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THE ACCURACY OF APPROXIMATE SOLUTIONS IN THE ANALYSIS OF FRACTURE OF COMPOSITES

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ABSTRACT

This paper concerns the accuracy of three related mathematical models (developed by Hedgepeth, Eringen and Sendeckyj and Jones) used in the stress analysis and in fracture studies of continuous-fiber composites. These models have particular application in the investigation of fiber and matrix stresses in unidirectional composites in the region near a crack tip. The interest in such models is motivated by the desire to be able to simplify the equations of elasticity to the point that they can be solved in a relatively easy manner.

DISCUSSION

This presentation is intended as an observation of the fundamental behavior of the solutions obtained from the three models, and as such, contains very little in the way of detailed numerical results. The knowledge of some results obtained from the models will, however, be assumed in some of the comparisons made. I have attempted to point out some of the most significant features and limitations of each model, and to indicate one approach used to investigate the usefulness and accuracy of the results. The reader is referred to the appropriate papers listed for more details and for examples of specific solutions.

In using any approximate model for a physical problem one is after a reduced, and presumably simpler, set of equations such that the important characteristics of the problem can be studied more easily. A major difficulty is often to make the proper simplifying (and usual restrictive) assumptions to give a tractable set of equations, while still retaining the essential physical behavior of the problem. One way to investigate the validity of solutions resulting from various assumptions is to compare solutions for special cases of the models with complete solutions. This will be the focus of this paper; that is, to investigate the above approximate models used in the stress and fracture analysis of composites and, by comparing some of the results with exact elasticity solutions, obtain a better understanding of the usefulness of the models. For example, we will see that for the models considered, some stress components are determined very accurately while others are so in error as to be of little value.

The three approximate models will be reviewed, and they will be applied to the special case of an idealized unidirectional composite containing a transverse notch as shown in Figure 1. The results from these models will then be compared with the related exact two-dimensional orthotropic elasticity solution. The comparison will be for the case of no crack-tip damage, as some very significant observations can easily be made. The use of these models to represent crack tip damage, that is, plasticity, fiber breaks and matrix splitting will also be discussed. This more complex form of crack-tip damage representation is perhaps the most important use of approximate models in investigating fracture of composites. A complete solution for the general case of crack tip damage has not yet been developed;

if so, we would have less need for the approximate models. One must then judge the accuracy of the models on the information obtained from comparisons between the special, simple cases, and infer their usefulness for more complex damage states.

One of the simplest models for the unidirectional composite is the classical shear-lag technique, first used by Kuhn[1] for stringer reinforced metal panels and later applied to composites by Hedgepeth[2,3]. This model uncouples the equilibrium equations and the resulting solution is found to give an accurate measure of the axial fiber stress, (σ_F) and the shear stress, (τ) between fibers, but a very poor measure of the transverse matrix normal stress, (σ_M). The model does not in fact, have enough freedom to remove the shear stresses from the crack surface and it is the presence of these stresses that introduces the large error in this matrix normal stress near the crack tip. The shear-lag model is however, relatively easy to extend to account for various forms of crack tip damage and it also provides the foundation for the next two models.

A modified form of the shear-lag model as presented by Eringen and Kim[4] does allow for satisfaction of the crack surface shear stress boundary condition but does not uncouple the equilibrium equations. This solution is found to give an inaccurate representation for both the matrix shear and transverse normal stresses near the crack tip but, as in the shear-lag model, does give an accurate value for the fiber stress. Because of the coupled equilibrium equations, it is more difficult to account for crack tip damage.

A third model has recently been developed by Sendeckyj and Jones[5] in which, as in reference 4, the equilibrium equations do not uncouple and the crack surface zero shear stress boundary condition is approximately satisfied. In contrast to both of the above models, an accurate value for the transverse normal stress is given. It was the need to calculate this stress that in part, to the development of this model. As with Eringen's model[4] the coupled equilibrium equations are more difficult to extend to include crack-tip damage.

