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16. Abstract The article is a survey of the purpose and methods of the unsteady lifting surface theory. Linearized Euler's equations are simplified by means of a Galileo-Lorentz transformation and a Laplace transformation so that the time and the compressibility of the fluid are limited to two constants. The solutions to this simplified problem are represented as integrals with a differential nucleus. These result in tolerance conditions, for which any exact solution must suffice. It is shown that none of the existent three-dimensional lifting surface theories in subsonic range satisfy these conditions.

As oscillating elliptic lifting surface which satisfies the tolerance conditions is calculated through the use of Lamé's functions. Numerical examples are calculated for the borderlinecases of infinitely stretched elliptic lifting surfaces and of circular lifting surfaces. These are compared with the theories accepted to date.

Out of the harmonic solutions any such temporal changes of the down current are calculated through the use of an inverse Laplace transformation.

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THREE-DIMENSIONAL UNSTEADY LIFTING SURFACE THEORY IN THE SUBSONIC RANGE

H. G. Küssner**

1. Introduction

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The aeroelastic problems of the modern airplane and aircraft are so numerous and so complex that a purely experimental treatment is bound to fail due to the multiplicity of the necessary parameter variations. The closer one comes to the boundaries of technology, the more necessary it becomes to apply mathematical and theoretical methods to solving a problem, even when making simplified assumptions. For this reason unsteady lifting surface theories (in spite of their recognized deficiencies) are used to a large extent in aeroelastic investigations of new airplane designs. In the leading airplane works they avoid purely theoretical or experimental directional oscillation flutter proofs, because every known process still has great deficiencies and because the only sufficient protection against directional oscillation flutter to date has been comparative investigations.

Since this pressing need exists, much work has been put into the development of unsteady lifting surface theory in the last 30 years. The two-dimensional problem has been solved completely for all Mach numbers. Many numerical tables have been calculated for the use of these solutions. In the case of the three- /41.3
dimensional problem, that is, in the case of lifting surfaces of finite wingspan, only minimum progress has been made to date, because the Kutta-condition in the subsonic range is a difficult boundary condition. First, I should like to present you with a

*Numbers in the margin indicate pagination in the original.

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survey of the general three-dimensional problem and solution assessment of same. Then I should like to report on my recent investigations in the subsonic range.

2. Basic Assumptions of the Lifting Surface Theory

The question how much stress is placed upon a winged aircraft with a sharp trailing edge when in flight could not be answered in the 19th century. Classical hydrodynamics supplied the paradoxical answer that, under ideal fluidity and stationary flow, stress is zero. Only at accelerated motion do stresses varying from zero -- the so-called Kelvin impulse -- occur.

On the basis of Prandtl's boundary layer theory, Kutta first conceived the idea of considering the viscosity of actual fluidity phenomenologically. He did this by assuming the smooth flow of the trailing edge, while otherwise figuring with ideal fluidity. Kutta's two-dimensional profile theory offered the first solutions which were also physically acceptable.

For three-dimensional flow, that is, for wings of finite wingspan, the Kutta method of conformal representation fails. For this reason also, simplified assumptions had to be made. It was necessary to limit oneself to infinitely small /41.4
disturbances, by which the problem is linearized. Accordingly, the thickness of the wing must be infinitely small, so that one can even talk of lifting surfaces.

At the trailing edge of this lifting surface the flow should stream away smoothly. The sharp front edge of the lifting surface, however, causes the flow to go around it, which leads to infinitely large disturbance speeds and pressures. As long as this specific pressure is integrable, we can nonetheless calculate total reactions upon the lifting surface, that is, upon lift, linear and higher torques.

For this reason, a lifting surface theory can be a theory only approaching the calculation of aerodynamic coefficients. It is desired, nonetheless, that under the given circumstances the mathematical boundary value problem of the lifting surface theory be solved exactly and that no further approximate assumptions be made. For the danger exists that accumulating approximate assumptions will result in loss of touch with physical reality.

First, we consider the boundary conditions of lifting surface theory. We shall assume frictionless compressible fluidity. The positive x-axis points in the direction of airspeed v, the y-axis in the direction of the wingspan (See Figure 1).

Further, t represents time, s = vt of the pattern of the middle of the wing, l is the largest half of the chord depth, c the sonic speed, $\beta = v/c$ the Mach number, ρ the air density, p the disturbance pressure and ϕ the velocity potential resulting from movement of the lifting surface. We shall make all valves dimensionless by setting

$$l = v = \rho = 1 \tag{1}$$

/41.5

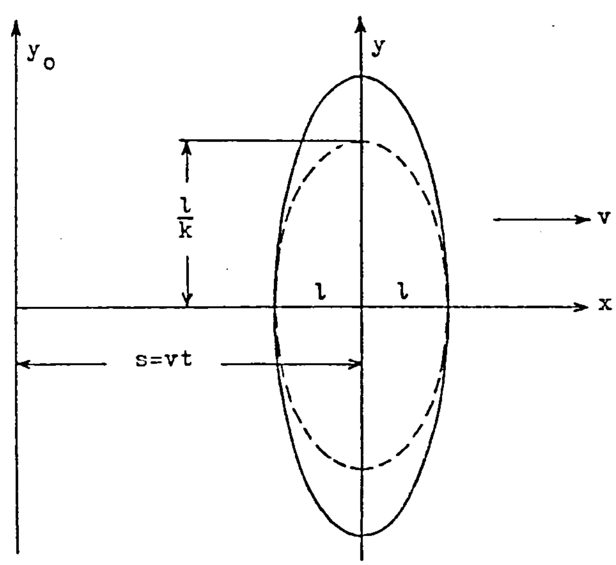

Figure 1. Moved lifting surface in the plane z = 0.

Further we assume that the lifting surface remains even up to infinitely small deviations and lies in the plane z = 0 (See Illustration 1.)

Because we have linearized our problem, the Euler equations become

$$\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial x}\right) \Phi(s,x,y,z) + p(s,x,y,z) = 0 \quad , \tag{2}$$

$$\left[\nabla^2 - \beta^2 \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial x}\right)^2\right] \Phi(s,x,y,z) = 0 \quad . \tag{3}$$

We take the vertical component of velocity at the lifting surface, the so-called down current, as given. With the exception of the leading edge, it is equal to the gradient of the velocity potential perpendicular to the lifting surface

$$\overline{w}(s,x,y) = \Phi_z(s,x,y,0) \tag{4}$$

Down current is determined either by the small /41.6 deformations of the lifting surface or given by an atmospheric squall area. According to the Kutta condition, disturbance pressure of the lifting surface at the trailing edge should always be p = 0. With that, our boundary value example is completely determined.

