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Abstract

This paper treats elliptic problems with corner singularities. Finite element approximations based on variational principles of the least squares type tend to display poor convergence properties in such contexts. Moreover, mesh refinement or the use of special singular elements do not appreciably improve matters. Here we show that if the least squares formulation is done in appropriately weighted space, then optimal convergence results in unweighted spaces like L^2.

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1. Introduction

Least squares methods have proven to be useful for indefinite elliptic systems. The Helmholtz equation is perhaps the most important example ([1]-[2]). The main advantage over standard Galerkin formulations is that least squares always yield Hermitian definite algebraic systems. Thus iterative methods like SOR can be used [3].

The main disadvantage of least squares is the extreme regularity requirement it has for optimal rates of convergence. For example, the standard least squares approximation to

$$\Delta\phi + q\phi = f \quad \text{in} \quad \Omega \tag{1.1}$$

$$\phi = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

is to require that

$$\int_\Omega \{|\operatorname{grad}\phi - \underline{u}|^2 + |\operatorname{div}\underline{u} + q\phi - f|^2\} \tag{1.3}$$

be minimized as ϕ and $\underline{u}$ vary over appropriate finite dimensional spaces. It has been shown that such an approach will give optimal L^2 convergence (i.e., second-order if linear elements are used, etc.) only in special circumstances [1]. The most restrictive condition being the existence of a number $0 < c < \infty$ such that for any f in the Sobolev space $H^1(\Omega)$ the solution ϕ of (1.1)-(1.2) lies in $H^3(\Omega)$ and

$$\|\phi\|_{H^3(\Omega)} \leq c\|f\|_{H^1(\Omega)}. \tag{1.4}$$

A regularity result of this type is valid only for smooth regions Ω, and in particular corners on $\partial\Omega$ are excluded. Numerical experiments suggest that this condition may in fact be necessary. For example, a series of calculations dealing with rectangular polygons in $\mathbb{R}^2$ having re-entrant corners showed that not only was the L^2 convergence suboptimal on uniform grids but it also remained suboptimal even in the presence of mesh refinement [4].

In this paper we consider an alternative least squares approximation in weighted Sobolev spaces. These are spaces where analogs of (1.4) are valid. The key feature of our analysis is that the error estimates are in <u>unweighted</u> norms like L^2. Selected numerical experiments with this type of formulation and with appropriate mesh refinement are also reported here.

The results obtained here generalize those in [5]. As in the latter the Hardy-Littlewood inequality plays a key role; however, in this paper the analysis takes a different direction in the sense that the discrete decomposition property introduced in [6] is also used extensively. In addition, the special mesh refinement introduced by Babuska, Kellogg, and Pitkäranta [7] is also exploited explicitly.

For simplicity we shall consider planar regions Ω having only one corner as is shown in Figure 1-1. Since existing proofs of the Hardy-Littlewood inequality [8] uses both exterior and interior cone conditions our results are restricted to interior angles θ_0 satisfying

$$0 < \theta_0 < 2\pi.$$

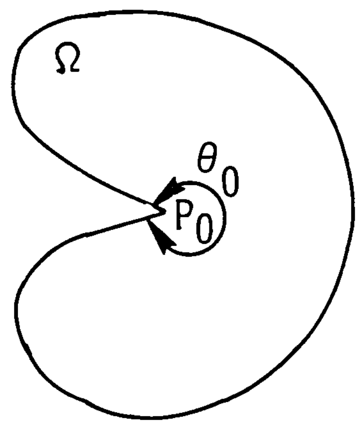

Figure 1-1: The planar region Ω.

2. Variational Formulation

Let r denote the distance to the corner at P_0 (see Figure 1-1) on $\partial\Omega$. For a nonnegative integer k and a nonnegative number β let

$$(2.1) \qquad \|\psi\|^2_{k,\beta} = \sum_{j=0}^{k} \int_\Omega r^{2(\beta+j-k)} |D^j \psi|^2,$$

and let $\vec{W}^{k,\beta}(\Omega)$ denote the closure of $[C^\infty(\bar{\Omega})]^2$ in this norm. We approach (1.1)-(1.2) with a least squares formulation in the space $\overset{o}{H}{}^1(\Omega) \times \vec{W}^{1,\beta}$ (where $\overset{o}{H}{}^1(\Omega)$ is space of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$). In particular, let

$$(2.2) \qquad S_h \subseteq \overset{o}{H}{}^1(\Omega), \qquad \vec{V}_h \subseteq \vec{W}^{1,\beta}(\Omega)$$

be finite dimensional subspaces parameterized by $h > 0$. We seek

(2.3) $$\phi_h \in S_h, \qquad u_h \in \vec{V}_h$$

which minimize

(2.4) $$\int_\Omega \{|\text{grad } \psi - \underline{v}|^2 + r^{2\beta} |\text{div } \underline{v} + q\psi - f|^2\}$$

over $(\psi,\underline{v}) \in S_h \times \vec{V}_h$. That is, we have (1.3) with a weight $r^{2\beta}$ on the most highly differential terms. Our goal is to find appropriate β for which $\phi - \phi_h, \underline{u} - \underline{u}_h$ converge in unweighted spaces like L^2 at the optimal rates.