All three models then give excellent agreement for the longitudinal fiber stress and it appears that this stress is insensitive to changes in the model. Reedy[6] presents a detailed comparison between the shear-lag model and a three-dimensional finite element elasticity solution for the two-phase material and finds that the fiber stress also agrees with the finite element solution.

The comments made above will now be illustrated by reviewing the three models and giving some predicted results for the particular problem of Figure 1. The related orthotropic and isotropic elasticity solution for the same geometry will then be considered in investigating the behavior of the approximate models.

All three models as well as the elasticity solutions start with the same two-dimensional equilibrium equations listed below. The difference in the solutions being the manner in which the derivatives of the stress components are approximated and in the assumptions made for the stress-displacement relations.

The equilibrium equations for a point in a planar elastic solid are

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 , \qquad \text{and} \qquad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad . \tag{1}$$

A. SHEAR-LAG MODEL, Hedgepeth:[2,3]

The equilibrium equations above are written for the finite width, 'typical' element containing a fiber with surrounding matrix as shown in Figure 1, and reduce to the following:

$$\frac{A_F}{t}\frac{d\sigma_F^n}{dy} + \tau^{n+1} - \tau^n = 0 \text{ , and} \tag{A.1}$$

$$\sigma_M^{n+1} - \sigma_M^n + \frac{h}{2}\frac{d}{dy}\{\tau^{n+1} - \tau^n\} = 0 \text{ .} \tag{A.2}$$

The stress-displacement relations assumed for this model are:

$$\sigma_F^n = E_F\frac{dv_n}{dy} \text{ ,} \tag{A.3}$$

$$\tau^{n+1} = \frac{G_M}{h}(v_{n+1} - v_n) \text{ , and} \tag{A.4}$$

$$\sigma_M^{n+1} = \frac{E_M}{h}(u_{n+1} - u_n) \text{ .} \tag{A.5}$$

Equation (A.4) is the basic shear-lag assumption. Substituting the stress-displacement equations into the equilibrium equations, (A.1) and (A.2) we have

$$\frac{E_F A_F h}{G_M t}\frac{d^2 v_n}{dy^2} + v_{n+1} - 2v_n + v_{n-1} = 0 \text{ , and} \tag{A.6}$$

$$\frac{E_M}{h}(u_{n+1} - 2u_n + u_{n-1}) + \frac{G_M}{2}\frac{d}{dy}(v_{n+1} - v_{n-1}) = 0 \tag{A.7}$$

It is seen that the axial equilibrium equation (A.6) does not contain the transverse displacement u_n and therefore can be solved independently of u_n, that is, the two equations uncouple. This is one of the significant features of the shear-lag model and it is also the source of some of the difficulties. Equation (A.6) is a second order differential-difference equation in the axial displacement v_n. With v_n determined from the solution of this equation both the fiber stress, σ_F, and the matrix shear stress, τ, are then completely specified. For the problem of Figure 1, the boundary conditions fully determine the displacement v_n and do not allow for any control over the shear stress on the crack surface. It is clear from equation (A.4) that the shear stress on the crack surface will not be zero unless all fibers displace the same amount, and this is not the case. Hedgepeth[2] develops the solution to equation (A.6) using an influence function technique. The transverse equilibrium equation, (A.7) is not considered by Hedgepeth. The solution to (A.7) is, however, presented in reference 7. Results from this model will be compared with the next solutions later in the paper.

B. MODIFIED SHEAR-LAG MODEL, Eringen and Kim[4]:

The same "typical" element of Figure 1 is used in this model as in section A and the resulting differential-difference equations are exactly the same as equations (A.1) and (A.2).

