(NASA-CR-175833) ALTERNATIVES FOR JET ENGINE CONTROL Semiannual Status Report, 1 Nov. 1983 - 30 Apr. 1984 (Notre Dame Univ.) 271 p HC A12/MF A01 CSCL 21E

N85-26715

Unclas
G3/07 21372

Department of

ELECTRICAL ENGINEERING

UNIVERSITY OF NOTRE DAME, NOTRE DAME INDIANA

Semi-Annual Status Report

to the

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

on

NASA Grant NSG-3048

ALTERNATIVES FOR JET ENGINE CONTROL*

November 1, 1983 - April 30, 1984

*This work has been carried out under the direction of

Dr. Michael K. Sain
Department of Electrical Engineering
University of Notre Dame
Notre Dame, Indiana
USA 46556

ABSTRACT

This report documents the technical progress of researches under National Aeronautics and Space Administration Grant NSG-3048, entitled "Alternatives for Jet Engine Control", during the six-month period from November 1, 1983 to April 30, 1984. NASA Technical Officer for the work was Dr. Bruce Lehtinen, at Lewis Research Center. Dr. Michael K. Sain was director of the investigation at the University of Notre Dame.

Since the last report, Mr. Daniel Bugajski has completed an M.S. Thesis on the topics discussed therein. This work includes extensive numerical testing, over and above the previous progress document. Results indicate that optimal inputs contribute significantly to the process of calculating tensor approximations for nonlinear systems, and that the resulting approximations may be order-reduced in a systematic way.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Inherent in the design of any control scheme is a model of the physical system or plant. The plant itself can be modelled in basically two ways. First, laws of nature can be applied to the system in question. Many of these models are time tested and still considered an adequate representation. Examples are the laws governing the mechanics of rigid bodies, and the set of equations governing the response of electric circuits. Or, second, if the system is far too complex for the relative simplicity of familiar equations, or if the governing scientific laws are too complicated to implement, system identification via excitation of the plant and measurement of the response can be performed [1]. It is this modelling technique we address here.

In the case of a linear approximation, this identification can be carried out without much computational difficulty. Unfortunately, because of system nonlinearities, in many instances the linear approximation, while adequate, leaves much room for improvement. In these cases, a model containing parameter estimates of higher degree terms (squared terms, cubic terms, and so forth) is an answer to characterizing the system nonlinearities.

Methods of calculating nonlinear models through the use of tensor algebraic ideas have been studied [2], and results have been good for academic examples [2] as well as for models of NASA's QCSE (Quiet, Clean, Shorthaul, Experimental) jet engine [3]. As we shall see, the tensor approach is invaluable in that nonlinear problems are solved using linear techniques. A challenge of this tensor method, as with all general nonlinear methods, is the problem of complexity. Obviously the size of the model will increase with the addition of states or control inputs; however a very noticeable increase in model size accompanies any increase in the degree of approximation. Presently, typical problems using square or cubic series approximations can be handled quite capably. As the need or desire to expand to fourth degree models, and beyond, surfaces, computational constraints will need increasing attention. Upper limits on addressable storage must be considered; and most matrix software routines have a vague upper limit on dimensions beyond which calculation becomes unreliable. In lieu of computational limitations, we can call on simple intuition to cite some disadvantages in the use of large models. A first obvious observation maintains that a model containing redundant information or one retaining useless information offers no advantages over the same model with the extraneous data disregarded. Both of these faults, however, are likely to occur as model size increases. A second, less compelling reason is that large models are simply more cumbersome than smaller, compact models.

In light of the above motives, clearly a scheme to reduce full size models could be nothing but beneficial to the modelling problem. This work then opens a pathway to the identification of such reduced systems by making use of a simple idea involving the comparison of squared errors. Finally, it is worth mentioning that state reduction is not addressed here, but rather reduction via the omission of selected terms in the series approximation is considered.

A brief review of the contents of the remaining chapters is appropriate. Chapter II presents some informal mathematical background. Among the items discussed are (1) overall problem formulation, (2) input design, (3) the singular value decomposition. Chapter III briefly summarizes the software package and the logical flow of each phase of the modelling/simulation procedure. In Chapter IV the scheme for model reduction is presented and discussed. It will be shown how the reduction method fits well into the present identification technique. Sample reductions and verifications are presented in Chapter V as a few example problems are studied. Finally, Chapter VI draws some conclusions and offers some pertinent suggestions.

CHAPTER II

PROBLEM BACKGROUND

The intent of this chapter is to provide a brief presentation of some of the principles on which the work in the following chapters is founded. The purpose here is not to relay detailed theory, but the material presented will be sufficient enough to gain a proper understanding of the problem.

The model structure formation, by means of a symmetric tensor method, is discussed in the first section along with a few ideas concerning the actual calculation of the model. Following this, the next section summarizes an information theoretic approach to input design used in the parameter identification. The final section presents an overview of the singular value decomposition because of its importance in the calculation of the model.

2.1 MODEL STRUCTURE

In a most general form, a nonlinear ordinary differential equation in x with input u can be written

$$\dot{x} = f(x,u)$$

where x is a real n-vector and u is a real m-vector.

Now suppose a Taylor series expansion is performed around some operating point $(\bar{x},\bar{u})$, then [2]

$$\begin{aligned}\dot{\hat{x}} = f(\hat{x},\hat{u}) &= f(\bar{x},\bar{u}) \\ &+ \left.\frac{\partial f}{\partial \hat{x}}\right|_{(\bar{x},\bar{u})} (\hat{x}-\bar{x}) \\ &+ \left.\frac{\partial f}{\partial \hat{u}}\right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})\end{aligned}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \, (\hat{x}-\bar{x})^T \left. \frac{\partial^2 f_i}{\partial \hat{x}^2} \right|_{(\bar{x},\bar{u})} (\hat{x}-\bar{x})$$

$$+ \sum_{i=1}^{n} e_i \, (\hat{x}-\bar{x})^T \left. \frac{\partial^2 f_i}{\partial \hat{x} \partial \hat{u}} \right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \, (\hat{u}-\bar{u})^T \left. \frac{\partial^2 f_i}{\partial \hat{u}^2} \right|_{(\bar{x},\bar{u})} (\hat{u}-\bar{u})$$

$$+ \ldots$$

where $e_i = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \leftarrow i^{th}$ position.

If the operating point $(\bar{x},\bar{u})$ is taken as a reference, then

$$x = \hat{x} - \bar{x},$$

$$u = \hat{u} - \bar{u},$$

and if

$$\dot{\bar{x}} = f(\bar{x},\bar{u}),$$

then

$$\dot{x} = \dot{\hat{x}} - f(\bar{x},\bar{u}).$$

Substituting these into the expansion expression, we obtain

$$\dot{x} = \left. \frac{\partial f}{\partial \hat{x}} \right|_{(\bar{x},\bar{u})} x + \left. \frac{\partial f}{\partial \hat{u}} \right|_{(\bar{x},\bar{u})} u$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \, x^T \frac{\partial^2 f_i}{\partial \hat{x}^2} \bigg|_{(\overline{x},\overline{u})} x$$

$$+ \sum_{i=1}^{n} e_i \, x^T \frac{\partial^2 f_i}{\partial \hat{x} \partial \hat{u}} \bigg|_{(\overline{x},\overline{u})} u$$

$$+ \frac{1}{2} \sum_{i=1}^{n} e_i \, u^T \frac{\partial^2 f_i}{\partial \hat{u}^2} \bigg|_{(\overline{x},\overline{u})} u$$

$$+ \ldots$$

It is at this point where tensor algebra becomes important. Specifically the bilinear symmetric tensor product, $\vee$, [4] is used to form the products of the higher degree terms. If $x \varepsilon X$ and $u \varepsilon U$, then

$$\vee : X \times X \to X \vee X$$
$$\vee : X \times U \to X \vee U$$
$$\vee : U \times U \to U \vee U$$
$$\vdots$$

We use these to write the differential equation as follows

$$\dot{x} = L_{10}x + L_{01}u + L_{20}(x \vee x) + L_{11}(x \vee u) + L_{02}(u \vee u) + \ldots$$

where the L_{ij} make up the model. Note that the first and second subscripts correspond to the number of times x and u respectively are used in the symmetric tensor products associated with a given model partition L_{ij}. Simplifying the differential equation again

$$\dot{x} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} L_{ij} \underbrace{(x \vee x \vee x \ldots)}_{i \text{ times}} \quad \underbrace{(u \vee u \vee u \ldots)}_{j \text{ times}} .$$