Our analysis can be also used for the case where one has weights on both terms in (2.4). It can be shown that the weight on the first term does not help, and to minimize technical details we anticipate this result and start with (2.4). An intuitive justification for (2.4) can be obtained from the nature of the corner singularity. Indeed, if θ_0 denotes the interior angle, then the solution ϕ to (1.1)-(1.2) will have a singularity of the form

(2.5) $$\phi \sim 0(r^{\pi/\theta_0}), \qquad \underline{u} = \text{grad } \phi \sim 0(r^{\pi/\theta_0 - 1})$$

and here we are concerned with the case or re-entrant corners where $\pi < \theta_0$. Thus if one were dealing with functions $(\psi,\underline{v})$ in $S,\vec{V}$ having this type of behavior (such as special singular elements), then

(2.6) $$\psi \sim 0(r^{\pi/\theta_0}), \qquad \underline{v} = 0(r^{\pi/\theta_0 - 1})$$

and so

(2.7) $$\text{div } \underline{v} = 0(r^{\pi/\theta_0 - 2}).$$

Observe that grad ψ and $\underline{v}$ are square integrable; hence the first term in (2.4) is well defined. However, div $\underline{v}$ is not square integrable, and the second term in (2.4) is finite only if β exceeds $1 - \pi/\theta_0$. Our analysis (Section 4) indicates that if β is slightly larger than this; i.e.,

(2.8) $$1 - \pi/2\theta_0 \leq \beta \leq 1;$$

then optional rates of convergence will result in L_2 under suitable conditions S_h and V_h.

An equivalent statement of the least squares formulation involves the bilinear form

(2.9) $$B_\beta((\psi,\underline{v}),(\xi,\underline{w})) = \int_\Omega (\text{grad } \psi - \underline{v})\cdot(\text{grad } \xi - \underline{w})$$
$$+ \int_\Omega r^{2\beta}(\text{div } \underline{v} + q\psi)(\text{div } \underline{w} + q\xi)$$

and the functional

(2.10) $$F_\beta(\xi,\underline{w}) = \int_\Omega r^{2\beta} f(\text{div } \underline{w} + q\xi)$$

defined for $(\psi,\underline{v})$ and $(\xi,\underline{w})$ defined in the finite dimensional subspaces $S_h \times \vec{V}_h$ of $\overset{o}{H}{}^1(\Omega) \times \vec{W}^{1,\beta}(\Omega)$. In particular, $(\phi_h,\underline{u}_h) \in S_h \times \vec{V}_h$ is the minimizer if and only if

(2.11) $$B_\beta((\phi_h,\underline{u}_h),(\xi,\underline{w})) = F_\beta(\xi,\underline{w}) \quad \text{all} \quad (\xi,\underline{w}) \in S_h \times \vec{V}_h$$

after selecting a basis for $S_h \times \vec{V}_h$, (2.11) reduces to a set of algebraic equations [9]. As noted in the first section, the chief virtue of this system is that it is Hermitian definite.

3. Approximation and Regularity

In this section we develop the approximation and regularity results that will be needed for the error analysis in the next section. It will be assumed throughout this paper that the function q is bounded away from eigenvalues of $-\Delta$ (with Dirichlet boundary conditions). Thus (1.1)-(1.2) has a unique solution. Of fundamental importance is the following result due to Kondratiev [10].

Theorem 3.1: Let ϕ satisfy (1.1)-(1.2) on the region Ω shown in Figure 1-1. Then for $t \geq 0$, $t + 1 \geq \beta > t - \frac{\pi}{\theta_0} + 1$

(3.1) $$\|\phi\|_{t+2,\beta} \leq C\|f\|_{t,\beta}$$

and

(3.2) $$\|\nabla\phi\|_{t+1,\beta} \leq C\|f\|_{t,\beta}.$$

We now turn to approximation. The starting point is a special triangulation first introduced by Babuska, Kellogg, and Pitkäranta [7]. Three conditions are needed in order to obtain the appropriate approximation results. The first is the standard angle condition on the individual triangles [9]. To describe the second let h be the maximum mesh spacing associated with the triangulation. Then, given h and weight factor β, each triangle T must satisfy

(3.3) $$d(T) = O(hr^{\beta})$$

where

$$d(T) = \max_{x,y \in T} |x-y|.$$

The third and most crucial rule governs the amount of refinement. Given h and β, the triangles which have the corner as a vertex must satisfy

(3.4) $$d(T) \leq Ch^{\frac{1}{1-\beta}}.$$

For a mesh refined according to these conditions, Pitkäranta [11] proves the following.

