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A MATHEMATICAL FORMULATION OF THE SCOLE CONTROL PROBLEM: PART 1

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SUMMARY

This is the first part of a multi-part report on a mathematical formulation of the SCOLE control problem, and deals primarily with various techniques of solution of the partial differential equations with delta-functions on the boundary, the most comprehensive of which is the formulation as a non-linear abstract wave equation which clarifies the concept of modes and leads immediately to a linear feedback law for stability augmentation. We also obtain a "closed-form solution" in a special case.

1. INTRODUCTION

This report is in several parts. In this the first part we deal mainly with techniques of solution of the partial differential equations with delta-functions on the boundary, as formulated in [1]. Several methods of solution are described. The most comprehensive of these is the formulation as a non-linear abstract wave-equation, which clarifies the concept of modes and allows us to develop a linear feedback that can stabilize the system, in the sense that the total energy cannot increase. We are also able to obtain a "closed-form solution" in a special case, using a "boundary input" approach following [3]. Finally, the mathematical formulation of the problem should help in the digital computer simulation of various control laws.

While this report closely follows [1] in notation, the beginning section, Section 2, restates the Beam Equations in such a way as to clarify the delta-function formulation in [1]. Section 3 outlines a Green's Function approach to the solution. Section 4 shows how the δ-functions can be removed, and replaced by time-varying boundary conditions. Section 5 shows how the Laplace Transform technique could be used and shows its equivalence to the Green's Function method. Finally Section 6 outlines the "Boundary Input" method patterened on the development in [3]. After specializing the boundary inputs to the specified boundary dynamics (in terms of ordinary differential equations) we go on to formulate an "abstract wave equation" in Section 8. Using this formulation, we are able to show that it is possible to stabilize the system (in terms of decreasing the total energy of the system) by a linear feedback. Moreover we also indicate how to calculate the modes, as well as a modal expansion of the solution leading to a nonlinear integral equation to take care of the nonlinearity introduced in the rotational motion. Finally in Section 9 we indicate one explicit solution using the "boundary input" technique of Section 6.

2. THE BEAM BENDING EQUATIONS

Let

$$U(t,s) = \begin{vmatrix} u_\phi(t,s) \\ u_\theta(t,s) \\ u_\psi(t,s) \end{vmatrix}, \qquad 0 \le t, \qquad -\infty < s < \infty$$

$$U(t,s) = 0 \qquad\qquad s \le 0, \quad s \ge L .$$

Superdots will denote derivatives with respect to time t, $0 \le t$. Primes will denote derivatives with respect to the space variable s. The basic beam deflection equations are:

Roll Beam Bending Equation

$$PA\,\ddot{u}_\phi(t,s) + EI_\phi u_\phi''''(t,s) = \sum_1^4 f_{\phi,n}(t)\,\delta(s-s_n) + \sum_1^4 g_{\phi,n}(t)\,\delta'(s-s_n) ,$$

$$0 \le t, \qquad -\infty < s < \infty$$

$$0 \le s_n \le L . \tag{2.1}$$

The delta-function and the delta-function derivative are to be interpreted as generalized functions. Hence for any C^1 function $h(s)$, $-\infty < s < \infty$, we have

$$\int_0^L (PA\ddot{u}_\phi(t,s) + EI_\phi u_\phi''''(t,s))\,h(s)\,ds = \sum_1^4 f_{\phi,n}(t)h(s_n) + \sum_1^4 g_{\phi,n}(t)h'(s_n) . \tag{2.2}$$

Pitch Beam Bending Equation

$$PA\,\ddot{u}_\theta(t,s) + EI_\theta u_\theta''''(t,s) = \sum_1^4 f_{\theta,n}(t)\,\delta(s-s_n) + \sum_1^4 g_{\theta,n}(t)\,\delta'(s-s_n)$$

$$0 \le t, \qquad -\infty < s < \infty$$

$$0 \le s_n \le L . \tag{2.3}$$

The interpretation of the delta-functions is the same as in the case of the previous Roll equations, so need not be repeated.

Yaw Beam Torsion Equation

$$PI_\psi \ddot{u}_\psi(t,s) \quad - \quad GI_\psi u''_\psi(t,s) \quad = \quad \sum_1^4 g_{\psi,n}(t)\ \delta(s-s_n) \qquad (2.4)$$

$$0 \le t, \qquad -\infty < s < \infty, \qquad 0 \le s_n \le L \quad .$$

Once again, the interpretation of the delta-function is the same as before.

3. METHOD OF SOLUTION I: GREEN'S FUNCTIONS

Roll Beam Bending Equations

We begin by constructing the Green's function for the homogeneous equations:

$$PA\ \ddot{u}_\phi(t,s) \quad + \quad EI_\phi u''''_\phi(t,s) \quad = \quad 0 \quad , \qquad 0 < t, \qquad 0 < s < L \quad .$$

with the boundary conditions ("free-free")

$$u''_\phi(t,0+) \quad = \quad u''_\phi(t,L-) \quad = \quad 0$$

$$u'''_\phi(t,0+) \quad = \quad u'''_\phi(t,L-) \quad = \quad 0$$

As is well-known [4], this can be done with the aid of the eigenfunctions $\phi_n(\cdot)$ which satisfy

$$\frac{d^4}{ds^4}\phi_n(s) \quad = \quad k_n^4\ \phi_n(s) \quad , \qquad 0 < s < L$$

and (to satisfy the boundary conditions) is of the form:

$$\phi_n(s) \quad = \quad A_n(\cosh k_n s \quad + \quad \cos k_n s) \quad + \quad B_n(\sinh k_n s \quad + \quad \sin k_n s)$$

where A_n, B_n are such that

$$\int_0^L \phi_n(s)^2 \, ds = 1$$

and, of course,

$$\int_0^L \phi_n(s) \, \phi_m(s) \, ds = \delta_n^m \quad .$$

The k_n satisfy

$$\cosh k_n L \quad \cos k_n L = 1 \quad .$$

It is well-known that [4]

$$k_0 = 0$$

$$k_n \sim \frac{(n+1)\pi}{2L} \quad .$$

Letting

$$\gamma_\phi^2 = \frac{EI_\phi}{PA} \quad ,$$

the Green's function is defined by

$$G_\phi(t,s,s') = \sum_0^\infty \frac{\sin \gamma_\phi k_n^2 t}{\gamma_\phi k_n^2} \phi_n(s) \, \phi_n(s') \quad , \qquad 0 \le t, \quad 0 \le s,s' \le L \, . \tag{3.1}$$

We may now apply Duhamel's principle [5] to obtain the solution to the non-homogeneous Roll Beam Bending Equations (2.1). Thus we have

$$u_\phi(t,s) = \int_0^t \sum_1^4 \int_0^L G_\phi(t-\sigma;\ s;\ s')(f_{\phi,n}(\sigma)\delta(s'-s_n') + g_{\phi,n}(\sigma)\delta'(s'-s_n'))\ d\sigma\ ds'$$

$$+ \sum_0^\infty a_n \sin\ (\gamma_\phi k_n^2 t + \beta_n)\ \phi_n(s)$$

$$= \int_0^t \sum_1^4 G_\phi(t-\sigma;\ s;\ s_n)\ f_{\phi,n}(\sigma)\ d\sigma + \int_0^t \sum_1^4 G_\phi'(t-\sigma;\ s,\ s_n)\ g_{\phi,n}(\sigma)\ d\sigma$$

$$+ \sum_0^\infty a_n \sin\ (\gamma_\phi k_n^2 t + \beta_n)\ \phi_n(s) \tag{3.2}$$

where

$$G_\phi'(t,s,s_n) = \frac{\partial}{\partial s'} G_\phi(t,s,s')\Big|_{s'=s_n} .$$

The term outside the integral in (3.2) takes care of the initial conditions:

$$u_\phi(0;s),\quad \dot{u}_\phi(0,s)$$

for suitable choice of $\{a_n\}$ and $\{\phi_n\}$, as is well known [4]. For simplicity, we shall often set the initial conditions to be zero.