Finally

$$\dot{x} = L\ z$$

where

$$L = [L_{10}\ L_{01}\ L_{20}\ L_{11}\ L_{02}\ L_{30}\ \ldots],$$

and z is a stacked vector of tensor term products:

$$z = \left[\begin{array}{c} x \\ \hline u \\ \hline x \vee x \\ \hline x \vee u \\ \hline u \vee u \\ \hline x \vee x \vee x \\ \hline \cdot \\ \cdot \\ \cdot \end{array}\right] .$$

It is interesting to note that the vector z is already reduced in one sense. Because of the properties of symmetric tensor algebra, redundant cross products are eliminated. For example

$$x \vee u = u \vee x\ ,$$

$$x \vee x \vee u = x \vee u \vee x = u \vee x \vee x\ .$$

Thus, when x and u have dimension greater than one, cross products of individual elements contain redundancies and are hence eliminated. Some example instances are

$$x_1x_2 = x_2x_1,$$

$$\begin{aligned} x_1x_2x_3 &= x_1x_3x_2 \\ &= x_2x_1x_3 \\ &= x_2x_3x_1 \\ &= x_3x_1x_2 \\ &= x_3x_2x_1\ , \end{aligned}$$

$$
\begin{aligned}
x_1x_2u_1 &= x_1u_1x_2 \\
&= x_2x_1u_1 \\
&= x_2u_1x_1 \\
&= u_1x_1x_2 \\
&= u_1x_2x_1 \ .
\end{aligned}
$$

The existing software package includes an efficient ordering algorithm [2,3] which insures that redundant products are not calculated. It is clear that a significant amount of reduction in size, and hence, calculation is inherent is the use of symmetric tensor algebra.

We can now address the question concerning the calculation of the model, L, given a "black box" system. In our case the black box consists of a QCSEE digital simulation routine or a system of mathematical equations that represent a physical system.

Recall the equation

$$\dot{x}(t) = L\ z(t)$$

where $\dot{x}(t)$ is a vector of state derivatives and $z(t)$ is a vector of tensor terms. Because L is the unknown, obviously we must know (or estimate) the values of $\dot{x}(t)$ and $z(t)$ to identify the parameters contained in L. To get a reliable determination of how $\dot{x}$ and z change with time, the system is perturbed from an equilibrium state, $(\bar{x},\bar{u})$, to a trajectory $(\hat{x},\hat{u})$ where [5]

$$
\begin{aligned}
x &= \hat{x} - \bar{x} \\
u &= \hat{u} - \bar{u}.
\end{aligned}
$$

The values of $x(t)$ and $u(t)$ and estimates of $\dot{x}(t)$ are then sampled and stored. From $x(t)$ and $u(t)$, $z(t)$ is formed. The problem now appears:

$$\begin{bmatrix} \dot{x}_1(t_1) & \dot{x}_1(t_2) & \cdots & \dot{x}_1(t_h) \\ \dot{x}_2(t_1) & \dot{x}_2(t_2) & \cdots & \dot{x}_2(t_h) \\ \vdots & \vdots & & \vdots \\ \dot{x}_n(t_1) & \dot{x}_n(t_2) & \cdots & \dot{x}_n(t_h) \end{bmatrix} = L \begin{bmatrix} z_1(t_1) & z_1(t_2) & \cdots & z_1(t_h) \\ z_2(t_1) & z_2(t_2) & \cdots & z_2(t_h) \\ \vdots & \vdots & & \vdots \\ z_p(t_1) & z_p(t_2) & \cdots & z_p(t_h) \end{bmatrix}$$

where h is the number of samples and p is the number of terms in the nonlinear approximation; p is dependent on the number of states, n, the number of inputs, m, and the degree of approximation. We shall write this as $\dot{X} = LZ$.

Fortunately, a very good routine for solving a problem of this form exists in the SPEAKEASY library of the IBM 370/3033. Using a singular value decomposition, discussed later, of Z, a least squares problem is solved [6] and L is calculated. The model then is ready for verification by means of digital simulation and comparison with true system solutions.

2.2 INPUT DESIGN

From the preceding section, it is clear that the model L is dependent on the excitation, $(x(0),u(t))$. Improper choice of perturbation might translate to an insufficient excitation of the nonlinearities, instability, or singularity. A careful method for choosing inputs could decrease the likelihood of encountering these defects. With this, the chance of identifying a "good" model, one with strong tracking ability and an acceptable region of stability, is then increased. In addition to better models, the capability of an input selection routine eliminates the need for an exhaustive search of input parameters.

For the purpose of this work, the theoretical detail of an input optimization derivation is not required. It is the intent here merely to present the basic members in the cost function calculation. For a theoretical and mathematical treatment, the reader is directed to [7]. The basic idea for the optimization comes from [8] and leads to the introduction of Fisher's Information Matrix, M. It is desired to get, in some sense, the minimum of a measure of M^{-1}. The formulation of Fisher's Information Matrix has its roots in probability, and many probabilistic rules and simplifications are used to express each element of M as

$$M_{ij} = \sum_{k=0}^{N} \left[\frac{\partial}{\partial \theta_i} y(k) \right]^T R^{-1} \left[\frac{\partial}{\partial \theta_j} y(k) \right],$$

where

$$i = 1,2,\ldots np,$$

$$j = 1,2,\ldots np,$$

and

N is the number of observation points,
y(k) is the model output sequence,
θ is a vector of parameters to be identified,
R is the covariance matrix of the measurement of y(k).

Now for our problem let

$$y = L\, z,$$

or for the discrete case

$$y(k) = L\, z(k).$$

Recall that θ is a vector of parameters to be identified, that is, each element of L. The partial derivatives can then be found. If

$$\underbrace{y}_{n \times 1} = \underbrace{L}_{n \times p}\ \underbrace{z}_{p \times 1},$$

then the partial derivative of y = Lz with respect to some element of L, say ℓ_{ij}, is an nx1 matrix with zeros in all but the i-th position where the entry is simply the j-th entry of z, z_j. Now let

$$R^{-1} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix};$$

upon the calculation of each M_{ij} and the building of M, the following observation is made [7]

$$M = R^{-1} \otimes \overline{M}$$

where $\overline{M}$ is symmetric and its upper triangular portion is shown in Figure 2.1.

Now that we have determined that $\overline{M}$ and hence M are calculable, their role in the cost function is examined. The objective function chosen is

$$\min \operatorname{tr}(M^{-1}).$$

We use a property of the Kronecker product [9] to obtain

$$M^{-1} = R \otimes \overline{M}^{-1}.$$

Another property allows

$$\operatorname{tr}(M^{-1}) = \operatorname{tr}(R)\operatorname{tr}(\overline{M}^{-1}).$$

But if $\overline{M}$ is an invertible matrix,

$$\operatorname{tr}(\overline{M}^{-1}) = \sum_{i=1}^{p} \frac{1}{\lambda_i}$$

where the λ_i are the eigenvalues of $\overline{M}$. Finally

$$\operatorname{tr}(M^{-1}) = \operatorname{tr}(R)\left(\sum_{i=1}^{p} \frac{1}{\lambda_i} \right).$$

It is worthwhile to note that, following from the above discussion, there are essentially two ways to generate the objective value; the eigenvalues of $\overline{M}$ or the inverse of M must be calculated. In the case of real, symmetric matrices, these are similar in software requirements. Because we only need the trace, and not the entire inverse, we choose the eigenvalue approach.

$$
\overline{M} = \left[\begin{array}{c|c|c|c|c|c}
\sum x \otimes x^T & \sum x \otimes u^T & \sum x \otimes (x \vee x)^T & \sum x \otimes (x \vee u)^T & \sum x \otimes (u \vee u)^T & \cdots \\ \hline
 & \sum u \otimes u^T & \sum u \otimes (x \vee x)^T & \sum u \otimes (x \vee u)^T & \sum u \otimes (u \vee u)^T & \cdots \\ \hline
 & & \sum (x \vee x) \otimes (x \vee x)^T & \sum (x \vee x) \otimes (x \vee u)^T & \sum (x \vee x) \otimes (u \vee u)^T & \cdots \\ \hline
 & & & \sum (x \vee u) \otimes (x \vee u)^T & \sum (x \vee u) \otimes (u \vee u)^T & \cdots \\ \hline
 & & & & \sum (u \vee u) \otimes (u \vee u)^T & \cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right]
$$

Figure 2.1. Upper Triangular Portion of the Matrix $\overline{M}$

This distinction becomes more important with increasing model size, which grows factorially with n, m, and degree of approximation. Despite the fact that matrix inversion is less acceptable, the current software makes provisions for both methods, for purposes of generality.

