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# DIFFERENTIATING MATRICES FOR ARBITRARILY SPACED GRID POINTS

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## Abstract

Differentiating matrices allow the numerical differentiation of functions defined at points of a discrete grid. The present work considers a type of differentiating matrix based on local approximation on a sequence of sliding subgrids. Previous derivations of this type of matrix have been restricted to grids with uniformly spaced points, and the resulting derivative approximations have lacked precision, especially at endpoints. The new formulation allows grids which have arbitrarily spaced points. It is shown that high accuracy can be achieved through use of differentiating matrices on non-uniform grids which include "near-boundary" points. Use of the differentiating matrix as an operator to solve eigenvalue problems involving ordinary differential equations is also considered.

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## Introduction

The present work on differentiating matrices has its origins in the study of the differential equations associated with rotating beam configurations which model the vibrations and aeroelastic stability of rotating structures such as helicopter rotor blades and propeller blades. The fourth-order boundary value problems associated with rotating beams do not, in general, have useful closed-form solutions. Consequently, most theoretical work on these problems has been asymptotic or numerical in nature.

In one approach to numerical solution of these problems, harmonic time dependence is assumed to reduce the governing partial differential equation to an ordinary differential equation in space which involves an eigenvalue. The fundamental derivative representing beam curvature is then taken as a new dependent variable, and the eigenvalue problem is rewritten as a second-order integro-differential equation (White & Malatino, 1975). This formulation can be conveniently expressed using appropriate integral and differential operators. In general, boundary operators will also be required.

The operator equation for the continuous solution may be converted into a matrix operator equation for a finite dimensional solution vector by evaluating the operator equation at a finite set of discrete grid points which span the interval of interest. The key question now becomes the manner in which the differential and integral operator matrices are approximated.

The present work examines one approach which approximates a differential operator through use of a differentiating matrix. The specific type of differentiating matrix considered uses a sequence of sliding subgrids to obtain local approximations at grid points. To maintain simplicity while illustrating the basic features of this technique, the discussion here will be

restricted to considering local approximations obtained through Lagrange interpolation. Similarly, the eigenvalue problems considered here are the simple historical test problems involving the harmonic equation. Applications of the present technique to a variety of rotating beam problems will be given in a later paper.

Differentiating matrices provide a means of numerically differentiating a function that is expressed in terms of the values of the function at increments of the independent variable, i.e., on a discrete grid with a finite number of points. When the values of the function at the grid points are arranged as a column vector, premultiplication by a differentiating matrix produces a vector containing approximate values of the derivative at the grid points.

An attractive feature of differentiating matrices is that their derivation requires only information about the grid points, and no explicit information is needed about the function to be differentiated. Thus, so long as the grid is not changed, the same differentiating matrix can be used to differentiate any function whose values are known at the grid points. This feature is shared by integrating matrices (Vakhitov, 1966; Hunter, 1970; Lakin, 1979) which may be used to numerically integrate functions on discrete grids. Differentiating matrices of the present type have previously been derived only for grids with equally spaced points. On these uniform grids, the same Newton forward difference interpolation formulas lead to both an integrating matrix and a differentiating matrix (Hunter & Jainchell, 1969). Indeed, these two types of matrices are natural complements. Integrating matrices have an identically zero row, and are thus intrinsically singular. Consequently, a differentiating matrix is necessary to undo an integration and acts as an approximate inverse.

The fact that these matrices depend only on the grid structure allows them to be used as approximations for differential and integral operators in eigenvalue problems. The result of this approximation is a matrix eigenvalue problem which can be solved by standard methods.

Work in recent years has focused almost exclusively on the development and use of integrating matrices to solve problems associated with rotating beams (White & Malatino, 1975; Kvaternik, White, & Kaza, 1978; Lakin, 1982). Despite its links to the integrating matrix, the differentiating matrix has been almost totally ignored. Two factors contributed to this situation. First, for uniform grids, a differentiating matrix was found by Hunter and Jainchell to give less accurate results than a corresponding integrating matrix in a simple test problem involving the harmonic equation. Further, for the vibrations of a rotating cantilevered beam, the problem of principal interest in earlier work, it was found that the matrix eigenvalue problem could be formulated using only the integrating matrix (White, 1978). However, this is not the case for other problems of current interest. For example, in the rotating beam model for vibrations of spokes of an energy-storing flywheel, the boundary conditions are such that reformulation as a matrix eigenvalue problem requires the use of both differentiating and integrating matrices. Indeed, this is the case for many beam problems which do not involve cantilevered boundary conditions.

