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RECENT ADVANCES IN HYBRID/MIXED FINITE ELEMENTS

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Summary

In two new formulations of Hybrid/Mixed finite element methods respectively by the Hellinger-Reissner principle and the Hu-Washizu principle, the stress equilibrium equations are brought in as conditions of constraint through the introduction of additional internal displacement parameters. The new approaches are more flexible and have better computing efficiencies. A procedure for the choice of assumed stress terms for 3-D solids is suggested. Example solutions are given for plates and shells using the present formulations and the idea of semiloof elements.

1 Introduction

Finite element methods can be considered as convenient methods for solving partial differential equations. In the case of solid mechanics the most common formulation is to use displacements as field variables in the differential equations. The so-called conventional finite element formulation is an approximate solution based on assumed displacements and the corresponding variational principle is the principle of the stationary potential energy. Such method can thus be called a Primal Finite Element Method. For solid mechanics problems a dual formulation can be made using stress functions as field variables and the corresponding complementary energy principle. Applications of such principle for the finite element method have been made, but are not popular.

At a rather early age of finite element development, short-comings of the assumed displacement method were discovered. The main one is the difficulty in constructing shape functions for thin plates and shells which require C' continuity-conditions. Other observations are that the so-called compatible elements are too rigid and, indeed, for some limiting cases, such as nearly incompressible materials or thin plates and shells which are formulated by taking transverse shearing strains into account, the corresponding finite element model may be completely locked. Another area that calls for improvement is to avoid the loss in accuracy when the evaluation of strains and hence stresses are accomplished by differentiation of the assumed displacements.

Many alternative finite element methods that have been developed are based on the introduction of additional field variables. Within the element more than one field variable can be introduced such as methods derived by the Hellinger-Reissner principle for which stresses and displacements are used and methods derived by the Hu-Washizu principle for which stresses, strains and displacements are used. Such models are named mixed models. When additional variables are introduced as Lagrange multipliers for the maintaining continuity conditions along the interelement boundary it is labelled as a hybrid model. These two models are not mutually exclusive and in many cases the same finite element method can be derived using different variational formulations. It is, thus, decided that a discussion of non-primal finite element method be lumped together under the name Hybrid/Mixed finite element method.

The assumed stress hybrid method which makes use of boundary displacements as additional variables, and hence leads to matrix displacement methods in the finite element formulation, can overcome the shortcomings of the assumed displacement methods that were mentioned above. However it also needs further improvements. The main ones are (1) the lack of a guideline for the choice of the assumed stress terms, (2) the excessive computing time needed in the inversion of a flexibility matrix during the process of constructing the element stiffness matrix and (3) the difficulty in the choice of equilibrating stresses in the case of complex problems such as shells.

The objectives of the present lecture are:

(1) To present two recently developed Hybrid/Mixed finite element formulations based, respectively on the Hellinger-Reissner principle and the Hu-Washizu principle for which stress equilibrium conditions need not be imposed initially but are brought in as conditions of constraint, instead, by including additional internal displacement parameters. These new approaches make the formulation more flexible and also improve the computing efficiency.

(2) To present a systematic procedure for the choice of the assumed stress terms that will

prevent the appearnace of kinematic deformation modes. Three dimensional elements are used as examples.
(3) To present some results on plates and shells using present approaches and the idea of semiLoof elements.[1,2]

2 Hybrid/Mixed Formulation by Hellinger-Reissner Principle

Element stiffness matrix can be constructed using different variational mothods. The initial step in common is to assign nodal displacements $\underset{\sim}{q}$ and corresponding boundary displacement $\tilde{\underset{\sim}{u}}$ which maintain compatibility with neighboring elements. Then by using any variational principle in solid mechanics a solution for such a problem with prescribed boundary conditions can be obtained. Such solutions may be in terms of displacements, stresses and/or strains. The element strain energy U can thus be expressed in terms of the nodal displacements q, and the element stiffness matrix $\underset{\sim}{k}$ can be obtained.