2.3 SINGULAR VALUE DECOMPOSITION

The discussion now turns to a brief presentation of the singular value decomposition (SVD). In this research effort, the SVD serves the important purpose of calculation of the generalized inverse of a matrix when the model is to be determined. The reader is directed to [10,11,12] for more formal and theoretically based discussions concerning the SVD and its computational implications and interpretations.

We first present the defining theorem of the SVD. If A is nxm with rank ρ, then there exist orthonormal matrices U (nxn) and V (mxm) such that

$$A = USV^T$$

where

T indicates transposition

and

S (nxm) is a "diagonal" matrix with diagonal elements s_i,

$$s_i > 0 \quad i = 1,2,\ldots\rho$$

$$s_i = 0 \quad i = (\rho+1)\ldots\min(n,m) \ .$$

The columns of U are the left singular vectors and are the orthonormal eigenvectors of AA*. The columns of V are the right singular vectors and are the orthonormal eigenvectors of A*A. As expected the diagonal elements of S are the singular values. The non-zero elements of $\{s_i\}$ are the positive square roots of the eigenvalues of A*A (or AA*). This may seem to suggest a simple way to calculate the singular values. This is not true, however, when finite precision arithmetic is involved. Because of the usually inexact representations due to truncation or rounding of numbers in a digital computer, the calculation of the eigenvalues of A*A can often lead to incorrect results concerning matrix rank [11,12].

Fortunately, SVD algorithms do exist that are computationally sound and not hindered by the above fault. Those that are generally considered among the best are the versions developed by Argonne National Laboratories. These include the SVD subroutine used by the SIMEQUAT algorithm of the SPEAKEASY library, IBM system 370/3033. The SIMEQUAT routine solves for the minimal solution x to the equation Ax=b. It does this by decomposing A and then finding the generalized inverse, A^I, (or A^{-1} if A is square) of A. Given the generalized inverse, the problem becomes trivial. If

$$A = USV^T$$

then

$$A^I = VS^IU^T$$

and

$$x = A^Ib$$

becomes

$$x = VS^IU^Tb.$$

Despite the existence of other methods for calculating the generalized inverse, A^I, the SVD remains the best. This is so because in problems where matrix rank is involved, as it is for matrix inversion, the SVD is the most reliable method of rank determination [11,12].

Finally, the concept of condition number is introduced. Very simply the condition number, k, is the ratio of the largest singular value to the smallest non-zero singular value. That is

$$k = \frac{\max s_i}{\min s_i} .$$

There is a lot of abstract theory involving the interpretation of the condition number, but for the purposes herein, we will regard the condition number as a measure of "nearness to singularity".

2.4 REMARKS

This chapter presents the major ideas involved in the present modelling scheme. A sound, qualitative understanding of section one is most important, since throughout the remainder of this thesis, references to these ideas will be made. In section two, a summary derivation of the input design is presented. Here we are concerned not with mathematical rigor but rather with the idea that an input can be designed at all. We shall see the importance of this design when the examples are discussed.

CHAPTER III

SOFTWARE OVERVIEW

The intent of this chapter is simply to present the logical flow of the software package. This includes both the modelling and the simulation segments. Most of the CATNAP [3] software still exists, although in a more flexible form. In addition, software has been added to provide for the input optimization as well as for the model reduction test as presented in Chapter IV. The interactive modelling and simulation packages make use of two computers. They are the IBM 370/3033 and the DEC PDP 11/44.

3.1 MODELLING

The modelling segment of the present software package ties together each of the three ideas found in Chapter II. The IBM 370/3033 computer is used here in light of its computational power and its extensive support software libraries. The routine SUPRVIZE is an IBM command language (CLIST) program which governs the entire modelling segment. It is in SUPRVIZE that necessary libraries are made accessible and control is passed from loader routine to identification routine.

The first major duty of SUPRVIZE is compilation and execution of the proper loader routine. The loader routine (usually written in extended precision FORTRAN) has the chore of set up for identification and/or optimization. It is first determined whether a reduced model is to be identified. If so, a set of column numbers corresponding to the column numbers of the full model is read from the data set REDUCE. The products that correspond to these column numbers are omitted from further calculation. Then if an optimization is desired, the proper arrays are initialized and the minimization routine is called. The minimization routine is supplied by IMSL (International Mathematical and Statistical Libraries) available on the IBM 370/3033. The IMSL routine requires a user supplied subroutine which calculates the cost function. In our case the cost function is computed using the eigenvalues of Fisher's Information Matrix as discussed in Section 2.2. Following the determination of inputs (whether by optimization or by user choice), the loader routine excites the system and forms the tensor term and state derivative matrices via sampling. These two matrices are loaded into the data set TEMPFILE for later use.

After termination of the loader execution, the next duty of SUPRVIZE is to invoke the high level language SPEAKEASY [6]. The routines written in SPEAKEASY have two tasks: model calculation, and reduction test, if desired. As we have mentioned in Section 2.1, the model is calculated using a least squares approximation and singular value decomposition in the SIMEQUAT function of the SPEAKEASY library. Following output of the model parameters, the model is stored in data set MODEL. Now the reduction test can be performed. After completion of the test, if model reduction/re-optimization is desired, a set of column numbers is written to the data set REDUCE. SPEAKEASY is then exited and control is passed back to SUPRVIZE.

The final tasks of SUPRVIZE are simple. If the model is to be kept, the user is prompted for a model name and storage area (one of two partioned data sets). The model then is stored in the desired partioned data set with the given name. If another model is to be identified, the procedure restarts. Otherwise, unnecessary data sets are deleted and modelling is complete. See Figure 3.1 for a flow diagram of the entire modelling scheme.