3. Transformation of the Euler Equations.

We want to simplify the solution of this task by first carrying out two transformations. First we shall perform a Galilei-Lorentz transformation of coordinates and time:

$$x' = x \; ; \quad y' = y\sqrt{1-\beta^2} \; ; \quad s' = s(1-\beta^2) - \beta^2 x \quad , \tag{5}$$
$$z' = z\sqrt{1-\beta^2} \; ; \quad t' = t(1-\beta^2) - \beta^2 x \quad .$$

Using $\beta = 0.7$ as an example, the given lifting surface countour reverts to the dotted line in Figure 1, which we shall continue to consider.

Further we perform a Laplace transformation; this changes the object function $F(s')$ which disappears with $s' < 0$ into the diagram function

$$\mathcal{L}\{F(s')|\,\omega^*\} = \int_0^\infty F(s') \exp(-\omega^* s')\, ds' \quad . \tag{6}$$

In this equation ω^* is a complex number.

By means of these two transformations (5) and (6), the Euler equations (2) and (3) become

$$(\omega^* - \frac{\partial}{\partial x})\; \phi^*(x,y,z) + p^*(x,y,z) = 0 \quad , \tag{7}$$
$$(\kappa^2 - \nabla^2)\; \phi^*(x,y,z) = 0 \quad . \tag{8}$$

The transformation lines have been left out here. The reduced frequencies are /41.7

$$\omega^* = \frac{\omega\, l}{v(1-\beta^2)} \quad ; \qquad \kappa = \frac{\omega\, l}{c(1-\beta^2)} = \beta\, \omega^* \quad . \tag{9}$$

In the case of harmonic vibrations, is the purely imaginary circuit frequency of the object function.

4. Integral representations of the solutions

By means of the Laplace transformation and the Lorentz transformation we have reduced the time and the compressibility of the fluidity down to the two constants $\omega *$ and $\varkappa$. The solution of our boundary value exercise can be represented generally by the following integrals over the lifting surface:

$$\Phi^*(x,y,z) = \iint w^*(x'y') \, K(x-x',y-y',z) \, dx'dy' \quad , \tag{10}$$

$$p^*(x,y,z) = \iint w^*(x',y') \, K_1(x-x',y-y',z) \, dx'dy' \quad . \tag{11}$$

Because we have linearized our problem, the nuclei K and K_1 are the differential nuclei of these integrals. For this reason we can transform equation (11) into:

$$p^*(-x',-y',z) = \iint w^*(-x,-y) \, K_1(x-x',y-y',z) \, dx \, dy \quad . \tag{12}$$

From these equations one can easily derive the theory of the opposite flow formulated by A. H. Flax, as I have shown in another of my works. This theory runs in our case at the lifting surface
$z = 0$

$$\iint [w^*(x,y) \, \overline{p^*(x,y,+0)} - \overline{w^*(x,y)} \, p^*(x,y,+0)] \, dx \, dy = 0 \quad . \tag{13}$$

Here the horizontal line means opposite flow. Now we can make the following general statements for the down current and the pressure in direct and opposite flow: /41.8

$$
\begin{aligned}
w^*(x,y) &= g(x,y) + f(x,y) \; ; \quad p^*(x,y) = p_g(x,y) - p_f(x,y) \, , \\
\overline{w^*(x,y)} &= g(x,y) - f(x,y) \; ; \quad \overline{p^*(x,y)} = p_g(x,y) + p_f(x,y) \, .
\end{aligned}
\tag{14}
$$

Here g means even and f uneven down current dispersion, p_g and p_f the corresponding pressure dispersion. If we substitute equation (14) for (13), then we attain the tolerance condition

$$\iint [f(x,y) \; p_g(x,y) + g(x,y) \; p_f(x,y)] \, dx \, dy = 0 \; . \tag{15}$$

If, for example, $f(x,y) = x$, $g(x,y) = 1$ is set, then we obtain the tolerance condition of the lift and torque coefficients from equation (15)

$$m_a + k_b = 0, \tag{16}$$

to which I shall return later. All exact lifting surface theories must satisfy these tolerance conditions.

The nuclei K and K_1 of the integral equations (10) and (11) are, in spite of their different characteristic, quite complicated, as we shall see. Easily calculable, on the other hand, is the nucleus M_1 of the inverse integral equation for the down current

$$w^*(x,y,z) = \frac{1}{2\pi} \iint p^*(x'.y',+0)\ M_1(x-x',y-y',z)\ dx'dy' \quad , \tag{17}$$

$$M_1(x,y,0) = \int_{-x}^{\infty} \exp[\omega^*(\alpha+x-\beta\sqrt{\alpha^2+y^2})]\ \frac{1+\beta\omega^*\sqrt{\alpha^2+y^2}}{(\alpha^2+y^2)\sqrt{\alpha^2+y^2}}\ d\alpha \ . \tag{18}$$

I derived the first approximation theory for oscillating lifting surfaces of finite wingspan from this integral equation in 1940 [1]. According to this theory, one calculates the pressure distribution first on the assumption of even flow in all cross sections in accordance with the two-dimensional /41.9 theory and then substitutes a new constant T ($\omega*$,k,y) for the integral constant T ($\omega*$). This new constant is calculated by solving a linear integral equation and contains the influence of the configuration of the lifting surface. The larger the span, the smaller the difference of both functions. Later, E. Reissner derived this same theory by other means and it appears to be the sole three-dimensional unsteady lifting surface theory to date which has been employed to a wide extent in calculating directional oscillation flutter.

In the steady boundary case $\omega* = 0$ this theory extends into Prandtl's theory of the supporting line. None of these theories, however, satisfy the tolerance conditions (15) and (16), not even in the case of infinite span. The basic assumption of even flow is, therefore, inapplicable. Lift will be approximately 18% too high, and there also exist experimental indications for a mistake of this magnitude. The essential difference between a two-dimensional and a three-dimensional solution consists of the way a disturbance fades in infinity. From this vantage point it is surprising that the assumption of even flow in the case of three-dimensional problems does not create even greater mistakes.

The simple integral equation (17), unfortunately, does not help us very much. because it isn't the pressure at the lifting surface which is given, but rather at the down current. If we wish to calculate the pressure from this integral equation, we must invert it. This, however, is a difficult mathematical procedure. Generally such exercises can be solved only numerically and approximately. One could put up the polynomials x and y with indefinite coefficients for the desired /41.10 pressure at the lifting surface, introduce this into equation (17), integrate and determine the coefficients such that the given down current is represented accurately.

In the subsonic range, the desired counter pressure dispersion p* is uniform at the leading edge of the lifting surface, as I have mentioned already. Because only regular functions can be integrated numerically, we would have to know the manner of the singularity beforehand, separate from the start and integrate analytically. To date the same manner of singularity as in the case of even flow has been assumed without proof.