The stress-displacement relations are

$$\sigma_F^{\,n} = E_F \frac{dv_n}{dy} , \tag{B.1}$$

$$\tau^{n+1} = \frac{G_M}{h} (v_{n+1} - v_n) + \frac{G_M}{2} \frac{d}{dy} (u_{n+1} + u_n) \text{ , and} \tag{B.2}$$

$$\sigma_M^{n+1} = \frac{E_M}{h} (u_{n+1} - u_n) \text{ .} \tag{B.3}$$

The shear stress assumption in (A.4) and (B.2) is the only difference in the two models. The differential term in (B.2), which accounts for matrix distortion due to fiber rotation, is the added term. The presence of the transverse displacement in this equation does not allow for uncoupling of the equilibrium equations, as seen below:

$$hE_F \frac{d^2 v_n}{dy^2} + G_M \left[\frac{1}{2} \frac{d}{dy}(u_{n+1} - u_{n-1}) + \frac{1}{h} (v_{n+1} - 2v_n + v_{n-1})\right] = 0, \text{ and} \tag{B.4}$$

$$\frac{E_M}{h}(u_{n+1} - 2u_n + u_{n-1}) + \frac{hG_M}{2}\left[\frac{1}{2}\frac{d^2}{dy^2}(u_{n+1} + 2u_n + u_{n-1})\right. \tag{B.5}$$

$$\left. + \frac{1}{h}\frac{d}{dy}\ v_{n+1} - v_{n-1})\right] = 0\ .$$

Using Fourier transforms, these equations reduce to a set of dual integral equations when the boundary conditions are applied. Due to the coupled equations, one does have the freedom to require that the shear stress vanish on the crack surface.

C. METHOD OF Sendeckyj and Jones[5]:

This model is not a direct modification of the above two; but rather, it is developed directly from the orthotropic elasticity equations. The displacement derivatives are written in terms of first order central difference formulas and the stress-displacement relations are approximated to account for the high stiffness in the fiber direction. This formulation accounts explicity for the transverse matrix normal stress and the Poisson's ratio effect.

The stiffness coefficients for the unidirectional laminate are approximated as

$$C_{11} = C_{11}^{m}/(1-\gamma)\ ,\ C_{22} = \gamma\, C_{22}^{f}\ ,\ C_{12} = \gamma C_{12}^{f} + (1-\gamma)C_{12}^{m}, C_{66} = C_{66}^{m}/(1-\gamma), \tag{C.1}$$

where γ is the fiber volume fraction and m and f indicate matrix and fiber properties. Defining additional constants as

$$E = C_{11}/C_{66},\ F = C_{22}/C_{66} \text{ and } H = 1 + C_{12}/C_{66} \tag{C.2}$$

the resulting stress-displacement equations are

$$\sigma_F^n = C_{22}\frac{dvn}{dy} + C_{12}\,(u_{n+1} - u_{n-1})/2h, \tag{C.3}$$

$$\tau^n = C_{66}\{(v_{n+1} - v_{n-1})/2h + \frac{du_n}{dy}\}, \tag{C.4}$$

$$\sigma_M^n = C_{11}\ (u_{n+1} - u_{n-1})/2h + C_{12}\frac{dv_n}{dy}, \text{ and} \tag{C.5}$$

the equilibrium equations are

$$hF\frac{d^2v_n}{dy2} + \frac{H}{2}\ \frac{d}{dy}(u_{n+1} - u_{n-1}) + \frac{1}{h}(v_{n+1} - 2v_n + v_{n+1}) = 0\ , \text{ and} \tag{C.6}$$

$$h\frac{d^2u_n}{dy2} + \frac{H}{2}\frac{d}{dy}(v_{n+1} - v_{n-1}) + \frac{E}{h}(u_{n+1} - 2u_n + u_{n-1}) = 0\ . \tag{C.7}$$

As in Eringen's work, Fourier transform methods are used to develop the solution.

