NASA
Technical
Paper
2416

1985

Relative Control Effectiveness Technique With Application to Airplane Control Coordination

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Summary

A technique called relative control effectiveness is developed to give a quantitative measure of the effectiveness of each of several controls upon the fundamental modes of a linear dynamic system. By postulating linear combinations of controls, a set of pseudo controls is formed. The relative control effectiveness measure is then used to formulate a set of optimization problems which yield the proportions of the controls to be assigned to each of the pseudo controls with the objective that each pseudo control strongly affects selected fundamental modes of the system while only weakly affecting the remaining modes.

The procedure is applied to the linearized lateral dynamics of a high-performance fighter airplane to obtain (1) a pseudo control which primarily affects the roll and spiral modes and (2) a second pseudo control which primarily affects the Dutch roll mode. Transient responses of the linear airplane model to lateral and directional aerodynamic controls and to the pseudo controls are presented. An example of the application of the procedure to airplane control with thrust vectoring is included.

Introduction

The usual definition of controllability of a linear system may be stated as follows:

> A linear system is controllable by a given control variable if that variable can excite each fundamental mode of the system. (See refs. 1 and 2.)

If this criterion is true for a given linear system, it is mathematically possible to devise a control scheme which influences all of its fundamental modes. However, this criterion does not indicate the degree of difficulty which would be experienced in generating such a control scheme. For example, a mode may be only slightly affected by reasonable, but limited, control deflections. Such a system would be "controllable" by the above definition but would be uncontrollable in practice.

What is needed is a quantitative measure of controllability which would indicate the "degree of controllability" of the system to control deflections. This measure could be used to compare the control effectiveness of multiple control variables upon each of the modes of the system and to suggest a preferred control structure; i.e., which control variables should be used to control each mode. Multiple control variables could be mixed in linear combinations, with the proportions of the mixings being determined by placing emphasis upon the quantitative controllability of specific modes. This approach could be useful in determining control-mixing schedules in the presence of changing operating conditions. The technique could also be used to reduce the number of control channels in the case of multiple redundant controls (for example, on airplane control using multiple aerodynamic controls and thrust vectoring).

A relative control effectiveness measure was developed and is described in this report. A method of applying this measure to obtain control-mixing combinations to provide control channels which can (more or less) independently affect the fundamental modes of a linear system is presented. The method is applied to an example of the lateral control of a modern high-performance fighter airplane in low-speed flight. This report also includes an example for thrust-vectoring nozzles.

Symbols

$\mathbf{A}$	state variable coefficient matrix ($n \times n$)
a_{ci}	real part of ith complex eigenvalue of $\mathbf{A}$
a_{ri}	ith real eigenvalue of $\mathbf{A}$
$\mathbf{B}$	control variable coefficient matrix ($n \times m$)
b_{ci}	imaginary part of ith complex eigenvalue of $\mathbf{A}$
$\mathbf{C}$	$m \times k$ matrix of control-mixing vectors ($\mathbf{c}_j$)
$\mathbf{C}_{\text{eff}}$	relative control effectiveness matrix (number of modes $\times$ m)
$\mathbf{c}_j$	jth control-mixing vector of dimension m
J	quadratic cost function for maximization problem
k	number of pseudo control variables
$\mathbf{M}$	similarity transformation matrix ($n \times n$)
m	number of control variables
n	number of state variables
p	body axis roll rate, deg/sec
r	body axis yaw rate, deg/sec
t	time, seconds
TAS	true airspeed, knots
$\mathbf{u}$	vector of control variables (u_i) of dimension m
$\tilde{\mathbf{u}}$	vector of scaled control variables ($\tilde{u}_i$) of dimension m
$\mathbf{v}$	vector of pseudo control variables (v_j) of dimension k

$\mathbf{W}$	$m \times m$ matrix of control variable authorities, $\mathbf{W} = \mathrm{diag}(w_i)$
$\tilde{\mathbf{W}}$	symmetric matrix for maximization problem ($m \times m$)
w_i	maximum value of control variable u_i
$\mathbf{x}$	vector of state variables (x_i) of dimension n
$\mathbf{y}$	vector of transformed state variables in modal coordinates (y_i) of dimension n
α	angle of attack, degrees
β	sideslip angle, degrees
$\mathbf{\Gamma}$	transformed system control variable coefficient matrix ($n \times m$)
γ_i	column vector formed of elements of the ith row of matrix $\mathbf{\Gamma}$
$\gamma_{i,j}$	element of matrix $\mathbf{\Gamma}$ in the ith row and jth column
δ_a	aileron deflection, degrees (positive, right aileron up)
δ_D	differential horizontal-tail deflection, degrees (positive, right tail trailing edge up)
δ_r	rudder deflection, degrees (positive, trailing edge left)
$\delta_{v,\mathrm{roll}}$	rolling thrust vector, degrees (positive, right nozzle up)
$\delta_{v,\mathrm{yaw}}$	yawing thrust vector, degrees (positive, nozzles left)
θ, Ψ	phase angles
$\mathbf{\Lambda}$	transformed system state variable coefficient matrix in block diagonal form ($n \times n$)
λ	Lagrange multiplier
$\boldsymbol{\mu}_{ci}$	eigenvector of dimension n corresponding to ith complex eigenvalue of $\mathbf{A}$
$\boldsymbol{\mu}_{ri}$	eigenvector of dimension n corresponding to ith real eigenvalue of $\mathbf{A}$
$\sum_{\mathrm{D}}$	summation over desired modes
$\sum_{\mathrm{ND}}$	summation over nondesired modes
τ	integration variable
ϕ	bank angle, degrees (positive, right wing down)

A dot over a variable (e.g., $\dot{x}$) denotes the derivative with respect to time.

Control Effectiveness Measure

In the analysis and design of multivariable control systems, it is customary to express the small-perturbation system dynamics in the form of a set of coupled linear first-order equations in the following matrix representation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

where

$\mathbf{x}$	vector of state variables of dimension n
$\mathbf{u}$	vector of control variables of dimension m
$\mathbf{A}$	$n \times n$ matrix of state variable coefficients
$\mathbf{B}$	$n \times m$ matrix of control variable coefficients

The state variable coefficient matrix, $\mathbf{A}$, determines the characteristics of the fundamental modes of the system. The control variable coefficient matrix, $\mathbf{B}$, specifies the influence of the control variables on the state variables. The matrix $\mathbf{B}$ does not directly indicate the controllability of the system. Moreover, a column of matrix $\mathbf{B}$ with large elements does not necessarily indicate that the system is influenced by the corresponding control variable any more than by any other control variable.

The controllability criteria of references 1 and 2 can be applied to equation (1). These criteria indicate whether it is mathematically possible to influence all the modes of the system by manipulation of the control variables. If not, the criteria identify the modes which cannot be altered by any means. Unfortunately, an ill-conditioned system may satisfy the controllability criteria but possess one or more modes which are weakly affected by all the control variables. Attempts to excite or modify such modes without unduly upsetting the remainder of the system may be futile in practice. This report presents an alternative method of measuring controllability which includes quantitative information in addition to the qualitative information of the conventional controllability criteria.