In order to test the validity and the convergence of such numerical collocation methods, it is desirable to possess at least an exact solution of the three-dimensional unsteady lifting surface theory. Such a possibility exists, first off, only for the elliptic lifting surface, because this is the sole type of even lifting surface, for which separable solutions of the wave equation or of the Laplace equation exist in orthogonal curved coordinates. For this reason I want to present to you a theory of the oscillating elliptic lifting surface in the following.

5. Theory of the oscillating elliptic lifting surface.

5.1. Orthogonal coordinates and the Lamé functions

We shall consider an elliptic lifting surface with the designations used in Figure 1. The Cartesian coordinates x,y,z can be expressed by the ellipsoidal coordinates /41.11 u,v,w as follows:

$$\begin{aligned} x &= \cosh u \cos v \cos w \quad &; \quad & 0 \leq u \leq \infty \quad , \\ y &= k^{-1}\sqrt{1+k^2\sinh^2 u}\sqrt{1-k^2\sin^2 v}\,\sin w \quad &; \quad & -\pi \leq v \leq \pi \quad , \\ z &= \sinh u \sin v\sqrt{1-k'^2\sin^2 w} \quad &; \quad & -\frac{\pi}{2} \leq w \leq \frac{\pi}{2} \quad . \end{aligned} \tag{19}$$

Where $k' = \sqrt{1-k^2}$. The square roots are always real and positive. In the boundary cases k = 0 and k' = 0 these coordinates extend into those of the elliptic cylinder and of the spheroid. The surface u = 0 represents the elliptic lifting surface, upper side and under side. Its leading edge is given by u = v = 0, its trailing edge by u = 0, v = π The Lamé auxiliary quantities of these coordinates are

$$\begin{aligned} \mho^2 &= \frac{1+k^2\sinh^2 u}{(\sinh^2 u+\sin^2 v)(1+k^2\sinh^2 u-k'^2\sin^2 w)} \quad , \\ \gamma^2 &= \frac{1-k^2\sin^2 v}{(\sinh^2 u+\sin^2 v)(1-k^2\sin^2 v-k'^2\sin^2 w)} \quad , \\ \varpi^2 &= \frac{k^2(1-k'^2\sin^2 w)}{(1+k^2\sinh^2 u-k'^2\sin^2 w)(1-k^2\sin^2 v-k'^2\sin^2 w)} \quad . \end{aligned} \tag{20}$$

The surface element of the upper surface u = const. is

$$d\sigma = \frac{dv\ dw}{V\ W} \quad . \tag{21}$$

Further we need the derivations according to z in the plane z = 0

$$\begin{aligned} \frac{\partial}{\partial z} &= U(0,v,w)\frac{\partial}{\partial u} \quad \text{on the lifting surface} \quad (u = 0) \;, \\ \frac{\partial}{\partial z} &= V(u,0,w)\frac{\partial}{\partial v} \quad \text{before the lifting surface} \quad (v = 0) \;. \end{aligned} \tag{22}$$

We shall limit ourselves in the following to incompressible fluids ($\varkappa$ = 0). For this reason we are seeking solutions of the Laplace equation $\Delta^2 \phi = 0$, which are regular and periodic upon surfaces u = const. and disappear in infinity. These solutions consist of products of the Lamé functions of the first and second order. The Lamé functions of the first order are polynominal in sin v, cos v and $\sqrt{1-k^2\sin^2 v}$. For that reason they can /41.12 be represented by associated Legendre spherical functions of the first order. For example, the Lamé polynomial is

$$\begin{aligned} Ec_3^1(v) &= \sin v\,[1 - c_3^1(k)\sin^2 v] \;, \\ &= \frac{1}{6}P_3^1(\cos v) + \frac{1}{60}[5 - 4c_3^1(k)]\,P_3^3(\cos v) \;, \\ \dot{E}c_3^1(w) &= \sqrt{1-k'^2\sin^2 w}\;[1 - c_3^2(k')\sin^2 w] \end{aligned} \tag{23}$$

The subscripts correspond to the nomenclature of Ince and Erdelyi; Ec_n^m are the functions which are even in cos v, Es_n^m the uneven. The coefficients c_n^m are algebraic functions of k. The Lamé functions of the second order are obtained from those of the first by insertion of the associated Legendre spherical functions of the second order (compare [2]):

$$Fc_3^1(u) = C_3^1(k)\left\{\frac{1}{6}\, Q_3^1(\cosh u) + \frac{1}{60}\,[5 - 4c_3^1(k)]\; Q_3^3(\cosh u)\right\}, \tag{24}$$

whenever $u > 0$ and $k > 0$

There is still a second, more complicated representation of the Lamé functions of the second order using elliptic functions and integrals, which are also valid for $u = 0$ and $k = 0$. By using this representation one generally obtains the coefficients

$$C_n^m(k) = [\, a_n^m(k)\; E'(k) + b_n^m(k)\; K'(k)\,]^{-1} \quad , \tag{25}$$

whereby a_n^m , b_n^m are algebraic functions and E', K' are complete elliptic normal integrations.

I have normalized all Lamé functions, such that at the position zero either the function itself or its first derivation equals one. I designate the product from normalized Lamé polynomials of the first order as ellipsoidal harmonics:

$$Sc_3^1(v,w) = Ec_3^1(v)\; \dot{E}c_3^1(w) \tag{26}$$

A table of the first 24 Lamé polynomials is to be found in the appendix of my printed report. /41.13

In order to calculate with these functions, we must be able to diffentiate and integrate. On the lifting surface $u = 0$ the derivations of the ellipsoidal harmonics are

$$\begin{aligned} \frac{\partial}{\partial x}\, Sc_{2n}^{2m}(v,w) &= U(0,v,w) \sum_{r=0}^{n} a_r^m\, Ss_{2n}^{2r} \; ; \; \sum a_r^m = 1 \quad , \\ \frac{\partial}{\partial y}\, Sc_{2n}^{2m}(v,w) &= U(0,v,w) \sum_{r=1}^{n} b_r^m\, Sc_{2n}^{2r-1} \; ; \sum b_r^m = 1 \quad . \end{aligned} \tag{27}$$

Corresponding formulas are valid for the remaining ellipsoidal harmonics. For their integration the orthogonality relation is valid

$$\frac{k}{4\pi\sqrt{1+k^2\sinh^2 u}} \; \gamma c_n^m(k) \iint Sc_n^m(v,w)\, Sc_s^r(v,w)\, \frac{U}{VW}\, dv\, dw = \begin{cases} 0, & \text{if } m \neq r \text{ or } n \neq s, \\ 1, & \text{if } m = r \text{ and } n = s. \end{cases} \tag{28}$$

The integrals are spread across the entire surface of the ellipsoid u = const. The coefficients γc_n^m are algebraic functions of k.