D. <u>TWO DIMENSIONAL ORTHOTROPIC ELASTICITY SOLUTION</u>, Paris and Sih[8]:

The stress-displacement relations for the x and y axes being the principal material directions are:

$$\sigma_y = \frac{E_y}{1-\nu_{xy}\ \nu_{yx}}\left(\frac{\partial v}{\partial y} + \nu_{xy}\frac{\partial u}{\partial x}\right), \tag{D.1}$$

$$\tau_{xy} = G_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \text{ and} \tag{D.2}$$

$$\sigma_x = \frac{E_x}{1-\nu_{xy}\ \nu_{yx}}\left(\frac{\partial u}{\partial x} + \nu_{yx}\frac{\partial v}{\partial y}\right). \tag{D.3}$$

The details of the solution to this problem are well known and are presented in [8] and in the text by Savin[9]. For the special case considered here, where the crack of length 2a is along the x-axis and the fibers are parallel to the y-axis, the stresses near the crack tip are [8]

$$\sigma_y = \sigma^\infty\sqrt{\frac{a}{2r}}\ \mathrm{Re}\left[\frac{1}{\mu_1 - \mu_2}(H_2\mu_1 - H_1\mu_2)\right], \tag{D.4}$$

$$\tau_{xy} = \sigma^{\infty} \sqrt{\frac{a}{2r}}\ \mathrm{Re}\left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} (H_2 - H_2)\right], \text{ and} \tag{D.5}$$

$$\sigma_x = \sigma^{\infty} \sqrt{\frac{a}{2r}}\ \mathrm{Re}\left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} (H_2 \mu_2 - H_1 \mu_1)\right], \tag{D.6}$$

where 'r' is the distance from the crack tip at an angle θ from the x-axis. The functions H_1 and H_2 are,

$$H_1 = (\cos\theta + \mu_1 \sin\theta)^{-1/2},\ H_2 = (\cos\theta + \mu_2 \sin\theta)^{-1/2}, \tag{D.7}$$

where μ_1 and μ_2 are material parameters.

COMPARISON OF RESULTS FROM THE ABOVE MODELS

I. Axial stress (fiber stress) near the crack tip.

All three approximate models give results for this stress within 2-3% of each other. The Hedgepeth shear-lag model is the only one in which a closed form expression can be obtained however, and for this reason it will be presented first.

Hedgepeth[2] deduces that the axial stress in the first unbroken fiber in front of the notch (at y=0) is

$$\sigma_F^{N+1} = \frac{\sigma^{\infty} (2)^{2m} (m+1)!(m)!}{(2m+1)!} \tag{2}$$

where m = 2N+1 = total number of broken fibers for the symmetric case of Figure 1. Hikami and Chou give a rigorous proof of equation (2) in reference 10.

The fiber stress given by equation (2) is not singular, as none of the stress components given by any of the approximate solutions are singular due to the discrete modeling used. It can be shown, however, that this stress does increase as the $\sqrt{a}$, which is the same form as the elasticity solution.

Further, using computed values from [2] for the stress in the unbroken fibers, it can be shown that these stresses decrease as $1/\sqrt{\ell}$ where $\ell = x - a$ is the distance away from the crack tip. This is also consistent with the elasticity solution as is its independence of elastic properties. The above statements are equivalent to the condition that the fiber stress in front of the crack behave as a modified nonsingular form of the classical Griffith crack solution[11] as:

$$\sigma_F = \frac{\sigma^\infty x}{\sqrt{x^2 - \tilde{a}^2}}, \qquad |x| \geq a, \; \tilde{a} = a - \alpha \tag{3}$$

where α is a small constant such that σ_F not be singular at $x = a$. A better way to write equation (3), in order to compare with equation (2), is as follows, (with $m = 2N+1$, $x = (N+1)d$);

$$\frac{\sigma_F^{N+1}}{\sigma^\infty} = \frac{2^{2m}(m+1)!(m)!}{(2m+1)!} = \frac{(N+1)d}{\sqrt{(N+1)^2 d^2 - (N+C_N)^2 d^2}} \tag{4}$$

where $n = N+1$ is the first unbroken fiber, d is the fiber spacing, and $N+C_N$ is an equivalent crack length. For large crack lengths and using Stirling's formula for (m)!, that is,

$$(m)! = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m, \text{ equation (4) gives} \tag{5}$$

$$C_N = \left(1 - \frac{1}{\pi}\right) = 0.6815 \tag{6}$$

For N=0 (one broken fiber), equation (4) gives $C_0 = 0.6614$ and for N=10, $C_{10} = 0.6805$ so that C_N is seen to be approximately constant for all N. If this value of C_N, equation (6), is used in a modified form of equation (4),