In using the Hellinger-Reissner principle to solve the above boundary value problem, the following functional should be maintained stationary,

$$\pi_R = \int_V [-\frac{1}{2}\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \underset{\sim}{\sigma} + \underset{\sim}{\sigma}^T(\underset{\sim}{D}\, \underset{\sim}{u})]\, dV - \int_{\partial V} \underset{\sim}{T}^T(\underset{\sim}{u} - \tilde{\underset{\sim}{u}})\, dS \qquad (1)$$

where $\underset{\sim}{S}$ is the elastic compliance matrix and

$$\underset{\sim}{T} = \underset{\sim}{\nu}\, \underset{\sim}{\sigma} \qquad (2)$$

relates the boundary tractions and the stresses. Here the element displacements are assumed to be, in general, not compatible with the prescribed boundary displacement $\tilde{u}$. If the element displacements $\underset{\sim}{u}$ are compatible with $\tilde{u}$, and there is no constraint for the assumed stresses $\underset{\sim}{\sigma}$ then the resulting element is identical to the compatible element, as was pointed out by Fraeijs de Veubeke.[3] If $\underset{\sim}{\sigma}$ satisfy the homogeneous equilibrium equations

$$\underset{\sim}{D}^T \underset{\sim}{\sigma} = 0 \qquad (3)$$

Then the second term in Eq. (1) can be changed to the boundary integral, resulting in

$$\pi_R = \int_V [-\frac{1}{2}\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \underset{\sim}{\sigma}]\, dV + \int_{\partial V} \underset{\sim}{T}^T \tilde{\underset{\sim}{u}}\, ds = -\pi_c \qquad (4)$$

This is the negative of the element complementary energy and is the basis for the initial development of the assumed stress hybrid element,[4] which may be considered as the result of a hybrid variational principle.[5] This also indicates that if the assumed stresses $\underset{\sim}{\sigma}$ in Eq. (1) satisfy Eq. (3) and $\underset{\sim}{u} = \tilde{\underset{\sim}{u}}$ the resulting element stiffness matrix by a mixed variational principle is identical to that by a hybrid variational principle.

With stresses expressed in terms of stress parameters $\underset{\sim}{\beta}$,

$$\underset{\sim}{\sigma} = \underset{\sim}{P}\, \underset{\sim}{\beta} \qquad (5)$$

and the element displacements $\underset{\sim}{u}$ or boundary displacements $\tilde{\underset{\sim}{u}}$ interpolated in terms of nodal displacements $\underset{\sim}{q}$, i.e.

$$u = \underset{\sim}{N}\, \underset{\sim}{q} \qquad (6)$$

$$u = \underset{\sim}{L}\, \underset{\sim}{q} \qquad (7)$$

then $\pi_c = -\pi_R = \frac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\, \underset{\sim}{\beta} - \underset{\sim}{\beta}^T \underset{\sim}{G}\, \underset{\sim}{q}$ (8)

where $\underset{\sim}{H} = \int_V \underset{\sim}{P}^T \underset{\sim}{S}\, \underset{\sim}{P}\, dV$ (9)

and $G = \int_V \underset{\sim}{P}^T \underset{\sim}{N}\, dV$ by π_R (10)

$= \int_{\partial V} \underset{\sim}{P}^T \underset{\sim}{\nu}^T \underset{\sim}{N} ds$ by π_c (11)

Then $\delta\pi = 0$ yields $\underset{\sim}{\beta}$ in terms of $\underset{\sim}{q}$. The resulting element stiffness matrix obtained from the strain energy expression is

$$\underset{\sim}{k} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} \qquad (12)$$

In a new formulation, the stress equilibrium conditions Eq. (3) are not satisfied but the element displacements $\underset{\sim}{u}$ are expressed in terms of nodal displacements $\underset{\sim}{q}$ and additional internal displacement parameters $\underset{\sim}{\lambda}$, i.e.

$$\underset{\sim}{u} = \underset{\sim}{u}_q + \underset{\sim}{u}_\lambda = \underset{\sim}{N}\underset{\sim}{q} + \underset{\sim}{M}\underset{\sim}{\lambda} \qquad (13)$$

Here $\underset{\sim}{u}_q$ are compatible with the boundary displacements $\tilde{\underset{\sim}{u}}$ but $\underset{\sim}{u}_\lambda$ are introduced so that $\underset{\sim}{u}$ are no longer compatible. By integrating the term with $\underset{\sim}{u}_\lambda$ and by recognizing that

$$\underset{\sim}{u}_\lambda = \underset{\sim}{u} - \tilde{\underset{\sim}{u}} \text{ on } \partial V$$

one obtains from Eq. (1)

$$\pi_R = \int_V [-\frac{1}{2}\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \underset{\sim}{\sigma} + \underset{\sim}{\sigma}^T(\underset{\sim}{D}\, \underset{\sim}{u}_q) - (\underset{\sim}{D}^T\underset{\sim}{\sigma})^T \underset{\sim}{u}_\lambda]\, dV \qquad (14)$$