5.2. Oscillating elliptic plates without approaching flow

Now we shall consider the problem of the oscillating elliptic plate without approaching flow. By means of the Lamé functions defined above we can make the following general statement for the regular velocity potential φ :

$$\varphi(u,v,w) = \lambda \iint \varphi_{u'}(0,v',w')\, U(0,v',w')\, d\sigma' \cdot \left[G(u,v,w;v',w') + H(u,v,w;v',w') \right] . \tag{29}$$

Where $\lambda = k/4\pi$. For down current dispersions of the plate, which are even in y' or sin w', we have the characteristic Green function /41.14

$$G(u,v,w;v',w') = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma c_{2n+1}^{2m+1}\, Fc_{2n+1}^{2m+1}(u)\, Sc_{2n+1}^{2m+1}(v,w)\, Sc_{2n+1}^{2m+1}(v',w') + \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma s_{2n+2}^{2m+2}\, Fs_{2n+2}^{2m+2}(u)\, Ss_{2n+2}^{2m+2}(v,w)\, Ss_{2n+2}^{2m+2}(v',w'), \tag{30}$$

and for down current dispersions which are uneven in y' or sin w'

$$H(u,v,w;v',w') = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma c_{2n+2}^{2m+1}\, Fc_{2n+2}^{2m+1}(u)\, Sc_{2n+2}^{2m+1}(v,w)\, Sc_{2n+2}^{2m+1}(v',w') + \sum_{n=0}^{\infty}\sum_{m=0}^{n} \gamma s_{2n+3}^{2m+2}\, Fs_{2n+3}^{2m+2}(u)\, Ss_{2n+3}^{2m+2}(v,w)\, Ss_{2n+3}^{2m+2}(v',w'). \tag{31}$$

At the plate u = 0 the given down current w* must be reproduced:

$$w^*(v,w) = \left.\frac{\partial\varphi}{\delta z}\right|_{z=0} = \varphi_u(0,v,w)\; U(0,v,w) \quad . \tag{32}$$

Now the down current w* or the function φ_u upon the lifting surface is any one regular function from v and w and is, as such, representable in ellipsoidal harmonics using a progression. Because I normalized the functions Fc, Fs such that their derivations are

$$Fc'(0) = Fs'(0) = 1 \tag{33}$$

and because the orthogonality relation equation (28) is valid, our statement (29) will be identically satisfied by the Green functions G and H in accord with equations (30) and (31). The first components of this statement for k' = 0 are to be found in other representations already in the textbook by Lamb [3]. Thus it is possible to call the statement (29) a classical statement.

The derivations of the characteristic functions G and H according to z at the plate u = 0 are the so-called Dirac δ-functions:

$$G_u(0,v,w;v',w')\ U(0,v',w') = \begin{cases} 0, & \text{if } v = v' \text{ or } w = w', \\ \infty, & \text{if } v = v' \text{ and } w = w', \end{cases} \tag{34}$$

$$H_u(0,v,w;v',w')\ U(0,v',w') = \begin{cases} 0, & \text{if } v = v' \text{ or } w = w', \\ \infty, & \text{if } v = v' \text{ or } w = w'. \end{cases} \tag{35}$$

5.3. Determination of the singular potential

In order to attain a theory of the inflight lifting surfaces with Kutta conditions out of the classical statement (29), we must add a singular potential ψ . Its derivation $\partial\psi/\partial Z$ has to disappear upon the lifting surface u = 0, because we have already reproduced the down current w using the regular potential . This requirement cannot, however, be completely satisfied, since periodic functions, in particular Fourier-series, do not permit representation of zero in the entire range of a period, in which all of the series coefficients do not disappear. If the singular potential is not to disappear identically, then its derivation must become infinite at least in one point of the lifting surface.

We already know functions with this feature, which besides that satisfy the Laplace differential equation; their derivations are the k-functions seen in equations (34) and (35). If we wish to satisfy the Kutta condition of smooth air flow at the trailing edge, then we may lay the singular point at any one point v' = 0, $w' = w_0$ of the leading edge. Then we obtain the singular velocity potential

/41.16

$$\psi(u,v,w) = \lambda \iint \varphi_{u'}(0,v',w')\, U(0,v',w')\, d\sigma' \cdot$$
$$g(v',w') \lim_{v_o=0} G(u,v,w;v_o,w_o)\, U(0,v_o,w_o)$$
$$+ h(v',w') \lim_{v_o=0} H(u,v,w;v_o,w_o)\, U(0,v_o,w_o) \tag{36}$$

Here g and h are functions which are still to be determined. In order to obtain flow at all points of the leading edge, that is, singular positions of the pressure, we can multiply equation (36) by any weight function

$$f(w_o) = \sum_0^\infty a_n \cos n\, w_o + \sum_1^\infty b_n \sin n\, w_o \tag{37}$$

and integrate from $w_o = -c/2$ to $+c/2$. Now we have

$$Sc_1^1(v,w) \quad U(0,v,w) = 1 \tag{38}$$

Then the integrals of the individual series components become

$$\lim_{v=0} \int_{-\pi/2}^{\pi/2} \frac{Sc_{2n+1}^{2m+1}(v,w)}{Sc_1^1(v,w)}\, f(w)\, dw = A_{2n+1}^{2m+1} = Sc_{2n+1,v}^{2m+1}(0,w_o^i) \quad , \tag{39}$$

$$\lim_{v=0} \int_{-\pi/2}^{\pi/2} \frac{Sc_{2n+2}^{2m+1}(v,w)}{Sc_1^1(v,w)}\, f(w)\, dw = B_{2n+2}^{2m+1} = Sc_{2n+2,v,w}^{2m+1}(0,w_o^i) \quad . \tag{40}$$

Because the functions g and h are still undetermined, we can fix the first coefficient of the series (37) arbitrarily:

$$a_o = 1/c \;;\; b_1 = 2/c \tag{41}$$

Then the coefficients become $A_1^1 = B_2^1 = 1$. The expression to the right in the equations (39) and (40) is only a symbolic notation for the coefficients A_n^m, B_n^m. Then in accordance with equations (29), (36), (39) and (40) the entire velocity potential of the lifting surface becomes

/41.17

$$\phi^*(u,v,w;v';w') = \varphi + \psi = \lambda \iint w^*(v';w')\, K(u,v,w;v';w')\, do' ,$$
$$K(u,v,w;v';w') = G(u,v,w;v';w') + g(v';w')\, G_{v'}(u,v,w;0,w_o^1) \qquad (42)$$
$$+ H(u,v,w;v';w') + h(v';w')\, H_{v'w'}(u,v,w;0,w_o^1)$$