Pitch Beam Bending Equations

In a similar fashion, defining

$$\gamma_\theta^2 = \frac{EI_\theta}{PA}$$

and the Green's function

$$G_\theta(t,s,s') = \sum_0^\infty \frac{\sin\ \gamma_\theta k_n^2 t}{\gamma_\theta k_n^2}\ \phi_n(s)\ \phi_n(s')$$

with $\phi_n(\cdot)$ as before, we can express the solution to the Pitch Beam Bending

Equations (2.3) as:

$$u_\theta(t,s) = \int_0^t \sum_1^4 G_\theta(t-\sigma;\ s,\ s_n)\ f_{\theta,n}(\sigma)\ d\sigma + \int_0^t \sum_1^4 G'_\theta(t-\sigma;\ s,\ s_n)\ g_{\theta,n}(\sigma)\ d\sigma$$

$$+ \sum_0^\infty b_n \sin\ (\gamma_\theta k_n^2 t + \theta_n)\ \phi_n(s) \qquad (3.3)$$

<u>Yaw Beam Torsion Equations</u>

We begin by constructing the Green's function for the homogeneous equations

$$PI_\psi \ddot{u}_\psi(t,s) - GI_\psi u''_\psi(t,s) = 0 \qquad 0 < t, \quad 0 < s < L$$

with the boundary conditions:

$$u'(t,0+) = u'(t,L-) = 0$$

(and zero initial conditions). It is well known [4] that this can be done with the aid of the eigen functions $\psi_n(\cdot)$ which satisfy

$$\frac{d^2}{ds^2}\psi_n(s) = -\lambda_n^2\ \psi_n(s)$$

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 0, \pm 1, \pm 2 \ldots .$$

Let

$$\gamma_\psi^2 = \frac{GI_\psi}{PI_\psi} .$$

Then the Green's function $G_\psi(t,s,s')$ is given by

$$G_\psi(t,s,s') = \frac{1}{L}\sum_{-\infty}^{\infty} \frac{\sin \lambda_n \gamma_\psi t}{\gamma_\psi\ \lambda_n} \cos \lambda_n s \quad \cos \lambda_n s' .$$

By Duhamel's principle [5], as before, we can then express the solution to the non-homogeneous, Yaw Beam Torsion Equation (2.4) as:

$$u_\psi(t,s) = \int_0^t \int_0^L G_\psi(t-\sigma;\ s;\ s') \left(\sum_1^4 g_{\psi,n}(\sigma)\delta(s'-s_n)\right) d\sigma\ ds'$$

$$= \sum_1^4 \int_0^t G_\psi(t-\sigma;\ s,\ s_n)\ g_{\psi,n}(\sigma)\ d\sigma$$

and the general solution allowing for nonzero initial conditions is then

$$u_\psi(t,s) = \sum_1^4 \int_0^t G_\psi(t-\sigma;\ s;\ s_n)\ g_{\psi,n}(\sigma)\ d\sigma + \sum_{-\infty}^{\infty} c_n \sin(\lambda_n \zeta_\psi t + \psi_n) \cos \lambda_n s \tag{3.4}$$

4. METHOD OF SOLUTION II: TIMEVARYING BOUNDARY CONDITIONS

In this section we shall reformulate the basic equations (2.1) - (2.4) removing the delta-functions and replacing them by time-varying boundary conditions.

Roll Beam Bending Equations

We begin with the Roll Beam Bending equations (2.1), in the form (2.2). Choose $h(\cdot)$ such that

$$h(s) = 0 \qquad s > \varepsilon.$$

Remembering that

$$s_1 = 0$$

we obtain

$$\int_0^\varepsilon (PA\,\ddot{u}_\phi(t,s) + EI_\phi u_\phi''''(t,s))\,h(s)\,ds = f_{\phi,1}(t)\,h(0) + g_{\phi,1}(t)\,h'(0) \quad .$$

We integrate the second integral on the left by parts, twice in succession:

$$\int_0^\varepsilon u_\phi''''(t,s)\, h(s)\, ds \;=\; u_\phi'''(t,\varepsilon)\, h(\varepsilon) \;-\; u_\phi'''(t,0-)\, h(0) \;-\; u_\phi''(t,\varepsilon)\, h'(\varepsilon)$$
$$+\; u_\phi''(t,0-)\, h'(0) \;+\; \int_0^\varepsilon u_\phi''(t,s)\, h''(s)\, ds \quad .$$

Next we let ε go to zero. Then we obtain in the limit:

$$EI_\phi(u_\phi'''(t,0+) - u_\phi'''(t,0-))\, h(0) \;+\; EI_\phi(u_\phi''(t,0-) \;-\; u_\phi''(t,0+))\, h'(0)$$
$$=\; f_{\phi,1}(t)\, h(0) \;+\; g_{\phi,1}(t)\, h'(0) \quad .$$

Since $h(\cdot)$ can be chosen arbitrarily we obtain that:

$$EI_\phi(u_\phi'''(t,0+) - u_\phi'''(t,0-)) \;=\; f_{\phi,1}(t)$$
$$EI_\phi(u_\phi''(t,0+) - u_\phi''(t,0-)) \;=\; -g_{\phi,1}(t)$$

Since, by definition,

$$u_\phi''(t,0-) \;=\; u_\phi'''(t,0-) \;=\; 0$$

we obtain

$$\left.\begin{aligned} EI_\phi u_\phi'''(t,0+) &= f_{\phi,1}(t) \\ EI_\phi u''(t,0+) &= -g_{\phi,1}(t) \quad . \end{aligned}\right\} \tag{4.1}$$

In an entirely similar manner, noting that $u_\phi'''(t,L+) = u_\phi''(t,L+) = 0$, we obtain:

$$\left.\begin{aligned} EI_\phi u_\phi'''(t,L-) &= -f_{\phi,4}(t) \\ EI_\phi u_\phi''(t,L-) &= g_{\phi,4}(t) \quad . \end{aligned}\right\} \tag{4.2}$$

Also

$$\left.\begin{aligned} EI_\phi(u_\phi'''(t,s_2+) - u_\phi'''(t,s_2-)) &= f_{\phi,2}(t) \\ EI_\phi(u_\phi'''(t,s_3+) - u_\phi'''(t,s_3-)) &= f_{\phi,3}(t) \quad . \end{aligned}\right\} \qquad (4.3)$$

Hence we may replace (2.1) by:

$$PA\,\ddot{u}_\phi(t,s) + EI_\phi u_\phi''''(t,s) = 0\,, \qquad 0 < s < L, \quad 0 < t \qquad (4.4)$$

with the time-varying boundary conditions:

$$\left.\begin{aligned} EI_\phi u_\phi'''(t,0+) &= f_{\phi,1}(t) \\ EI_\phi u_\phi'''(t,L-) &= -f_{\phi,4}(t) \\ EI_\phi(u_\phi'''(t,s_2+) - u_\phi'''(t,s_2-)) &= f_{\phi,2}(t) \\ EI_\phi(u_\phi'''(t,s_3+) - u_\phi'''(t,s_3-)) &= -f_{\phi,3}(t) \\ EI_\phi u_\phi''(t,0+) &= -g_{\phi,1}(t) \\ EI_\phi u_\phi''(t,L-) &= g_{\phi,4}(t) \end{aligned}\right\} \qquad (4.4a)$$

and any initial conditions in addition that may be imposed:

$$u_\phi(0+,s) \quad \text{given}$$

$$\dot{u}_\phi(0+,s) \quad \text{given}$$

<u>Pitch Beam Bending Equations</u>

In an entirely similar manner, we may replace (2.3) by:

$$PA\,\ddot{u}_\theta(t,s) + EI_\theta u_\theta''''(t,s) = 0\,, \qquad 0 < s < L, \quad 0 < t \qquad (4.5)$$

with the time-varying boundary conditions

$$\left.\begin{aligned}
EI_\theta u_\theta'''(t,0+) &= f_{\theta,1}(t) \\
EI_\theta u_\theta'''(t,L-) &= -f_{\theta,4}(t) \\
EI_\theta(u_\theta'''(t,s_2+) - u_\theta'''(t,s_2-)) &= f_{\theta,2}(t) \\
EI_\theta(u_\theta'''(t,s_3+) - u_\theta'''(t,s_3-)) &= f_{\theta,3}(t) \\
EI_\theta u_\theta''(t,0+) &= -g_{\theta,1}(t) \\
EI_\theta u_\theta''(t,L-) &= g_{\theta,4}(t)
\end{aligned}\right\} \qquad (4.5a)$$

and any initial conditions imposed

$$u_\theta(0+,s) \quad \text{given}$$

$$\dot{u}_\theta(0+,s) \quad \text{given}$$

<u>Yaw Beam Torsion Equations</u>

Analogously we may replace (2.4) by:

$$PI_\psi \ddot{u}_\psi(t,s) - GI_\psi u_\psi''(t,s) = 0, \qquad 0 < s < L, \quad 0 < t \qquad (4.6)$$

with the time-varying boundary conditions

$$\left.\begin{aligned}
GI_\psi u_\psi'(t,0+) &= -g_{\psi,1}(t) \\
GI_\psi u_\psi'(t,L-) &= +g_{\psi,4}(t)
\end{aligned}\right\} \qquad (4.6a)$$

and any initial conditions imposed:

$$u_\psi(0+,s) \quad \text{given}$$

$$\dot{u}_\psi(0+,s) \quad \text{given} \quad .$$

5. METHOD OF SOLUTION III: LAPLACE TRANSFORMS

In this section we shall develop the Laplace Transform technique of solution of the Beam equations using the boundary-condition formulation of the previous section. We shall take the initial conditions to be zero. Let

$$\bar{u}(z,s) = \int_0^\infty e^{-zt}\, u(t,s)\, dt \ , \qquad \text{Re } z > 0$$