In the present work, differentiating matrices are first derived for grids with arbitrarily spaced points. This is a natural generalization, as physical problems often have spatial variations in material properties which necessitate the use of unequal increments in the independent variable. However, in the present context, removal of the uniform grid restriction also

serves another, more fundamental, purpose. For differentiating matrices on uniform grids, relatively large approximation errors are found to occur at the endpoints of the interval being considered. Errors at interior points are considerably smaller. This situation is not surprising given the one-sided nature of the approximation at an endpoint and the intrinsically two-sided character of the derivative operation. In a formulation which is not restricted to uniform grids, one or more additional interior grid points can be added "close" to each endpoint. It will be shown that, at the cost of a slight increase in the size of the differentiating matrix, the addition of even a single "near-boundary" point dramatically reduces the errors in the derivative approximations. A reconsideration of the test problem of Hunter and Jainchell further shows that the results obtained using the differentiating matrix with near-boundary points as an operator in an eigenvalue context are now fully comparable in accuracy with the earlier integrating matrix results. The reliability of the generalized differentiating matrix is further demonstrated by considering two additional test problems involving both integrating and differentiating matrices.

## 2. Derivation of the Differentiating Matrix

Suppose that  $f(x)$  is a function whose values are known on a grid  $G$  consisting of the  $N+1$  discrete points  $x_0, x_1, \dots, x_N$ . The spacings  $x_j - x_{j-1}$  ( $j = 1, 2, \dots, N$ ) must be non-negative, but are otherwise arbitrary. Let  $\{f\}$  denote the  $N+1$ -dimensional column vector  $(f_0, f_1, \dots, f_N)^T$  where  $f_j = f(x_j)$ . A differentiating matrix on the grid  $G$  is then a square  $N+1$ -by- $N+1$  matrix  $[D]$  such that

$$[D] \{f\} = \{f'\} \quad (2.1)$$

where  $\{f'\} = (f'_0, f'_1, \dots, f'_N)^t$ , with  $f'_j = f'(x_j)$ , is a column vector of the derivative values at the grid points. In particular, let  $[D_i]$  denote the  $i$ -th row vector of the matrix  $[D]$ . Then, by (2.1), the dot product  $[D_i] \cdot \{f\} = f'_i$ . Hence,  $[D_i]$  can be obtained directly from any expression for  $f'_i$  which can be written as a linear combination of the values  $f_j$  for  $j = 0, \dots, N$ . In practice, the row vectors of  $[D]$  will be obtained from approximations to  $f'_i$  rather than exact expressions. Consequently, a differentiating matrix on  $G$  will not be unique.

Differentiating matrices on  $G$  may be obtained in a number of ways including interpolation and least-squares polynomial approximation, and the form of the matrix depends on both the underlying technique and the degree of the approximating polynomials. In the present derivation, rows of  $[D]$  will be obtained using Lagrange interpolation polynomials of degree  $n \leq N$  on a sequence of appropriate subgrids with  $n+1$  points.

As a first step in obtaining a suitable approximation to  $f'_i$ , and hence,  $[D_i]$ ,  $f(x)$  will be approximated by an interpolating polynomial of degree  $n$  on an appropriate subgrid  $G_\gamma$  of  $G$  which contains the point  $x_i$ . In particular, suppose that  $G_\gamma$  consists of the  $n+1$  grid points  $x_\gamma, x_{\gamma+1}, \dots, x_{\gamma+n}$ . As the eventual goal is an approximation to  $f'_i$ ,  $x_i$  should be centered as much as possible in  $G_\gamma$ . This is not a problem if  $n$  is even as  $x_i$  will be exactly centered in  $G_\gamma$  if  $\gamma = i - n/2$ . If  $n$  is odd, however, two choices of  $\gamma$  are possible:  $\gamma = i - (n+1)/2$  or  $\gamma = i - (n-1)/2$ . It will, of course, not be possible to center the endpoints  $x_0$  or  $x_N$  or points  $x_i$  which either have values of  $\gamma$  which are less than

zero, or values of  $\gamma + n$  greater than  $N$ . Consequently,  $\gamma$  must be restricted to the range  $0 \leq \gamma \leq N-n-1$ .