An indication of the influence of the control variables upon the system modes may be obtained by transforming the system equations to a block diagonal form

$$\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y} + \mathbf{\Gamma}\tilde{\mathbf{u}} \tag{2}$$

$$\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} a_{r1} & & & & & \\ & a_{r2} & & & & \\ & & \ddots & & & \\ & & & a_{c1} & b_{c1} & \\ & & & -b_{c1} & a_{c1} & \\ & & & & & \ddots \end{bmatrix} \tag{3}$$

$$\Gamma = \mathbf{M}^{-1}\mathbf{B}\mathbf{W} \tag{4}$$

where

$$\mathbf{x} = \mathbf{M}\mathbf{y} \tag{5}$$

$$\mathbf{u} = \mathbf{W}\tilde{\mathbf{u}} \tag{6}$$

$$\mathbf{W} = \mathrm{diag}(w_i) \tag{7}$$

The similarity transformation matrix, $\mathbf{M}$, is chosen such that it is analogous to the modal matrix used in references 1 and 2, except that it is a real matrix. The matrix $\mathbf{M}$, which is sometimes called the right eigenvalues of $\mathbf{A}$, has real coefficients because of the different method of handling the complex modes. This choice results in the transformed state variables $\mathbf{y}$ (the modal coordinates) and the coefficient matrices Λ and Γ also being real.

The similarity transformation matrix, $\mathbf{M}$, is assembled from the eigenvectors of $\mathbf{A}$. For each real eigenvalue of $\mathbf{A}$, one column of $\mathbf{M}$ is set equal to the corresponding eigenvector as is done in constructing the modal matrix. For each complex conjugate pair of eigenvalues of $\mathbf{A}$, one column of $\mathbf{M}$ is set equal to the real part of the corresponding eigenvectors, and the adjacent column is set equal to the imaginary part of either eigenvector. In mathematical notation, this is

$$\mathbf{M} = [\boldsymbol{\mu}_{r1},\ \boldsymbol{\mu}_{r2},\ \ldots,\ \mathrm{Re}(\boldsymbol{\mu}_{c1}),\ \mathrm{Im}(\boldsymbol{\mu}_{c1}),\ \ldots] \tag{8}$$

where $\boldsymbol{\mu}_{ri}$ is the eigenvector corresponding to the ith real eigenvalue and $\boldsymbol{\mu}_{ci}$ is a complex eigenvector corresponding to the ith complex conjugate pair of eigenvalues.

The matrix $\mathbf{W}$ has the maximum value (authority) of each of the control variables ($w_i = \max(u_i)$) as its diagonal elements. The transformed system is driven by the scaled control variables $\tilde{u}_i$ which have unity ranges (± 1).

In the transformed state coefficient matrix of equation (3), a_{ri} is the ith real eigenvalue, and a_{ci} and b_{ci} are the real and imaginary parts, respectively, of the ith complex pair of eigenvalues. The elements of $\mathbf{y}$, which are commonly referred to as modal coordinates, are indications of the excitation of the modes. The influences of each control variable upon the fundamental modes are given by the elements of the transformed control coefficient matrix, Γ, of equation (4). The absolute sizes of the elements of Γ have little significance in themselves. However, they are useful in assessing the relative influence of the various control variables upon each of the modes. Since scaled control variables are being used, the elements of Γ give an indication of the control effectivenesses upon the fundamental modes, which is consistent with respect to the authorities of the control variables.

For a real mode, one may compare the elements of the corresponding row of the matrix Γ to get an immediate comparison of the relative effects of the control variables upon that mode. For a complex mode, there is a pair of transformed state variables, y_i and y_{i+1}, which are coupled. Therefore, there are two rows of the matrix Γ to be considered. Consider the following second-order example:

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \sum_j \begin{Bmatrix} \gamma_{1,j} \\ \gamma_{2,j} \end{Bmatrix} \tilde{u}_j \tag{9}$$

where the subscripts on a and b have been omitted for brevity. The general solution of this system may be written as

$$\begin{aligned} \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} &= \sqrt{y_1^2(0) + y_2^2(0)}\, e^{at} \begin{Bmatrix} \cos(bt-\theta) \\ \sin(bt-\theta) \end{Bmatrix} \\ &\quad + \sum_j \sqrt{(\gamma_{1,j})^2 + (\gamma_{2,j})^2} \\ &\quad \times \int_0^t e^{a(t-\tau)} \begin{Bmatrix} \cos[b(t-\tau)-\Psi] \\ \sin[b(t-\tau)-\Psi] \end{Bmatrix} \tilde{u}_j(\tau)\, d\tau \end{aligned} \tag{10}$$

where

$$\theta = \tan^{-1}\left(\frac{y_2(0)}{y_1(0)}\right) \tag{11}$$

$$\Psi = \tan^{-1}\left(\frac{\gamma_{2,j}}{\gamma_{1,j}}\right) \tag{12}$$

It is evident that the control variables influence the two transformed state variables in equivalent fashions and that this influence is proportional to $\sqrt{(\gamma_{1,j})^2 + (\gamma_{2,j})^2}$.

A relative control effectiveness matrix which incorporates these concepts can now be defined. For each real mode, the relative control effectiveness of the jth control variable upon the ith first-order mode is taken to be the magnitude of the (i,j) element of Γ divided by the Euclidean norm of the ith row of Γ,

$$\mathbf{C}_{\text{eff}}(i,j) = \frac{|\gamma_{i,j}|}{\sqrt{\sum_j (\gamma_{i,j})^2}} \tag{13}$$

For each complex mode, the relative control effectiveness of the jth control upon the $(i,i+1)$th second-order mode is taken to be the square root of the sum of the squares of the (i,j) and $(i+1,j)$ elements of Γ

divided by the Euclidean norm of the ith and $(i+1)$th rows of $\mathbf{\Gamma}$ combined,

$$\mathbf{C}_{\text{eff}}(i,j) = \frac{\sqrt{(\gamma_{i,j})^2 + (\gamma_{i+1,j})^2}}{\sqrt{\sum_j \left[(\gamma_{i,j})^2 + (\gamma_{i+1,j})^2\right]}} \tag{14}$$

The elements of the control coefficient matrix, $\mathbf{\Gamma}$, provide a somewhat direct indication of the effectiveness of each of the control variables in influencing each of the fundamental modes of the system. However, the magnitudes of the elements are dependent upon the lengths of those eigenvectors used to transform the system into block diagonal form. The elements of the relative control effectiveness matrix, $\mathbf{C}_{\text{eff}}$, are limited to positive values between zero and unity and are independent of the eigenvector lengths. However, they do not provide any absolute information about the effectiveness of the controls.

Specification of Control Crossfeeds

Systems having multiple control effectors provide numerous paths by which control law schemes may direct signals from the operator controls and the system output variables to the control variables. Multivariable design techniques, such as those implemented in reference 3, generate a large number of control gains. Only a few of these may be significant. Attempts to systematically eliminate the ineffectual gains have been made in references 4 through 6.

The approach taken in the present study is first to find control-mixing combinations (control crossfeeds). A number of "pseudo" controls are formed. The activation of each of the pseudo controls commands proportional deflections of the system controls. Once a suitable set of pseudo controls is established, the design of control laws can be conducted by means of conventional techniques.