Roll Beam Bending Equations

The Roll Beam equations (4.4) may be Laplace transformed to yield:

$$z^2\, \bar{u}_\phi(z,s) + \gamma_\phi^2\, \bar{u}_\phi''''(z,s) = 0 \ , \qquad 0 < s < L \ .$$

As is well known, the general solution to this equation for each fixed z can be written

$$\bar{u}_\phi(z,s) = A_1(z) \cosh\sqrt{\frac{z}{\gamma_\phi}}\, s + A_2(z) \cos\sqrt{\frac{z}{\gamma_\phi}}\, s + A_3(z) \sinh\sqrt{\frac{z}{\gamma_\phi}}\, s$$
$$+ A_4(z) \sin\sqrt{\frac{z}{\gamma_\phi}}\, s \quad . \qquad (5.1)$$

The functions $A_1(z)$, $A_2(z)$, $A_3(z)$ and $A_4(z)$ are then to be determined by the imposed boundary conditions (4.4a). Thus let

$$\bar{f}_{\phi,n}(z) = \int_0^\infty e^{-zt}\, f_{\phi,n}(t)\, dt$$

$$\bar{g}_{\phi,n}(z) = \int_0^\infty e^{-zt}\, g_{\phi,n}(t)\, dt$$

First, for simplicity, we omit the "proof-mass" conditions at s_2 and s_3. Then we have

$$\bar{f}_{\phi,1}(z) = EI_\phi \bar{u}_\phi'''(z,0+)$$

$$= EI_\phi \left(\sqrt{\frac{z}{\gamma_\phi}}\right)^3 (A_3(z) - A_4(z)) \quad .$$

Similarly

$$\bar{f}_{\phi,4}(z) = -EI_\phi \left(\sqrt{\frac{z}{\gamma_\phi}}\right)^3 \left\{A_1(z) \sinh\sqrt{\frac{z}{\gamma_\phi}}\, L - A_2(z) \sin\sqrt{\frac{z}{\gamma_\phi}}\, L \right.$$

$$\left. + A_3(z) \cosh\sqrt{\frac{z}{\gamma_\phi}}\, L - A_4(z) \cos\sqrt{\frac{z}{\gamma_\phi}}\, L\right\}$$

$$\bar{g}_{\phi,1}(z) = -EI_\phi \left(\frac{z}{\gamma_\phi}\right)(A_1(z) - A_2(z))$$

$$\bar{g}_{\phi,4}(z) = EI_\phi \left(\frac{z}{\gamma_\phi}\right) \left\{A_1(z) \cosh\sqrt{\frac{z}{\gamma_\phi}}\, L - A_2(z) \cos\sqrt{\frac{z}{\gamma_\phi}}\, L \right.$$

$$\left. + A_3(z) \sinh\sqrt{\frac{z}{\gamma_\phi}}\, L - A_4(z) \sin\sqrt{\frac{z}{\gamma_\phi}}\, L\right\} \quad .$$

There are four linear equations, which can be solved to determine the unknown functions. $A_1(z)$, $A_2(z)$, $A_3(z)$, $A_4(z)$, in terms of $\bar{f}_{\phi,1}(z)$, $\bar{f}_{\phi,4}(z)$, $\bar{g}_{\phi,1}(z)$ and $\bar{g}_{\phi,4}(z)$. The solution will be unique, as can easily be seen (by the non-singularity at the determinant).

To include the proof-mass conditions at s_2 and s_3 we must subdivide the interval $[0,L]$ into three: $[0,s_2]$, $[s_2,s_3]$ and $[s_3,L]$. In each of these intervals we will express the solution in terms of the coefficient functions which will differ now from interval to interval. We omit the details.

Pitch Beam Bending Equations

Since the Pitch Beam Bending Equations (4.5) so completely parallel the Roll Beam Bending Equation, we shall not need to go into the details of Laplace

Transformation. Thus, if we omit the "proof-mass" equations, we can write:

$$\bar{u}_\theta(z,s) = B_1(z)\cosh\sqrt{\frac{z}{\gamma_\theta}}\,s + B_2(z)\cos\sqrt{\frac{z}{\gamma_\theta}}\,s + B_3(z)\sinh\sqrt{\frac{z}{\gamma_\theta}}\,s$$

$$+ B_4(z)\sin\sqrt{\frac{z}{\gamma_\theta}}\,s \tag{5.2}$$

where the functions $B_1(z)$, $B_2(z)$, $B_3(z)$, $B_4(z)$ are determined as linear transformations of:

$$\bar{f}_{\theta,n}(z) = \int_0^\infty e^{-zt} f_{\theta,n}(t)\,dt$$

$$\bar{g}_{\theta,n}(z) = \int_0^\infty e^{-zt} g_{\theta,n}(t)\,dt$$

just as in the Roll Beam Bending case. To include "proof-mass" conditions, we proceed also exactly as in that case.

<u>Yaw Beam Torsion Equations</u>

Here Laplace transforming (4.6), we have:

$$z^2\bar{u}_\psi(z,s) - \gamma_\psi^2\bar{u}''_\psi(z,s) = 0\,, \qquad 0 < s < L, \quad \mathrm{Re}\ Z > 0\ .$$

The general solution is of course

$$\bar{u}_\psi(z,s) = C_1(z)\cosh\frac{zs}{\gamma_\phi} + C_2(z)\sinh\frac{zs}{\gamma_\psi}\ .$$

The substitution of the boundary conditions yield:

$$-\bar{g}_{\psi,1}(z) = GI_\psi\bar{u}'_\psi(z,0+) = GI_\psi\left(\frac{z}{\gamma_\psi}\right)C_2(z)$$

$$\bar{g}_{\psi,4}(z) = GI_\psi\left(\frac{z}{\gamma_\psi}\right)\left(C_1(z)\sinh\frac{zL}{\gamma_\psi} + C_2(z)\cosh\frac{zL}{\gamma_\psi}\right)$$

where

$$\bar{g}_{\psi,1}(z) = \int_0^\infty e^{-zt}\, g_{\psi,1}(t)\, dt$$

$$\bar{g}_{\psi,4}(z) = \int_0^\infty e^{-zt}\, g_{\psi,4}(t)\, dt \quad .$$

The boundary conditions are simple enough so that we can write the explicit solution:

$$\bar{u}_\psi(z,s) = \frac{\bar{g}_{\psi,1}(z) \cosh \frac{z}{\gamma_\psi}(L-s)}{z \sinh \frac{zL}{\gamma_\psi}} + \frac{\bar{g}_{\psi,4}(z) \cosh \frac{zs}{\gamma_\psi}}{z \sinh \frac{zL}{\gamma_\psi}} \tag{5.3}$$

where of course we have set all the initial conditions to zero. It is not difficult to see that (5.3) is exactly the Laplace transform of the time-domain solution expressed by (3.2). We omit the details, except to remark that we need to use the fact that $\sinh \frac{zL}{\gamma_\psi}$ is an entire function with zeros at

$$z = i\, \frac{n\pi}{L}\, \gamma_\psi = i\, \lambda_n\, \gamma_\psi \quad .$$

We can make the same statement for the other Beam equation solutions: that (5.1) is precisely the Laplace transform of the time-domain solution (3.2) and similarly also (5.2) is the Laplace transform version of (3.3).