If  $G$  is a uniform grid,  $f(x)$  can be approximated on  $G_\gamma$  by a Newton forward or backward difference interpolating polynomial which can then be rewritten as a linear combination of  $f_\gamma, f_{\gamma+1}, \dots, f_{\gamma+n}$ . Newton forward difference formulas were used by Hunter and Jainchell in their initial derivation of the differentiating matrix. These difference formulas may be generalized to arbitrary grids, if desired, using divided differences. However, it rapidly becomes quite difficult on non-uniform grids to perform the required separation of function values and grid information. Hence, the present derivation utilizes Lagrange interpolating polynomials which are valid for arbitrary grids and, as defined, separate function values and grid information. The required approximation to  $f(x)$  on  $G_\gamma$  is now

$$f(x) = \sum_{j=0}^n L_j^{(\gamma)}(x) f_{\gamma+j} \quad (2.2)$$

where the Lagrange coefficients  $L_j^{(\gamma)}(x)$  are given by

$$L_j^{(\gamma)}(x) = \left\{ \prod_{\substack{m=0 \\ m \neq j}}^n (x - x_{\gamma+m}) \right\} / \left\{ \prod_{\substack{m=0 \\ m \neq j}}^n (x_{\gamma+j} - x_{\gamma+m}) \right\}. \quad (2.3)$$



Approximations to  $f'(x)$  on  $G_\gamma$  can be obtained directly from (2.2) by differentiation. Evaluation at  $x_i$  now gives

$$f'_i = \sum_{j=0}^n L_j^{(\gamma)'}(x_i) f_{\gamma+j} \quad (2.4)$$

where

$$L_j^{(\gamma)'}(x_i) = \sum_{\substack{m=0 \\ m \neq j}}^n \frac{L_j^{(\gamma)}(x_i)}{x_{\gamma+j} - x_{\gamma+m}}. \quad (2.5)$$

In fact, the general form (2.5) of  $L_j^{(\gamma)'}(x_i)$  can be considerably simplified. Suppose that  $i = \gamma + k$  with  $0 \leq k \leq n$ . Then, for  $j$  not equal to  $k$

$$L_j^{(\gamma)'}(x_i) = \left\{ \prod_{\substack{m=0 \\ m \neq k, j}}^n (x_i - x_{\gamma+m}) \right\} / \left\{ \prod_{\substack{m=0 \\ m \neq j}}^n (x_{\gamma+j} - x_{\gamma+m}) \right\} \quad (2.6)$$

and for  $j = k$

$$L_k^{(\gamma)'}(x_i) = \sum_{\substack{m=0 \\ m \neq k}}^n \frac{1}{x_i - x_{\gamma+m}}. \quad (2.7)$$

The  $i$ -th row vector  $[D_i]$  of  $[D]$  can now be obtained in a straightforward manner from equation (2.4). If the elements of this row vector are  $d_{ij}$ , then  $d_{ij} = 0$  for  $j < \gamma$  and  $j > N - \gamma - n$  while  $d_{i, \gamma+m} = L_m^{(\gamma)'}(x_i)$  for  $m = 0, \dots, n$ .

The introduction of near-boundary points can drastically change the form of the differentiating matrix on a grid. For example, if  $G$  is a uniform grid of five points with  $x_0 = 0$  and  $x_4 = 4$ , the differentiating matrix based on fourth-degree polynomials is

$$[D] = \frac{1}{12} \begin{bmatrix} -25 & 48 & -36 & 16 & -3 \\ -3 & -10 & 18 & -6 & 1 \\ 1 & -8 & 0 & 8 & -1 \\ -1 & 6 & -18 & 10 & 3 \\ 3 & -16 & 36 & -48 & 25 \end{bmatrix}. \quad (2.8)$$