If $\underset{\sim}{u}_\lambda$ are bubble functions, i.e. $\underset{\sim}{u}_\lambda$ along the element boundary are zero, then $\underset{\sim}{u} - \tilde{\underset{\sim}{u}} = 0$ and the above equation still holds. It is seen that the last term in Eq.(2)contains the stress equilibrium condition with $\underset{\sim}{u}_\lambda$ as Lagrange multipliers. Thus, in finite element formulation the stresses need not satisfy the equilibrium condition initially while the introduction of the internal displacements $\underset{\sim}{u}_\lambda$ will enforce this condition. With $\underset{\sim}{\sigma}$, $\underset{\sim}{u}_q$ and $\underset{\sim}{u}_\lambda$ represented by Eqs. (5) and (13),

$$\pi_R = -\frac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\, \underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{\beta}^T \underset{\sim}{R}\, \underset{\sim}{\lambda} \qquad (15)$$

where $\underset{\sim}{H}$ and $\underset{\sim}{G}$ are given in Eqs. (9) and (10) and

$$\underset{\sim}{R} = \int_V (\underset{\sim}{D}^T \underset{\sim}{P})^T \underset{\sim}{M}\, dV \qquad (16)$$

The stationary condition of π_R then yields

$$\beta = H^{-1}\,(G\,q - R\,\lambda) \tag{17}$$

$$R^T\beta = 0 \tag{18}$$

Eliminating λ and substituting β into the strain energy expression one obtains

$$k = G^T H^{-1} G - G^T H^{-1} R\,(R^T H^{-1} R)^{-1}\, R^T H^{-1} G \tag{19}$$

Equation (18) is, of course, the equilibrium constraint for the stress parameters β. When problems are set up in terms of Cartesian coordinates, such constraint equations can be identified directly. For thin plates and shells for which compatible shape functions are difficult to construct, π_c of Eq. (4) is used by the existing hybrid stress method. Tong has suggested the use of uncoupled stresses while adding a constraint term $\beta^T R\,\lambda$ to Eq. (4).[6] The corresponding variational functional may be written as

$$\pi_R^* = \int_V [-\tfrac{1}{2}\,\sigma^T S\,\sigma - (D^T\sigma)^T\,u_\lambda]\,dV + \int_{\partial V} T^T\bar{u}\,dS \tag{20}$$

and the resulting element stiffness matrix is the same as Eq. (19). The motivation by Tong is that when the P matrix in Eq. (5) are not coupled, the inversion of the flexibility matrix H can be simplified [7,8]. For example, in the case of 3-D solid, with $\sigma = \{\sigma_x\ \sigma_y\ \sigma_z\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{xz}\}$, if the expansion $P_1\beta$ for all stress components are of identical form then the H-matrix Eq. (9) will take the form

$$H = \begin{bmatrix} S_{11}\phi & \cdots & S_{16}\phi \\ \vdots & \ddots & \vdots \\ S_{16}\phi & & S_{66}\phi \end{bmatrix} \tag{21}$$

where S is the elastic compliance matrix and

$$\phi = \int P_1^T P_1\,dv \tag{22}$$

The inversion of H then becomes simply

$$H^{-1} = \begin{bmatrix} C_{11}\psi & \cdots & C_{16}\psi \\ \vdots & \ddots & \vdots \\ C_{16}\psi & & C_{66}\psi \end{bmatrix} \tag{23}$$

where $C = S^{-1}$ and $\psi = \phi^{-1}$ (24)

Although in using uncoupled stresses the size of H is increased and the inversion of another matrix must be performed when Eq. (19) is used to derive the element stiffness matrix, there could be considerable savings in computation when the orders of ψ are much smaller than that of the H matrix of a coupled stress ... It will be noted that for isotropic ... are coupled in the stress-strain relation it is only necessary to use the same stress expansion for the normal components.