For systems which have multiple, redundant controls, this procedure reduces the number of controls which must be dealt with in designing control laws. The control system structure is simplified, and the number of control gains is reduced. If the pseudo controls are formulated so that each has its effect concentrated upon a single selected mode, then the control law design problem decouples into a series of first- and second-order problems. In general, the decoupling will not be complete but may be adequate to allow the utilization of a simplified control structure. A technique for calculating the required relationships between the pseudo controls and the system control variables is developed next.

Define a set of pseudo control variables, v_j, which are linearly related to the scaled control variables, $\tilde{u}_i$, through the matrix $\mathbf{C}$ such that

$$\tilde{\mathbf{u}} = \mathbf{C}\mathbf{v} = \sum_j \mathbf{c}_j v_j \tag{15}$$

where the columns of $\mathbf{C}$ have unity length; i.e., $\mathbf{c}_j^T\mathbf{c}_j = 1$. Combining equations (15) and (2) yields the following form for the transformed system:

$$\dot{\mathbf{y}} = \mathbf{\Lambda}\mathbf{y} + \sum_j \begin{Bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_n^T \end{Bmatrix} \mathbf{c}_j v_j \tag{16}$$

where γ_i^T is the ith row of $\mathbf{\Gamma}$. The vectors $\mathbf{c}_j$ are control-mixing combination vectors which specify the proportions in which the system control effectors are actuated.

For each pseudo control, v_j, a control-mixing vector, $\mathbf{c}_j$, is found. The effect of the pseudo control upon one or more modes is maximal while the effect on other modes is minimal. To accomplish this, the following maximization problem is solved. Find the control-mixing vector, $\mathbf{c}_j$, which maximizes

$$J = \mathbf{c}_j^T \sum_D \gamma_i\gamma_i^T \mathbf{c}_j - \mathbf{c}_j^T \sum_{\text{ND}} \gamma_i\gamma_i^T \mathbf{c}_j \tag{17}$$

subject to the constraint

$$\mathbf{c}_j^T\mathbf{c}_j = 1 \tag{18}$$

where the first term of J includes the mode(s) to be controlled by v_j (desired), and the second term of J includes the modes to be unaffected by v_j (not desired).

This constrained maximization problem is solved by introducing a Lagrange multiplier, λ, and finding the combination of $\mathbf{c}_j$ and λ which maximizes the following expression:

$$J = \mathbf{c}_j^T \left[\sum_{\text{D}} \gamma_i\gamma_i^T - \sum_{\text{ND}} \gamma_i\gamma_i^T\right] \mathbf{c}_j - \lambda(\mathbf{c}_j^T\mathbf{c}_j - 1) \tag{19}$$

Taking the derivatives of this expression with respect to $\mathbf{c}_j$ and λ yields necessary conditions for the solution

$$\frac{1}{2}\frac{\partial J}{\partial \mathbf{c}_j} = \tilde{\mathbf{W}}\mathbf{c}_j - \lambda\mathbf{c}_j = 0 \tag{20}$$

$$\frac{\partial J}{\partial \lambda} = \mathbf{c}_j^T \mathbf{c}_j - 1 = 0 \tag{21}$$

where

$$\tilde{\mathbf{W}} = \sum_{\mathrm{D}} \gamma_i \gamma_i^T - \sum_{\mathrm{ND}} \gamma_i \gamma_i^T \tag{22}$$

Equations (20) and (21) are in the form of an eigenvector decomposition problem. The control mixing vector, $\mathbf{c}_j$, is a normalized eigenvector of matrix $\tilde{\mathbf{W}}$, with λ being its associated eigenvalue. Since $\tilde{\mathbf{W}}$ is real and symmetric (eq. (22)), its eigenvalues and eigenvectors are real and orthogonal (ref. 2). By combining equations (19) through (22), the value of J is found to be equal to the eigenvalue, as follows:

$$J = \mathbf{c}_j^T \tilde{\mathbf{W}} \mathbf{c}_j = \mathbf{c}_j^T \lambda \mathbf{c}_j = \lambda \tag{23}$$

Therefore, the maximizing solution for $\mathbf{c}_j$ is the one for which the corresponding eigenvalue, λ, has the maximum positive value.

The technique is applied by partitioning the system modes into groups. A pseudo control which has its principal effect upon each group is to be found. The control-mixing matrix for each pseudo control is then found by solving the above eigenvector problem. After this is accomplished, any of the usual multivariable control design techniques may be used to complete the design of the control system.

Throughout the above analysis, it is implicitly assumed that the fundamental modes were equally important. If this is not the case, the procedure could be extended in a straightforward way to apply a set of weights to differentiate the fundamental modes.

Example Applications

The concept of relative control effectiveness and a technique to select control-mixing combinations for generating those pseudo control variables which principally affect selected modes of a system have been developed. This section applies these developments to two example flight control problems.

Lateral-Directional Airplane Controls

Consider the lateral control of a modern high-performance fighter airplane in level flight at a true airspeed of 165 knots with an angle of attack of 10°. This flight condition approximates the normal landing approach speed for the example airplane. The lateral perturbation model for this example (cf. eq. (1)) is

$$\frac{d}{dt}\begin{Bmatrix} p \\ \phi \\ r \\ \beta \end{Bmatrix} = \begin{bmatrix} -2.13 & 0 & 2.19 & -12.7 \\ 1 & 0 & 0.176 & 0 \\ 0.0646 & 0 & -0.559 & 1.44 \\ 0.174 & 0.114 & -0.985 & -0.160 \end{bmatrix} \begin{Bmatrix} p \\ \phi \\ r \\ \beta \end{Bmatrix} + \begin{bmatrix} 4.38 & 1.10 & 4.09 \\ 0 & 0 & 0 \\ -0.212 & -1.20 & 0.222 \\ -0.00155 & 0.0330 & -0.0103 \end{bmatrix} \begin{Bmatrix} \delta_a \\ \delta_r \\ \delta_D \end{Bmatrix} \tag{24}$$

where

p roll rate, deg/sec

ϕ bank angle, degrees

r yaw rate, deg/sec

β sideslip angle, degrees

δ_a aileron deflection, degrees

δ_r rudder deflection, degrees

δ_D differential horizontal-tail deflection, degrees

The block diagonal (modal) form of this system (cf. eqs. (2) through (8)) is

$$\dot{\mathbf{y}} = \begin{bmatrix} -1.37 & 0 & 0 & 0 \\ 0 & -0.101 & 0 & 0 \\ 0 & 0 & -0.693 & 1.56 \\ 0 & 0 & -1.56 & -0.693 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 156 & -124 & 110 \\ 43.0 & -120 & 51.4 \\ 38.7 & -3.67 & 21.4 \\ 147 & 212 & 23.3 \end{bmatrix} \begin{Bmatrix} \delta_a/40 \\ \delta_r/30 \\ \delta_D/20 \end{Bmatrix} \tag{25}$$

where the similarity transform matrix is

$$\mathbf{M} = \begin{bmatrix} 0.805 & -0.116 & -0.654 & 0.548 \\ -0.591 & 0.988 & 0.437 & 0.233 \\ 0.017 & 0.092 & 0.013 & -0.107 \\ -0.046 & 0.034 & 0.144 & 0 \end{bmatrix} \tag{26}$$

The authorities of the aileron, rudder, and differential tail are 40°, 30°, and 20°, respectively. The first real mode of this system is the roll mode, the second real mode is the spiral mode, and the complex mode is the Dutch roll mode.