The constants w_o^I contained in this statement can only be calculated later. First we must determine the functions g(v',w') and h(v',w'). To this end we differentiate the nucleus K according to z on the plane z = 0 in front of the lifting surface, that is, for v = 0, and obtain

$$\frac{\partial K}{\partial z} = V(u,0,w)\,\frac{\partial K}{\partial v} = \frac{1}{\sinh u}\lim_{v=0} K(u,v,w;v';w')\, U(0,v,w) . \qquad (43)$$

In order to reach the necessary symmetry of the nucleus in both the coordinates which are factored out and those which are not, we multiply equation (43) and integrate from w = $-\pi/2$ to $+\pi/2$. By solving linear differential equation (7) on the positive x-axis in front of the lifting surface, we obtain the additional condition

$$\int_{x_o}^{\infty} K_z \exp(-\omega^* x)dx = 0 \qquad (44)$$

From the equations (42) to (44) there finally follows the nucleus of the velocity potential

$$K(u,v,w;v',w') = \int_0^\infty \exp(-\omega^* \cosh u')\, du' \left\{ \lambda_1 \begin{vmatrix} G(u,v,w;v'w') & , G_v(u';0,w_o^1;v';w') \\ G_{v'}(u,v,w;0,w_o^1) & , G_{vv'}(u';0,w_o^1;0,w_o^1) \end{vmatrix} + \lambda_2 \begin{vmatrix} H(u,v,w;v';w') & , H_{vw}(u';0,w_o^1;v';w') \\ H_{v'w'}(u,v,w;0,w_o^1) & , H_{vwv'w'}(u';0,w_o^1;0,w_o^1) \end{vmatrix} \right\} \quad (45)$$

If K is to be a differential nucleus, then the following tolerance conditions with any one constant C_1 and C_2 must be valid:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right) G(u,v,w;v';w') = \begin{vmatrix} G_v(0,0,w_o^1;v';w') & , G_{v'}(u,v,w;\pi,w_o^1) \\ G_v(0,\pi,w_o^1;v';w') & , G_{v'}(u,v,w;0,w_o^1) \end{vmatrix}$$
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right) H(u,v,w;v';w') = \begin{vmatrix} H_{vw}(0,0,w_o^1;v';w'), & H_{v'w'}(u,v,w;\pi,w_o^1) \\ H_{vw}(0,\pi,w_o^1;v';w'), & H_{v'w'}(u,v,w;0,w_o^1) \end{vmatrix} \quad (46)$$

Every differential nucleus satisfies the differential equation /41.18

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right) K(x-x',y-y',z-z') = 0 \quad . \quad (47)$$

From equations (7) and (47) we obtain the nucleus of the integral representation of the pressure

$$K_1 = (-\omega^* + \frac{\partial}{\partial x})K = -(\omega^* + \frac{\partial}{\partial x'})K \tag{48}$$

We insert equation (45) into (48), integrate partially according to x and employ the tolerance condition (46). Then we finally obtain the integral representation of the pressure

$$\begin{aligned} p^*(u,v,w) &= -\lambda \iint w^*(v';w')\, K_1(u,v,w;v';w')\, d\sigma' \quad , \\ K_1(u,v,w;v';w') &= (\omega^* + \frac{\partial}{\partial x'})\, G(u,v,w;v';w') + G_{v'}(u,v,w;0,w_o^i) \cdot \qquad (49) \\ &\quad \cdot \frac{1}{C_1}[G_v(0,\pi,w_o^i;v';w')T_1 - G_v(0,0,w_o^i;v';w')] \\ &\quad + (\omega^* + \frac{\partial}{\partial x'})\, H(u,v,w;v';w') + H_{v'w'}(u,v,w;0,w_o^i) \cdot \qquad (50) \\ &\quad \cdot \frac{1}{C_2}[H_{vw}(0,\pi,w_o^i;v';w')T_2 - H_{vw}(0,0,w_o^i;v'w')] \quad . \end{aligned}$$

The constants T_1 and T_2 are given by

$$\begin{aligned} T_1 &= \frac{\int_0^\infty G_{vv'}(u,0,w_o^i;\pi,w_o^i)\exp(-\omega^* \cosh u)\, du}{\int_0^\infty G_{vv'}(u,0,w_o^i;0,w_o^i)\exp(-\omega^* \cosh u)du} \quad , \\ T_2 &= \frac{\int_0^\infty H_{vwv'w'}(u,0,w_o^i;\pi,w_o^i)\exp(-\omega^* \cosh u)\, du}{\int_0^\infty H_{vwv'w'}(u,0,w_o^i;0,w_o^i)\exp(-\omega^* \cosh u)\, du} \quad . \end{aligned} \tag{51}$$

Now we must still determine the missing constants. We begin

Page 22 is missing from original

According to this, the constants A^{2m+1}_{2n+1} and B^{2m+1}_{2n+2} are independent of C_1 and C_2, while the constants A^{2m+2}_{2n+2} are proportional to C_1 and the constants B^{2m+2}_{2n+3} are proportional to C_2. If we insert these values into equation (46), then the constants C_1 and C_2 cancel out on both sides; for this reason they are not determinable by means of our statement.

A further difficulty exists in the fact, that the set of equations (54) for $n > 1$ yields more conditions than are necessary for the calculation of the constants A^{2m+1}_{2n+1}, ... For this reason all of the tolerance conditions cannot be satisfied.

These difficulties probably can be avoided, if the extended statement for the nucleus of the velocity potential is made:

/41.20

$$
\begin{aligned}
K &= G(u,v,w;v',w') + H(u,v,w;v',w') \\
&+ \sum_{r=0}^{\infty} \sum_{s=0}^{r} \gamma c^{2s+1}_{2r+1}\, Sc^{2s+1}_{2r+1}(v',w') \lim_{v'=0} \int_{-\pi/2}^{\pi/2} G(u,v,w;v',w')\, U(0,v',w')\, f^{2s+1}_{2r+1}(w')\, dw' \\
&+ \sum \sum \cdots \qquad (54a)
\end{aligned}
$$

By insertion of the derivative $\partial K/\partial z$ for $z = 0$ in equation (44) and integration on skew lines y = const. one obtains for every Wichtung function $f^{2s+1}_{2r+1}(w')$, $f^{2s+2}_{2r+2}(w')$, ... an integral equation in y

$$\int_{x_0}^{\infty} \exp(-\omega^* x)\, dx\, V(u,0,w) \Big\{ Fc_{2r+1}^{2s+1}(u)\, Sc_{2r+1,v}^{2s+1}(0,w)$$
$$+ \lim_{v'=0} \int_{-\pi/2}^{\pi/2} G_v(u,0,w;v',w')\, U(0,v',w')\, f_{2r+1}^{2s+1}(w')\, dw' \Big\} = 0 \qquad (54b)$$
$$w = \arc\sin\left(\frac{ky}{\sqrt{1+k^2\sinh^2 u}}\right); \quad x = \cosh u \sqrt{1 - \frac{k^2 v^2}{1+k^2\sinh^2 u}}$$
$$-1 \leq ky \leq 1 \;; \quad x = x_0 : u = 0 \;; \quad x = \infty : u = \infty$$

Solutions of these integral equations do not yet exist. It would be necessary to test if these solutions were unique and if the nucleus thusly calculated were a difference nucleus.