Adding two near-boundary points at .01 and 3.99 to the uniform grid to give a new non-uniform grid of seven points, the differentiating matrix based on sixth-degree polynomials has the form

$$[D] = \begin{bmatrix} -102.33 & 102.37 & -.05 & .03 & -.02 & .25 & -.25 \\ -97.69 & 97.65 & .05 & -.03 & .02 & -.25 & .25 \\ 18.55 & -19.18 & -.16 & 1.12 & -.50 & 6.35 & -6.18 \\ -8.27 & 8.51 & -.89 & 0 & .89 & -8.51 & 8.27 \\ 6.18 & -6.35 & .50 & -1.12 & .16 & 19.18 & -18.55 \\ -.25 & .25 & -.18 & .03 & -.05 & -97.65 & 97.69 \\ .25 & -.25 & .18 & -.03 & .05 & -102.37 & 102.33 \end{bmatrix} \quad (2.9)$$

Matrices which approximate higher derivatives may also be defined. For example, the second derivative matrix  $[D_2]$  on  $G$  is an  $N+1$ -by- $N+1$  matrix such that

$$[D_2] \{f\} = \{f''\} \quad (2.10)$$

where  $\{f''\}$  is a column vector of values of  $f''(x)$  at the grid points. If  $n = N$ , then as noted by Hunter and Jainchell,

$$[D_2] = [D]^2. \quad (2.11)$$

However, if  $n < N$ , then the second derivative matrix cannot be obtained simply by squaring  $[D]$  as the leading and/or trailing zeros in the rows of  $[D]$  lead to inaccurate approximations. Rather, the second derivative matrix must be computed directly from the subgrid approximations

$$f''(x_i) = \sum_{j=0}^n L_j^{(\gamma)''}(x_i) f_{\gamma+j} \quad (2.12)$$

where

$$L_j^{(\gamma)''}(x_i) = \sum_{\substack{p=0 \\ p \neq j}}^{n-1} \sum_{\substack{m=p+1 \\ m \neq j}}^n \frac{2L_j^{(\gamma)}(x_i)}{(x_{\gamma+j} - x_{\gamma+p})(x_{\gamma+j} - x_{\gamma+m})}. \quad (2.13)$$

As is the case with the first derivative matrix, considerable simplification of the coefficients in (2.13) is possible. For example, if  $i = \gamma + k$ , then

$$L_k^{(\gamma)''}(x_i) = \sum_{\substack{p=0 \\ p \neq k}}^{n-1} \sum_{\substack{m=p+1 \\ m \neq k}}^n \frac{2}{(x_i - x_{\gamma+p})(x_i - x_{\gamma+m})}. \quad (2.14)$$

Similar results are easily derived for third and higher derivatives.

### 3. Accuracy of the Differentiating Matrix

The accuracy of derivative approximations obtained using differentiating matrices based on uniform grids has been investigated by Hunter and Jainchell. They found that, in general, if  $n$ -th degree polynomials are used to obtain approximations to the values  $f'_i (i = 1, \dots, N)$ , then the errors are of order  $h^n$  where  $h = (x_N - x_0)/N$  is the uniform spacing. However, in assessing this error estimate, it must be noted that in actual applications to problems in rotor dynamics,  $h$  is usually  $O(1)$  or even  $O(10)$ . In practice, the use of differentiating matrices on uniform grids leads to large errors in derivative approximations at endpoints. These errors can be clearly seen from the data in the first three rows of Table 1. The first row of this table gives the exact derivatives of  $f(x) = \sin(k\pi x/4)$  and  $g(x) = \cos(k\pi x/4)$  at  $x = 0$  for  $k = 1, 2, 3$ . Rows two and three give approximations to  $f'(0)$  and  $g'(0)$  obtained using a differentiating matrix based on a uniform grid of five points with  $x_0 = 0$ ,  $x_N = 4$ ,  $h = 1$ , and interpolating polynomials of degree three and four. The errors are particularly pronounced for the two higher modes.

The final eight rows of Table 1 show approximations to  $f'(0)$  and  $g'(0)$  obtained by using a non-uniform grid of seven points consisting of the previous uniform grid plus two near-boundary points at  $\Delta x$  and  $x_N - \Delta x$ . Differentiating matrices based on fifth and sixth degree polynomials for  $\Delta x = .05, .01, .001$ , and  $.0001$  were used. At the smallest value of  $\Delta x$ , approximations to both  $f'(0)$  and  $g'(0)$  agree with the exact values to at least four decimal places for all three values of  $k$ .