The relative control effectiveness matrix, $\mathbf{C}_{\mathrm{eff}}$, is calculated by applying equation (13) to the first two rows of the control coefficient matrix, $\mathbf{\Gamma}$, appearing in the second term of equation (25) and by applying equation (14) to the second two rows as follows:

$$\mathbf{C}_{\text{eff}} = \begin{array}{c} \begin{array}{ccc} \delta_a & \delta_r & \delta_D \end{array} \\ \begin{bmatrix} 0.684 & 0.544 & 0.486 \\ 0.312 & 0.874 & 0.373 \\ 0.577 & 0.808 & 0.120 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array} \tag{27}$$

Examination of this matrix indicates that the effects of the three controls upon the roll mode are comparable. The rudder is dominant in its influence on the spiral and Dutch roll modes, and the ailerons strongly affect the Dutch roll mode. Unless the controls are carefully proportioned, those control commands generated by either cockpit controller movements or by feedback control mechanisms will excite all the modes of the system and result in potentially undesirable intermode couplings.

Two pseudo control variables were calculated for this system. It was intended that the first would have its principal effect upon the roll and spiral modes, and the second, upon the Dutch roll mode. The matrices for the eigenvalue decomposition problems, $\tilde{\mathbf{W}}$, are calculated according to equation (22). For the roll and spiral pseudo control variable, v_1, the first term of equation (22) includes the first two rows of $\mathbf{\Gamma}$, and the second term includes the second two rows. The calculated $\tilde{\mathbf{W}}$ matrix follows:

$$\tilde{\mathbf{W}} = \begin{bmatrix} 3\,070 & -55\,400 & 15\,200 \\ -55\,400 & -15\,300 & -24\,700 \\ 15\,200 & -24\,700 & 13\,900 \end{bmatrix} \tag{28}$$

The eigenvalues of equation (28) are 65 100, −0.1, and −63 400. The control-mixing vector for the roll and spiral modes is the eigenvector of equation (28) corresponding to the largest positive eigenvalue (namely, 65 100).

For the Dutch roll pseudo control variable, v_2, the terms of equation (22) are interchanged for this example. The matrix $\tilde{\mathbf{W}}$ is equal to the negative of equation (28). Therefore, the control-mixing vector is the eigenvector of equation (28) corresponding to the most negative eigenvalue (namely, −63 400). It is fortuitous that both control-mixing vectors can be found by solving a single eigenvalue decomposition problem for this example. In general, a separate decomposition problem must be solved for each pseudo control variable.

The eigenvectors found above are arranged to form the control-mixing matrix (cf. eq. (15)) as follows:

$$\begin{Bmatrix} \delta_a \\ \delta_r \\ \delta_D \end{Bmatrix} = \begin{bmatrix} 25.9 & 24.7 \\ -17.8 & 23.3 \\ 9.56 & 2.54 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \tag{29}$$

Note that the control scale factors have been included in equation (29), so that δ_a, δ_r, and δ_D are in terms of degrees of deflection, whereas the pseudo control variables are dimensionless with ranges of approximately ± 1. The numerical procedure used to calculate the eigenvectors of equation (28) might cause sign inversions in any of the columns of equation (29). It may be necessary to change the signs of the columns in order to obtain the desired phasing of the controls.

Use of equation (25) and the control-mixing matrix of equation (29) gives the relative control effectiveness matrix for the airplane,

$$\mathbf{C}_{\text{eff}} = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.998 & 0.062 \\ 0.899 & 0.438 \\ 0.161 & 0.987 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array} \tag{30}$$

The roll and spiral modes are now principally affected by v_1 with small coupling of the Dutch roll mode. This pseudo control causes the aerodynamic controls to be activated in a coordinated fashion. A positive value of v_1 yields (1) aileron and differential tail, left surfaces trailing edge down and (2) rudder, trailing edge right. The Dutch roll mode is principally affected by v_2 with some coupling of the spiral mode. This pseudo control causes crossed controls. A positive value of v_2 yields (1) ailerons and differential tail, left surfaces trailing edge down and (2) rudder, trailing edge left. The partial decoupling of the system modes improves the conditioning of the system for manual control and for the design of feedback control laws. It has also reduced the number of control variables which must be considered in controlling the airplane.

Transient responses were obtained for control step inputs for the two pseudo controls and for lateral (ailerons with differential horizontal tail) and directional (rudder) aerodynamic controls. The control effector deflections used are given in table I. These deflections are 10 percent of the available values. The time histories of the airplane state variables are presented in figures 1 and 2. These data were obtained from analog computer simulations of the linearized dynamics of equation (24). In each case, the controls were deflected for approximately 6 seconds and then returned to their neutral positions.

Figure 1(a) shows the airplane open-loop response to simultaneous deflections of the ailerons and differential tail with a 2-to-1 gearing ratio. The presence of the Dutch roll mode appears as a damped oscillation in the roll rate and sideslip responses. The slow exponential character of the spiral mode can be seen in all the state variables. Figure 1(b) shows the airplane response to rudder deflection. The airplane reaches a fairly steady roll rate and sideslip after an initial overshoot. The spiral mode is also present in all the state variables.

Figure 2(a) shows the airplane response to the pseudo control variable v_1, which was selected for maximum influence on the roll and spiral modes. The responses are quite different from those of figure 1(a). The presence of the Dutch roll mode is not apparent, the roll rate achieves a significantly greater magnitude, the sideslip smoothly follows the bank angle, and the yaw rate responds immediately in the desired direction.

Figure 2(b) shows the airplane response to the pseudo control v_2, which was selected for maximum influence on the Dutch roll mode. These responses are different from those of figure 1(b). The roll rate is seen to settle to a smaller value, and the sideslip angle is increased and achieves a steady value.

Aerodynamic and Thrust-Vectoring Controls

In this example, the mathematical model of the airplane was modified to include ideal thrust-vectoring nozzles at the extreme rear end of the airplane to provide additional rolling- and yawing-moment control capability. Each nozzle was assumed to be capable of changing the direction of thrust 15° in any direction. Rolling moments are produced by vectoring one nozzle upward and the other nozzle downward for an available deflection ($\delta_{v,\mathrm{roll}}$) of 30°. Yawing moments are produced by vectoring the nozzles simultaneously to the side for an available deflection ($\delta_{v,\mathrm{yaw}}$) of 15°.

The airplane was trimmed for steady level flight at an angle of attack of 15° and an airspeed of 133 knots. This airspeed is approximately 32 knots less than the normal landing approach speed for flaps up, and the angle of attack corresponds to the maximum pitch attitude allowed at touchdown to avoid tail skid.

The control effectiveness technique was applied to this model to find control-mixing combinations which maximize the responses of the roll and spiral modes with minimum excitation of the Dutch roll mode. The technique was applied to the case in which both the aerodynamic and the thrust-vectoring controls are active and to the case in which only the aerodynamic controls are active. The calculated control-mixing matrices and relative control effectiveness matrices for this example are given in tables II and III.