$$\sigma_y^n = \frac{\sigma^\infty n}{\sqrt{n^2-(N+C_N)^2}} \quad n \geq N+1 \text{ and } y = 0, \tag{7}$$

it is found that the numerical values of σ_y^n from Hedgepeth[2] agree very closely with equation (7) for all $n \geq N+1$. Therefore, the axial fiber stress has the behavior of a discrete model of the singular stress given by equation (3), and is of the same form as the orthotropic elasticity solution given by equation (D.4), both near the crack tip and away from the crack.

As mentioned at the beginning of this section, all three models give essentially the same results for the fiber stress. Therefore, we conclude that all models give an accurate representation for this stress component.

II. Shear stress between fibers near the crack tip.

The maximum shear stress at the crack tip given by the shear-lag model has a closed form representation in much the same way as the fiber stress. Fichter[12] deduces this value and, again, Hikami and Chou[10] rigorously prove it to be

$$\tau^{N+1} = -\sigma_\infty \sqrt{\frac{G_M A_F}{E_F ht}} \; \frac{\pi\,(2m-1)!}{2^{2m}[(m-1)!]^2} \tag{8}$$

For large crack lengths, with m=2N+1 and using Stirling's formula

$$\tau^{N+1} = -\sigma_\infty \sqrt{\frac{G_M A_F}{E_F ht}} \; \sqrt{\frac{\pi N}{2}} \tag{9}$$

and, as with the fiber stress, the shear stress at the crack tip increases as the square root of the crack half-length N. Also, in the same manner as the fiber stress, the numerical value for the shear stress away from the crack tip is found to decrease as $(1/\sqrt{y})$ with 'y' being the vertical distance from the crack between the last broken and first unbroken fibers.

While the three models predict essentially the same results for axial fiber stress, they do not give the same shear stress values. The significant difference is that the shear-lag model gives the maximum shear stress to be at the crack tip (y=0), while the other two models require the shear stress to be zero at the crack tip.

The shear-lag model then gives a shear stress value closer to a discrete form of a square-root singular function than either of the two modified methods. Admittedly, the shear-lag model does not remove the shear stresses from the crack surfaces, but this fact appears to give more difficulty in the calculation of the transverse matrix stress than to the shear stress between fibers.

III. Transverse matrix normal stress between fibers in front of crack tip

None of the three models give an explicit representation for this stress but it is very clear from the numerical results of the referenced papers that both the Hedgepeth and the Eringen models are inconsistent with the orthotropic elasticity solution. In particular, the shear-lag model[2] gives a tensile stress at the crack tip and a decaying compressive stress between fibers as one proceeds away from the crack tip. The Eringen model[4] gives alternating normal stresses between fibers; that is, compressive at the crack tip, tensile between the first and second unbroken fibers, compressive between the third and fourth, etc. Neither of these agree with the correct form of a positive (singular) stress[8,11] that decays as the square-root of the distance from the crack tip. The model developed by Sendeckyj and Jones does have the proper behavior.

CONCLUSIONS REGARDING THE THREE APPROXIMATE MODELS

I. All three models give good agreement for the fiber stress. The form of this stress is a discrete model of a square-root singular stress at the crack tip with a stress in the fibers away from the crack tip that decays as the square-root of the distance from the crack-tip. This is in agreement with the isotropic and orthotropic elasticity solutions and it is felt that this stress component does represent the actual stress in the fibers. The results presented in references 13-15 also indicate very good agreement with experimental studies for predicted fiber failure and fracture strength for unidirectional fiber-critical boron/aluminum notched laminates. It then appears that the shear-lag model continues to give an accurate fiber stress prediction when the model is extended to account for crack-tip damage.