A simple example will serve to illustrate the reason that employing a non-uniform grid with near-boundary points serves to drastically reduce

derivative approximation errors at the endpoints of the grid. Consider first the uniform grid approximation (with spacing  $h$ ) to the derivative of an unspecified function  $f(x)$  at the right hand endpoint  $x_N$ . Using a third degree Lagrange interpolating polynomial on the subgrid  $G_Y$  consisting of the four points  $x_{N-3}$  through  $x_N$  gives as an approximation to  $f'_N$  the expression

$$[D_{N+1}] \cdot \{f\} = \frac{11f_N - 18f_{N-1} + 9f_{N-2} - 2f_{N-3}}{6h}. \quad (3.1)$$

If the function values  $f_k (k = N-3, \dots, N-1)$  are now expanded in Taylor series relative to the endpoint  $x_N$ , then

$$[D_{N+1}] \cdot \{f\} = f'(x_N) + O(h^3) \quad (3.2)$$

which is the result found by Hunter and Jainchell.

Consider next the approximation to the derivative at  $x_N$  obtained using a fourth degree Lagrange interpolating polynomial on the non-uniform grid  $\hat{G}_Y$  consisting of  $G_Y$  plus the near-boundary point  $\hat{x} = x_N - \Delta x$ . Let  $\hat{f} = f(\hat{x})$ . Then

$$[\hat{D}_{N+1}] \cdot \{f\} = \frac{f_N - \hat{f}}{\Delta x} \left\{ 1 + \frac{11}{6} \frac{\Delta x}{h} + O\left(\frac{\Delta x}{h}\right)^2 \right\} \\ + \frac{\Delta x}{h^2} (108f_{N-1} - 27f_{N-2} + 4f_{N-3}) \left\{ 1 + O\left(\frac{\Delta x}{h}\right) \right\}. \quad (3.3)$$

By the nature of the near-boundary point of the non-uniform grid,  $\Delta x$  is small compared to  $h$ . Expanding the function values  $\hat{f}$  and  $f_k (k = N-3, \dots, N-1)$  in Taylor series relative to  $x_N$  now gives the result

$$\left[ \hat{D}_{N+1} \right] \cdot \{f\} = f'(x_N) + O\left(\frac{\Delta x}{h}, \frac{\Delta x}{h^2}\right). \quad (3.4)$$

Unlike error estimates for the uniform grid, the non-uniform grid error estimate (3.4) is essentially independent of the degree of the interpolating polynomials.

An equation similar to (3.4) predicts that the errors in the derivative approximation at  $x_0$  will also be  $O\left(\frac{\Delta x}{h}, \frac{\Delta x}{h^2}\right)$ . A measure of the extent to which this is valid in practice can be obtained by examining the data in Table 1. The actual errors for the non-uniform grid approximation are completely consistent. For  $h = 1$  and all four values of  $\Delta x$ , approximations to both  $f'(0)$  and  $g'(0)$  are valid with errors of order  $\Delta x$ .

#### 4. The Differentiating Matrix as an Operator

As a test of integrating and differentiating matrices, Hunter and Jainchell considered the harmonic equation

$$y'' + \omega^2 y = 0 \quad 0 \leq x \leq 4 \quad (4.1)$$

with the homogeneous boundary conditions

$$y(0) = y(4) = 0. \quad (4.2)$$

The exact solution of this problem is simply

$$y_k(x) = \sin \omega_k x \quad \text{with} \quad \omega_k = \frac{k\pi}{4}. \quad (4.3)$$

If equation (4.1) is written at each of the  $N+1$  points of a discrete grid  $G$  with  $x_0 = 0$  and  $x_N = 4$ , the differentiating matrix may be used as an operator to rewrite the problem in matrix notation. Let  $\{0\}$  be the  $N+1$  dimensional zero column vector. The differentiating matrix formulation of (4.1) is then

$$[D_2] \{y\} + \omega^2 \{y\} = \{0\} \quad (4.4)$$

where  $[D_2]$  is the second derivative matrix. This may be written in the eigenvalue form

$$[A] \{y\} = \lambda \{y\} ; [A] = [D_2]^{-1} \quad (4.5)$$

where

$$\lambda = 1/\omega^2 \quad (4.6)$$

so that the dominant eigenvalue corresponds to the lowest frequency. Both boundary conditions in (4.2) have not been explicitly used in obtaining the eigenvalue problem (4.5). However, since  $y_0 = y_N = 0$ , equation (4.5) reduces to the  $(N-1)$ -by- $(N-1)$  dimensional problem

$$[\bar{A}] \{\bar{y}\} = \lambda \{\bar{y}\} ; [\bar{A}] = [\bar{D}_2]^{-1} \quad (4.7)$$

where  $[\bar{D}_2]$  is the  $(N-1)$ -by- $(N-1)$  matrix obtained from  $[D_2]$  by removing the first and last row and column and  $\{\bar{y}\}$  is the  $(N-1)$ -dimensional column vector  $(y_1, y_2, \dots, y_{N-1})^t$ .

