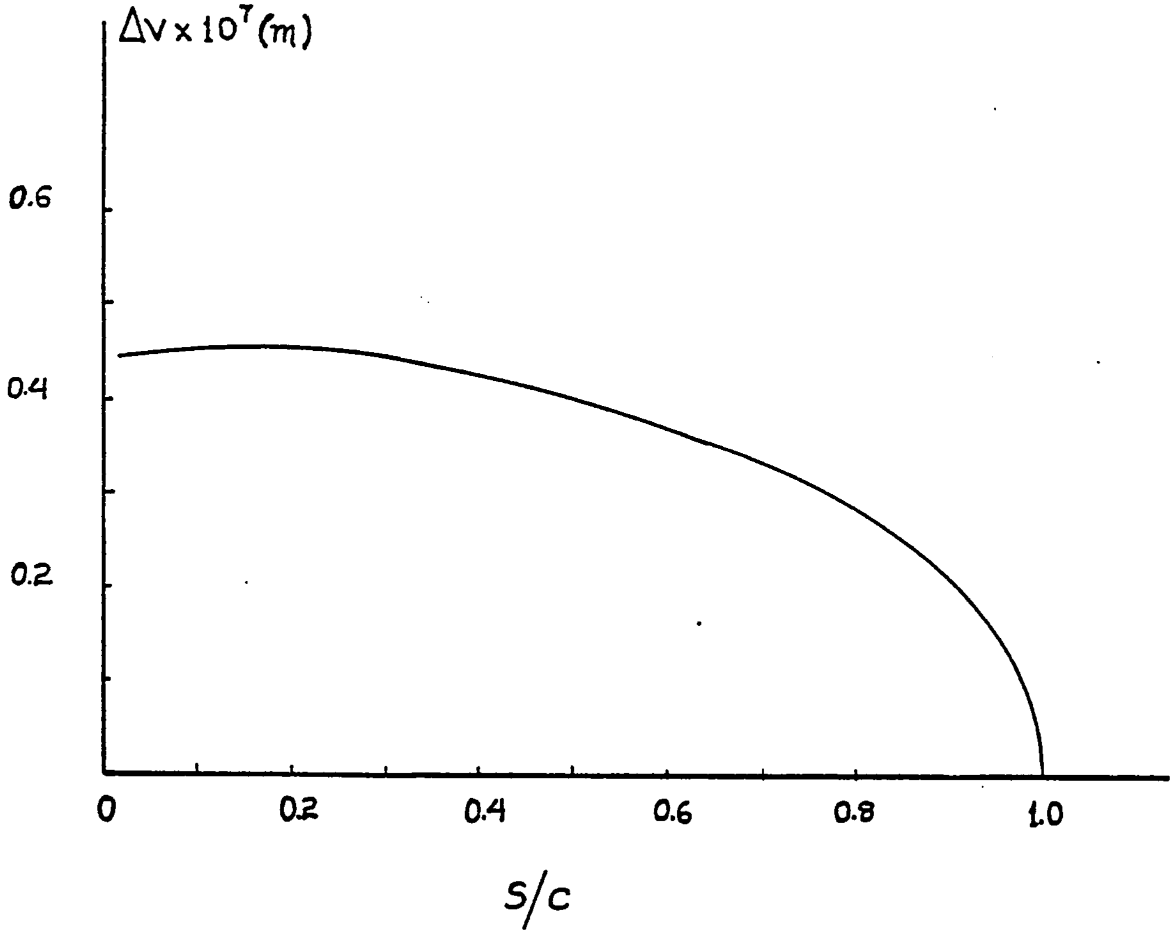

Figure 6. The crack opening displacement Δv for $c/a = 1$, $K = 10^7$ 1/m.