Theorem 3.2: Let $0 \leq \beta < 1$. Let S_h be the space of continuous piecewise linear polynomials. Let ψ be defined on Ω such that

(3.5) $$\int_{\Omega} r^{2\beta} \, |D^2 \psi|^2 < \infty.$$

Then there exists a constant C depending on β such that

(3.6) $$\min_{\psi_h \in S_h} \{\int_{\Omega} |D^1(\psi - \psi_h)|^2 + h^{-1} \int_{\Gamma} r^{-\beta} \, |\psi - \psi_h|^2\} \leq Ch^2 \int_{\Omega} r^{2\beta} \, |D^2 \psi|^2.$$

Proof: See [11].

Equation (3.6) implies that

(3.7) $$\min_{\psi_h \in S_h} \|\psi - \psi_h\|_1 \leq Ch\|\psi\|_{2,\beta}.$$

We will need a similar result for approximation in other norms.

Lemma 3.1: For $0 \leq \beta < 1$

(3.8) $$\min_{\psi_h \in S_h} \{\int_\Omega |\psi - \psi_h|^2 + h^2 \int_\Omega r^{2\beta} |D^1(\psi - \psi_h)|^2\} \leq Ch^4 \int_\Omega r^{4\beta} |D^2 \psi|^2.$$

Proof: See [12].

Lemma 3.1 contains two approximation results, namely

(3.9) $$\|\psi - \psi_h\|_0 \leq Ch^2 \|\psi\|_{2,2\beta}$$

and

(3.10) $$\|\psi - \psi_h\|_{1,\beta} \leq Ch\|\psi\|_{2,2\beta}.$$

Our analysis will require estimates in dual norms. In particular, the following will be important:

(3.11) $$\|\psi\|_{*,\beta} = \sup_{\eta \in W^{1,2\beta}} \int_\Omega \frac{r^{2\beta} \eta\psi}{\|\eta\|_{1,2\beta}}.$$

The following is an inequality that will be useful in the sequel.

Lemma 3.2: $\|\mathrm{div}\ v\|_{*,\beta} \leq C\|v\|_0.$

Proof: Follows from (3.11), Schwarz's inequality, and the inequality

$$\|\nabla(r^{2\beta}\eta)\|^2 \leq C\|\eta\|^2_{1,2\beta}. \tag{3.12}$$

The final result that will be needed is the analog of the Grid Decomposition Property introduced in [6]. This is a condition on $\vec{V}_h$, and not all finite elements spaces have this property as we will indicate in the sequel. The version we will need can be stated as follows. There is a number C, $0 < C < \infty$ and independent of $\vec{V}_h$ such that for each $\underline{v}_h$ in $\vec{V}_h$ there are w_h, z_h in $\vec{V}_h$ for which

$$v_h = w_h + z_h$$

with

$$\operatorname{div} z_h = 0 \qquad \int_\Omega z_h \cdot w_h = 0$$

and

$$\|w_h\|_0 \leq C\|\operatorname{div} v_h\|_{*,\beta}. \tag{3.13}$$

4. Error Estimates

The analysis given here has a structure similar to that found in the analysis of mixed methods (see, for example, [6]). The first step is use the basic orthogonality property derived from (2.11) to get an estimate in a nonstandard norm; in this case it is

$$|||(\psi,\underline{v})||| = B_\beta((\psi,\underline{v}),(\psi,\underline{v}))^{1/2}, \tag{4.1}$$

where $B_\beta(\cdot,\cdot)$ is the bilinear form defined by (2.9). One then uses this to derive estimates in the negative norm (3.11), from which one can obtain L^2 estimates for $\phi - \phi_h$. These plus the grid decomposition property yield L^2 estimates for $u - u_h$. In the sequel we let

(4.2) $$\varepsilon = \phi - \phi_h$$

and

(4.3) $$e = \underline{u} - \underline{u}_h.$$

In addition throughout we shall assume

(4.4) $$1 - \pi/2\theta_0 < \beta \leq 1.$$

Lemma 4.1:

(4.5) $$|||(\varepsilon,\underline{e})||| \leq Ch\{\|f\|_{0,\beta} + \|f\|_{1,2\beta}\}.$$