The value of the pseudo control variable v_1, which primarily affects the roll and spiral modes, was selected so that the deflections of all the airplane controls would be less than or equal to 15 percent of their authorities. These deflections are presented in table IV. Also included for reference is a baseline example which has the ailerons and differential tail deflected to 15 percent of their authorities. Transient responses of the airplane (linear model) were obtained for step inputs of the lateral controls with values as given in the table. These step responses are presented in figure 3.

The baseline example (fig. 3(a)) is objectionable for several reasons: (1) the oscillatory nature of the roll rate, which causes the bank angle to "ratchet," (2) the initial negative response of the yaw rate, (3) the rapid rise of the sideslip angle, and (4) the small constant value reached by the bank angle, which implies the necessity of holding a substantial lateral-control input in order to maintain a desired bank angle. This behavior of the baseline example can be attributed to the adverse yaw characteristic of the ailerons which excites the Dutch roll mode.

Smooth increases in bank angle with well-behaved yaw rate and reduced sideslip are obtained by using mixed aerodynamic controls (fig. 3(b)) and mixed aerodynamic and thrust-vectoring controls (fig. 3(c)). The airplane turn coordination has been improved. The limiting control for these cases is the rudder. The case of figure 3(c) uses a relatively large amount of the yawing thrust vector to supplement the rudder. The rolling thrust vector is ineffective because of the small distance between the engine nozzles. Since it is ineffective, the deflection of the rolling thrust vector is made small in comparison with the more effective controls.

The addition of thrust-vectoring controls (to the airplane of this example) has increased the roll rate capabilities of the airplane by approximately 60 percent with only a small increase in sideslip while using moderate deflection angles. The improvement of the airplane response was achieved by appropriately gearing the controls without using any feedback loops.

Concluding Remarks

A method for selecting mixing combinations (crossfeeds) for the control variables of a linear system has been developed. The mixing combinations are chosen so that the resulting control channels have their principal influences on selected fundamental modes of the system. In the ideal case, each fundamental mode would be controlled by one control channel independent of the others. The control design problem would then reduce to a series of first- and second-order design problems. This ideal decoupling is not possible in general. A series of algebraic maximization problems is used to find the control-mixing combinations which maximize the effects of the control channels on selected modes while simultaneously minimizing the effects on the remaining modes.

The technique allows a reduction in the number of channels in the case of multiple, redundant controls. The most effective control effectors are selected in the proper combination for each operating condition. Given a judicious selection of modes to be associated with a number of pseudo controls, the resultant control-mixing combinations may transform the system into one which is better conditioned for the subsequent application of feedback control law design procedures.

A quantitative measure of relative controllability is a product of the procedure. This measure can be used to assess the influence of each control variable on the modes of the system as well as on the relative effectiveness of the control-mixing combinations.

The method was applied to the lateral and directional control of an example airplane trimmed in level flight at an angle of attack of 10°. Two pseudo control variables were calculated for this system. The first has its principal effect on the roll and spiral modes, and the second, on the Dutch roll mode. Use of these pseudo control variables was found to eliminate the oscillations present in the roll rate for a step lateral-control input and to improve the sideslip response with reduced rolling motions for a step directional-control input.

The airplane model was modified to include thrust-vectoring nozzles which provide rolling and yawing control moments for use in very slow flight at moderate angles of attack. Yawing moments generated by thrust vectoring significantly improve the roll rate capability of the airplane. The rolling moment generated by thrust vectoring was found to be ineffective for the example studied.

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December 11, 1984

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TABLE I. CONTROL DEFLECTIONS: LATERAL-DIRECTIONAL

[$\alpha = 10°$; TAS = 165 knots]

Control variable (authority)	Control deflection, degrees			
	Baseline system		Pseudo controls	
	Lateral (fig. 1(a))	Directional (fig. 1(b))	v_1 (fig. 2(a))	v_2 (fig. 2(b))
δ_a (40°)	*4		*4	3.18
δ_r (30°)		*3	−2.75	*3
δ_D (20°)	*2		1.48	0.328

*10 percent of authority.

TABLE II. RELATIVE CONTROL EFFECTIVENESS: AERODYNAMIC CONTROLS

[$\alpha = 15°$; TAS = 133 knots]

(a) Relative control effectiveness matrix, original controls

$$\begin{array}{cc} & \begin{array}{ccc} \delta_a & \delta_r & \delta_D \end{array} \\ \mathbf{C}_{\text{eff}} = & \begin{bmatrix} 0.220 & 0.886 & 0.409 \\ 0.040 & 0.965 & 0.258 \\ 0.660 & 0.716 & 0.229 \end{bmatrix} \end{array} \begin{array}{l} \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array}$$

(b) Control-mixing matrix*

$$\begin{Bmatrix} \delta_a \\ \delta_r \\ \delta_D \end{Bmatrix} = \begin{bmatrix} 12.2 & 31.4 \\ -25.1 & 14.7 \\ 9.03 & 7.57 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

*All values are given in degrees.

(c) Relative control effectiveness matrix, pseudo controls

$$\begin{array}{cc} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \mathbf{C}_{\text{eff}} = & \begin{bmatrix} 0.994 & 0.107 \\ 0.914 & 0.407 \\ 0.297 & 0.955 \end{bmatrix} \end{array} \begin{array}{l} \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array}$$

TABLE III. RELATIVE CONTROL EFFECTIVENESS: AERODYNAMIC AND THRUST-VECTORING CONTROLS

[$\alpha = 15°$; TAS = 133 knots]

(a) Relative control effectiveness matrix, original controls

$$\mathbf{C}_{\text{eff}} = \begin{array}{c} \begin{array}{ccccc} \delta_a & \delta_r & \delta_D & \delta_{v,\text{roll}} & \delta_{v,\text{yaw}} \end{array} \\ \begin{bmatrix} 0.178 & 0.719 & 0.332 & 0.029 & 0.583 \\ 0.032 & 0.768 & 0.205 & 0.001 & 0.606 \\ 0.583 & 0.633 & 0.202 & 0.074 & 0.461 \end{bmatrix} \end{array} \begin{array}{l} \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array}$$

(b) Control-mixing matrix*

$$\begin{Bmatrix} \delta_a \\ \delta_r \\ \delta_D \\ \delta_{v,\text{roll}} \\ \delta_{v,\text{yaw}} \end{Bmatrix} = \begin{bmatrix} 12.1 & 31.2 \\ -20.0 & 11.7 \\ 7.98 & 8.07 \\ 1.35 & 3.07 \\ -8.24 & 3.84 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

*All values are given in degrees.