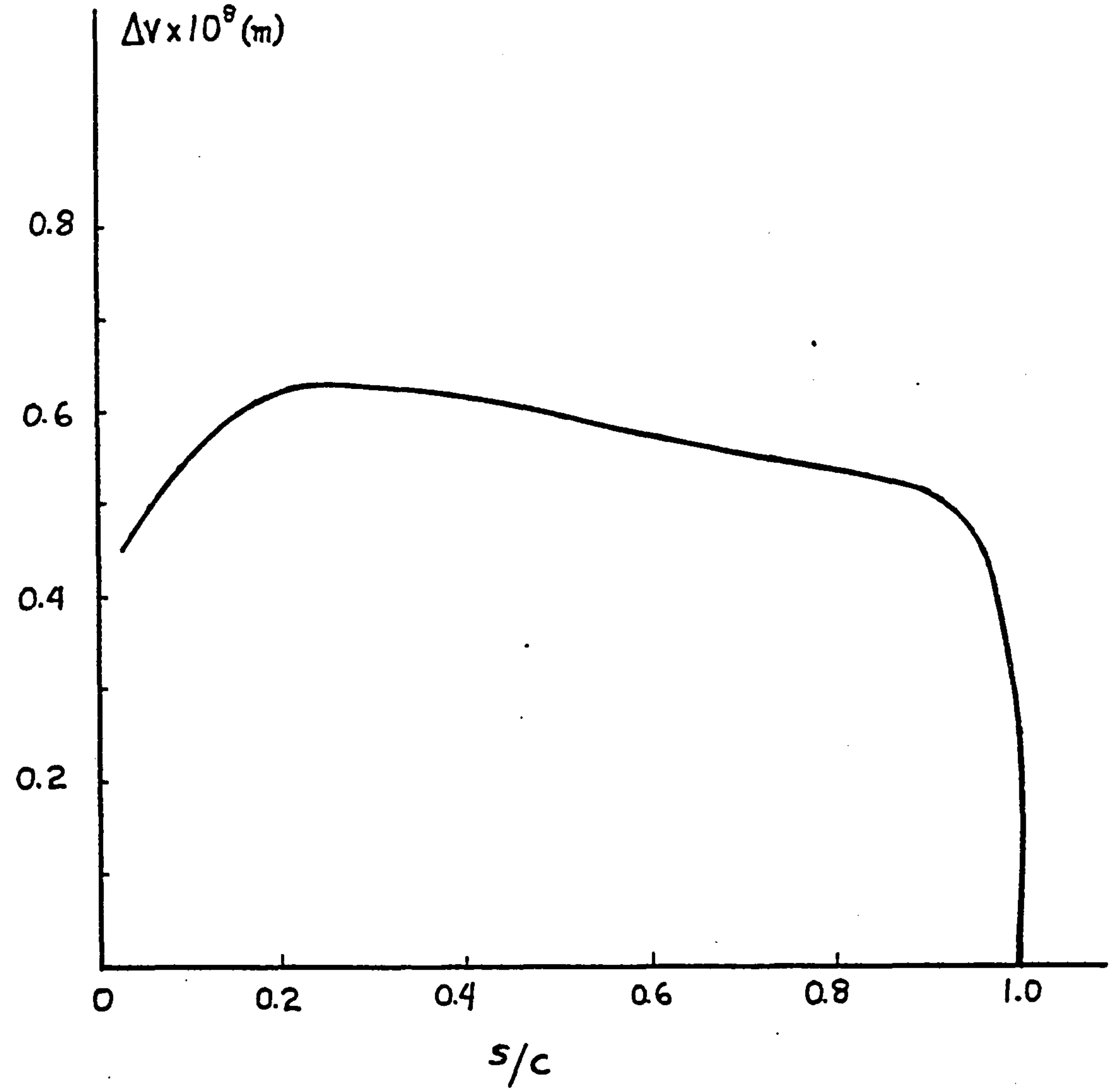

Figure 7. The crack opening displacement Δv for $c/a = 5$, $K = 10^8$ 1/m.

1. Report No. NASA CR-172563	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Fracture Analysis of Notched Composites		5. Report Date February 1985
		6. Performing Organization Code
7. Author(s) F. Delale		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address Drexel University Philadelphia, PA 19104		11. Contract or Grant No. NAG1-480
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		14. Sponsoring Agency Code 534-06-23-03
15. Supplementary Notes Langley Technical Monitor: C. A. Bigelow		

16. Abstract

In this study the problem of part-through cracks emanating from a central notch in a composite laminate is considered. The composite laminate is modeled as an orthotropic plate and along the branches, it is assumed that the stresses are carried by the fibers only. The stresses along the uncracked portions are assumed to be proportional to the crack opening displacement at that point. After formulating the branched crack problem, the model is applied to a $(\pm 45°)_{2s}$ composite laminate. The stress concentration factors for the fibers are computed for various values of the spring constant K. Sample results showing the crack opening displacement are also displayed in the figures.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
integral equations stress-intensity factors stress concentration factors part-through cracks inclined cracks	Unclassified-Unlimited Subject Category 24

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	36	A03