The present formulation by using additional internal displacement parameters is a more general way to introduce the stress equilibrium conditions. For example, the stress terms may be expanded in terms of natural coordinates such as isoparametric coordinates for quadrilateral and hexahedral elements and area and volume coordinates for triangles and tetrahedrons. Also, a limited number of additional terms may be used. In these cases the stress equilibrium equations are satisfied approximately but the formulation of the problem is simplified. It should be remarked that in the formulation by the Hellinger-Reissner principle the equilibrium conditions need not be satisfied for the assumed stresses.

3 Hybrid/Mixed Formulation by Hu-Washizu Principle

In using the Hu-Washizu principle to solve the above boundary value problem the following functional should be maintained stationary,

$$\pi_{HW} = \int_V [\tfrac{1}{2}\,\varepsilon^T C\,\varepsilon - \sigma^T\varepsilon + \sigma^T(D\,u)]\,dV - \int_{\partial V} T^T(u - \bar{u})\,dS \tag{25}$$

In 1973, Wolf[7] used the Hu-Washizu principle for finite element formulations, but the resulting method has not been used widely. L.M. Tang and his colleagues[10] at the Dalian Institute of Technology have initiated the so-called quasi-conforming elements or string-net elements for the construction of element stiffness matrices and have discussed the connection of their method with the Hu-Washizu principle[11]. An interpretation of their method is that in applying the Hu-Washizu principle each stress component and corresponding strain component are expressed in terms of the same polynomial expansion, and the resulting matrix to be inverted then becomes one with only diagonal submatrices. This approach can again be combined with the use of additional internal displacement parameters to enforce the stress equilibrium conditions. Again, using the 3-D problem as an example, in the finite element implementation, let

$$\varepsilon = P\,\alpha \tag{26}$$

$$\sigma = P\,\beta \tag{27}$$

with identical P for stresses and strains but the expansions for different components may be entirely independent. With element displacements again expressed by Eq. (13), equation (25) then becomes

$$\pi_{HW} = \frac{1}{2}\alpha^T J \alpha - \beta^T \bar{H} \alpha + \beta^T G q - \beta^T R \lambda \qquad (28)$$

where $$J = \int_V P^T C P \, dV \qquad (29)$$

$$\bar{H} = \int_V P^T P \, dV = \begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_6 \end{bmatrix} \qquad (30)$$

and G and R are given in Eqs. (10) and (16).

The stationary condition of π_{HW} with respect to β α and λ then leads to three sets of equations, which can be used to obtain α in terms of q. The resulting element stiffness matrix is

$$k = G^T W G - G^T W R \, (R^T W R)^{-1} \, R^T W G \qquad (31)$$

where $$W = \bar{H}^{-1} J \bar{H}^{-1} \qquad (32)$$

As shown in Eq. (30) $\bar{H}$ matrix consists of only diagonal submatrices. Thus, its inversion always involves only the inversion of H_i even if the material property is anisotropic.

For general quadrilateral and hexahedral elements it is more convenient to express strains, stresses and displacements in the isoparametric coordinates. In that case, the Jacobian $|J|$ appears in the area or the volume integrals. Here, however, the strain terms can be expressed as

$$|J| \, \varepsilon = P \, \alpha \qquad (33)$$

and for properly chosen P, the resulting submatrices H_i may all become diagonal matrices and $\bar{H}$ will also be a simple diagonal matrix.

4 Choice of Assumed Stress Terms for Hexahedral Elements

The guidelines for the assumed stresses in the hybrid/mixed finite element formulations are to include adequate terms to suppress all possible kinematic deformation modes and to avoid excessive stress terms in order to prevent the forming of an overly rigid element. For hexahedral elements of general distorted geometry kinematic deformation modes can, in general, be suppressed when the number of stress terms is equal or larger than the total number of degrees of freedom minus six. However, for regular brick-shaped elements the possible displacement modes are of simple shapes, not all assumed stress terms are useful for suppressing the kinematic modes. Thus, their selections become a very important step in the finite element formulation.