(c) Relative control effectiveness matrix, pseudo controls

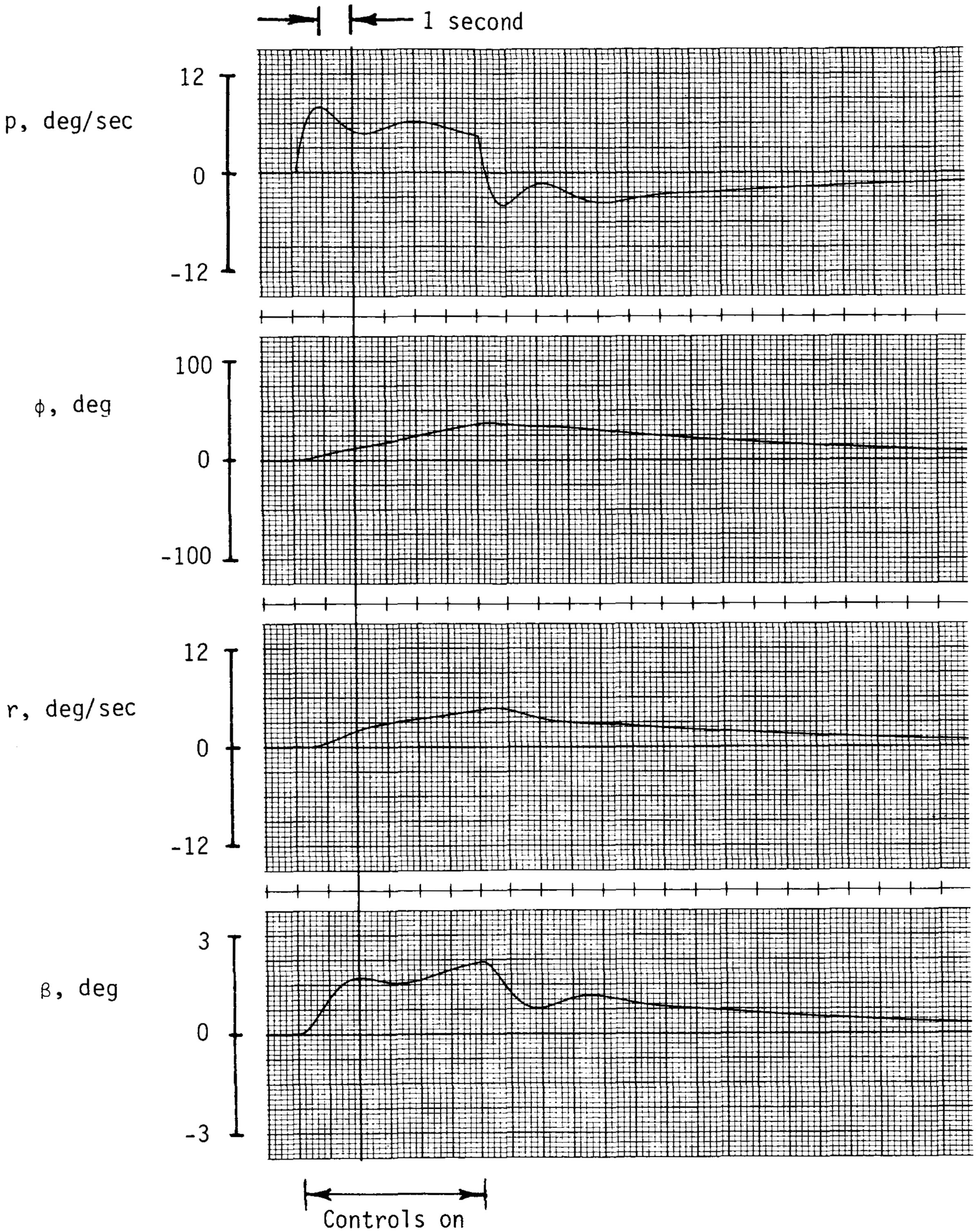
$$\mathbf{C}_{\text{eff}} = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 0.988 & 0.155 \\ 0.918 & 0.398 \\ 0.416 & 0.909 \end{bmatrix} \end{array} \begin{array}{l} \text{Roll} \\ \text{Spiral} \\ \text{Dutch roll} \end{array}$$

TABLE IV. CONTROL DEFLECTIONS: AERODYNAMIC AND THRUST-VECTORING

[$\alpha = 15°$; TAS = 133 knots]

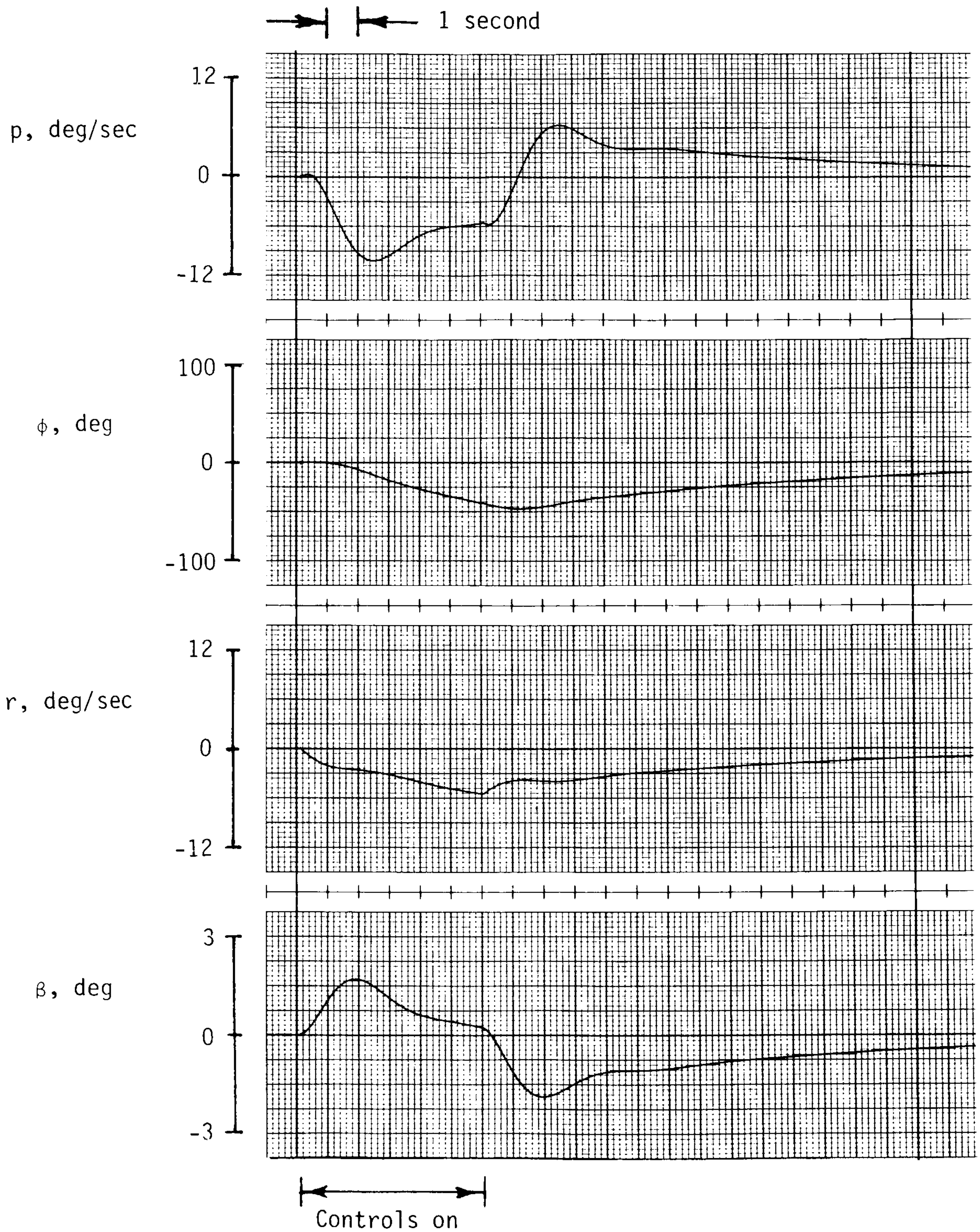
Control variable (authority)	Control deflection, degrees		
	Baseline system (fig. 3(a))	Aerodynamic controls (fig. 3(b))	Thrust-vectoring controls (fig. 3(c))
δ_a (40°)	*6	2.19	2.72
δ_r (30°)		*−4.5	*−4.5
δ_D (20°)	*3	1.62	1.79
$\delta_{v,\text{roll}}$ (30°)			0.304
$\delta_{v,\text{yaw}}$ (15°)			−1.85

*15 percent of authority.



