NASA Contractor Report 172538

ICASE REPORT NO. 85-8

NASA-CR-172538
19850012419

ICASE

THE EIGENVALUES OF THE PSEUDOSPECTRAL FOURIER APPROXIMATION TO THE OPERATOR $\sin(2x)\frac{\partial}{\partial x}$

Hillel Tal-Ezer

Contract No. NAS1-17070
February 1985

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

THE EIGENVALUES OF THE PSEUDOSPECTRAL FOURIER APPROXIMATION TO THE OPERATOR $\sin(2x)\frac{\partial}{\partial x}$

Hillel Tal-Ezer
School of Mathematical Sciences, Tel-Aviv University

Abstract

In this note we show that the eigenvalues Z_i of the pseudospectral Fourier approximation to the operator $\sin(2x)\frac{\partial}{\partial x}$ satisfy

$$R_e\, Z_i = \pm 1 \quad \text{or} \quad R_e\, Z_i = 0.$$

Whereas this does not prove stability for the Fourier method, applied to the hyperbolic equation

$$U_t = \sin(2x)U_x \qquad -\pi < x < \pi;$$

it indicates that the growth in time of the numerical solution is essentially the same as that of the solution to the differential equation.

To be submitted for publication in Mathematics of Computation.

Research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at the Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23665.

1. Introduction

Let us consider the problem

$$U_t - GU = 0 \qquad 0 \leq x \leq 2\pi$$
$$U(x,0) = U^o(x) \tag{1.1}$$

where

$$G = a(x)\frac{\partial}{\partial x} . \tag{1.2}$$

In the Fourier pseudospectral (collocation) method, we seek a trigonometric polynomial of degree N , U_N, that satisfies

$$(U_N)_t - G_N U_N = 0$$
$$U_N(x,0) = U_N^o(x) \tag{1.3}$$

where

$$G_N = P_N G ;$$

P_N is the pseudospectral projection operator [5]. It is known [2] that when $a(x)$ does not change sign in the interval, the semidiscrete solution of (1.3) is stable. When $a(x)$ changes sign in the interval, the situation is much more complicated. Gottlieb, Orszag and Turkel [1] have proved stability for the case where $a(x)$ is of the form

$$a(x) = \alpha \sin(x) + \beta \cos(x) + \gamma . \tag{1.4}$$

In [4], Tadmor argues that this stability proof results from the special form of $a(x)$ in (1.4) and cannot be extended. In the next section we prove a theorem related to the problem of stability of (1.1) where $a(x)$ is a second degree trigonometric polynomial.

2. The Theorem and Its Proof

Theorem: Considering (1.1), (1.2), where $a(x) = \sin(2x)$, then the eigenvalues λ_i^N of G_N satisfy

$$R_e\lambda_i^N = -1 \quad \text{or} \quad R_e\lambda_i^N = 0 \quad \text{or} \quad R_e\lambda_i^N = 1. \tag{2.1}$$

Proof:

The projected subspace V_N that results from using the operator P_N is spanned by the following $2N$ basis functions

$$V_N = S_p\{1,\cos(x),\ldots,\cos(Nx),\ \sin(x),\ldots,\sin(N-1)x)\}. \quad (N \text{ even}) \tag{2.2}$$

Define the following four subspaces of V_N

$$\begin{aligned} W_1 &= S_p\{\cos(x),\ \cos(3x),\ldots,\cos((N-1)x)\} \\ W_2 &= S_p\{\sin(x),\ \sin(3x),\ldots,\sin((N-1)x)\} \\ W_3 &= S_p\{\sin(2x,\ \sin(4x),\ldots,\sin((N-2)x)\} \\ W_4 &= S_p\{1,\cos(2x),\ldots,\cos(Nx)\}. \end{aligned} \tag{2.3}$$

It is easily verified that

$$V_N = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \tag{2.4}$$

and each W_i is invariant of G_N; therefore we can discuss separately the four matrices which represent G_N in each one of the subspaces W_i.