6. METHOD OF SOLUTION IV: BOUNDARY INPUTS

In this technique of solution for the Beam equations, we seek to express the solution as the sum of two. We begin by considering the case where the "proof-mass" conditions are omitted.

Roll Beam Bending Equation

Let $u^o_\phi(t,s)$ denote the solution of the non-homogeneous equations

$$PA\,\ddot{u}^o_\phi(t,s) + EI_\phi u^{o\,''''}_\phi(t,s) = z(t,s)\ , \qquad 0 < s < L, \quad 0 < t$$

with the zero "free-free" end conditions:

$$u^{o\,'''}_\phi(t,0+) = u^{o\,'''}_\phi(t,L-) = 0$$

$$u^{o\,''}_\phi(t,0+) = u^{o\,''}_\phi(t,L-) = 0 \quad .$$

Let

$$D_\phi(t,s) = a_3(t)s^3 + a_2(t)s^2 + a_1(t)s + a_0(t) \quad .$$

Note that

$$\frac{\partial^4}{\partial s^4} D_\phi(t,s) \equiv 0 \ , \qquad 0 < s < L$$

where the functions $a_i(t)$, $i = 0,1,2,3$, are yet to be determined.

We consider the "time-varying boundary conditions" version of the Roll Beam Bending Equations: viz. (4.4), and seek a solution in the form of the sum

$$u^o_\phi(t,s) + D_\phi(t,s) \quad .$$

It follows that we must have

$$PA(\ddot{u}^o_\phi(t,s) + \ddot{D}_\phi(t,s)) + EI_\phi u^{o\,''''}_\phi(t,s) = 0\ , \qquad 0 < s < L,$$

from which we see that:

$$z(t,s) = -PA\ddot{D}_\phi(t,s)$$

$$PA\,\ddot{u}^o_\phi(t,s) + EI_\phi u^{o\,''''}_\phi(t,s) = -PA\ddot{D}_\phi(t,s) \quad .$$

Hence

$$u^{o}(t,s) = \int_0^t G_\phi(t-\sigma;\ s;\ s')\ (-PA\ \ddot{D}_\phi(\sigma,s')\ d\sigma$$

$$= (-PA) \left\{ \int_0^t G_\phi(t-\sigma;\ s;\ s') \int_0^L \sum \ddot{a}_k(\sigma)\ (s')^k\ d\sigma\ ds' \right\}$$

Hence the total solution is:

$$\sum_0^3 a_k(t) s^k \ - \ (PA) \sum_0^3 \int_0^t \int_0^L G_\phi(t-\sigma;\ s;\ s')\ \ddot{a}_k(\sigma)\ (s')^k\ d\sigma\ ds' \tag{6.1}$$

The coefficient functions $\{a_k(\cdot)\}$ are then to be determined from the time-varying boundary conditions, as we shall describe below.

Pitch Beam Bending Equations

Here we can write the total solution analogously as:

$$\sum_0^3 b_k(t) s^k \ - \ (PA) \sum_0^3 \int_0^t \int_0^L G_\theta(t-\sigma;\ s;\ s')\ \ddot{b}_k(\sigma)\ (s')^k\ d\sigma\ ds' \tag{6.2}$$

where again the coefficient functions $b_k(\cdot)$ are to be determined later from the time-varying boundary conditions.

Yaw Beam Torsion Equations

Here we seek the solution in the form:

$$u_\psi^{o}(t,s) \ + \ D_\psi(t,s)$$

where $u_\psi^{o}(t,s)$ satisfies:

$$PI_\psi \ddot{u}^o_\psi(t,s) \;-\; GI_\psi u^{o\,\prime\prime}_\psi(t,s) \;=\; z(t,s) \;, \qquad 0 < s < L \;, \quad 0 < t \;.$$

$$u^{o\,\prime}_\psi(t,0+) \;=\; u^{o\,\prime}_\psi(t,L-) \;=\; 0$$

and

$$D_\psi(t,s) \;=\; C_1(t)s \;+\; C_0(t) \;.$$

Since we must have that:

$$PI_\psi(\ddot{u}^o_\psi(t,s) + \ddot{D}_\psi(t,s)) \;-\; GI_\psi u^{o\,\prime\prime}_\psi(t,s) \;=\; 0 \;, \qquad 0 < s < L \;,$$

it follows that

$$z(t,s) \;=\; -PI_\psi \ddot{D}_\psi(t,s) \;.$$

Hence the solution is:

$$C_0(t) \;+\; C_1(t)s \;-\; PI_\psi \int_0^t \int_0^L G_\psi(t-\sigma;\; s;\; s')(\ddot{C}_0(\sigma) + \ddot{C}_1(\sigma)s')\; d\sigma\; ds' \qquad (6.3)$$

where the coefficient functions $C_0(t)$ and $C_1(t)$ will need to be determined by the time-varying boundary conditions, as we shall show.

To accommodate proof-mass conditions we will need to break up the interval $[0,L]$ into three sub-intervals $[0,s_2]$, $[s_2,s_3]$ and $[s_3,L]$ and choose different sets of the $\{a_k(t)\}$, $\{b_k(t)\}$, $\{c_k(t)\}$ coefficient functions as we shall illustrate later (see Section 9).

7. FURTHER REDUCTION OF THE BOUNDARY CONDITIONS

To proceed further with any of the methods of solution it is necessary to specialize the boundary functions $f_{\phi,i}(t)$, $f_{\theta,i}(t)$, $g_{\phi,i}(t)$, $g_{\theta,i}(t)$, $g_{\psi,i}(t)$, $i = 1,2,3,4$ to their given values. It is convenient to introduce some vector notation for this purpose. Let

$$\omega_1(t) = \begin{vmatrix} \dot{u}'_\phi(t,0+) \\ \dot{u}'_\theta(t,0+) \\ \dot{u}_\psi(t,0+) \end{vmatrix} \tag{7.1}$$

$$\omega_4(t) = \begin{vmatrix} \dot{u}'_\phi(t,L-) \\ \dot{u}'_\theta(t,L-) \\ \dot{u}_\psi(t,L-) \end{vmatrix} \quad . \tag{7.2}$$

These are the angular velocity vectors of the shuttle and the antenna respectively. Let

$$g_1(t) = \begin{vmatrix} g_{\phi,1}(t) \\ g_{\theta,1}(t) \\ g_{\psi,1}(t) \end{vmatrix} \tag{7.3}$$

$$g_4(4) = \begin{vmatrix} g_{\phi,4}(t) \\ g_{\theta,4}(t) \\ g_{\psi,4}(t) \end{vmatrix} \tag{7.4}$$

and let the force applied at reflector center of mass be

$$F_r = [F_x, F_y, 0]^T \quad . \tag{7.4a}$$

Then

$$g_1(t) = -(I_1\dot{\omega}_1 + \omega_1 \otimes I_1\omega_1 - M_1(t) - M_D(t)) \tag{7.5}$$

$$g_4(t) = -(\hat{I}_4\dot{\omega}_4 + \omega_4 \otimes \hat{I}_4\omega_4 - M_4(t) - r \otimes F_r(t)) - m_4 r \otimes \ddot{\zeta}_4 \quad . \tag{7.6}$$

where

$$\hat{I}_4 = I_4 + m_4 \begin{vmatrix} r_y^2 & -r_x r_y & 0 \\ -r_x r_y & r_x^2 & 0 \\ 0 & 0 & r_x^2 + r_y^2 \end{vmatrix} \quad , \tag{7.6a}$$

$$\zeta_4 = \begin{vmatrix} -u_\phi(L-) \\ u_\theta(L-) \\ z(L-) \end{vmatrix} = \text{coordinates of beam tip,}$$

and

$$r = (r_x, r_y, 0)^T = \text{coordinates of reflector center of mass with respect to axes through beam tip.}$$

Let

$$f_1(t) = \begin{vmatrix} f_{\phi,1}(t) \\ f_{\theta,1}(t) \end{vmatrix} \tag{7.7}$$

$$f_2(t) = \begin{vmatrix} f_{\phi,2}(t) \\ f_{\theta,2}(t) \end{vmatrix} \tag{7.8}$$

$$f_3(t) = \begin{vmatrix} f_{\phi,3}(t) \\ f_{\theta,3}(t) \end{vmatrix} \tag{7.9}$$

$$f_4(t) = \begin{vmatrix} f_{\phi,4}(t) \\ f_{\theta,4}(t) \end{vmatrix} \tag{7.10}$$