The solution equation (50) obtained from the statement equation (42) is only approximately valid; it requires only the solution of finite linear equation sets, whose coefficients are easily calculable. The approximate solution equation (50) is applicable, although it is not unique.

I am not of the opinion that the three-dimensional lifting surface theory in frictionless fluidity ought to be given up, because it isn't unique. One could choose the constants C_1 and C_2 in such a way that measured and calculated aerodynamic coefficients agree as well as they can. Further, in boundary cases of infinite tension (k = 0), one could adapt them to the two-dimensional theory. The following constants correspond to this assumption:

$$C_1 = -\gamma c_1^1\, Fc_1^1(0) = \frac{3}{E'(k)},$$
$$C_2 = -\gamma c_2^1\, Fc_2^1(0) = \frac{15(1-k^2)}{(1-2k^2)E' + k^2 K'}. \qquad (55)$$

With these constants I have calculated the aerodynamic coefficients, to which I shall return again. I cannot yet name any corresponding numerical values for the constants T_1 and T_2. On the basis of certain symmetric observations one can guess that the following boundary values are valid for stationary flow ($\omega^* = 0$):

$$T_1 = T_2 = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k = 1 \end{cases} \quad (56)$$

The solution obtained in equations (49) to (51) has the same form as those of the two-dimensional theory. It is valid for $\varkappa = 0$, can, however, be carried over in sections to $\varkappa > 0$, if the Lamé wave function is inserted in place of the Lamé potential function. I carried out a corresponding transformation for the two-dimensional theory [4]. In the case of stationary flow ($\omega^* = 0$) the solution obtained above is also valid for compressible fluids, because $\varkappa = \omega^* B = 0$, if one of the two factors disappears.

/41.22

6. Aerodynamic coefficients of the oscillating elliptic lifting surface.

We now want to calculate an example showing the aerodynamic coefficients for an oscillating elliptic lifting surface with the bilinear down current dispersion

$$\begin{aligned} w^*(x,y) &= a + bx + ck^{-1}y + ek^{-1}xy \quad , \\ w^*(v,w) &= [aSc_1^1(v,w) + bSs_2^2(v,w) + cSc_2^1(v,w) \\ &\quad + eSs_3^2(v,w)] \; U(0,v,w) \end{aligned} \quad (57)$$

The dimensionless aerodynamic coefficients, based on the elliptic surface $\pi l^2/k$ and its half axis l and l/k as lever arm, are for

$$\text{lift:} \quad k_{a,b} = \frac{k}{\pi} \iint p_{a,b}(v,w)\ Sc_1^1(v,w)\ U(0,v,w)\ d\sigma ,$$

$$\text{pitching torque:} \quad m_{a,b} = \frac{k}{\pi} \iint p_{a,b}(v,w)\ Ss_2^2(v,w)\ U(0,v,w)\ d\sigma ,$$

$$\text{rolling torque:} \quad n_{c,e} = \frac{k}{\pi} \iint p_{c,e}(v,w)\ Sc_2^1(v,w)\ U(0,v,w)\ d\sigma , \tag{58}$$

$$\text{deviation torque:} \quad q_{c,e} = \frac{k}{\pi} \iint p_{c,e}(v,w)\ Ss_3^2(v,w)\ U(0,v,w)\ d\sigma .$$

From equations (49), (50), (55), (57) and (58) we obtain the surprisingly simple pattern of the aerodynamic coefficients

$$\begin{aligned} k_a &= A[\omega^* + 1 + T_1] \quad ; \quad n_c = B[\omega^* + 1 + T_2] , \\ -k_b = m_a &= A\ [\tfrac{1}{2}(1+T_1)] \quad ; \quad -n_e = q_c = B\ [\tfrac{1}{2}(1+T_2)] , \\ m_b &= A\ [\alpha\omega^* + \tfrac{1}{4}(1-T_1)] ; \quad q_e = B\ [\beta\omega^* + \tfrac{1}{4}(1-T_2)] . \end{aligned} \tag{59}$$

This pattern is valid with other constants even for the two-dimensional theory and perhaps also for other non-elliptic lifting surface-contours. The constants in the two-dimensional theory are:

/41.23

$$A = \frac{\pi}{2} ; \quad \alpha A = \frac{\pi}{16} ; \quad B = \beta = 0 \tag{60}$$

and in the case of elliptic lifting surfaces

$$
\begin{aligned}
A &= -\frac{4}{\gamma c_1^1}\, Fc_1^1(0) = \frac{4}{3}\ \frac{1}{E'} , \\
\alpha A &= -\frac{4}{\gamma s_2^2}\, Fs_2^2(0) = \frac{4}{15}\ \frac{1-k^2}{(2-k^2)E' - k^2K'} , \\
B &= -\frac{4}{\gamma c_2^1}\, Fc_2^1(0) = \frac{4}{15}\ \frac{1-k^2}{(1-2k^2)E' + k^2K'} , \\
\beta B &= -\frac{4}{\gamma s_3^2}\, Fs_3^2(0) = \frac{4}{105}\ \frac{(1-k^2)^2}{2(1-k^2+k^4)E' - (k^2+k^4)K'} .
\end{aligned}
\qquad (61)
$$

I would particularly like to point to the fact that the constants in equations (60) and (61) already arise from the classic theory of plates without oncoming flow oscillating in ideal fluidity. What is essentially new about the lifting surface theory is only the integration constants T_1 and T_2, which can still be functions of the configuration of the lifting surface and of the reduced frequency $\omega *$, in the case of compressible fluidity as the Machnumber β.