<u>Proof</u>: It follows from (2.11) that

(4.6) $$B_\beta((\varepsilon,\underline{e}),(\psi_h,\underline{v}_h)) = 0$$

for all $(\psi_h,\underline{v}_h) \in S \times \vec{V}_h$. Thus

(4.7) $$|||(\varepsilon,\underline{e})||| \leq \inf|||(\phi - \psi_h,\underline{u} - \underline{v}_h)|||$$

where the inf is taken over all $(\phi_h,\underline{v}_h) \in S_h \times V_h$. Using the approximation properties in Section 3 we obtain

$$(4.8)\qquad |||(\varepsilon,\underline{e})||| \le Ch\{\|u\|_{1,\beta} + \|\phi\|_{2,\beta} + \|\underline{u}\|_{2,2\beta}\}.$$

The inequality (4.5) follows from (4.8) using Theorem 3.1. We now establish an error estimate for $\operatorname{div} e + q\varepsilon$ in the dual norm $\|\cdot\|_{*,\beta}$.

Lemma 4.2:

$$(4.9)\qquad \|\operatorname{div} e + q\varepsilon\|_{*,\beta} \le C|||(e,\varepsilon)|||.$$

<u>Proof</u>: For $\eta \in W^{1,2\beta}$ solve

$$(4.10)\qquad \Delta\xi + q\xi = \eta \quad \text{in} \quad \Omega$$

$$(4.11)\qquad \xi = 0 \quad \text{on} \quad \partial\Omega$$

with

$$(4.12)\qquad p = \operatorname{grad} \xi.$$

Observe that by orthogonality and (4.12)

$$(4.13)\qquad B((\varepsilon,e),(\xi - \xi_h, p - p_h)) = \int r^{2\beta}\, \eta(\operatorname{div} e + q\varepsilon).$$

Therefore,

$$(4.14)\qquad \int r^{2\beta}\, \eta(\operatorname{div} e + q\varepsilon) \le |||(\varepsilon,e)|||\ \ |||(\xi - \xi_h, p - p_h)|||$$

$$\le |||(\varepsilon,e)|||\ \ \{\|\xi - \xi_h\|_1 + \|p - p_h\|_{1,\beta}\}.$$

We choose ξ_h, p_h so that

$$\|\xi - \xi_h\|_1 \leq Ch\|\xi\|_{2,\beta} \leq Ch\|\xi\|_{3,2\beta} \tag{4.15}$$

and

$$\|p - p_h\|_{1,\beta} \leq Ch\|p\|_{2,2\beta}. \tag{4.16}$$

Using regularity (3.1) and (3.2), the estimate (4.9) is obtained by taking the sup over η in (4.14) with $\|\eta\|_{1,2\beta} \leq 1$.

We now state and prove our two main results.

Theorem 4.1: Assume that q is bounded on $\overline{\Omega}$. Then

$$\|\varepsilon\|_0 \leq Ch|||(\varepsilon,\underline{e})|||. \tag{4.17}$$

<u>Proof</u>: Solve

$$\Delta\eta + q\eta = \varepsilon \quad \text{in} \quad \Omega \tag{4.18}$$

$$\eta = 0 \quad \text{on} \quad \partial\Omega \tag{4.19}$$

for η. For any $\eta_h \in S_h$ we have

$$B_\beta((\varepsilon,\underline{e}),(0,\eta - \eta_h)) = B_\beta((\varepsilon,\underline{e}),(0,\eta)). \tag{4.20}$$

But

(4.21)

$$B_\beta((\varepsilon,\underline{e}),(0,\eta)) = \int_\Omega [\text{grad }\varepsilon\cdot\text{grad }\eta - e\cdot\text{grad }\eta + r^{2\beta}\, q\eta(\text{div }\underline{e} + q\varepsilon)].$$

On the other hand

$$(4.22) \qquad \int_\Omega \varepsilon^2 = \int_\Omega \epsilon\,(\Delta\eta + q\eta) = \int[-\mathrm{grad}\ \varepsilon\cdot\mathrm{grad}\ \eta + q\eta\varepsilon].$$

Since

$$(4.23) \qquad \int_\Omega e\cdot\mathrm{grad}\ \eta = -\int_\Omega \mathrm{div}\ e\eta$$

putting (4.21) into (4.22) gives

$$(4.24) \qquad \int_\Omega \varepsilon^2 = -B_\beta((\varepsilon,\underline{e}),(0,\eta - \eta_h)) + \int_\Omega (1 + r^{2\beta}\ q)\cdot(\mathrm{div}\ \underline{e} + q\varepsilon).$$