Then

$$f_1(t) = - \begin{vmatrix} m_1\ddot{u}_\phi(t,0+) \\ m_1\ddot{u}_\theta(t,0+) \end{vmatrix} \tag{7.11}$$

$$f_2(t) = - \begin{vmatrix} m_2\ddot{u}_\phi(t,s_2) + m_2\ddot{\Delta}_{\phi,2} \\ m_2\ddot{u}_\theta(t,s_2) + m_2\ddot{\Delta}_{\theta,2} \end{vmatrix} \tag{7.12}$$

$$f_3(t) = - \begin{vmatrix} m_3\ddot{u}_\phi(t,s_3) + m_3\ddot{\Delta}_{\phi,3} \\ m_3\ddot{u}_\theta(t,s_3) + m_3\ddot{\Delta}_{\theta,3} \end{vmatrix} \tag{7.13}$$

$$f_4(t) = -m_4 \begin{bmatrix} 1 & 0 & r_x \\ 0 & 1 & r_y \end{bmatrix} \begin{vmatrix} \ddot{u}_\phi(L-) \\ \ddot{u}_\theta(L-) \\ \ddot{u}_\psi(L-) \end{vmatrix} + \begin{vmatrix} F_y \\ -F_x \end{vmatrix} . \tag{7.14}$$

8. METHOD II CONTINUED: ABSTRACT FORMULATION

In this section we continue with method II, and develop an abstract (wave-equation) formulation of the total problem. First we begin by substituting for the boundary functions as in the previous section. We obtain then the following ensemble of partial differential equations and ordinary differential equations.

$$\left.\begin{aligned} PA\,\ddot{u}_\phi(t,s) + EI_\phi u_\phi''''(t,s) &= 0 \\ PA\,\ddot{u}_\theta(t,s) + EI_\theta u_\theta''''(t,s) &= 0 \end{aligned}\right\} \quad \begin{aligned} &0 < s < s_2 \\ &s_2 < s < s_3 \\ &s_3 < s < L \end{aligned} \tag{8.1}$$

$$PI_\psi \ddot{u}_\psi(t,s) - GI_\psi u_\psi''(t,s) = 0 , \qquad 0 < s < L \tag{8.2}$$

$$\left.\begin{aligned} EI_\phi u_\phi'''(t,0+) &= -m_1 \ddot{u}_\phi(t,0+) \\ EI_\theta u_\theta'''(t,0+) &= -m_1 \ddot{u}_\theta(t,0+) \end{aligned}\right\} \tag{8.3}$$

$$\begin{vmatrix} EI_\phi u_\phi'''(t,L-) \\ EI_\theta u_\theta'''(t,L-) \end{vmatrix} = m_4 \begin{bmatrix} 1 & 0 & r_x \\ 0 & 1 & r_y \end{bmatrix} \begin{vmatrix} \ddot{u}_\phi(L-) \\ \ddot{u}_\theta(L-) \\ \ddot{u}_\psi(L-) \end{vmatrix} + \begin{vmatrix} -F_y(t) \\ F_x(t) \end{vmatrix} \tag{8.4}$$

$$\begin{vmatrix} EI_\phi u_\phi''(t,0+) \\ EI_\theta u_\theta''(t,0+) \\ GI_\psi u_\psi'(t,0+) \end{vmatrix} = I_1 \dot{\omega}_1 + \omega_1 \otimes I_1 \omega_1 - M_1(t) - M_D(t) \tag{8.5}$$

$$\begin{vmatrix} EI_\phi u_\phi''(t,L-) \\ EI_\theta u_\theta''(t,L-) \\ GI_\psi u_\psi'(t,L-) \end{vmatrix} = -(\hat{I}_4 \dot{\omega}_4 + \omega_4 \otimes \hat{I}_4 \omega_4 - M_4(t) - r \otimes F_r(t)) - m_4 r \otimes \ddot{\zeta}_4 \tag{8.6}$$

$$\left.\begin{aligned}
EI_\phi(u_\phi'''(t,s_2+) - u_\phi'''(t,s_2-)) &= -(m_2\ddot{u}_\phi(t,s_2) + m_2\ddot{\Delta}_{\phi,2}(t)) \\
EI_\theta(u_\theta'''(t,s_2+) - u_\theta'''(t,s_2-)) &= -(m_2\ddot{u}_\theta(t,s_2) + m_2\ddot{\Delta}_{\theta,2}(t)) \\
EI_\phi(u_\phi'''(t,s_3+) - u_\phi'''(t,s_3-)) &= -(m_3\ddot{u}_\phi(t,s_3) + m_3\ddot{\Delta}_{\phi,3}(t)) \\
EI_\theta(u_\theta'''(t,s_3+) - u_\theta'''(t,s_3-)) &= -(m_3\ddot{u}_\theta(t,s_3) + m_3\ddot{\Delta}_{\theta,3}(t)) \quad .
\end{aligned}\right\} \quad (8.7)$$

Let D denote the class of 3×1 functions $u(\cdot)$:

$$u(s) = \begin{vmatrix} u_\phi(s) \\ u_\theta(s) \\ u_\psi(s) \end{vmatrix}$$

such that

$$u_\phi, \quad u_\phi', \quad u_\phi'', \quad u_\phi''' \quad \in \quad L_2[0,L]$$

and u_ϕ''' has L_2-derivatives in $[0,s_2]$, $[s_2,s_3]$ and $[s_3,L]$;

$$u_\theta, \quad u_\theta', \quad u_\theta'', \quad u_\theta''' \quad \in \quad L_2[0,L]$$

and u_θ''' has L_2-derivatives in $[0,s_2]$, $[s_2,s_3]$ and $[s_3,L]$;

$$u_\psi, \quad u_\psi', \quad u_\psi'' \quad \in \quad L_2[0,L] \quad .$$

Introduce the following inner products on D:

$$\begin{aligned}
[u,v] = &\int_0^L u_\phi(s)\, v_\phi(s)\, ds + \int_0^L u_\theta(s)\, v_\theta(s)\, ds + \int_0^L u_\psi(s)\, v_\psi(s)\, ds \\
&+ u_\phi(0+)\, v_\phi(0+) + u_\theta(0+)\, v_\theta(0+) + u_\psi(0+)\, v_\psi(0+) \\
&+ u_\phi(L-)\, v_\phi(L-) + u_\theta(L-)\, v_\theta(L-) + u_\psi(L-)\, v_\psi(L-) \\
&+ u_\phi'(0+)\, v_\phi'(0+) + u_\theta'(0+)\, v_\theta'(0+) + u_\phi'(L-)\, v_\phi'(L-) + u_\theta'(L-)\, v_\theta'(L-) \\
&+ u_\phi(s_2)\, v_\phi(s_2) + u_\theta(s_2)\, v_\theta(s_2) + u_\phi(s_3)\, v_\phi(s_3) + u_\theta(s_3)\, v_\theta(s_3) \quad .
\end{aligned}$$

Complete the space under this inner product. It is readily seen that the completed space denoted X will be $L_2(0,T)^3 \times R^{14}$. Define the operator A on D by:

$$y = Ax$$

where

$x =$	$y =$
$u_\phi(\cdot)$	$EI_\phi u_\phi''''(\cdot)$
$u_\theta(\cdot)$	$EI_\theta u_\theta''''(\cdot)$
$u_\psi(\cdot)$	$-GI_\psi u_\psi''(\cdot)$
$u_\phi(0+)$	$EI_\phi u_\phi'''(0+)$
$u_\theta(0+)$	$EI_\theta u_\theta'''(0+)$
$u_\phi(L-)$	$-EI_\phi u_\phi'''(L-)$
$u_\theta(L-)$	$-EI_\theta u_\theta'''(L-)$
$u_\phi'(0+)$	$-EI_\phi u_\phi''(0+)$
$u_\theta'(0+)$	$-EI_\theta u_\theta''(0+)$
$u_\psi(0+)$	$-GI_\psi u_\psi'(0+)$
$u_\phi'(L-)$	$EI_\phi u_\phi''(L-)$
$u_\theta'(L-)$	$EI_\theta u_\theta''(L-)$
$u_\psi(L-)$	$GI_\psi u_\psi'(L-)$
$u_\phi(s_2)$	$EI_\phi(u_\phi'''(s_2+) - u_\phi'''(s_2-))$
$u_\theta(s_2)$	$EI_\theta(u_\theta'''(s_2+) - u_\theta'''(s_2-))$
$u_\phi(s_3)$	$EI_\phi(u_\phi'''(s_3+) - u_\phi'''(s_3-))$
$u_\theta(s_3)$	$EI_\theta(u_\theta'''(s_3+) - u_\theta'''(s_3-))$