Hunter and Jainchell used a differentiating matrix based on fourth degree polynomials on a uniform five-point grid with  $h = 1$  to solve the 3-by-3 eigenvalue problem corresponding to (4.7). They obtained as approximations to the first three modes the eigenvalues and eigenvectors

$$\begin{aligned}\omega_1 &= 0.7893 & \{\bar{y}_1\} &= (0.70, 1.00, 0.78)^t, \\ \omega_2 &= 1.4142 & \{\bar{y}_2\} &= (1.00, 0.00, -1.00)^t, \\ \omega_3 &= 1.7920 & \{\bar{y}_3\} &= (-0.27, 1.00, -0.27)^t.\end{aligned}$$

By contrast, exact values for the first three modes are

$$\begin{aligned}\omega_1 &= 0.7854 & \{\bar{y}_1\} &= (0.71, 1.00, 0.79)^t, \\ \omega_2 &= 1.5708 & \{\bar{y}_2\} &= (1.00, 0.00, -1.00)^t, \\ \omega_3 &= 2.3562 & \{\bar{y}_3\} &= (-0.71, 1.00, -0.71)^t.\end{aligned}$$

The differentiating matrix results on the uniform grid thus compared poorly with the exact solutions.

Hunter and Jainchell also solved this eigenvalue problem using a formulation based only on integrating matrices on the uniform five-point grid. They obtained the results

$$\begin{aligned}\omega_1 &= 0.7840 & \{\bar{y}_1\} &= (0.71, 1.00, 0.71)^t, \\ \omega_2 &= 1.5492 & \{\bar{y}_2\} &= (1.00, 0.00, -1.00)^t, \\ \omega_3 &= 2.8522 & \{\bar{y}_3\} &= (-0.83, 1.00, -0.83)^t.\end{aligned}$$

For uniform grids, the integrating matrix formulation of the eigenvalue problem was thus clearly superior.

The situation is quite different when the eigenvalue problem (4.5) is solved using a second derivative matrix based on a grid which includes near-boundary points. In particular, consider the non-uniform, seven-point grid obtained from the previous uniform grid by adding two near-boundary points at



0.01 and 3.99. Using a differentiating matrix on this grid based on sixth degree interpolating polynomials, the eigenvalues and eigenvectors of the 5-by-5 problem corresponding to (4.7) are now

$$\begin{aligned}\omega_1 &= 0.7855 & \{\bar{y}_1\} &= (0.71, 1.00, 0.71)^t, \\ \omega_2 &= 1.5499 & \{\bar{y}_2\} &= (1.00, 0.00, -1.00)^t, \\ \omega_3 &= 2.1724 & \{\bar{y}_3\} &= (-0.61, 1.00, -0.61)^t.\end{aligned}$$

These results are fully comparable with the integrating matrix approximations. This conclusion is not changed if the integrating matrix method is also enhanced through the addition of near-boundary points.

Two additional eigenvalue problems involving equation (4.1) will further serve to illustrate the utility of differentiating matrix operators on grids which include near-boundary points. Consider first the eigenvalue problem consisting of equation (4.1) with boundary conditions

$$y'(0) = y(4) = 0. \quad (4.8)$$

Exact solutions are

$$y(x) = \cos \hat{\omega}_k x \quad \text{with} \quad \hat{\omega}_k = \frac{(2k+1)\pi}{8}, \quad (4.9)$$

and thus

$$\hat{\omega}_1 = 0.3927, \quad \hat{\omega}_2 = 1.1781, \quad \text{and} \quad \hat{\omega}_3 = 1.9635.$$