Now consider a 20-node brick element with the x, y and z axes coincide with the axes of symmetry as shown in Figure 1. The element displacements can be

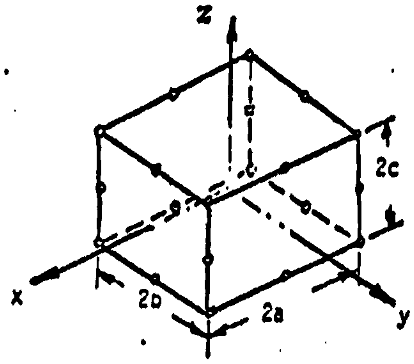

Figure 1 20-Node Brick Element

expressed in terms of the nodal displacements u_i v_i and w_i ($i = 1$ to 20) as

$$u = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{20} N_i(x,y,z) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \qquad (34)$$

One can conclude that the shape functions $N_i(x,y,z)$ can be represented by 20 independent and simple polynomial terms as follows:

Constant, x, y, z, x^2, y^2, z^2, xy, yz, xz, x^2y, xy^2, y^2z, yz^2, x^2z, xz^2, xyz, x^2yz, xy^2z and xyz^2

For a regular brick element, then, all possible element displacements can be represented by 60 independent displacement modes. Here, the six rigid body displacement modes correspond to zero strains, hence, can not be suppressed by any stresses and one only needs to consider the remaining 54 displacements.

It is seen that each displacement mode may correspond up to three non-zero strains. Let ε be one of these strains then, to suppress a particular mode it is necessary that a stress term σ exists for the stress component corresponding to ε such that the integral $\int_V \sigma \varepsilon dV$ does not vanish. Each stress term, however, can be used to suppress only one individual displacement mode. An ideal situation, thus, would be to choose 54 assumed stress terms each of which is associated with one individual displacement mode.

One of the possible choices is shown in Table 1. Here, in the various block spaces the polynomial expansions for the six stress components and the corresponding displacements to be suppressed are indicated. There are 54 independent β-terms. In many other blocks stress terms are introduced to maintain the equilibrium conditions. They are not used to suppress any of the 54 displacement modes.

It should be noted that some displacement terms may each correspond to three strains and some strain terms may be arrived from more than one displacement mode. The net result is that some of the 54

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Table 1

Stress Terms and Corresponding Displacement Modes to be Suppressed for 20-Node Brick Element

	σ_x	σ_y	σ_z	σ_{xy}	σ_{yz}	σ_{zx}
Const	β_1 $u=x$	β_2 $v=y$	β_3 $w=z$	β_4 $u=y, v=x$	β_5 $v=z, w=y$	β_6 $u=z, w=x$
x	$-\beta_{14}$ $-\beta_{21}$	β_9 $v=xy$	β_{11} $w=xz$	β_{13} $v=x^2$	β_{16} $w=xy$	β_{19} $w=x^2$
y	β_7 $u=xy$	$-\beta_{13}$ $-\beta_{18}$	β_{12} $w=yz$	β_{14} $u=y^2$	β_{17} $w=y^2$	β_{20} $u=yz$
z	β_8 $u=xz$	β_{10} $v=yz$	$-\beta_{17}$ $-\beta_{19}$	β_{15} $v=xz$	β_{18} $v=z^2$	β_{21} $u=z^2$
xy	β_{22} $u=x^2y$	β_{27} $v=xy^2$	β_{32} $w=xyz$		β_{39} $w=xy^2$	β_{41} $w=x^2y$
yz	β_{23} $u=xyz$	β_{28} $v=y^2z$	β_{33} $w=yz^2$	β_{37} $u=y^2z$		β_{42} $u=yz^2$
xz	β_{24} $u=x^2z$	β_{29} $v=xyz$	β_{34} $w=xz^2$	β_{38} $v=x^2z$	β_{40} $v=xz^2$	
x^2		β_{30} $v=x^2y$	β_{35} $w=x^2z$	$-\beta_{27}/2$ $-\beta_{40}/2$		$-\beta_{34}/2$ $-\beta_{29}/2$
y^2	β_{25} $u=xy^2$		β_{36} $w=y^2z$	$-\beta_{22}/2$ $-\beta_{42}/2$	$-\beta_{33}/2$ $-\beta_{41}/2$	
z^2	β_{26} $u=xz^2$	β_{31} $v=yz^2$			$-\beta_{28}/2$ $-\beta_{38}/2$	$-\beta_{24}/2$ $-\beta_{[illegible]}/2$
xyz	β_{49} $u=x^2yz$	β_{50} $v=xy^2z$	β_{51} $w=xyz^2$			
xy^2		$\underline{3\beta_{52}}$	β_{47} $w=xy^2z$			
x^2y			β_{48} $w=x^2yz$	$\underline{-3\beta_{52}}$		$\underline{-\beta_{51}/2}$
yz^2	β_{43} $u=xyz^2$		$\underline{3\beta_{53}}$			
y^2z	β_{44} $u=xy^2z$			$\underline{-\beta_{49}/2}$	$\underline{-3\beta_{53}}$	
xz^2		β_{45} $v=xyz^2$			$\underline{-\beta_{50}/2}$	$\underline{-3\beta_{54}}$
x^2z	$\underline{3\beta_{54}}$	β_{46} $v=x^2yz$				
x^3	β_{52} $u=x^2$					
y^3		β_{53} $v=y^2$				
z^3			β_{54} $w=z^2$			