Then for u,v in D, we note that u_ϕ, v_ϕ, u_θ, v_θ, u_ϕ', v_ϕ', u_θ', v_θ' are continuous in any closed subinterval of $[0,L]$, and can integrate by parts to obtain:

$$\int_0^{s_2} u_\phi''''(s)v_\phi(s)\,ds = u_\phi'''(s_2-)v_\phi(s_2) - u_\phi'''(0+)v_\phi(0+) - u_\phi''(s_2-)v_\phi'(s_2) + u_\phi''(0+)v_\phi'(0+) + \int_0^{s_2} u_\phi''(s)\,v_\phi''(s)\,ds ,$$

$$\int_{s_2}^{s_3} u_\phi''''(s)\,v_\phi(s)\,ds = u_\phi'''(s_3-)v_\phi(s_3) - u_\phi'''(s_2+)v_\phi(s_2) - u_\phi''(s_3-)v_\phi'(s_3) + u_\phi''(s_2+)v_\phi'(s_2) + \int_{s_2}^{s_3} u_\phi''(s)\,v_\phi''(s)\,ds ,$$

$$\int_{s_3}^{L} u_\phi''''(s)\,v_\phi(s)\,ds = u_\phi'''(L-)v_\phi(L-) - u_\phi'''(s_3+)v_\phi(s_3) - u_\phi''(L-)v_\phi'(L-) + u_\phi''(s_3+)v_\phi'(s_3) + \int_{s_3}^{L} u_\phi''(s)\,v_\phi''(s)\,ds ,$$

with similar expressions for the functions $u_\theta(\cdot)$, $v_\theta(\cdot)$. Also

$$\int_0^L u_\psi''(s)\,v_\psi(s)\,ds = u_\psi'(L-)v_\psi(L-) - u_\psi'(0+)v_\psi(0+) - \int_0^L u_\psi'(s)\,v_\psi'(s)\,ds .$$

From these relations it readily follows that (we omit the algebra)

$$[Au,v] = \int_0^L u_\psi'(s)\,v_\psi'(s)\,ds + \int_0^L u_\theta''(s)\,v_\theta''(s)\,ds + \int_0^L u_\phi''(s)\,v_\phi''(s)\,ds .$$

In particular, for u in D:

$$[Au,u] = \int_0^L (u_\psi'(s)^2 + u_\phi''(s)^2 + u_\theta''(s)^2)\,ds$$

and is thus

$$\geq 0 .$$

It readily follows from these calculations that A is a closed self-adjoint and nonnegative definite operator, and of course the domain of A is dense in x by the very construction of x.

The set of equations (8.1) - (8.7) can then be expressed as an "abstract" or "function space" equation, with $x(t)$ taking values in X (see [3] for the general theory)

$$M\ddot{x}(t) + Ax(t) + Bu(t) + FN(t) + K(\dot{x}(t)) = 0 \tag{8.8}$$

where M is the 17×17 matrix specified by:

$$M = \{m_{i,j}\}$$

where all the terms are zero except

$$m_{1,1} = PA$$

$$m_{2,2} = PA$$

$$m_{3,3} = PI_\psi$$

$$m_{4,4} = m_1$$

$$m_{5,5} = m_1$$

$$m_{6,6} = m_4$$

$$m_{7,7} = m_4$$

$$m_{13,6} = m_{6,13} = m_4 r_x$$

$$m_{13,7} = m_{7,13} = m_4 r_y$$

$$\begin{vmatrix} m_{8,8} & m_{8,9} & m_{8,10} \\ m_{9,8} & m_{9,9} & m_{9,10} \\ m_{10,8} & m_{10,9} & m_{10,10} \end{vmatrix} = I_1$$

$$\begin{vmatrix} m_{11,11} & m_{11,12} & m_{11,13} \\ m_{12,11} & m_{12,12} & m_{12,13} \\ m_{13,11} & m_{12,13} & m_{13,13} \end{vmatrix} = \hat{I}_4$$

$$m_{14,14} = m_2$$

$$m_{15,15} = m_2$$

$$m_{16,16} = m_3$$

$$m_{17,17} = m_3 .$$

We assume that M is nonsingular. Pictorially, M takes the form:

Let M = 17×17 matrix. Blanks indicate zero.

$$
\begin{array}{c|ccccccccccccccccc|}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} \\
\hline
x_1 & PA & 0 & 0 & & & & & & & & & & & & & & \\
x_2 & 0 & PA & 0 & & & & & & & & & & & & & & \\
x_3 & 0 & 0 & PI_{\psi} & & & & & & & & & & & & & & \\
x_4 & & & & m_1 & 0 & & & & & & & & & & & & \\
x_5 & & & & 0 & m_1 & & & & & & & & & & & & \\
x_6 & & & & & & m_4 & 0 & 0 & 0 & 0 & 0 & 0 & m_4 r_x & & & & \\
x_7 & & & & & & 0 & m_4 & 0 & 0 & 0 & 0 & 0 & m_4 r_y & & & & \\
x_8 & & & & & & 0 & 0 & & & & 0 & 0 & 0 & & & & \\
x_9 & & & & & & 0 & 0 & & I_1 & & 0 & 0 & 0 & & & & \\
x_{10} & & & & & & 0 & 0 & & & & 0 & 0 & 0 & & & & \\
x_{11} & & & & & & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
x_{12} & & & & & & 0 & 0 & 0 & 0 & 0 & & \hat{I}_4 & & & & & \\
x_{13} & & & & & & m_4 r_x & m_4 r_y & 0 & 0 & 0 & & & & & & & \\
x_{14} & & & & & & & & & & & & & & m_2 & 0 & & \\
x_{15} & & & & & & & & & & & & & & 0 & m_2 & & \\
x_{16} & & & & & & & & & & & & & & & & m_3 & 0 \\
x_{17} & & & & & & & & & & & & & & & & 0 & m_3 \\
\hline
\end{array}
$$

We note that (because it is of importance) I_1 can be taken to be diagonal, the off-diagonal term being small. The "control" $u(t)$ is 10×1:

$$u(t) = \begin{vmatrix} M_1(t) \\ M_4(t) \\ m_2\ddot{\Delta}_{\phi,2} \\ m_2\ddot{\Delta}_{\theta,2} \\ m_3\ddot{\Delta}_{\phi,3} \\ m_3\ddot{\Delta}_{\theta,3} \end{vmatrix}$$

and B is correspondingly a 17×10 constant matrix given by

$$B = \begin{vmatrix} 0_{7\times10} \\ I_{10\times10} \end{vmatrix}$$

($0_{7\times10}$ denotes 7×10 zero matrix)

($I_{10\times10}$ denotes 10×10 identity matrix)

$N(t)$ is the noise disturbance which is 3×1, originally denoted in (7.5) by $M_D(t)$ so that F is 17×3:

$$F = \begin{vmatrix} 0_{7\times3} \\ I_{3\times3} \\ 0_{7\times3} \end{vmatrix} .$$

Finally $K(\dot{x})$ is a nonlinear function of $\dot{x}(t)$ given by

$$K(\dot{x}(t)) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega_1 \otimes I_1\omega_1 \\ \omega_4 \otimes I_4\omega_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \tag{8.9}$$