To estimate the right hand side of (4.24) we note that η_h can be chosen such that

$$(4.25) \qquad B_\beta((\varepsilon,\underline{e}),(0,\eta - \eta_h) \leqq |||(\varepsilon,\underline{e})|||\ \ |||(0,\eta - \eta_h)|||$$

$$\leq Ch\|\eta\|_{2,\beta}\ |||(\varepsilon,\underline{e})|||.$$

Also

$$(4.26) \qquad \|\eta\|_{2,\beta} \leq c\|\varepsilon\|_{0,\beta} \leq C\|\varepsilon\|_0.$$

Finally,

$$(4.27) \qquad \int_\Omega (1 + r^{2\beta}\ q)\eta(\mathrm{div}\ e + q\varepsilon) \leq c\|\mathrm{div}\ e + q\varepsilon\|_{*,\beta}\ \|r^{-2\beta}\ \eta\|_{1,2\beta}.$$

Moreover,

$$(4.28) \qquad \|r^{-2\beta}\ \eta\|_{1,2\beta} \leq C\|\eta\|_{1,0} \leq C\|e\|_{-1,0} \leq C\|\varepsilon\|_0.$$

Combining these we obtain (4.17).

Corollary: Assume q is bounded on $\overline{\Omega}$. Then

$$\|\operatorname{div} \underline{e}\|_{*,\beta} \leq Ch|||(\varepsilon,\underline{e})|||. \tag{4.29}$$

Theorem 4.2: Suppose q is bounded on $\overline{\Omega}$ and suppose $S_h \times V_h$ satisfies the grid decomposition property (3.13). Then

$$\|\underline{e}\|_0 \leq Ch|||(\varepsilon,\underline{e})||| + C \inf_{\hat{\underline{u}}_h \in V_h} \|\underline{u} - \hat{\underline{u}}_h\|_0. \tag{4.30}$$

Proof: Let $\hat{\underline{u}}_h \in V_h$ be given. Use (3.13) to decompose $\underline{u}_h - \hat{\underline{u}}_h$ as follows:

$$\underline{u}_h - \hat{\underline{u}}_h = \underline{w}_h + \underline{z}_h, \tag{4.31}$$

where

$$\operatorname{div} \underline{z}_h = 0, \qquad \int_\Omega \underline{w}_h \cdot z_h = 0 \tag{4.32}$$

and

$$\|\underline{w}_h\|_0 \leq C\|\operatorname{div}(\underline{u}_h - \hat{\underline{u}}_h)\|_{*,\beta}. \tag{4.33}$$

Note that for any $(\psi_h,\underline{v}_h) \in S_h \times V_h$ we have

$$0 = B_\beta((\varepsilon,\underline{e}),(\psi_h,\underline{v}_h)) = \int_\Omega (\operatorname{grad} \varepsilon - \underline{e})(\operatorname{grad} \phi_h - \underline{v}_h) + \int_\Omega r^{2\beta}(\operatorname{div} \underline{e} + q\varepsilon)(\operatorname{div} \underline{v}_h + q\phi_h). \tag{4.34}$$

Letting $\phi_h = 0$ and $\underline{v}_h = \underline{z}_h$ we have

$$\int_\Omega \underline{z}_h \cdot \underline{e} = 0. \tag{4.35}$$

Thus

$$\int_\Omega \underline{z}_h \cdot \underline{z}_h = \int \underline{z}_h(\underline{u}_h - \hat{\underline{u}}_h) = \int \underline{z}_h \cdot (\underline{u} - \hat{\underline{u}}_h), \tag{4.36}$$

i.e.,

$$\|\underline{z}_h\|_0 \leq \|\underline{u} - \hat{\underline{u}}_h\|_0. \tag{4.37}$$

But from (4.33) and Lemma 3.2

$$\begin{aligned} \|\underline{w}_h\|_0 &\leq C\|\mathrm{div}(\underline{u}_h - \hat{\underline{u}}_h)\|_{*,\beta} \leq C\|\mathrm{div}\ \underline{e}\|_{*,\beta} + C\|\mathrm{div}(\underline{u} - \hat{\underline{u}}_h)\|_{*,\beta} \\ &\leq C\|\mathrm{div}\ \underline{e}\|_{*,\beta} + C\|\underline{u} - \hat{\underline{u}}_h\|_0. \end{aligned} \tag{4.38}$$

It follows that

$$\|e\|_0 \leq \|\underline{u} - \hat{\underline{u}}_h\|_0 + \|\underline{u}_h - \hat{\underline{u}}_h\|_0 \leq \|u - \hat{\underline{u}}_h\|_0 + \|\underline{w}_h\|_0 + \|\underline{z}_h\|_0. \tag{4.39}$$

Combining (4.37)-(4.39) with the above Corollary we obtain (4.30).