7. Numerical example for the boundary case k = 0 and k = 1

In the boundary case of the elliptic lifting surface of infinite span (k = 0) and in the circular lifting surface (k = 1) there occur the constants contained in Table I

Table I. Constants for k = 0 and k = 1.

k	E'	K'	A	αA	B	βB	A_1^1	A_2^2	B_2^1	B_3^2	γc_1^1	γs_2^2	γc_2^1	γs_3^2
0	1	∞	$\frac{4}{3}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{2}{105}$	1	1	1	1	3	15	15	105
1	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{8}{3\pi}$	$\frac{16}{45\pi}$	$\frac{16}{45\pi}$	$\frac{64}{1575\pi}$	1	$\frac{3}{4}$	1	$\frac{5}{8}$	3	15	15	105

In order to compare available theories at least in the case of stationary flow, I have calculated the coefficients for k = 0 in accord with Prandtl's hypothesis of even flow: /41.24

$$k_a = \pi \;;\quad k_b = -\frac{\pi}{2} \;;\quad m_a = \frac{4}{3} \;;\quad m_b = 0 \;,$$
$$n_c = \frac{\pi}{4} \;;\quad n_e = -\frac{\pi}{8} \;;\quad q_c = \frac{4}{15} \;;\quad q_e = 0 \;. \tag{62}$$

For the circular lifting surface (k = 1) the numerical values from Krienes and Schade [5] are available. Further I have calculated the aerodynamic coefficients in accordance with equation (59) and Table I, whereby I inserted the values (56) for the T functions. The results are contained in Table II.

Table II. Stationary aerodynamic coefficients for k = 0 and k = 1.

k	k_a	$-k_b$	m_a	m_b	n_c	$-n_e$	q_c	q_e	Procedure
0	3,1416	1,5708	1,3333	0	0,7854	0,3927	0,2667	0	Prandtl
	2,6667	1,3333	1,3333	0	0,5333	0,2667	0,2667	0	Küssner
	+17,8	+17,8	0	0	+47,3	+47,3	0	0	% Difference
1	0,8992	0,4718	0,4659	0,2191	0,1276	0,0582	0,0554	0,0299	Krienes [5]
	0,8488	0,4244	0,4244	0,2122	0,1132	0,0566	0,0566	0,0283	Küssner
	+5,9	+11,2	+9,8	+3,3	+12,7	+2,8	-2,1	+6,0	% Difference

For the elliptic lifting surface of very large spans ($k = 0$) there are substantial differences between both theories at the lift and at rolling torque, while pitching torque and deviation torque agree. According to Prandtl the coefficients do not satisfy tolerance conditions

$$k_b + m_a = 0 \; ; \; n_e + q_c = 0 \tag{63}$$

In the case of the circular lifting surface the differences are smaller; but the coefficients from Krienes do not satisfy the equation (63) either. The deviations represent 1.3% /41.25 and 5.1% and are larger than the probable calculation errors.

8. Experimental testing and dispersion of the singularities.

8.1. Displacement of the singularities at the edge of the lifting surface.

It is not easy to test the lifting surface theory experimentally. Discrepancies occur as a result of the boundary layer and the finite airfoil thickness. If one uses very thin plates, discrepancies result from turbulence behind the leading edge. These deviations can be calculated only roughly or guessed at. The measurement values thusly corrected agree quite well with the new theory, less well with the old.

I should like to mention a suggestion which was made by B.T. George, N. Rott and by me. In the case of oscillating wings in awater channel I have observed also that the trailing edge is weakly flowed around. The solution (50) with singularities on the leading edge could therefore be superimposed by a corresponding solution with singularities on the trailing edge. The components of both solutions are to be determined in such a manner that the measurement results are represented as well as possible. If both portions are alike, then a solution with the stationary lift zero is obtained.

A similar procedure is to be employed in the case of a thrusting lifting surface, whose flight direction is inclined at the angle α' against the symmetric plane y = 0. The Euler equation then becomes

/41.26

$$(\omega^* - \cos\alpha \frac{\partial}{\partial x} - \sin\alpha \frac{\partial}{\partial y})\, \phi^*(x,y,z) + p^*(x,y,z) = 0 \,. \tag{64}$$

The leading edge of the lifting surface is shifted around the wing tips at the valve

$$w_1 = \text{arctg}\,(k\ \text{tg}\,\alpha') \tag{65}$$

The singularities which are to be ordered along the new leading edge are once again to be determined such that the nuclei K and K_1 become difference nuclei. I cannot discuss this any further here.

8.2. Infinite pressure increase on the leading edge

Much has been said about the infinite pressure increase near the leading edge of the lifting surface. Although the purpose of the lifting surface theory can only exist in the approximate calculation of the aerodynamic coefficients, this question has a certain interest. As long as no closed expression for the characteristic Green function is known, as in the two-dimensional case, the integral representation of the pressure equations (49) and (50) offer no answer to this question, because the series development of G for u = 0 becomes divergent.

One could try to calculate the pressure in a different manner out of the velocity potential. Corresponding to equation (48) one could, for example, differentiate the regular components of K according to (-x'), the singular ones according to x. One then obtains from equations (42) and (45) the pressure on the lifting surface u = 0

$$p^*(0,v,w) = -\omega^*\varphi + f(v,w) + A\, U(0,v,w)\, [g_0(v,w)+g_1(v;w)\cos v] \quad (66)$$

Here f and φ are finite amounts of ellipsoidal harmonics and A is a constant, all of which are dependent upon the given down current g_0 and g_1 are even functions in cos v, which are independent of the down current. /41.27

All of the statements of the three-dimensional lifting surface theory known to date are of the type given in equation (66) and this is the sole reason why I have presented this statement to you, because it is unacceptable. If we differentiate the function G_v, $(0,v,w;0,w_o^{\prime})$ using x, then we lose the characteristics, which are required. The Kutta condition is no longer satisfied and the singularity at the edges is no longer integratable. One can already be convinced of this fact in the two-dimensional lifting surface theory for which the Green function is

$$\begin{aligned} G(u,v,v') &= -\sum_{1}^{\infty} \frac{1}{n} \exp(-nu) \sin nv \sin nv' , \\ &= \frac{1}{4} \ln \frac{\cosh u - \cos(v-v')}{\cosh u - \cos(v+v')} \end{aligned} \quad (67)$$

Kinner and Krienes consider the functions g_o and g_1, which are deduced from the Green function, to be freely selected with the side conditions, that p* of the Laplace equation suffices and that the Kutta condition and the down current condition be satisfied as best as possible. This is, however, only practical for a few points of the lifting surface. If the functions g_o $(0,w) = g_1\ (0,w)$ were final, then we would obtain the same kind of singularity of the pressure from equation (66) as in the case of even flow, that is, 1/sin v. Since, however, the statement (66) is unacceptable, this type of consideration has no convincing power. The oscillating plate without oncoming flow has a singularity like 1/sin v at the edge.