Because of the form of the boundary conditions, the problem (4.1) and (4.8) cannot be reduced to a matrix eigenvalue problem for  $\{y\}$  itself using

only the differentiating matrix. Rather, a combination of differentiating and integrating matrices can be used to obtain a matrix eigenvalue problem for the first derivative vector  $\{y'\}$ . To do this, a differentiating matrix is used to express the second derivative vector as  $\{y''\} = [D] \{y'\}$ . The boundary condition at  $x_N$  is now used to express  $y(x)$  in terms of  $y'(x)$  as

$$y(x) = -\int_x^4 y'(s) ds. \quad (4.10)$$

If  $[J_1]$  is an integrating matrix on  $G$  which approximates integrals from  $x$  to  $x_N$ , (4.10) has the matrix form

$$[D] \{y'\} - \hat{\omega}^2 [J_1] \{y'\} = \{0\}. \quad (4.11)$$

Equation (4.1) thus has the matrix eigenvalue formulation

$$[A] \{y'\} = \hat{\lambda} \{y'\} \quad \text{with} \quad [A] = [D]^{-1} [J_1] \quad (4.12)$$

and  $\hat{\lambda} = 1/\hat{\omega}^2$ . The boundary condition that  $y'_0 = 0$  has not been explicitly used in deriving (4.12). However, this condition implies solutions can be obtained from the  $N$ -by- $N$  problem  $[\bar{A}] \{\bar{y}'\} = \{\bar{y}'\}$ , where  $[\bar{A}]$  is the  $N$ -by- $N$  matrix obtained from  $[A]$  by deleting the first row and column and  $\{\bar{y}'\} = (y'_1, y'_2, \dots, y'_N)^t$ .

To solve the matrix eigenvalue problem,  $G$  was taken to be the same seven-point grid as above with near boundary points at 0.01 and 3.99. A differentiating matrix  $[D]$  based on fourth degree polynomials and an integrating matrix  $[J_1]$  based on fifth degree polynomials were used. This

even/odd degree scheme was chosen so that grid points or subintervals would be centered to the maximum extent possible in the approximation subgrids. Approximations for the first three eigenvalues were found to be

$$\hat{\omega}_1 = 0.3926, \quad \hat{\omega}_2 = 1.1750, \text{ and } \hat{\omega}_3 = 2.0883.$$

Approximations to the eigenvectors  $\{\bar{y}_k\}$  for these eigenvalues, obtained by using the eigenvectors  $\{\bar{y}_k\}$  of (4.12) in (4.11), agreed well with exact results.

In an alternate procedure, the eigenvalue problem with boundary conditions (4.8) can be solved for  $\{y\}$  directly using a combination of the integrating matrix  $[J_1]$  and the integrating matrix  $[J_0]$  which approximates integrals from  $x_0$  to  $x$ . This formulation is somewhat simpler and also fractionally more accurate than the mixed differentiating-integrating matrix formulation above. However, if the boundary conditions associated with equation (4.1) are changed to

$$y'(0) = y'(4) = 0, \tag{4.13}$$

it is no longer possible to derive a viable matrix eigenvalue formulation of the type  $[A] \{y\} = \lambda \{y\}$  using integrating matrices alone. The lack of information on  $y$  itself at either endpoint forces use of a boundary matrix operator which leads to a singular matrix at the next-to-last step of the reformulation. This is not the case if a combination of integrating and differentiating matrices is used to reformulate (4.1) and (4.13) as a matrix eigenvalue problem for  $\{y'\}$ .

If equation (4.1) is integrated from  $x$  to  $x_N$  using the condition  $y'_N = 0$ , then

$$-y' + \omega^2 \int_x^4 y(s) ds = 0. \quad (4.14)$$

The function  $y(x)$  may also be expressed in terms of  $y'(x)$  as

$$y(x) = \int_0^x y'(s) ds + y(0) \quad (4.15)$$

and, from equation (4.1), the unknown boundary value  $y_0$  can be written as  $y_0 = -\lambda y''(0)$  with  $\lambda$  as in (4.6). Consequently,

$$-y'(x) + (x - 4) y''(0) + \omega^2 \int_x^4 \int_0^s y'(\xi) d\xi ds = 0. \quad (4.16)$$