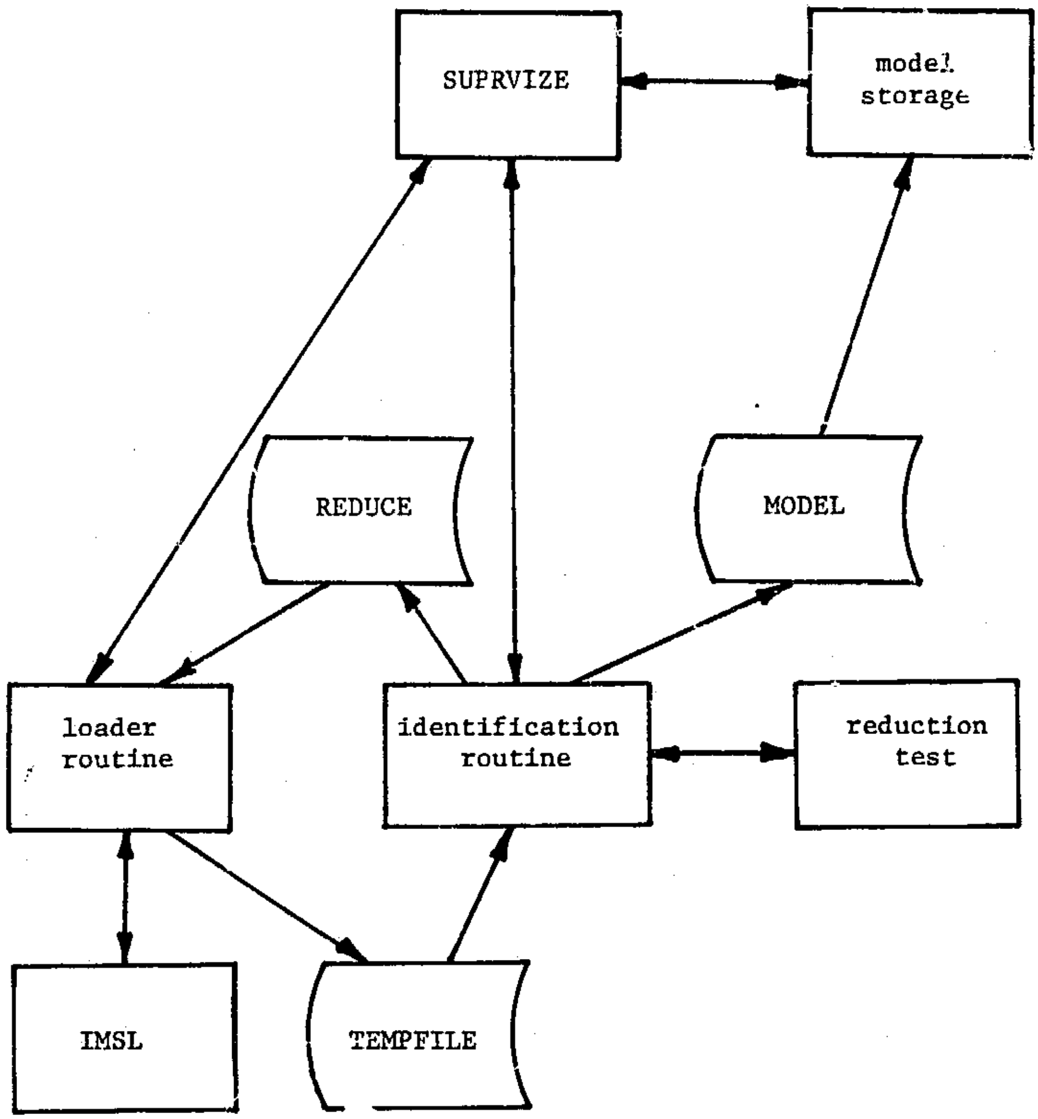


Figure 3.1. Logical Structure of Modelling Scheme

3.2 SIMULATION

Simulation of the nonlinear models can take place on either the IBM 370/3033 or the DEC PDP 11/44. For bulk tabular verification, the IBM 370/3033 is generally preferred due to its computational speed. Plot capability is handled with SAS (Statistical Aralysis Software). For plot comparison, though, it is preferred to use the DEC PDP 11/44. While computationally slower than the IBM, the DEC machine has high speed CRT plot capability with a hard copy unit for instant quality plots.

Simulation on the IBM 370/3033 is governed by the CLIST program COMPARE. COMPARE has several duties, the first of which is compilation of the simulation routine. The simulation routine (usually written in single precision FORTRAN) has the chore of solving the true system and integrating one or two models for a specified time interval. Furthermore, the simulation routine stores the data in the case that plotting is desired, and performs an error analysis between the models (if two models are simulated simultaneously). The error criteria is a simple ratio of mean square errors of the two models for each state.

After the simulation routine has been compiled, control passes back to COMPARE where the set up for simulation takes place. If a bulk tabular simulation is desired, COMPARE compiles and runs a short program that builds any specified table and stores it in data set SIMPNT. Upon exit from COMPARE, SIMPNT can be copied to a partitioned data set, TABLES, which acts as a library of simulation tables. If the desired simulation table is already built, COMPARE will ask for its name along with the names of the desired models. Following this, execution of the simulation routine is started. If the simulation is at a single point, the user is prompted for the input parameters.

Otherwise, the table is read for the input parameters of each simulation. After all simulations have been completed, control is passed back to COMPARE. If plotting is desired, control is passed to the SAS plotting routine. Otherwise the option to do another simulation is made. If simulation is complete, COMPARE is exited. See Figure 3.2 for the logical flow of simulation on the IBM 370/3033.

For simulation on the PDP 11/44, the models must first be transferred via magnetic tape from the IBM 370/3033 . Once the model is on the DEC machine, an updated version of the CATNAP simulation routine is used. See Section 3.3 of [3] for an explanation of this simulation structure. Here we list only major changes to the old CATNAP software. They are

1) comparison ability between any two models and the true solution (as opposed to strictly true solutions, linear model and some nonlinear model),
2) instead of a separate Versatec hard copy routine, a hard copy unit which transfers the CRT image to paper is used,
3) the error criteria is now the ratio of mean square errors,
4) ability to blow up a portion of any plot for closer inspection of model trajectories.

In the writing of the current software, every effort was made to keep the programs general and flexible. See [15] for current IBM 370/3033 software listings. See also [16] for current listings of PDP 11/44 software.

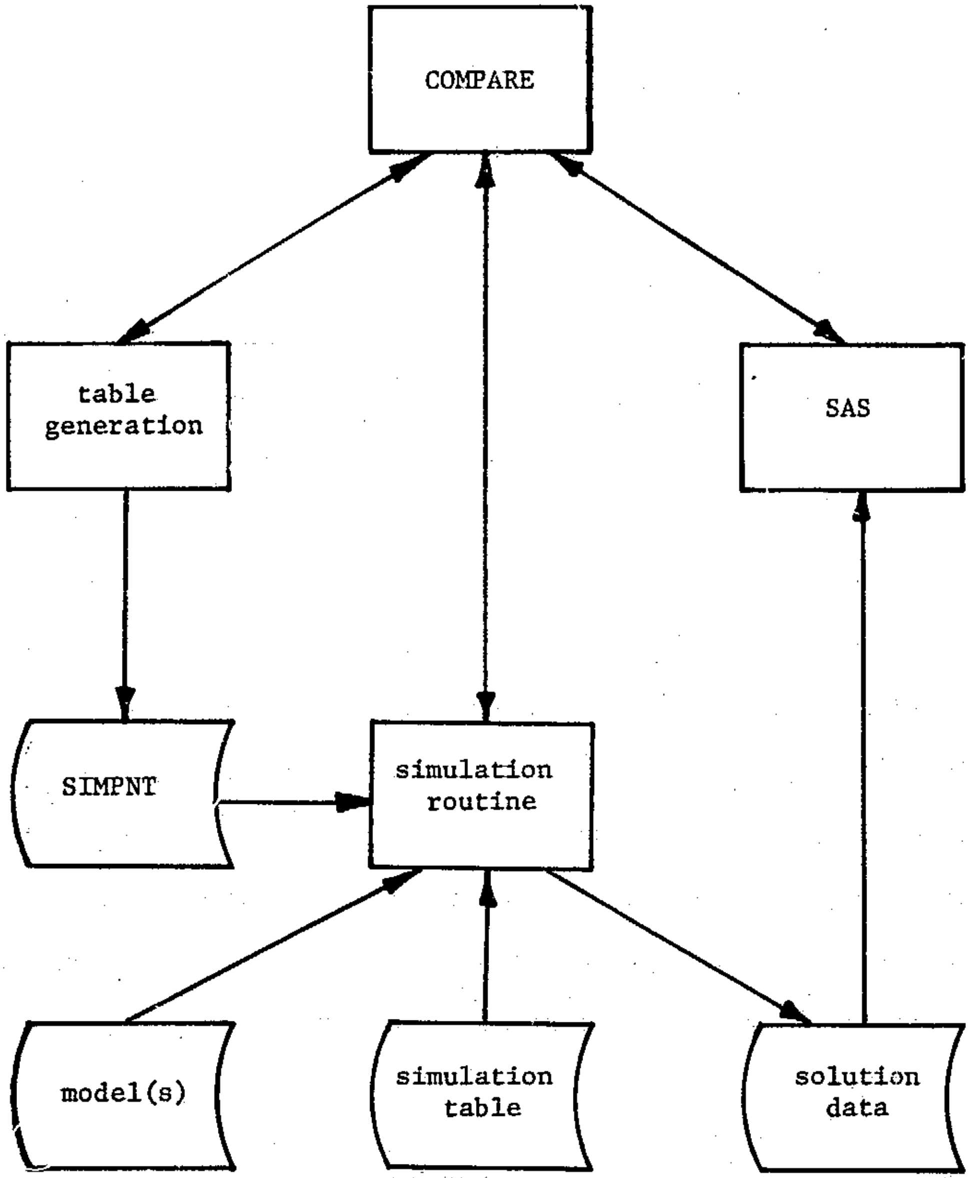


Figure 3.2. Logical Struction of Simulation on IBM 370/3033

CHAPTER IV

MODEL REDUCTION

In Chapter II some mathematical ideas were set forth that are at the center of the entire problem of nonlinear modelling via tensor parameterization. It is hoped in any nonlinear modelling exercise that the addition of higher degree terms will benefit the model in that the system nonlinearities will be more easily characterized by these higher degree terms or combinations thereof. But these models with the obvious advantage of higher degree lead us to a challenge. This challenge is the dimension or size of the model, p, and the arrays associated with its computation. Figure 4.1 shows how the size of model increases with the increase of the three determining factors, number of states, number of controls and degree. As the model gets larger, several issues may need to be examined, such as addressable storage limits, limits on computational reliability, and time of computation. Clearly a way of making the models easier to handle is very desirable. In short, if modelling is to proceed to higher degree terms, the models must have unnecessary monomials eliminated. It is to this end that we devote the remainder of this chapter to the introduction of one such size reduction method.