5. Numerical Results

In this section we report results of computations which demonstrate the weighted least squares method and confirm the analytical results of the previous section. All numerical experiments were performed on a VAX 11-750 computer. Special attention is given to the roles played by mesh refinement and the weight.

All of the examples deal with the Laplace equation

(5.1) $$\Delta\phi = f.$$

We actually solve the equivalent first-order system

(5.2) $$\operatorname{div} \underline{u} = f$$

(5.3) $$\underline{u} - \operatorname{grad} \phi = 0.$$

The insensitivity of least squares to type of boundary condition (Dirichlet, Neumann, or Mixed) has already been demonstrated [2]. Thus it is sufficient for the examples reported here to use the Dirichlet condition

(5.4) $$\phi = g \quad \text{on} \quad \Gamma.$$

Consider the L-shaped membrane shown in Figure 5-1.

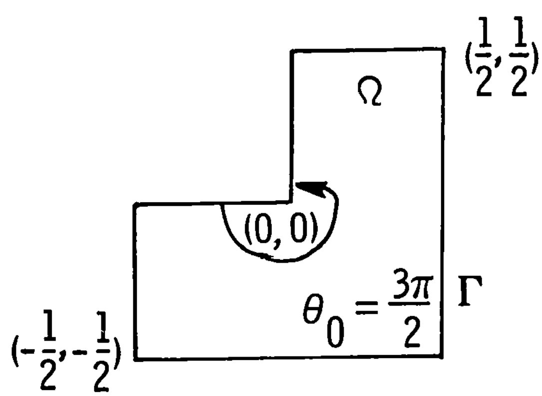

Figure 5-1: L-shaped Membrane.

The re-entrant corner has measure $\theta_0 = \frac{3\pi}{2}$. Thus from [13] we know that the solution of (5.1) with homogeneous boundary condition has a singularity with leading term

$$\psi = r^{2/3} \sin[\tfrac{2}{3}(\theta - \pi)] \tag{5.5}$$

where (r,θ) are standard polar coordinates. Therefore, we use ψ in (5.5) as our test solution. Analysis in Section 4 indicates that optimal rates will be assured in the weighted least squares solution by the proper choice of weight exponent β and correct mesh refinement. The approximating space for both ϕ and $\underline{u}$ is the space of continuous piecewise linear polynomials. For $\theta_0 = \frac{3\pi}{2}$, (3.4) and (4.4) tell us we need

$$h_{min} = O(h^3) \quad \text{and} \quad \frac{2}{3} < \beta < 1.$$

Symmetry allows us to solve on the region shown in Figure 5-2, with a tangency condition imposed in the line of symmetry, as shown.

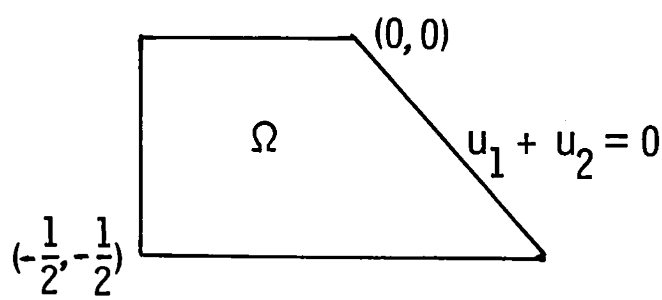

Figure 5-2: Computational Region for L-shaped Membrane.

Figures 5-3 and 5-4 display two of the finite element grids used in the numerical experiments. Note that each triangulation is constructed by subdividing the basic criss-cross grid so that every element is an isosceles right triangle. This type of refinement was chosen instead of the coordinate stretching from a uniform mesh since the latter contained some elements with large aspect ratios. To assure accurate determination of convergence rates, the meshes were constructed so that the number of points N varied with the maximum mesh spacing h according to the relation

$$N = O(h^{-2}).$$

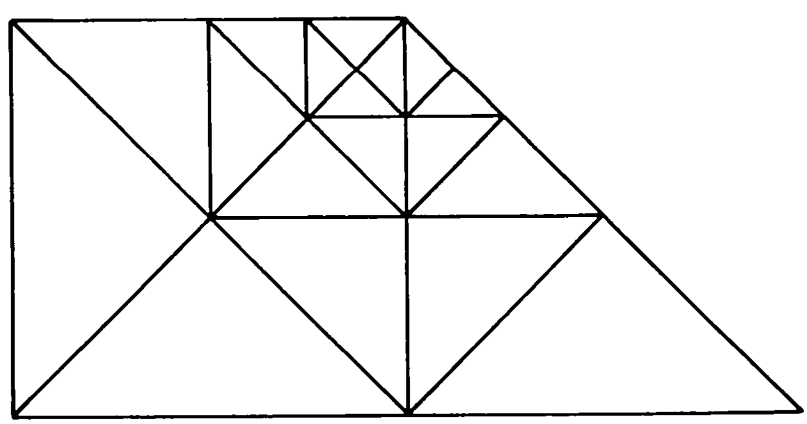

Figure 5-3: Refined Mesh, $h = 1/2$.