The value of  $y''(0)$  may now be obtained from  $y'(x)$  using a matrix formulation on a grid  $G$  involving both the differentiating matrix  $[D]$  and the left boundary matrix  $[B_0]$ . In particular,  $[B_0]$  is an  $(N+1)$ -by- $(N+1)$  matrix with first column vector  $\{1\}$  and all other entries zero so that  $[B_0] \{f\} = \{f(0)\}$ . If  $[E]$  is a diagonal matrix with entries  $e_{ii} = x_{i-1} - x_n$ , the matrix eigenvalue formulation of (4.1) and (4.13) is now  $[A] \{y'\} = \lambda \{y'\}$  with

$$[A] = \{[E] [B_0] [D] + [I]\}^{-1} [J_1] [J_0] \quad (4.17)$$

and  $[I]$  is the identity matrix. The condition that  $y'(0) = 0$  implies that the first row and column of  $[A]$  may be removed and solutions obtained from the resulting  $N$ -by- $N$  eigenvalue problem. Once  $\omega_k$  and  $\{y'_k\}$  have been

determined,  $\{y_k\}$  itself may be obtained using (4.15) and (4.16) Exact solutions are given by (4.3) with sine replaced by cosine.

The matrix formulation for this eigenvalue problem was solved on the seven-point non-uniform grid with near-boundary points  $G = (0., 0.001, 1., 2., 3., 3.999, 4.)$ . The differentiating matrix was based on sixth degree polynomials, and both integrating matrices were based on fifth degree polynomials. Eigenvalues were found to be

$$\omega_1 = 0.7872, \quad \omega_2 = 1.7663, \text{ and } \omega_3 = 3.7402.$$

Results for the higher modes can be improved by increasing the number of points in  $G$ .

## 5. Conclusions

Generalization of the differentiating matrices to grids with non-uniformly spaced points removes many of the difficulties associated with previous versions of these matrices which were restricted to uniform grids. The inclusion of appropriate near-boundary points allows for accurate approximation of derivative values, including derivatives at endpoints. Greatly improved accuracy is also achieved when the present differentiating matrices are used as operators to express second order eigenvalue problems in matrix form.

For eigenvalue problems which can be reformulated as matrix problems using only the integrating matrix, such as the fourth order models for vibrations of rotating cantilevered beams, alternate formulations involving

the integrating and/or differentiating matrices will probably not be used in practice. This is because a pure integrating matrix approach, if feasible, allows for a slightly simpler treatment of boundary conditions. However, boundary conditions for many eigenvalue problems are such that a mixed formulation involving both types of matrices is unavoidable. Differentiating and integrating matrices on grids with near-boundary points should provide the basis for a fast and efficient numerical technique for solving these problems.

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**Table 1.** Comparison of the differentiating matrices on a uniform grid of five points with  $h = 1$  and on a non-uniform grid obtained by adding two near-boundary points with spacing  $\Delta x$  to the uniform grid. The test functions are  $f(x) = \sin \pi x/4$  and  $g(x) = \cos \pi x/4$ . Approximations to  $f'(0)$  and  $g'(0)$  are obtained using interpolating polynomials of degree  $m$  as indicated.

		k = 1			k = 2			k = 3		
		$f'(0)$	$g'(0)$	$f'(0)$	$g'(0)$	$f'(0)$	$g'(0)$	$f'(0)$	$g'(0)$	$g'(0)$
Exact		0.785398	0.0	1.570796	0.0	2.356195	0.0	0.0	0.0	0.0
Uniform Grid										
$\Delta x = 5 \cdot 10^{-2}$	m = 3	0.857023	0.052285	2.66667	-0.33333	3.857023	-3.718952			
	m = 4	0.771236	0.052285	2.66667	0.66667	6.771236	-3.718952			
	m = 5	0.785079	-0.000155	1.556718	0.010521	2.370428	0.128625			
	m = 6	0.785429	-0.000155	1.556718	-0.006087	2.259033	0.128625			
	m = 5	0.785334	-0.000031	1.567969	0.002107	2.358892	0.025608			
	m = 6	0.785404	-0.000031	1.567969	-0.001224	2.336776	0.025608			
$\Delta x = 10^{-2}$	m = 5	0.785392	-0.000003	1.570513	0.000211	2.356461	0.002558			
	m = 6	0.785399	-0.000003	1.570513	-0.000123	2.354253	0.002558			
$\Delta x = 10^{-3}$	m = 5	0.785398	0.0	1.570768	0.000021	2.356221	0.000256			
	m = 6	0.785398	0.0	1.570768	-0.000012	2.356000	0.000256			