displacement modes may be suppressed by alternative stress terms and there exist many other combinations of 54 stress terms that can suppress all possible kinematic modes.

Another observation is that the stress expansion is complete only up to linear terms, hence, the element properties will be dependent on the reference axes for the assumed stresses. Spilker et al.[12] has shown that a hybrid stress element may behave excellently under one coordinate system but will contain a kinematic deformation mode under another reference system. For a regular brick element, the reference axes should obviously be the axes of symmetry. For elements of general geometric shape it is still required to choose a local reference coordinate system in order to achieve an optimal element behavior.

It is also noted that there are alternative choices of the stress terms for maintaining the stress equilibrium conditions. In Table 1 complete symmetric conditions are not maintained in this particular stress selection. If such conditions are to be maintained and all the stress equilibrium equations are to be satisfied the number of independent stress terms will necessarily exceed 54. Alternatively, some stress equilibrium conditions can be relaxed in order to maintain complete symmetric conditions. It can be seen that in Table 1 if we exclude all the underlined terms which are introduced to maintain the equilibrium conditions of all cubic stress terms, the symmetry condition is satisfied. An example solution of a sphere with a hole at its center subjected to varying temperatures along the radial direction shows, in fact, that by using fewer stress terms in such a way the results are better than those obtained by using all terms in Table 1, and even better than that by the conventional displacement method. It has been found, however, that the equilibrium conditions for the lower order stress terms should be maintained in order to achieve an optimal element behavior. Also excellent results can be obtained by using an element with 57 β-terms obtained by replacing the β_{52}, β_{53} and β_{54} terms in Table 1 by $\sigma_x = \beta_{52}xy^2 + \beta_{53}xz^2$, $\sigma_y = \beta_{54}x^2y + \beta_{55}yz^2$, $\sigma_z = \beta_{56}y^2z + \beta_{57}x^2z$ without any shear stress terms for maintaining equilibrium conditions. Details of these solutions are given in Ref. 13.

For brick elements with 8, 12 and 16 nodes the independent displacement modes can all be identified from the 60 modes of the 20-node element. Thus, the corresponding stress terms can also be identified immediately from Table 1. For example, the independent displacements for the 8-node brick element are

$$u,v,w = \text{constant},\ x, y, z, xy, yz, xz, xyz$$

and the corresponding assumed stresses are

$$
\begin{aligned}
\sigma_x &= \beta_1 + \beta_2 y + \beta_3 z + \beta_4 yz \\
\sigma_y &= \beta_5 + \beta_6 x + \beta_7 z + \beta_8 xz \\
\sigma_z &= \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} xy \\
\sigma_{xy} &= \beta_{13} + \beta_{14} z \\
\sigma_{yz} &= \beta_{15} + \beta_{16} x \\
\sigma_{xz} &= \beta_{17} + \beta_{18} y
\end{aligned} \tag{35}
$$

In this case the three quadratic stress terms are necessary in order to suppress the displacement modes $u, v, w = xyz$. It is also noted that even the linear terms are not complete here. In fact, if the linear terms are complete with altogether 24 β's, the resulting element will be almost as rigid as an assumed-displacement element based on trilinear shape functions.