The most important thing to note about the function $K(\dot{x}(t))$ is that

$$[K(\dot{x}(t)), \dot{x}(t)] = 0 \tag{8.10}$$

Indeed

$$[\omega_1 \otimes I_1\omega_1, \omega_1] = 0 \tag{8.11}$$

$$[\omega_4 \otimes I_4\omega_4, \omega_4] = 0 \quad . \tag{8.12}$$

The abstract formulation has many advantages beyond the immediate one of providing a succinct statement of the essence of the problem. First the total energy in the system is

$$E(t) = \frac{1}{2}[M\dot{x}(t), \dot{x}(t)] + \frac{1}{2}[Ax(t), x(t)] \quad , \tag{8.13}$$

the time-derivative of which is:

$$\frac{d}{dt} E(t) \quad = \quad [M\ddot{x}(t) + Ax(t), \quad \dot{x}(t)]$$

which, using (8.8)

$$= \quad -[Bu(t), \ \dot{x}(t)] - [K(\dot{x}(t)), \ \dot{x}(t)] - [FN(t), \ \dot{x}(t)]$$

$$= \quad -[Bu(t), \ \dot{x}(t)] - [FN(t), \ \dot{x}(t)]$$

since

$$[K(\dot{x}(t)), \ \dot{x}(t)] \quad = \quad 0 \quad .$$

Hence, if we ignore the noise term for the moment, we see immediately that the system can be "stabilized" in the sense that the system energy $E(t)$ decreases (or does not increase) by taking the feedback control:

$$\begin{vmatrix} M_1(t) \\ M_4(t) \\ m_2\ddot{\Delta}_{\phi,2}(t) \\ m_2\ddot{\Delta}_{\theta,2}(t) \\ m_3\ddot{\Delta}_{\phi,3}(t) \\ m_3\ddot{\Delta}_{\theta,3}(t) \end{vmatrix} \quad = \quad P \begin{vmatrix} \omega_1(t) \\ \omega_4(t) \\ \dot{u}_\phi(t,s_2) \\ \dot{u}_\theta(t,s_2) \\ \dot{u}_\phi(t,s_3) \\ \dot{u}_\theta(t,s_3) \end{vmatrix} \tag{8.14}$$

where P is positive definite. In particular it is enough to take P in the form

$$P \quad = \quad \begin{vmatrix} P_{6\times 6} & 0_{6\times 4} \\ 0_{4\times 6} & 0_{4\times 4} \end{vmatrix} \tag{8.15}$$

where the 6×6 matrix $P_{6\times 6}$ is positive definite. We can thus dispense with the "proof-mass" controllers and use only "feeding back" appropriately the angular velocities.

To proceed further with solving the abstract equation (8.8), let

$$Y(t) = \begin{vmatrix} x(t) \\ \dot{x}(t) \end{vmatrix} \tag{8.16}$$

Then we can rewrite (8.8) in the usual "state space" form as

$$\dot{Y}(t) = \mathcal{A}Y(t) + \mathcal{B}U(t) + \mathcal{K}(Y(t)) + \mathcal{F}N(t) \tag{8.17}$$

where

$$\mathcal{A} = \begin{vmatrix} 0 & I \\ -M^{-1}A & 0 \end{vmatrix} \tag{8.18}$$

$$\mathcal{B}U = \begin{vmatrix} 0 \\ -BU \end{vmatrix} \tag{8.19}$$

Let

$$Y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}$$

Then

$$\mathcal{K}(Y) = \begin{vmatrix} 0 \\ -K(y_2) \end{vmatrix} \tag{8.20}$$

$$\mathcal{F}N(t) = \begin{vmatrix} 0 \\ -FN(t) \end{vmatrix}$$

We next introduce the energy inner product (see [3] for this)

$$[Y,Z]_E = [My_2,z_2] + [\sqrt{A}\,y_1, \sqrt{A}\,z_1]$$

where

$$Y = \begin{vmatrix} y_1 \\ y_2 \end{vmatrix}, \qquad Z = \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} .$$

Then in this inner product

$$A = -A^*$$

so that A is dissipative:

$$[AY, Y]_E + [Y, AY]_E \leq 0 \quad .$$

Moreover A has a compact resolvent and the spectrum of A consists entirely of the "point" spectrum (eigenvalues). Now the eigenvectors of A given by

$$AY = \lambda Y \tag{8.21}$$

or, using (8.18), we have

$$y_2 = \lambda y_1$$

$$-M^{-1} A y_1 = \lambda y_2 = \lambda^2 y_1$$

or,

$$Ay_1 = -\lambda^2 M y_1 \tag{8.22}$$

Since A is self-adjoint and nonnegative definite and so is M it follows that we can express the eigenvalues as

$$\lambda_n = \pm i \, \mu_n \quad , \qquad \mu_n > 0$$

and further let Φ_n^+ denote the eigenvector corresponding to $i\mu_n$, and Φ_n^- that correspond to $-i\mu_n$. Then

$$[My_1, y_1] = \| \|^2 + mx_6^2 - 2ax_{67} + mx_7^2 + \cdots$$

$$2a >> m$$

$$\phi_n^+ = \begin{vmatrix} e_n \\ +i\mu_n en \end{vmatrix} \tag{8.23}$$

$$\phi_n^- = \begin{vmatrix} e_n \\ -i\mu_n e_n \end{vmatrix} \tag{8.24}$$

where

$$Ae_n = \mu_n^2 M e_n \tag{8.25}$$

and the $\{e_n\}$ is an M-orthogonal basis in X. In other words we are guaranteed to find functions $e_n(s)$, $0 < s < L$ such that, writing

$$e_n = \begin{vmatrix} \phi_n(s) \\ \theta_n(s) \\ \psi_n(x) \end{vmatrix}$$

$$\left.\begin{aligned} \phi_n''''(s) &= \gamma_\phi^2 \mu_n^2 \ \phi_n(s) \\ \theta_n''''(s) &= \gamma_\theta^2 \mu_n^2 \ \theta_n(s) \end{aligned}\right\} \quad \begin{array}{l} \textit{in the open intervals} \\ (0,s_2), \quad (s_2,s_3), \quad (s_3,L) \end{array}$$

$$-\psi_n''(s) = \gamma_\psi^2 \mu_n^2 \ \psi_n(s) \qquad 0 < s < L$$

$$EI_\phi \ \phi_n'''(0+) = m_1 \mu_n^2 \Phi_n(0+)$$

$$EI_\theta \ \theta_n'''(0+) = m_1 \mu_n^2 \theta_n(0+)$$

$$EI_\phi \ \phi_n'''(L-) = -\mu_n^2\left(m_4\phi_n(L-) + m_4 r_x \psi_n(L-)\right)$$

$$EI_\theta \ \theta_n'''(L-) = -\mu_n^2\left(m_4\theta_n(L-) + m_4 r_y \psi_n(L-)\right)$$

$$\begin{vmatrix} EI_\phi \ \phi_n''(0+) \\ EI_\theta \ \theta_n''(0+) \\ GI_\psi \ \psi_n'(0+) \end{vmatrix} = -\mu_n^2 \ I_1 \begin{vmatrix} \phi_n'(0+) \\ \theta_n'(0+) \\ \psi_n(0+) \end{vmatrix}$$

$$\begin{vmatrix} EI_\phi \ \phi_n''(L-) \\ EI_\theta \ \theta_n''(L-) \\ GI_\psi \ \psi_n'(L-) \end{vmatrix} = \mu_n^2 \left\{ \hat{I}_4 \begin{vmatrix} \phi_n'(L-) \\ \theta_n'(L-) \\ \psi_n(L-) \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ m_4 r_x \phi_n(L-) + m_4 r_y \theta_n(L-) \end{vmatrix} \right\}$$

$$EI_\phi(\phi_n'''(s_2+) - \phi_n'''(s_2-)) = m_2 \mu_n^2 \phi_n(s_2)$$

$$EI_\theta(\theta_n'''(s_2+) - \theta_n'''(s_2-)) = m_2 \mu_n^2 \theta_n(s_2)$$

$$EI_\phi(\phi_n'''(s_3+) - \phi_n'''(s_3-)) = m_3 \mu_n^2 \phi_n(s_3)$$

$$EI_\theta(\theta_n'''(s_3+) - \theta_n'''(s_3-)) = m_3 \mu_n^2 \theta_n(s_3) \quad .$$

Moreoever for distinct eigenvalues μ_n, μ_m

$$[Ae_n, e_m] = 0 = [Me_n, e_m]$$

and we may orthonormalize so that $[Me_n, e_n] = 1$. Note that we do not need to verify whether any matrices are nonsingular as in [2] for exmaple! We shall go into the details of actually calculating the eigen functions in Part 2.