9. Solutions at any one temporal change of the down current.

In closing I should like to discuss briefly the unsteady lifting surface theory at any one temporal change of the down current. At the outset we undertook a Laplace transformation of this general problem, in order to facilitate our task. After having solved the simplified task, we can cancel this transformation again by means of an inverse Laplace transformation. This occurs when we multiply the harmonic solution by the integral operator

$$L = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\omega^* \ (s-\beta^2(s+x))] \frac{d\omega^*}{\omega^*} \tag{68}$$

If we employ this operator on the integral equations (17) and (18) for the down current, then we obtain the result

$$w(s,x,y) = \frac{\sqrt{1-\beta^2}}{2\pi} \iint dx'dy' \int_{s'=0}^{s'=s} dp(s,x,y,+0)\; M_2(s-s',x-x',y-y',0), \tag{69}$$

$$M_2(s,x,y,0) = \begin{cases} \dfrac{s+x+\sqrt{(s+x)^2+y^2/(1-\beta^2)}}{y^2\sqrt{(s+x)^2+y^2/(1-\beta^2)}} & \text{,wenn } s < \beta\sqrt{(s+x)^2+y^2/(1-\beta^2)}, \\[2ex] \dfrac{x+\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}} & \text{,wenn } s > \beta\sqrt{(s+x)^2+y^2/(1-\beta^2)}. \end{cases} \tag{70}$$

It is surprising, that the nucleus M_2 of this most common integral equation is such a simple algebraic function of the Cartesian coordinate. In order to eliminate the quadratic singularity of the nucleus M_2, one must integrate equation (69) partially according to y' and find the Cauchy principal value.

A special, but equally interesting result is /41.29
gained if we employ the integral operator (68) on the pressure of the harmonically oscillating elliptic lifting surface equations (49) and (50). Then we obtain the pressure in time point t = s

$$\begin{aligned} p(u,v,w,s) = -\lambda \iint d\sigma' \Big\{ & (G+H)\frac{\partial}{\partial s}\,\bar{w}(s,v',w') + \bar{w}(s,v',w')\frac{\partial}{\partial x'}\,(G+H) \\ & + \frac{1}{C_1}\, G_v(u,v,w;0,w_o^1)\Big[G_v(0,\pi,w_o^1;v',w')\int_{s'=0}^{s'=s} U_1(s-s')\,d\bar{w}(s',v',w') \\ & \qquad - G_v(0,0,w_o^1;v',w')\;\bar{w}(s,v',w')\Big] \\ & + \frac{1}{C_2}\, H_{vw}(u,v,w;0,w_o^1)\Big[H_{vw}(0,\pi,w_o^1;v',w')\int_{s'=0}^{s'=s} U_2(s-s')\,d\bar{w}(s',v',w') \\ & \qquad - H_{vw}(0,0,w_o^1;v',w')\;\bar{w}(s,v',w')\Big] \Big\}. \end{aligned} \tag{71}$$

The integrations constants T_1 and T_2 are the picture functions of the object functions

$$U_{1,2}(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} T_{1,2}(\omega^*) \exp\omega^* s \frac{d\omega^*}{\omega^*} \quad , \quad \text{if } s > 0 \qquad (72)$$

The function $U_1(s)$ is to date only known in the two-dimesnional theory and was first calculated in 1924 by H. Wagner.

We now consider the special wind dispersion

$$\overline{w}(s,x,y) = \overline{w}(s+x,y) \quad . \qquad (73)$$

This down current occurs when an airplane flies into a stationary squall area. If we introduce this down current into equation (71), then only the singular components with the factors $1/C_1$ and $1/C_2$ yield an amount different from zero to the pressure. For this reason there are in this case only two possible types of pressure dispersions at the lifting surface, a symmetrical proportional $G_{v'}(0,v,w;0,w_o')$ and an antisymmetrical proportional $H_{v',w'}(0,v,w;0,w_o')$. I determined the corresponding statement of the two-dimensional theory already in 1940.

10. References.

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Entered on August 3, 1956

The first twenty-four Lamé polynomials

	$E_n^m(v)$	$\dot{E}_n^m(w)$
Ec_0^0	1	1
Ec_1^0	d	s
$Ec_2^{0,2}$	$1 - c_2^{0,2}(k)\ s^2$	$1 - c_2^{2,0}(k')\ s^2$
$Ec_3^{0,2}$	$d\ [\ 1 - c_3^{0,2}(k)\ s^2\ ]$	$s\ [\ 1 - c_3^{3,1}(k')\ s^2\ ]$
Es_1^1	c	c
Es_2^1	$c\ d$	$c\ s$
$Es_3^{1,3}$	$c\ [\ 1 - s_3^{1,3}(k)\ s^2\ ]$	$c\ [\ 1 - s_3^{3,1}(k')\ s^2\ ]$
$Es_4^{1,3}$	$c\ d\ [\ 1 - s_4^{1,3}(k)\ s^2\ ]$	$c\ s\ [\ 1 - s_4^{4,2}(k')\ s^2\ ]$
Ec_1^1	s	d
Ec_2^1	$s\ d$	$d\ s$
$Ec_3^{1,3}$	$s\ [\ 1 - c_3^{1,3}(k)\ s^2\ ]$	$d\ [\ 1 - c_3^{2,0}(k')\ s^2\ ]$
$Ec_4^{1,3}$	$s\ d\ [\ 1 - c_4^{1,3}(k)\ s^2\ ]$	$d\ s\ [\ 1 - c_4^{3,1}(k')\ s^2\ ]$
Es_2^2	$s\ c$	$d\ c$
Es_3^2	$s\ c\ d$	$d\ c\ s$
$Es_4^{2,4}$	$s\ c\ [\ 1 - s_4^{2,4}(k)\ s^2\ ]$	$d\ c\ [\ 1 - s_4^{3,1}(k')\ s^2\ ]$
$Es_5^{2,4}$	$s\ c\ d\ [\ 1 - c_2^{0,2}(k)\ s^2]$	$d\ c\ s\ [\ 1 - c_2^{2,0}(k')\ s^2\ ]$

Abbreviations:

$s = \sin v$ or $\sin w$

$c = \cos v$ or $\cos w$

$d = 1-k^2\sin^2 v$ or $1-k'^2\sin^2 w$

$$
\begin{aligned}
c_2^{0,2}(k) &= 1 + k^2 \mp \sqrt{1 - k^2 + k^4} \\
c_3^{0,2}(k) &= 1 + 2k^2 \mp \sqrt{1 - k^2 + 4k^4} \\
s_3^{1,3}(k) &= 2 + k^2 \mp \sqrt{4 - k^2 + k^4} \\
s_4^{1,3}(k) &= 2 + 2k^2 \mp \sqrt{4 + k^2 + 4k^4} \\
c_3^{1,3}(k) &= \frac{1}{3}\left(2 + 2k^2 \mp \sqrt{4 - 7k^2 + 4k^4}\right) \\
c_4^{1,3}(k) &= \frac{1}{3}\left(2 + 3k^2 \mp \sqrt{4 - 9k^2 + 9k^4}\right) \\
s_4^{2,4}(k) &= \frac{1}{3}\left(3 + 2k^2 \mp \sqrt{9 - 9k^2 + 4k^4}\right)
\end{aligned}
$$