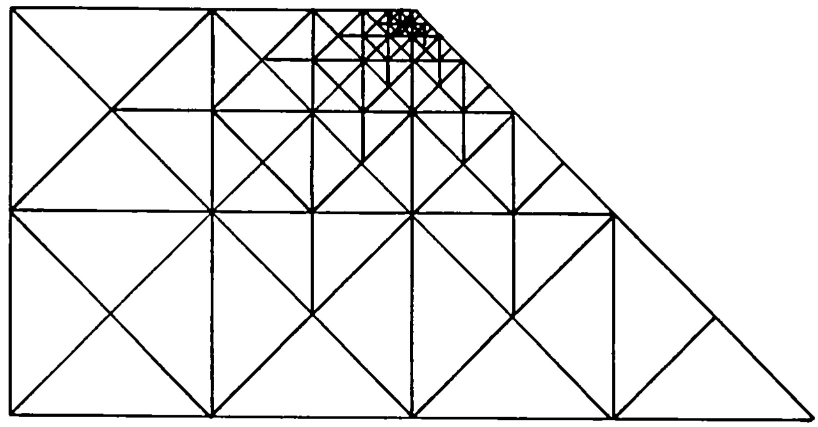

Figure 5-4: Refined Mesh, $h = 1/4$.

The L_2 errors in $\underline{u}$ and ϕ are displayed in Figure 5-5. Results from the weighted least squares scheme on a refined mesh are contrasted with those using the standard least squares scheme on a uniform grid.

We also applied the weighted least squares scheme to Laplace's equation on a square region with a crack, illustrated in Figure 5-6. This model is characteristic of such physical problems as torsion of a cracked beam and flow over a very thin airfoil.

It must be noted that the analytical results do not hold for this problem because the cone condition used in the regularity result is not satisfied. However, the results hold if the crack is replaced by a re-entrant corner with measure $\theta_0 = 2\pi - \varepsilon$. For this problem, (3.4) and (4.4) indicate that the mesh refinement and weight parameters must satisfy

$$h_{min} = O(h^4) \quad \text{and} \quad \frac{3}{4} < \beta < 1.$$

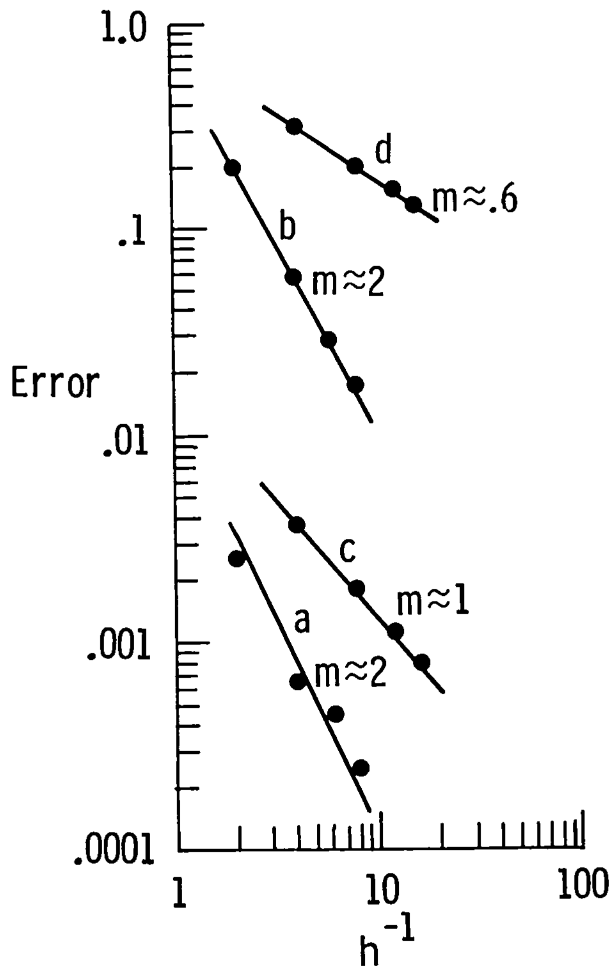

Figure 5-5: L_2 error in weighted least squares approximation to ϕ (a) and $\underline{u} = \nabla\phi$ (b) contrasted to standard (unweighted) least squares approximation to ϕ (c) and u (d).