Define now

$$B_i^M = [G_N]_{w_i} \qquad 1 \leq i \leq 4 \qquad (M = \frac{N}{2}); \tag{2.5}$$

then by using elementary trigonometric relations we get that B_i^M are tridiagonal matrices whose elements are:

$$B_1^M = \frac{1}{2} \begin{pmatrix} -1 & -3 & & & & \\ 1 & 0 & -5 & & & \\ & 3 & \cdot & \cdot & & \\ & & \cdot & \cdot & -N+3 & \\ & & & \cdot & 0 & -N+1 \\ & & & & N-3 & N-1 \end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}} ;$$

$$B_2^M = \frac{1}{2} \begin{pmatrix} 1 & -3 & & & & \\ 1 & 0 & -5 & & & \\ & 3 & \cdot & \cdot & & \\ & & \cdot & \cdot & -N+3 & \\ & & & \cdot & 0 & -N+1 \\ & & & & N-3 & -N+1 \end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}} ;$$

$$B_3^M = \frac{1}{2} \begin{pmatrix} 0 & -4 & & & & \\ 2 & 0 & -6 & & & \\ & 4 & \cdot & \cdot & & \\ & & \cdot & \cdot & -N+4 & \\ & & & \cdot & 0 & -N+2 \\ & & & & N-4 & 0 \end{pmatrix}_{\left(\frac{N}{2} -1\right) \times \left(\frac{N}{2} -1\right)} ;$$

$$B_4^M = \frac{1}{2} \begin{pmatrix} 0 & -2 & & & & \\ 0 & 0 & -4 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & -N+2 & \\ & & & \cdot & 0 & 0 \\ & & & & N-2 & 0 \end{pmatrix}_{\left(\frac{N}{2} +1\right) \left(\frac{N}{2} +1\right)} ;$$

let A by any tridiagonal matrix:

$$A = \begin{pmatrix} a_1 & c_1 & & & & \\ b_2 & a_2 & c_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & b_{n-1} & a_{n-1} & c_{n-1} \\ & & & & b_n & a_n \end{pmatrix} \tag{2.6}$$

and let A_k by the submatrix

$$A_k = \begin{pmatrix} a_1 & c_1 & & & & \\ b_2 & a_2 & c_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & b_{k-1} & a_{k-1} & c_{k-1} \\ & & & & b_k & a_k \end{pmatrix} . \tag{2.7}$$

Upon defining

$$q_k(A) = \det A_k \tag{2,8}$$

it is easily verified that

$$q_{k+1}(A) = a_{k+1}\, q_k(A) - b_{k+1}\, c_k\, q_{k-1}(A) \tag{2.9}$$

and

$$q_n(A) = \det A.$$

In the following we treat each one of the matrices B_i^M, $i = 1,2,3,4$ separately.

Lemma 1: The matrix B_1^M has one zero eigenvalue, and all its other eigenvalues λ_i satisfy $R_e \lambda_i = 1$.

Proof: For any M define

$$C_M = 2B_1^M - \lambda I .$$

The characteristic polynomial of $2B_1^M$ is given by

$$Q_M(\lambda) = \det C_M \tag{2.10}$$

and using (2.8)

$$Q_M(\lambda) = q_M(C_M).$$

We define now the following family of polynomials (in the variable λ)

$$\begin{aligned} &P_0 = 1 \qquad P_1 = -(\lambda + 1) \\ &P_{k+1} = -\lambda P_k + (4k^2 - 1) P_{k-1} \qquad 1 \le k < \infty . \end{aligned} \tag{2.11}$$

Note that from (2.9) and the structure of C_M

$$P_k = q_k(C_M) \qquad 2 \le k < M ; \tag{2.12}$$

however (2.12) is not true for $k = M$; rather we have

$$Q_M(\lambda) = (2M - 1 - \lambda)\ P_{M-1} + (4(M-1)^2 - 1)\ P_{M-2} \qquad 2 < M\ . \qquad (2.13)$$

From (2.11) we get

$$Q_M(\lambda) = (2M - 1)\ P_{M-1} + P_M \qquad 2 < M. \qquad (2.14)$$

Using (2.14) and (2.13) results in

$$Q_{M+1}(\lambda) = -\lambda\ P_M + (2M+1)\ Q_M \qquad 2 < M. \qquad (2.15)$$

Finally we solve (2.15) for P_M in terms of $Q_M(\lambda)$, $Q_{M+1}(\lambda)$ and substitute the result in (2.14). We thus get the polynomials $Q_M(\lambda)$, $M \geq 2$ that satisfy the following recursion formula

$$\begin{aligned} &Q_2(\lambda) = \lambda(\lambda-2)\ ;\ Q_3(\lambda) = -\lambda(\lambda^2-4\lambda + 13) \\ &Q_{M+1}(\lambda) = (2-\lambda)\ Q_M(\lambda) + (2M-1)^2\ Q_{M-1}(\lambda) \qquad 3 < M\ . \end{aligned} \qquad (2.16)$$