II. Only the shear-lag model gives the maximum longitudinal shear stress to occur at the crack tip. The other two models require the shear stress to be zero at the crack tip. The shear stress from the shear-lag model, as with the fiber stress, can be represented as a discrete model of a square-root-singular, square-root-decaying stress and is a proper approximation to the elasticity solution. After a distance of 4-5 fiber spacings away from the crack-tip (measured vertically between the last broken and first unbroken fibers), all three models give approximately the same shear stress. So except for the maximum value, the three models agree reasonably well. The shear-lag model appears to be more appropriate to use to predict matrix split initiation, although longitudinal splitting is also strongly influenced by transverse matrix stresses. An investigation of this behavior is given in [15].

III. The transverse matrix normal stress as given by the Hedgepeth model [2] and by the Eringen model [4] is very much in error and of little use. The stress given by the Sendeckyj-Jones [5] model does have the proper form and should be useful in estimating this stress component.

IV. As mentioned earlier, the shear-lag model is relatively easy to extend to account for various forms of crack-tip damage and the papers [13 - 15] make use of this fact. No corresponding studies have been presented using the other models.

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Fig 1

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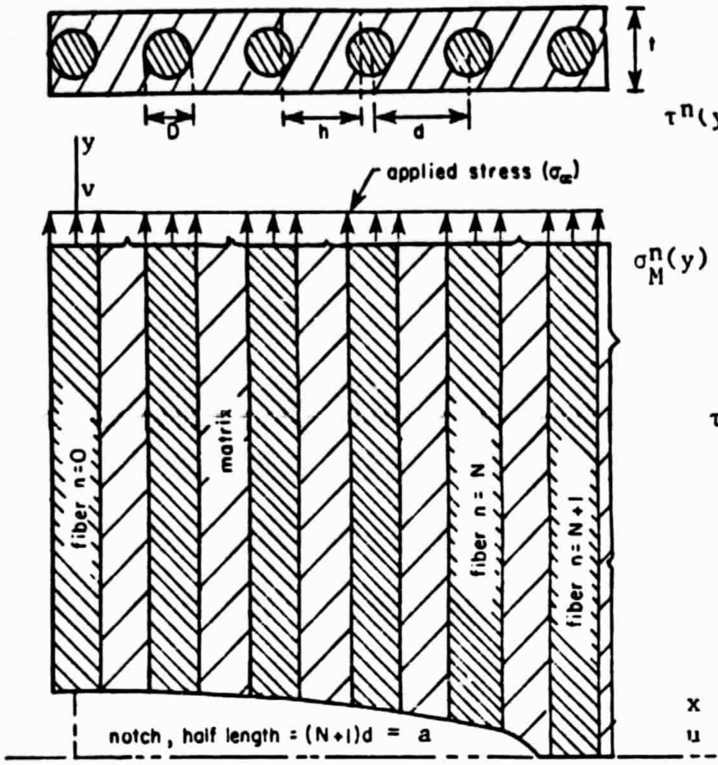

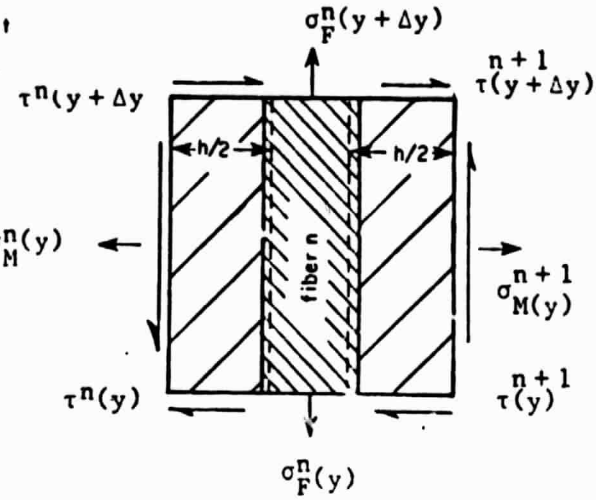

Free-Body Diagram of
a Typical Fiber/Matrix Element

ILLUSTRATIONS

Figure 1. First quadrant of a notched, unidirectional lamina.

This is the only figure for the paper entitled "The Accuracy of Approximate Solutions in the Analysis of Fracture of Composites", author: James G. Goree.