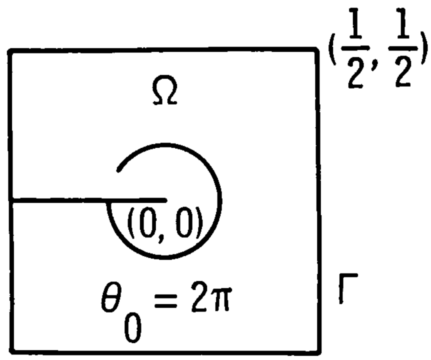

Figure 5-6: Cracked Square.

For the test solution we again use the leading term in the singularity which is

(5.6) $$\psi = r^{1/2} \sin[\tfrac{1}{2} (\theta - \pi)].$$

As before, by symmetry we will solve only on the region shown in Figure 5-7.

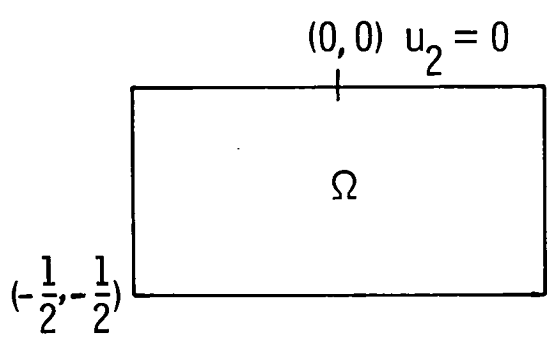

Figure 5-7: Computational Region for Crack Problem.

The L_2 errors in the weighted least squares solution to the crack problem are displayed in Figure 5-8.

The following conclusions can be drawn from these computations. The accuracy lost in the least squares solution due to a corner-type singularity is fully restored by the use of a weighted scheme on a refined mesh. Specific criteria, which depend on the measure of the corner, have been developed to determine the correct weight and order of refinement.

Moreover, the weighted scheme inherits all the advantages associated with least squares. Second-order accuracy is achieved using the same finite element spaces for ϕ and $\underline{u}$. The associated matrix system is always symmetric and positive definite, allowing solutions by standard iterative techniques. The essential boundary conditions can be included in the variational principle instead of being imposed directly on the approximating finite dimensional space.

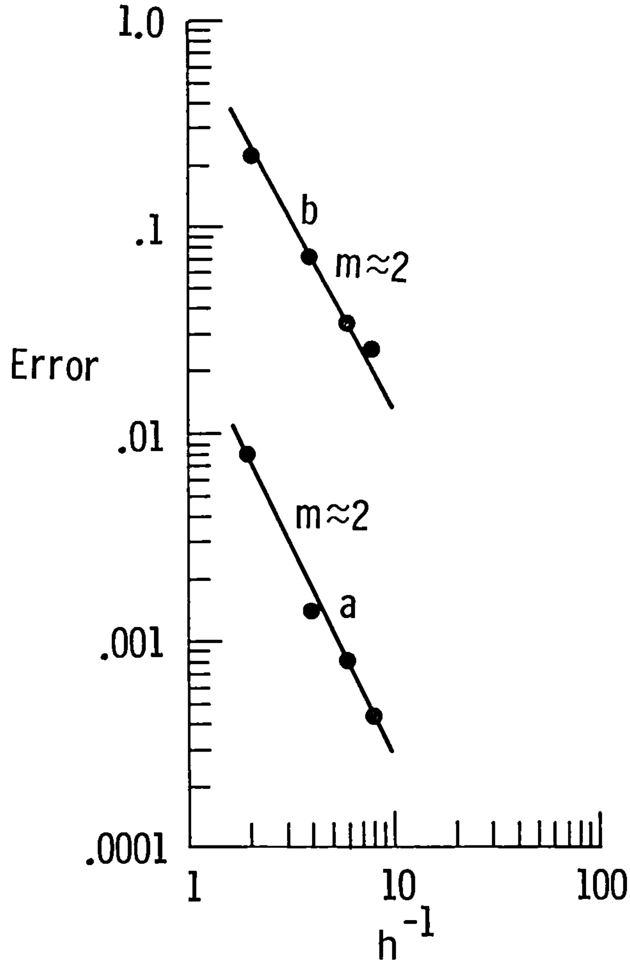

Figure 5-8: L_2 errors in weighted least squares solution to crack problem for ϕ (a) and $\underline{u} = \nabla\phi$ (b).

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16. Abstract

This paper treats elliptic problems with corner singularities. Finite element approximations based on variational principles of the least squares type tend to display poor convergence properties in such contexts. Moreover, mesh refinement or the use of special singular elements do not appreciably improve matters. Here we show that if the least squares formulation is done in appropriately weighted space, then optimal convergence results in unweighted spaces like L^2.

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