Next we can write a "modal expansion" for the solution of (8.8). Thus let

$$x(t) = \sum_0^\infty \bar{a}_n(t) \, e_n \quad , \tag{8.26}$$

where we take the eigenvalues μ_n^2 in increasing order, the smallest eigenvalue being μ_0. Substituting (8.26) into (8.8). we obtain

$$\sum_0^\infty (\ddot{\bar{a}}_n(t)\, Me_n + \mu_n^2\, \bar{a}_n(t)\, Me_n) = \sum_0^\infty h_n(t)\, e_n \tag{8.27}$$

where we take

$$h_n(t) = [h(t), Me_n]$$

$$h(t) = -FN(t) - Bu(t) - K(\dot{x}(t)) \quad .$$

Hence

$$[\ddot{\bar{a}}_n(t) + \mu_n^2 \bar{a}_n(t)] = h_n(t)\, [e_n, e_n]$$

so that

$$\bar{a}_n(t) = \int_0^t \sin \frac{\mu_n(t-\sigma)}{\mu_n}\, h_n(\sigma)\, d\sigma \quad [e_n, e_n] \tag{8.28}$$

taking initial conditions to be zero. Hence we can write (in terms of the operator valued Green's Function):

$$x(t) = \int_0^t G(t-\sigma)\, h(\sigma)\, d\sigma \tag{8.29}$$

where $G(t)$ is defined by

$$G(t)x = \sum_0^\infty \frac{\sin \mu_n t}{\mu_n} \quad [e_n, e_n]\, [x, Me_n]\, e_n$$

for each x in X. To allow for the nonlinearity we must next solve the integral equation:

$$x(t) + \int_0^t G(t-\sigma)\ K(\dot{x}(\sigma))\ d\sigma + \int_0^t G(t-\sigma)\ FN(\sigma)\ d\sigma + \int_0^t G(t-\sigma)\ Bu(\sigma)\ d\sigma$$

$$= 0 \quad .$$

We shall examine this equation in more detail in Part 2.

9. AN EXAMPLE OF AN EXPLICIT SOLUTION OBTAINED BY THE BOUNDARY INPUT METHOD

In this section we develop one explicit solution to the Beam Equations by means of the Boundary Input technique outlined in Section 6.

The simplest such solution is obtained by setting, using the notation of that section,

$$\left.\begin{aligned} \ddot{D}_\phi(t,s) &= 0 \\ \ddot{D}_\theta(t,s) &= 0 \\ \ddot{D}_\psi(t,s) &= 0 \end{aligned}\right\} \tag{9.1}$$

so that the total solution (6.1), (6.2), (6.3) can be expressed very simply as

$$\left.\begin{aligned} u_\phi(t,s) &= \sum_0^3 a_k(t)\ s^k \\ u_\theta(t,s) &= \sum_0^3 b_k(t)\ s^k \\ u_\psi(t,s) &= c_1(t)s + c_0(t) \end{aligned}\right\} \tag{9.2}$$

All we need to do now is to determine the coefficients in (9.2) (taking into account (9.1)) by substituting into the equations (8.3) through (8.7). Since we are only interested in exhibiting one particular solution, we shall simplify the problem by neglecting the proof-mass conditions. We shall also neglect noise, so that

$$M_D(t) = 0 \quad .$$

Next we demand a solution in which the angular accelerations are zero:

$$\dot{\omega}_1 = 0$$

$$\dot{\omega}_4 = 0 \quad .$$

Condition (8.3) yields that

$$a_3(t) \equiv 0$$

$$b_3(t) \equiv 0 \quad .$$

Condition (8.4) is satisfied by taking

$$F_y(t) = F_x(t) = 0 \quad . \tag{9.3}$$

As for (8.5), we note first that

$$\omega_1 = \begin{vmatrix} \dot{a}_1(t) \\ \dot{b}_1(t) \\ \dot{c}_0(t) \end{vmatrix} = \begin{vmatrix} \omega_{\phi,1} \\ \omega_{\theta,1} \\ \omega_{\psi,1} \end{vmatrix} \quad .$$

Since ω_1 is constant, this implies that

$$\left.\begin{aligned} a_1(t) &= t\omega_{\phi,1} + a_1 \\ b_1(t) &= t\omega_{\theta,1} + b_1 \\ c_0(t) &= t\omega_{\psi,1} + c_0 \end{aligned}\right\} \tag{9.4}$$

But

$$\omega_4 = \begin{vmatrix} \omega_{\phi,4} \\ \omega_{\theta,4} \\ \omega_{\psi,4} \end{vmatrix}$$

and

$$\omega_{\phi,4} = 2\dot{a}_2(t)L + \dot{a}_1(t)$$

$$\omega_{\theta,4} = 2\dot{b}_2(t)L + \dot{b}_1(t)$$

$$\omega_{\psi,4} = \dot{c}_1(t)L + \dot{c}_0(t) \quad .$$

Hence

$$\dot{a}_2(t) = \frac{(\omega_{\phi,4} - \omega_{\phi,1})}{2L}$$

$$\dot{b}_2(t) = \frac{(\omega_{\theta,4} - \omega_{\theta,1})}{2L}$$

$$\dot{c}_1(t) = \frac{(\omega_{\psi,4} - \omega_{\psi,1})}{L} \quad .$$

Setting the initial "position" conditions:

$$\left.\begin{aligned} u_\phi(0,s) &= 0 \\ u_\theta(0,s) &= 0 \\ u_\psi(0,s) &= 0 \end{aligned}\right\} \qquad (9.5)$$

yields

$$D_\phi(0,s) = 0$$

$$D_\theta(0,s) = 0$$

$$D_\psi(0,s) = 0$$

So that, in particular, in (9.4):

$$a_1 = b_1 = c_0 = 0 .$$

Also

$$a_2(0) = b_2(0) = c_1(0) = 0 ,$$

thus determining completely the functions $a_2(t)$, $b_2(t)$, $a_1(t)$, $b_1(t)$, $c_1(t)$, $c_0(t)$. To satisfy (8.5) and (8.6), we define the control moments $M_1(t)$ and $M_4(t)$ by

$$M_1(t) = \begin{vmatrix} t(EI_\phi) \dfrac{(\omega_{\phi,4} - \omega_{\phi,1})}{L} \\ t(EI_\theta) \dfrac{(\omega_{\theta,4} - \omega_{\theta,1})}{L} \\ t(GI_\psi) \dfrac{(\omega_{\psi,4} - \omega_{\psi,1})}{L} \end{vmatrix} - \omega_1 \otimes I_1\omega_1 \tag{9.6}$$

$$M_4(t) = -M_1(t) . \tag{9.7}$$

Thus the final solution is:

$$\left.\begin{aligned} u_\phi(t,s) &= t\,\frac{(\omega_{\phi,4} - \omega_{\phi,1})}{2L}\,s^2 + t\omega_{\phi,1}s + a_\phi t \\ u_\theta(t,s) &= t\,\frac{(\omega_{\theta,4} - \omega_{\theta,1})}{2L}\,s^2 + t\omega_{\theta,1}s + b_\theta t \\ u_\psi(t,s) &= t\,\frac{(\omega_{\psi,4} - \omega_{\psi,1})}{L}\,s + t\omega_{\psi,1} \end{aligned}\right\} \tag{9.8}$$

where a_ϕ, b_θ are arbitrary constants, and can be chosen to satisfy initial time-derivative conditions for $u_\phi(t,s)$ and $u_\theta(t,s)$. Note that the total energy in the system, defined by (8.13) is a constant. Also, the initial "rates" are

$$\dot{u}_\phi(0,s) = \frac{(\omega_{\phi,4} - \omega_{\phi,1})}{2L} s^2 + \omega_{\phi,1} s + a_\phi$$

$$\dot{u}_\theta(0,s) = \frac{(\omega_{\theta,4} - \omega_{\theta,1})}{2L} s^2 + \omega_{\theta,1} s + a_\theta$$

$$\dot{u}_\psi(0,s) = \frac{(\omega_{\psi,4} - \omega_{\psi,1})}{L} s + \omega_{\psi,1}$$

which are of course nonzero. The control is chosen, roughly speaking, to maintain this rate. Our aim here has been limited to obtaining one explicit solution. In the forthcoming work we shall enlarge further on this technique.

CONCLUDING REMARKS

Several techniques of solution for the partial differential equation formulation of the SCOLE control problem have been described which should facilitate conceptual understanding as well as numerical computation of feedback laws. Additional work is required before the relative merits of the various approaches can be evaluated.

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16. Abstract
A mathematical formulation of the SCOLE control problem in terms of a continuous model described by partial differential equations with delta-functions on the boundary is presented along with three techniques of solution. The abstract wave-equation approach leads immediately to a linear feedback law that can ensure (strong) stability. The "boundary-control" approach yields an explicit solution, albeit in a simple case.

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