In using the present methods by π_R for isotropic materials, the stress expansion of the three normal stress components must be identical while the shearing stress components may be kept the same as those in Table 1. For a formulation similar to the 57-β element described earlier, the size of the $\underset{\sim}{H}$-matrix is 75 x 75 and the evaluation of an element stiffness matrix will require the inversion of one 18 x 18 ($\underset{\sim}{R}^T \underset{\sim}{H}^{-1} \underset{\sim}{R}$) matrix, one 17 x 17 $\underset{\sim}{c}$-matrix and three 8 x 8 matrices instead of the inversion of a 57 x 57 flexibility matrix. When the formulation is by π_{HW} and the stress terms are the same as that by π_R, the result will be identical to that by π_R. However, in this case the stress expansions for different normal components need not be the same and one may use fewer stress terms than that by π_R.

5 SemiLoof Elements for Plates and Shells

For thin plates and shells, Irons has pointed out the advantage of using semiLoof elements[2], for which the derivatives of lateral displacement w are not maintained compatible at the corners of the elements while normal derivatives $w_{,n}$ at some points along the interelement boundary are kept the same. By using the assumed displacement approach the derivation of a semiLoof element is very complicated. However, it is rather straightforward using the assumed stress hybrid method[12]. Within a semiLoof element the derivatives of boundary displacement w are not continuous at the corner nodes but the interelement compatibility conditions are completely satisfied.

Triangular plate elements with 12 D.O.F. as shown in Figure 2, have been derived using 9, 13 and 17 independent moment terms. The Loof-nodes are located at 1/3 points. The 9 moment terms are the complete linear terms, hence, resulting element by the hybrid method is identical to the equilibrium element by Fraeijs de Veubeke[14]. It is necessarily very flexible

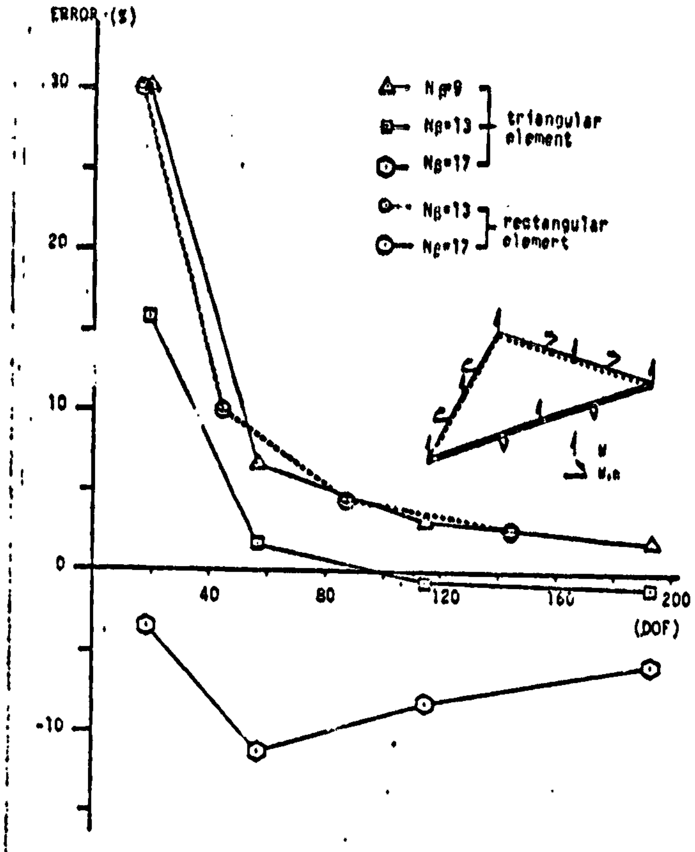

Fig. 2 Center Deflection of Clamped Square Plate Under a Concentrated Load

as shown by the solutions (Fig. 2) of the center deflection of a clamped square plate under a concentrated load. The use of four additional moment terms improves the accuracy of the solution. The addition of eight moment terms, however, makes the element too rigid. Two 16-dof semiLoof elements of rectangular shape have also been formulated using 13- and 17-β terms. These elements are necessarily more flexible than the corresponding triangular elements as shown by the results given in Figure 2.

For shell elements the equilibrium equations will always couple the membrane stresses and the moment stresses. However, for isotropic materials, they are not coupled in the stress-strain relations. Thus, in this case, by using the new approach by π_R the H-matrix can always be partitioned into four sub-matrices along the diagonal. In the new approach by π_{HW}, of course, different components in the membrane and moment stresses are always decoupled. Another added advantage of the new formulations is that the resulting $\underset{\sim}{H}$ matrices by both variational principles do not contain the geometry of the shell, making the implementation simplified.