4.1 REDUCTION TECHNIQUE

As mentioned in past chapters, the model is calculated using a least squares approximation given times series state derivative data and time series tensor product data. Because this calculation uses a least squares approximation, and a truncation of the series approximation there will be an inherent error in the model. We can characterize the error, E, as

$$E = \dot{X} - L\, Z.$$

number of states	number of controls	degree of approximation	model size
2	2	1	2 x 4
2	2	2	2 x 14
2	2	3	2 x 34
2	2	4	2 x 69
2	3	1	2 x 5
2	3	2	2 x 20
2	3	3	2 x 55
2	3	4	2 x 125
2	4	1	2 x 6
2	4	2	2 x 27
2	4	3	2 x 83
2	4	4	2 x 209
3	2	1	3 x 5
3	2	2	3 x 20
3	2	3	3 x 55
3	2	4	3 x 125
3	3	1	3 x 6
3	3	2	3 x 27
3	3	3	3 x 83
3	3	4	3 x 209
3	4	1	3 x 7
3	4	2	3 x 35
3	4	3	3 x 119
3	4	4	3 x 315
4	2	1	4 x 6
4	2	2	4 x 27
4	2	3	4 x 83
4	2	4	4 x 209
4	3	1	4 x 7
4	3	2	4 x 35
4	3	3	4 x 119
4	3	4	4 x 315
4	4	1	4 x 8
4	4	2	4 x 44
4	4	3	4 x 164
4	4	4	4 x 474

Figure 4.1. Variation of Model Sizes with Various Parameters

Because $\dot{X}$ and Z are time series data, E will become a time series matrix of state derivative errors. This simple calculation will become the base for the reduction method.

The idea of using an error matrix analysis is a modified approach from a treatment of nonlinear model reduction given by A.A. Desrochers [13]. Desrochers poses the problem with only a linear control input and in discrete time

$$x(k+1) = A\,F(k) + B\,u(k)$$

where
- x is a state vector,
- u is an input,
- A and B constitute the model,
- F is q-vector of states and state combinations, that is, $F(k) = \tilde{F}(x(k))$.

To determine which columns[1] of A are most dominant, a modified state error vector $\tilde{e}$ is formed

$$\tilde{e}_j(k) = x(k+1) - [A(,j)B]\begin{bmatrix} F(j,k) \\ \hline u(k) \end{bmatrix} \quad j = 1,2,\ldots,q.$$

Note that the error is modified in the sense that it is formed using only one term at a time in addition to the control input. The state error vectors are then squared and summed for all h time points

$$\sum_{k=0}^{h-1} \tilde{e}_j^T(k)\,\tilde{e}_j(k).$$

Based on this squared error number, the most important terms can be kept and the others discarded.

Now in our more general problem, we have nonlinear input terms in addition to the linear terms, and we choose to test the effect of all terms in-

[1]Henceforth, given a matrix A, we denote the ij-th element as A(i,j), the i-th column of A as A(,i), and the i-th row as A(i,).

cluding those corresponding to the linear control inputs. (Recall Desrochers decided to keep all linear controls.) Our test error matrix E_j is

$$E_j = \dot{X} - L_j Z_j , \quad j = 1,2,\ldots,p$$

where E_j is the time series error matrix with the j-th term removed,
p is the number of terms in the approximation,
L_j is the model with the j-th column removed,
Z_j is the time series tensor term data with the j-th row removed.

Note that now, in addition to the error incurred by the averaging of the least squares algorithm, there is error due the loss of the j-th term. We see that each column of E_j is the error vector at a certain time. Then we can use the same squared error calculation,

$$\sum_{k=0}^{h-1} e_j^T(k)\, e_j(k) \qquad j = 1,2,\ldots,p$$

where $E_j = [e_j(0) \mid e_j(1) \mid \ldots \mid e_j(h-1)]$, to determine the effect of a certain term on the model. The influence of a term on the model is measured by checking the difference of the squared error of the full model and the squared error of the model shortened by that term. As an aside, it is interesting to note a philosophical difference between the problem formulation of Desrochers and this problem formulation. Desrochers chose to pick a significant term by including only that term in the error test. On the other hand, our reduction test checks a term's significance when that term is removed.

A simple example here will illustrate the calculations. Suppose we have a two state, one control system which we sample for four time points and calculate a linear approximation. Let

$$\dot{X} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}.$$

First we calculate the state error matrix E of the full model, $E = \dot{X} - LZ$,

$$E = \begin{bmatrix} -7 & -12 & -5 & -5 \\ -5 & -9 & 3 & -9 \end{bmatrix}.$$

Now calculate the squared error for the full model, Ψ,

$$\Psi = \sum_{k=0}^{3} E^T(,k)\; E(,k) = [-7 \quad -5] \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$+ [-12 \quad -9] \begin{bmatrix} -12 \\ -9 \end{bmatrix}$$

$$+ [-5 \quad 3] \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$+ [-5 \quad -9] \begin{bmatrix} -5 \\ -9 \end{bmatrix}$$

$$= 439.$$

Following this calculation, we start checking the errors of the shortened models. We want first to calculate E_1 which is the state error matrix with the first term of the model, x_1, and its corresponding data removed. According to the method, the first row of z is removed, the first column of L is removed, and the error calculations are repeated:

$$E_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -11 & -3 & 1 \\ -5 & -7 & 7 & -3 \end{bmatrix}.$$

$$\Psi_1 = \sum_{k=0}^{3} E_1^T(,k)\; E_1(,k) = [-7 \quad -5] \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$= [-11 \quad -7] \begin{bmatrix} -11 \\ -7 \end{bmatrix}$$

$$+ [-3 \quad 7] \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$
$$= 312.$$

Then the relative error due to term x_1 is $\Psi - \Psi_1 = 312-439 = -127$.

Now we repeat the calculations for the removal of the second term in the model, x_2. Here

$$E_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{bmatrix}.$$
$$= \begin{bmatrix} -5 & -8 & 1 & -2 \\ -6 & -11 & 0 & -9 \end{bmatrix}.$$

The squared error number, Ψ_2,

$$\Psi_2 = \sum_{k=0}^{3} E_2^T(,k)\ E_2(,k) = 332,$$

and the relative error is $332-439 = -107$.

Finally the last term, corresponding to u, is checked. Thus

$$E_3 = \begin{bmatrix} -1 & -3 & -5 & 1 \\ 3 & 3 & 3 & -5 \end{bmatrix},$$

$$\Psi_3 = \sum_{k=0}^{3} E_3^T(,k)\ E_3(,k) = 88,$$

and the relative error is $88-439 = -351$.

In most cases, examination and comparison of the computed relative errors will give a fair indication of which terms in the model are the most influential. Sometimes this is not the case and we must extend this squared error concept to look at more data. The extension is a simple and informative one.

Recall that each error matrix, E and the E_j's, is a time series matrix representation of the errors in the states. For example, in the previous example the time series error in the first state of the full model is

$$E(1,) = [-7 \quad -12 \quad -5 \quad -5],$$

the first row of E. Now use the squared error calculation for each individual state to get

$$\psi_i = \sum_{k=0}^{h-1} E(i,k)\, E(i,k) \qquad i = 1,2,\ldots,n.$$

This says that the error in the i-th state, ψ_i, is the sum of the squares of each of the entries of the i-th row of E. Furthermore note that the error in the entire system, Ψ, is the sum of the ψ_i's. That is

$$\Psi = \sum_{i=1}^{n} \psi_i,$$

and for the reduced cases

$$\Psi_\ell = \sum_{i=1}^{n} \psi_{\ell i},$$

where ℓ is the number of the term removed from the model. With these additional calculations, now, we can compare not only the entire system errors (the $\Psi - \Psi_\ell$), but also the errors in the states $\psi_i - \psi_{\ell i}$.