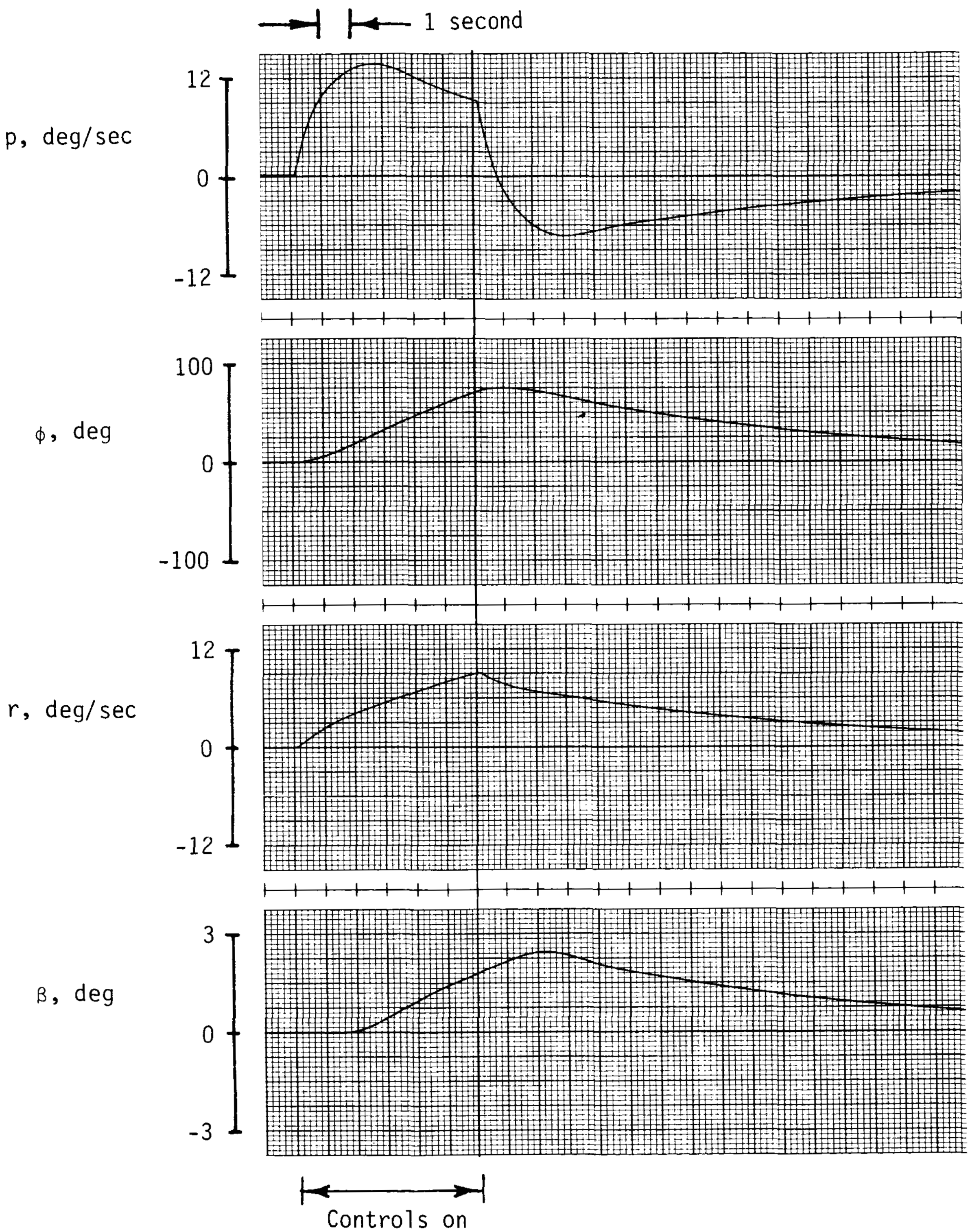
(a) Ailerons and differential tail (lateral controls).

Figure 1. Response of linear airplane model to baseline controls. $\alpha = 10°$.



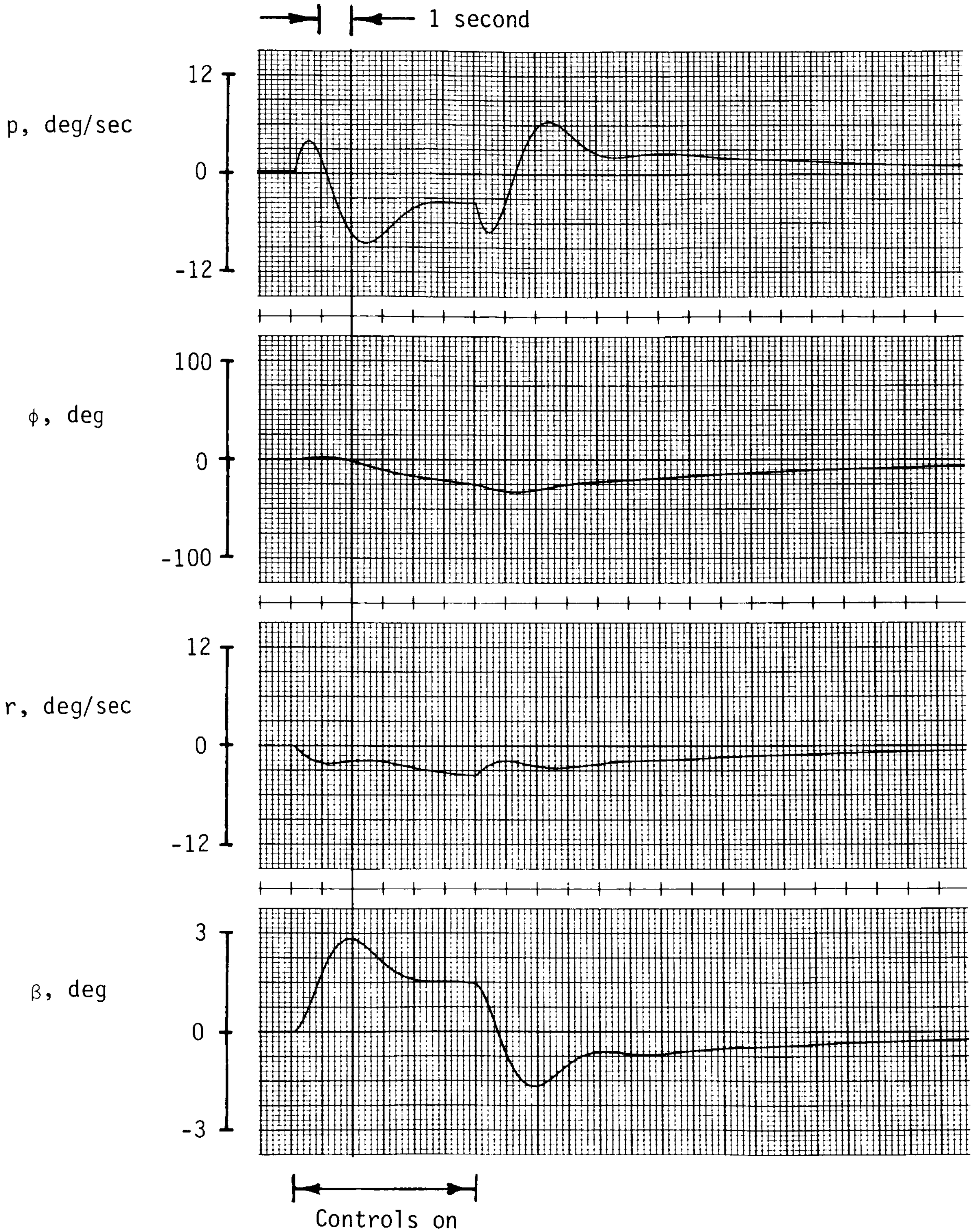
(b) Rudder (directional control).