It is easy to verify now that $\lambda = 0$ is an eigenvalue of $2B_1^M$. In fact $\lambda = 0$ is a root of $Q_2(\lambda)$ and $Q_3(\lambda)$ and therefore of any $Q_M(\lambda)$. We define now

$$x = i(2 - \lambda) \qquad (a)\ (2.17)$$

and

$$R_M(x) = \frac{1}{\lambda}\ Q_M(\lambda) \cdot (i)^{M-1} \qquad (b)$$

to get

$$R_2 = -x\ ;\ R_3 = x^2 - 9$$

and

$$R_{M+1} = x\ R_M - (2M-1)^2\ R_{M-1} \qquad M \geq 3. \qquad (2.18)$$

The relation (2.18) defines $R_M(x)$ as a family of orthogonal polynomials on the real axis. Therefore, for every M the roots of $R_M(x)$ are real, which implies by (2.17)(a) that $2 - \lambda$ are imaginary. Therefore, the eigenvalues of the matrices $2B_1^M$ for any M have real part equal to 2. This completes the proof of Lemma 1.

<u>Lemma 2</u>: For any M the matrix B_2^M has one zero eigenvalue and the real part of the others is -1.

<u>Proof</u>: The proof is an immediate result of the fact that in view of (2.9)

$$q_k(-B_2^M - \lambda I)$$

satisfy the same recurrence formula as $q_k(B_1^M - \lambda I)$.

<u>Lemma 3</u>: The eigenvalues of B_3^M are purely imaginary.

<u>Proof</u>: Define the matrix

$$D = \begin{pmatrix} 1/\sqrt{2} & & & & \\ & 1/\sqrt{4} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1/\sqrt{N-2} \end{pmatrix} .$$

Then it is clear that

$$D^{-1} \; B_3^M \; D$$

is a skew symmetric matrix, and therefore its eigenvalues are purely imaginary. The same is of course true for B_3^M.

<u>Lemma 4</u>: The eigenvalues of B_4^M are purely imaginary.

Proof: From the definition of B_3^M and B_4^M it follows that if P_k is characteristic polynomial of $(B_3^M)_{k\times k}$ then $\lambda^2 P_k$ is the characteristic polynomial of $(B_4^M)_{(k+2)\times(k+2)}$. Thus the eigenvalues of B_4^M are purely imaginary.

The proof of Lemma 4 concludes the proof of the theorem.

References

[1] D. Gottlieb, S. Orszag, E. Turkel, Stability of pseudospectral and finite difference methods for variable coefficient problems, Math. Comp., Vol. 37, No. 156, (1981), pp. 293-305.

[2] D. Gottlieb, S. Orszag, Numerical Analysis of Spectral Methods; Theory and Applications, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM Publisher, Philadelphia, PA 1977.

[3] E. Issacson, H.B. Keller, Analysis of Numerical Methods, John Wiley and Sons, Inc., New York. (1966).

[4] E. Tadmor, Finite-difference, spectral and Galerkin methods for time-dependent problems, ICASE Report No. 83-22, NASA CR-172149, 1983.

[5] R.G. Voigt, D. Gottlieb, M.Y. Hussaini, Spectral Methods for Partial Differential Equations, Proceedings of a Symposium. August 16-18, 1983, SIAM, Philadelphia, PA, 1984.

1. Report No. NASA CR-172538 ICASE Report No. 85-8	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle The Eigenvalues of the Pseudospectral Fourier Approximation to the Operator $\sin(2x) \frac{\partial}{\partial x}$		5. Report Date February 1985
		6. Performing Organization Code
7. Author(s) Hillel Tal-Ezer		8. Performing Organization Report No. 85-8
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		10. Work Unit No.
		11. Contract or Grant No. NAS1-17070
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-31-83-01

15. Supplementary Notes

Langley Technical Monitor: J. C. South, Jr.
Final Report

16. Abstract

In this note we show that the eigenvalues Z_i of the pseudospectral Fourier approximation to the operator $\sin(2x) \frac{\partial}{\partial x}$ satisfy

$$R_e Z_i = \pm 1 \quad \text{or} \quad R_e Z_i = 0.$$

Whereas this does not prove stability for the Fourier method, applied to the hyperbolic equation

$$U_t = \sin(2x) U_x \qquad -\pi < x < \pi;$$

it indicates that the growth in time of the numerical solution is essentially the same as that of the solution to the differential equation.

17. Key Words (Suggested by Author(s)) pseudospectral stability Fourier variable coefficients	18. Distribution Statement 64 - Numerical Analysis Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	11	A02

For sale by the National Technical Information Service, Springfield, Virginia 22161

NASA-Langley, 1985