To illustrate these calculations, we return to the example. We begin with the full model. The error in the first state is

$$\begin{aligned}\psi_1 &= \sum_{k=0}^{3} E(1,k)E(1,k)\\ &= (-7)^2 + (-12)^2 + (-5)^2 + (-5)^2\\ &= 243,\end{aligned}$$

and the error in the second state is

$$\begin{aligned}\psi_2 &= \sum_{k=0}^{3} E(2,k)E(2,k)\\ &= (-5)^2 + (-9)^2 + (3)^2 + (-9)^2\end{aligned}$$

$$= 196.$$

Thus

$$\Psi = \psi_1 + \psi_2 = 243 + 196 = 439.$$

After removing the first term in the model, the error in the first state is

$$\psi_{11} = \sum_{k=0}^{3} E_1(1,k)E_1(1,k)$$

$$= (-7)^2 + (-11)^2 + (-3)^2 + (1)^2$$

$$= 180,$$

and the error in the second state is

$$\psi_{12} = \sum_{k=0}^{3} E_1(2,k)E_1(2,k)$$

$$= (-5)^2 + (-7)^2 + (7)^2 + (-3)^2$$

$$= 132.$$

So

$$\Psi_1 = \psi_{11} + \psi_{12} = 180 + 132 = 312.$$

Now, as we calculated before, the change in the overall error is $\Psi_1 - \Psi = -127$. In addition to this we check the change in the state errors. That is,

$$\psi_{11} - \psi_1 = 180 - 243 = -63$$

and

$$\psi_{12} - \psi_2 = 132 - 196 = -64.$$

Now, repeat the calculations when only the second term of the model is removed.

$$\psi_{21} = \sum_{k=0}^{3} E_2(1,k)E_2(1,k)$$

$$= (-5)^2 + (-8)^2 + (1)^2 + (-2)^2$$

$$= 94$$

$$\psi_{22} = \sum_{k=0}^{3} E_2(2,k)E_2(2,k)$$

$$= (-6)^2 + (-11)^2 + (0)^2 + (-9)^2$$

$$= 238.$$

Then
$$\Psi_2 = \psi_{21} + \psi_{22} = 94 + 238 = 332,$$

$$\Psi_2 - \Psi = 332 - 439 = -107,$$

and

$$\psi_{21} - \psi_1 = 94 - 243 = -149$$

$$\psi_{22} - \psi_2 = 238 - 196 = 42.$$

Finally, the third term is removed and the calculations are performed. We get

$$\psi_{31} = 36$$

$$\psi_{32} = 52$$

and $\Psi_3 = 88$, so $\Psi_3 - \Psi = -351$.

Also
$$\psi_{31} - \psi_1 = -207$$

$$\psi_{32} - \psi_2 = -144,$$

which finishes the computation.

Arranging this data in a table will make it readable and usable. See Figure 4.2 for the reduction data of the sample problem. Naturally, as the model increases in size, the testing will increase to include all terms in the model approximation. These, then, are the simple calculations concerning the reduction method. When a reduction is done in the following chapter, an explanation of how to use these numbers will be presented.

Assume for now that the above test has indicated a set of terms are likely candidates for removal. There are several steps that can now be taken.

Full model system error: 439

Full model state errors: 243 196

Term: x_1

Change in system error: −127

Change in state errors: −63 −64

Term: x_2

Change in system error: −107

Change in state errors: −149 42

Term: u_1

Change in system error: −351

Change in state errors: −207 −144

Figure 4.2 Reduction Data for Sample Exercise

One option is to remove those terms from the model and retain the remainder of the terms thus producing the reduced model. This, although simple, is not a logical choice. For when we omit terms in a model, we are overlooking some dynamics. Regardless of their significance, these lost dynamics may be modelled adequately using the remaining terms, if they are given a chance. Hence, it is advantageous to regenerate the derivative and tensor data followed by a re-calculation of the now reduced model. But, by removing terms, we have changed the model structure and, intuitively, the system will need to be excited differently. Therefore, it would be a further advantage if a new input is designed.

The decision to re-optimize after terms are omitted seems sound, but it is by no means trivial. It was argued that a new input design is the sensible next step after the decision is made as to which terms are kept. But if the input changes (due to the optimization), the full model associated with this new excitation may contain influential terms at those places we have already made unavailable by reduction. Because of this difficulty, an assumption is made. It is assumed that a relatively insignificant term will remain relatively insignificant if the change in input excitation is not extreme. To ensure that the change in excitation is not too great, terms are not discarded in large groups. This is so, because it is hypothesized that a large change in model structure (that is, many terms removed) will cause the optimum input to move farther away from the present input parameters. Furthermore, the input optimization routine converges to a local minimum which will usually ensure the parameters do not move too much. To summarize, problems in reduction may be encountered if there is significant change between the input parameters of the model to be reduced and the input parameters of the reduced model.

As a result of the previous discussion, we have made a minor rule for reduction. It claims that terms to be omitted should not be discarded in large groups. There is another rule implicit in the preceding paragraph which is worth stating. The model to be reduced must have been identified at an input parameter set which is locally optimum. If it is not, optimization for the reduced model may change the input parameters significantly, and we have already argued that in general, this is not desired.

It is worthwhile to pause here to make some observations. First, by way of reasoning, a "multiple pass" reduction method has evolved. That is, an optimum input is chosen, the model is calculated, and the least significant terms in the model are determined. These terms are removed from further calculation, and an optimum input is re-chosen, the model is re-calculated, and so on. Obviously this process has a limit, and as the model becomes smaller, more care must be taken when choosing terms to be omitted. In some cases, the final pass is obvious. In these instances, the reduction test will indicate that all remaining terms are relatively significant. In other cases, the final pass is not so clear. Usually after a few passes, the model is adequately shortened, and the reduction test may indicate only a single term or two to be insignificant. It is in these cases that care must be taken when omitting terms. This is intuitively sensible since, as a model is made smaller, the dynamics in a single term are harder pressed to appear in the remaining terms. Because of this, model performance may suffer. It is in such cases that after a model has been reduced, some sample simulations should be run to check if model performance is still acceptable. If it is not, then the previous step was the final one. Otherwise reduction can continue. A second observation is that in requiring only a small change in input parameter design from step to step in the reduction process, we have taken advantage of the fact that the IMSL optimization routine yields a local minimum. While it would be more advantageous to have a global minimization routine if full model identification was being done, in the case of iteratively reducing models, the local optimization is actually more desirable.

Returning to the reduction method, we can use the rules from the previous arguments to state the scheme in full:

a) identify a model at an input which is locally optimum,
b) determine a set of terms which seem to be the least significant.
c) remove the terms in the set and re-optimize,
d) identify reduced model,
e) model simulation to test performance,
f) if reduction data indicates more insignificant terms, go to step b), otherwise stop.

This simple method works quite well as the examples in the following chapter will show.

4.2 REMARKS

This chapter summarizes the present reduction method, including demonstration of the calculations and a step by step outline of the technique. Difficulties arising from the combination of input design and term removal are discussed and a suitable compromise is obtained by the application of some basic rules. The following chapter contains concrete examples where the qualitative ideas of this chapter become clearer.

CHAPTER V

EXAMPLE REDUCTIONS

It is in this chapter that a few examples are studied to illustrate the power and usefulness of the model reduction method presented in the previous chapter. The first two systems were initially investigated in [2]. The first system is a two state, two control input set of nonlinear differential equations. The second system is a three state, three control set of strictly polynomic nonlinear differential equations. The final system is a three state, two input example. The first system will be modelled and reduced for second and third degree approximations. The other two examples will demonstrate second degree reductions.

The true test of a model's validity is borne out in simulation. Therefore much of the chapter is comprised of simulation data tables and plots. With this in mind, we establish two general criteria for judging a reduced model. They are:

a) the reduced model should approximate the full model of like degree of approximation,

b) the reduced model should outperform the full model of less degree.

Both of these are sensible goals and the examples herein do well with respect to them. Finally, it is worth restating a weak assumption necessary for model reduction. This is that the model should be an acceptable representation of the system. This is fairly obvious: since the motivation of a reduced model is to approximate the full model, we would hope the full model accurately describes the true system.

5.1 TWO STATE SYSTEM

The first system is given by the following equations

$$\dot{x}_1 = u_2\cosh(x_1x_2) - e^{2u_1}\sinh(2x_1) - 3\sinh(x_2),$$
$$\dot{x}_2 = e^{u_1u_2}\sinh(x_1) - e^{u_1}u_1\cosh(x_1^2) + \sinh(x_2).$$

These equations were studied in depth to demonstrate the power of the input optimization in identifying models that improve in quality as the degree of approximation increased. This study appears in Appendix A. It is from this report that "good" second and third degree models are taken as starting points for the reduction.

5.1.1 DEGREE TWO APPROXIMATION

For this system, the point of expansion is the origin because it is a stable equilibrium point. In the case of a second degree expansion, 14 model parameters corresponding to the tensor products

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1x_2,\ x_2x_2,$$
$$x_1u_1,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_1,$$
$$u_1u_2,\ u_2u_2$$

must be identified.