Figure 1. Concluded.



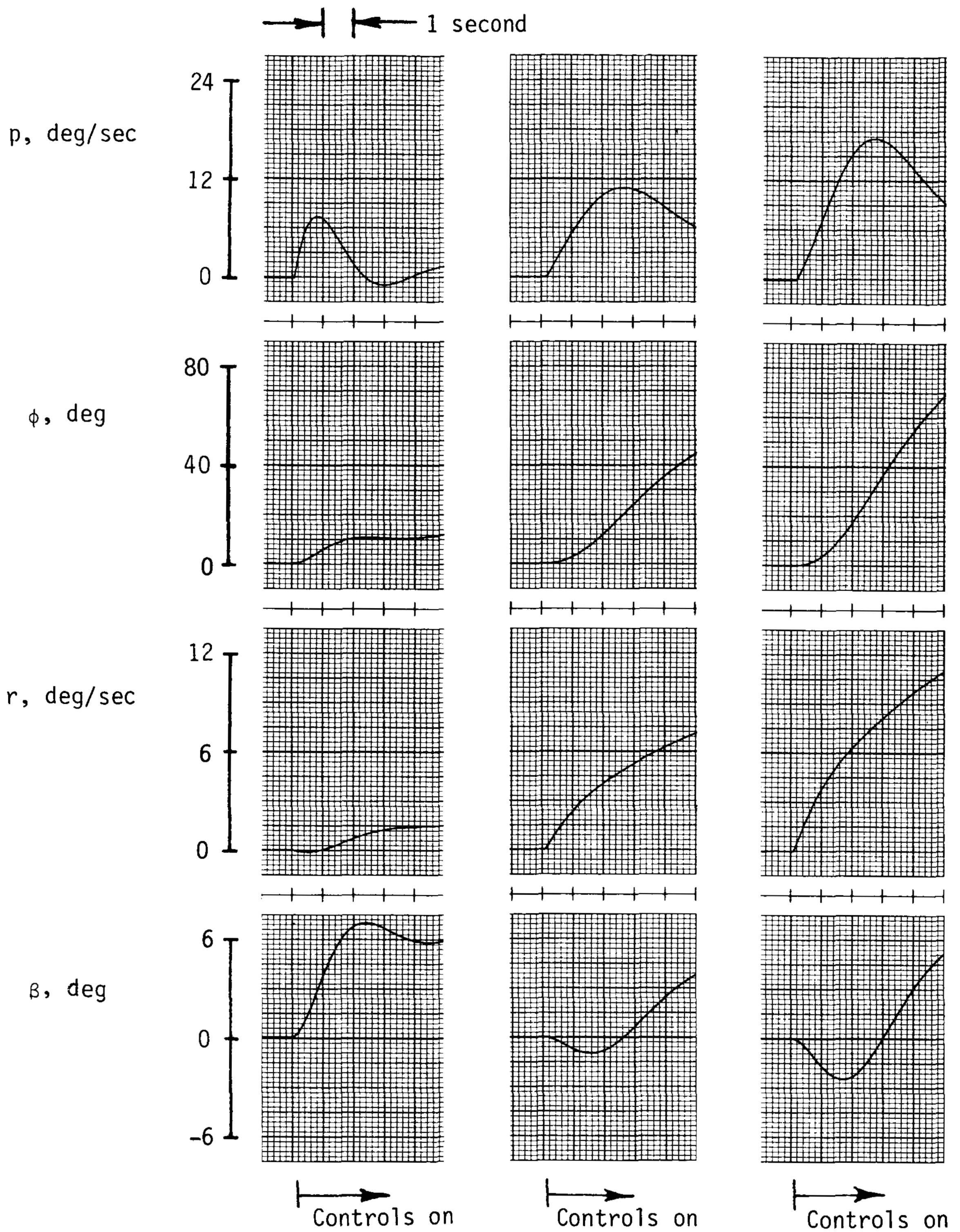
(a) Pseudo control v_1 (lateral control).

Figure 2. Response of linear airplane model to pseudo controls. $\alpha = 10°$.



(b) Pseudo control v_2 (directional control).

Figure 2. Concluded.



(a) Baseline system. (b) Aerodynamic controls. (c) Aerodynamic and thrust-vectoring controls.

Figure 3. Responses to step lateral-control commands. $\alpha = 15^{\circ}$.

1. Report No. NASA TP-2416	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Relative Control Effectiveness Technique With Application to Airplane Control Coordination		5. Report Date April 1985
		6. Performing Organization Code 505-34-03-02
7. Author(s) Frederick J. Lallman		8. Performing Organization Report No. L-15864
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665		10. Work Unit No.
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546		13. Type of Report and Period Covered Technical Paper
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

A method for selecting optimal combinations of the control variables (crossfeeds) of a linear system is reported. The combinations are chosen so that the control channels have their principal influences on selected fundamental modes of the system. A series of algebraic maximization problems is used to maximize the effects of the control channels on selected modes while simultaneously minimizing the effects on the remaining modes. The method is applied to the lateral and directional control of a linearized airplane model having ailerons, a rudder, and differential tail surfaces. Integration of these controls eliminates oscillations present in the roll rate for a step lateral-control input and improves the sideslip response with reduced rolling motions for a step directional-control input. Inclusion of thrust-vectoring engine nozzles improves the roll rate capability of the airplane.

17. Key Words (Suggested by Authors(s))		18. Distribution Statement	
Airplane control Integrated control Thrust vector control Optimization Controllability		Unclassified—Unlimited Subject Category 05	
19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 16	22. Price A02

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