Both triangular and quadrilateral shallow shell elements have been constructed for semiLoof boundary nodes by the present methods based on both the Hellinger-Reissner principle and the Hu-Washizu principle[15]. The assumed stress terms which for the two

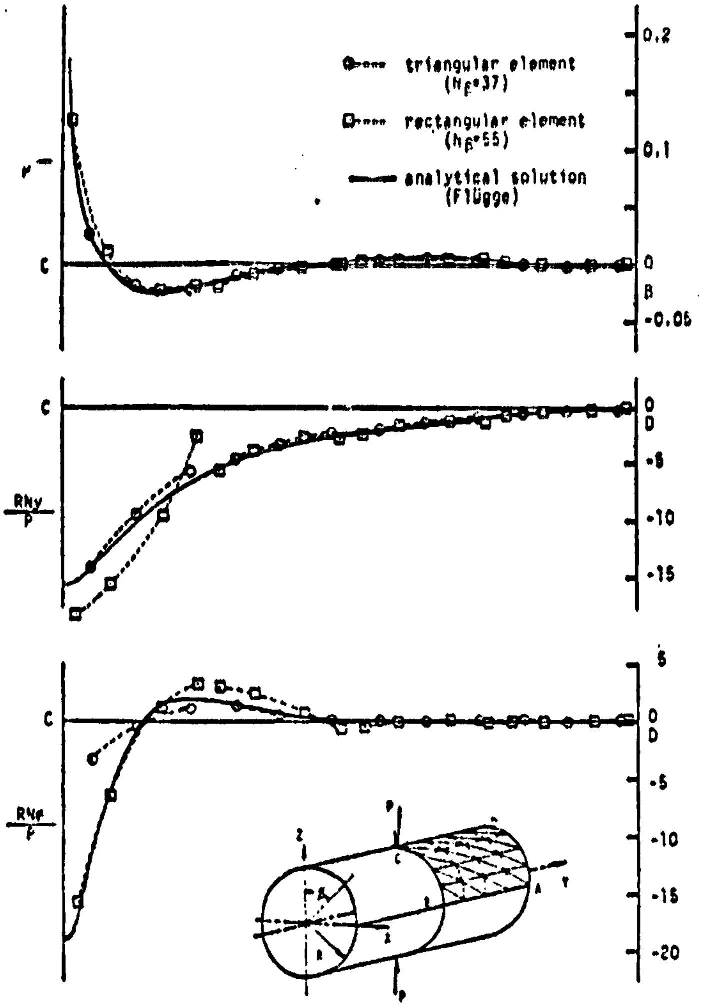

Fig. 3 Stress Distribution of a Pinched Cylinder by SemiLoof Elements

element, consist of 37 independent β's initially, and 12 constraint equatic for β are then introduced reducing the number of independent β's to 25. For the 32-dof quadrilateral shell element 55 independent β's were used initially and 22 constraint equations for β are then introduced reducing the number of independent β's to 33.

A problem of the pinched cylindrical shell with freely supported ends was analyzed using 4 x 4 mesh. It was found that the displacements and stresses obtained by these two variational methods differ so little the the resulting plots of the membrane and moment distributions (Fig. 3) can not be distinguished from one to the other. An analyses of the computing efforts for these two approaches indicates that the formulation by the Hu-Washizu principle has an edge over that by the Hellinger-Reissner principle. Both methods are simpler to formulate than the original hybrid stress method for which the membrane and moment stress terms are coupled by the equilibrium equations.

6 Conclusions

New applications of the Hellinger-Reissner principle and the Hu-Washizu principle to finite element formulations have led to a more flexible and most efficient finite element method by the assumed stresses. The possibility of using natural coordinates and the relaxation of some equilibrium conditions are the flexibility of the new hybrid/mixed stress method. The decoupling of the flexibility matrix H leads to a saving in computing time. A systematic procedure for the choice of stress terms in certain hybrid/mixed finite elements has been established based on the requirement for suppressing the kinematic deformation modes. Numerical results indicate the advantage of of maintaining the equilibrium of the lower order stress terms while relaxing that of the higher order terms. Hybrid/Mixed formulations are suitable to plates and shells using semiLoof elements.

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