Now, to begin the reduction, a good full model that is locally optimum is necessary. The first appendix gives us such a model. That particular second degree model was identified using an initial state perturbation vector of

$$x_0 = (0.005,\ -0.005).$$

In addition the excitation for each input was taken to be a sum of two sinusoids:

$$u_1(t) = 0.025\sin(2\pi t\cdot\phi_1) + 0.025\sin(2\pi t\cdot 2\phi_1),$$
$$u_2(t) = 0.025\sin(2\pi t\cdot\phi_2) + 0.025\sin(2\pi t\cdot 2\phi_2).$$

In this case $\phi = (\phi_1, \phi_2)$ was determined by optimization to be (0.7418, 0.9022) given the starting values (0.75, 1.0). The system was run for 4 seconds and 100 samples were taken at evenly spaced intervals.

Given this locally optimum model, the reduction test can be performed. See Figure 5.1.1 for the full model and its reduction test. The decision as to which terms should be omitted (or equivalently which terms should be kept) must now be made. Upon inspection of the reduction test, it should be noted that with the removal of any term, the change in the squared error number may increase or decrease significantly, or stay about the same. In this particular application, we choose to ignore the sign of the square error numbers and look upon their magnitudes as quantifiers of the effect of a term on the model. This is reasonable since removal of a term with a large change in squared error (either an increase or decrease) will most likely cause a large change in model performance. This, then, directs our attention to the terms with small changes in their errors. These terms are in some sense the least significant, so it is hoped that removal of them will have the least effect on model performance.

As a conservative first attempt, it is decided that error changes (both system and state errors) with magnitudes less than about 4 will be omitted. The set of these terms is

$$x_1x_1,\ x_1x_2,\ x_2x_2,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_2.$$

Equivalently, the terms kept are

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1u_1,\ u_1u_1.$$

Note that all linear terms were kept. In general, saving the linear terms is desired, since they hold the good local behavior.

Now, an input optimization is performed using only the six significant terms given by the reduction test The system is again perturbed with initial conditions

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0050     -0.0050

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2           2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.24999999999999996E-01  0.24999999999999996E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.7418      0.9022

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N

ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y

THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  14 ROWS AND 100 COLUMNS.

NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N

TSO SPEAKEASY III PI+  7:18 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED

PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
```

Figure 5.1.1a Full Second Degree Model and Reduction Test (First Pass)

```
 -2.0009 -2.9978
  .9998   .99982

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50055+.86378i -.50055-.86378i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.7287E-4  .99934
 -1.0003    -3.5733E-5


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  .0019401  -.080981    .034007
  .0052751   9.3478E-4 -3.8931E-4


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0686    -.0062571 -.030538   -.024848
 -.018969   -.011543  -.01223    -.011688


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.033944   -6.0501E-4  .0049369
 -.99912     .0066456   8.6412E-5


  S (A 14 COMPONENT ARRAY)
  9.7385  6.5841  6.4438  5.5233  3.802   3.1743  2.3796  2.1075  1.8207
  1.5919  1.1872  .83463  .51123  .74939
 THE MAXIMUM SINGULAR VALUE:
MAX =  9.7385
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .51123

 ...AND THEIR RATIO:
RTIO =  19.049


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     195.7138
FULL STATE ERRORS:     171.4433       24.2706

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:   187.8448
CHANGE IN STATE ERRORS:   145.1516       42.6933

TERM:          X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:    90.6012
CHANGE IN STATE ERRORS:    82.8124        7.7888

TERM:          U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:     8.0872
CHANGE IN STATE ERRORS:     0.0621        8.0251

TERM:          U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:    39.0776
CHANGE IN STATE ERRORS:    39.0775        0.0001

TERM:       X1.X1.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:     0.0306
```

Figure 5.1.1b Full Second Degree Model and Reduction Test (First Pass)

```
CHANGE IN STATE ERRORS:       -0.0430       0.0736

TERM:       X1.X2.  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:       -2.0255
CHANGE IN STATE ERRORS:       -2.0145      -0.0111

TERM:       X2.X2.  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:       -0.6560
CHANGE IN STATE ERRORS:       -0.6514      -0.0046

TERM:       X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:     -122.4182
CHANGE IN STATE ERRORS:     -122.3798      -0.0384

TERM:       X1.U2.  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:       -0.0639
CHANGE IN STATE ERRORS:        0.0211      -0.0849

TERM:       X2.U1.  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:        0.9927
CHANGE IN STATE ERRORS:        0.9675       0.0253

TERM:       X2.U2.  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:        0.2700
CHANGE IN STATE ERRORS:        0.1913       0.0787

TERM:       U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:      -15.4397
CHANGE IN STATE ERRORS:        0.2414     -15.6811

TERM:       U1.U2.  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:       -0.0333
CHANGE IN STATE ERRORS:       -0.0192      -0.0141

TERM:       U2.U2.  COLUMN #:  14.
CHANGE IN SYSTEM ERROR:       -0.1777
CHANGE IN STATE ERRORS:       -0.1797       0.0019

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 6

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
WHEN ASKED, ENTER A AS AN ARRAY.
A = ARRAY(6:1,2,3,4,8,12)

MANUAL MODE
 SPACE USED   63 K NOW,   72 K PEAK,   SIZE    500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:Y
```

Figure 5.1.1c Full Second Degree Model and Reduction Test (First Pass)

$$x_0 = (0.005, -0.005).$$

The control inputs are again given by

$$u_1(t) = 0.025\sin(2\pi t\cdot\phi_1) + 0.025\sin(2\pi t\cdot 2\phi_1),$$

$$u_2(t) = 0.025\sin(2\pi t\cdot\phi_2) + 0.025\sin(2\pi t\cdot 2\phi_2).$$

Now as starting frequencies for the optimization, we use the frequency set determined by the first pass optimimation, $\phi = (0.7418, 0.9022)$. The optimum frequencies are computed to be $\phi = (0.6499, 0.9014)$. Note that there is not much change in the optimum frequencies which is what we desire. The model is re-calculated and the reduction test again performed. See Figure 5.1.2 for the "second pass" output. Inspection of the reduction data indicates that the remaining terms are all fairly significant. So the model given in Figure 5.1.2 is the final reduced model. It contains 6 terms of a possible 14; a 57% reduction. Furthermore, by taking all the partial derivatives, the exact second degree approximation would be

$$L_{10} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad L_{01} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$L_{20} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$L_{11} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_{02} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Obviously, the reduction method singled out the appropriate columns to be kept. In addition the error test for the individual states indicates the correct insignificant entries in each column. The reduction method seems to

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
      0.0050     -0.0050

ENTER SAMPLING PARAMETERS:
           APPROXIMATION DEGREE
           # SAMPLE POINTS
           INTERVAL BETWEEN SAMPLES
           INTEGRATION STEPSIZE
   2  100     0.0400     0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
           2           2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.2499999999999999996E-01  0.2499999999999999996E-01

ENTER THE 2 FREQUENCY WEIGHTS:
      0.7418        0.9022

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
           1 - NO OPTIMIZATIONS
           2 - AMPLITUDES ONLY
           3 - FREQUENCIES ONLY
           4 - INITIAL CONDITIONS ONLY
           5 - AMPLITUDES AND FREQUENCIES
           6 - AMPLITUDES AND INITIAL CONDITIONS
           7 - FREQUENCIES AND INITIAL CONDITIONS
           8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
            3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
           4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2    200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.6872968D+00  0.9476237D+00
THE CONDITION NUMBER:  0.126D+05
OBJECTIVE FUNCTION VALUE: 0.1975D+00

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.6782293D+00  0.9468474D+00
THE CONDITION NUMBER:  0.118D+05
OBJECTIVE FUNCTION VALUE: 0.1918D+00

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.6495584D+00  0.8921068D+00
THE CONDITION NUMBER:  0.107D+05
OBJECTIVE FUNCTION VALUE: 0.1745D+00
```

Figure 5.1.2a Reduced Second Degree Model and Reduction Test (Second Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.6498634D+00  0.9013817D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

 50 CALLS OF FCN.....
THE PARAMETERS:
 0.6498652D+00  0.9013841D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

 60 CALLS OF FCN.....
THE PARAMETERS:
 0.6498653D+00  0.9013842D+00
THE CONDITION NUMBER:  0.104D+05
OBJECTIVE FUNCTION VALUE: 0.1719D+00

CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y




THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS   6 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:       63

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.650     FREQ(2)=  0.901     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N




TSO SPEAKEASY III PI+  7:33 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 6 COMPONENT ARRAY)
  1   2   3   4   8   12


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -2.003   -3.0012
  .9999    1.0001

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50146+.8639i -.50146-.8639i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  2.3743E-4  .9991
 -1.0005    -1.5637E-5
```

Figure 5.1.2b Reduced Second Degree Model and Reduction Test (Second Pass)

```
PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  0  0  0
  0  0  0


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0049   0         0         0
  .016344  0         0         0


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.016019  0         0
 -.99954   0         0


  S (A 6 COMPONENT ARRAY)
  7.3342  2.012   2.2395  4.5216  5.5657  6.1157
 THE MAXIMUM SINGULAR VALUE:
MAX =  7.3342
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  2.012

 ...AND THEIR RATIO:
RTIO =  3.6452



WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:      200.9931
FULL STATE ERRORS:      176.7734        24.2198

TERM:            X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     236.6909
CHANGE IN STATE ERRORS:     194.4798        42.2111

TERM:            X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:      67.8517
CHANGE IN STATE ERRORS:      60.9716         6.8801

TERM:            U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:      11.0693
CHANGE IN STATE ERRORS:       0.0157        11.0535

TERM:            U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:     -16.1350
CHANGE IN STATE ERRORS:     -16.1348        -0.0002

TERM:         X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:    -120.4635
CHANGE IN STATE ERRORS:    -120.5174         0.0539

TERM:         U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -15.9942
CHANGE IN STATE ERRORS:       0.2470       -16.2412

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N

MANUAL MODE
 SPACE USED  43 K NOW,  47 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.2c Reduced Second Degree Model and Reduction Test (Second Pass)

have done about as well as could be expected. The true proof of a model's validity though lies in its simulation.

For the purpose of model verification, simulation consists of the excitation of the true system and the model at points about the point of identification. The control input excitation in this case is a single cosinusoid. That is

$$u_i(t) = \alpha_i \cos(2\pi t \cdot \phi_i) \quad i = 1,2,\ldots,m.$$

Furthermore, an initial condition for each state is required. The true system and each model are run for a specified time and sampled at evenly spaced points throughout the interval. If two models are simulated simultaneously, a mean square error (with respect to the true solution) analysis is performed. The error criterion, then, is just the ratio of the mean square errors of the two models for each state. For the following simulations, the error ratio is

$$R_i = \frac{\mathrm{MSE}(\mathrm{MODEL1}_i)}{\mathrm{MSE}(\mathrm{LINEAR}_i)}, \quad i = 1,2,\ldots,n$$

where in this case $\mathrm{MODEL1}_i$ refers to the i-th state of the six term reduced model. Obviously then, we desire the error ratios to be less than one. But, the following tabular data will show for some cases the error ratio between two models becomes very large. In these simulations it is necessary to check the normalized error data and closely examine the graphical output. Most of the time the large numbers are due to the fact that one model's trajectory lies on top of the true solution trajectory. Despite the fact that the other model may also be very close to the true solution, the error ratio becomes large.

Recall that two goals were set for performance evaluation of reduced models. The first is to outperform models of lesser degree, and the second is to approximate full model behavior. Tables 5.1.1 through Table 5.1.8 address the first criterion. In each of these tables, the heading MODEL1 refers to the reduced second degree model. The first three tables (Table 5.1.1 through Table 5.1.3) originally appear in [2] and test the model performance very close to the identification point. Note that throughout the trio of tables, according to the error ratios, the reduced model seems to be outperformed when the input amplitudes are zero. This behavior of the error criterion was explained above and a representative plot set is given in Figure 5.1.3.[1] Representative plot sets of the simulation data tables follow each table. Note that even with the loss of terms, the second degree model exhibits exceptional behavior over the linear approximation. In Table 5.1.4, the model's region of validity is tested farther away from the origin as the input amplitudes are increased substantially over those in previous tables. Table 5.1.5 gives a set of simulations whose initial state conditions are (-0.01, -0.01), and whose input amplitudes and frequencies are randomly chosen in the ranges (-0.2, 0.2) and (2, 6) respectively. The following three tables (Tables 5.1.6 through Table 5.1.8) test the low frequency and d.c. behavior of the models. Table 5.1.6 has various small frequencies and steps. Note that it was previously determined in [2] that the true system is unstable in the first state when excited with steps of magnitude more than 0.1. Table 5.1.7 is another random table of smaller steps whose initial state conditions are in the range (-0.01, 0.01) and whose amplitudes are in the range (-0.025, 0.025). Table 5.1.8 has some larger steps with the initial state conditions randomly chosen in the range (-0.05, 0.05) and input amplitudes also randomly chosen in the range (-0.075, 0.075).

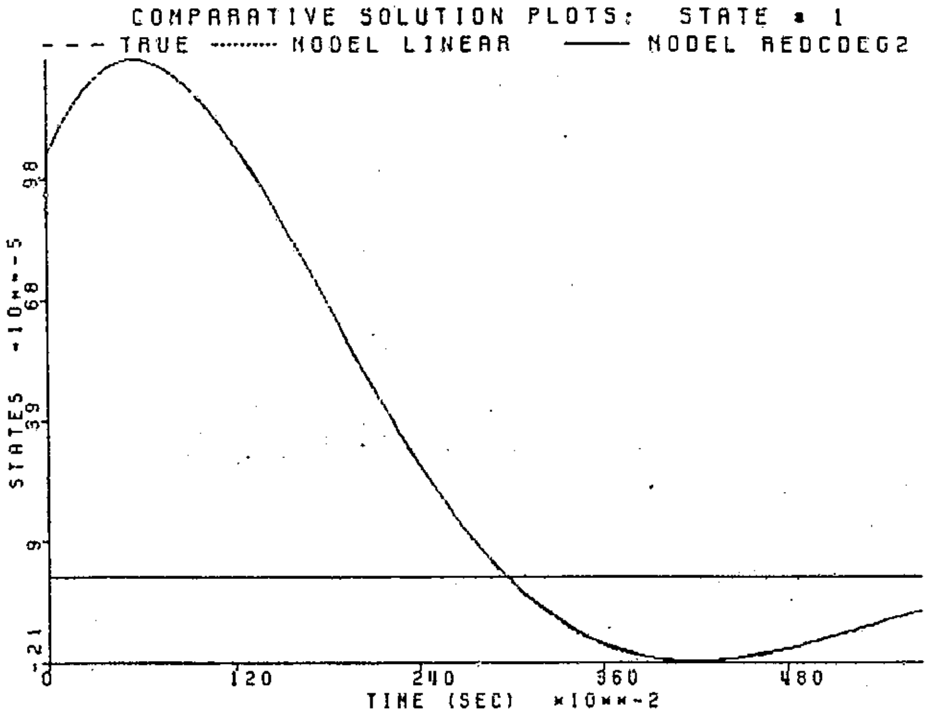
---

[1]In all plots, if a curve is indistinguishable, it over-lays another curve.

```
****************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
       # OF TERMS IN MODEL 1:   6
     DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.748E+06	0.606E+06
2	0.001	0.001	0.050	0.050	0.75	1.00	0.488E-05	0.549E-04
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.269E-05	0.452E-04
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.261E-05	0.362E-04
5	0.001	0.001	-0.050	0.050	0.75	1.00	0.638E-05	0.729E-04
6	0.001	0.001	0.150	0.150	0.75	1.00	0.257E-03	0.162E-02
7	0.001	0.001	0.150	-0.150	0.75	1.00	0.258E-03	0.237E-02
8	0.001	0.001	-0.150	-0.150	0.75	1.00	0.485E-03	0.316E-02
9	0.001	0.001	-0.150	0.150	0.75	1.00	0.702E-03	0.367E-02
10	0.001	-0.001	0.000	0.000	0.75	1.00	0.588E+07	0.759E+07
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.277E-05	0.335E-04
12	0.001	-0.001	0.050	-0.050	0.75	1.00	0.218E-05	0.374E-04
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.121E-05	0.194E-04
14	0.001	-0.001	-0.050	0.050	0.75	1.00	0.545E-05	0.568E-04
15	0.001	-0.001	0.150	0.150	0.75	1.00	0.320E-03	0.188E-02
16	0.001	-0.001	0.150	-0.150	0.75	1.00	0.285E-03	0.251E-02
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	0.581E-03	0.362E-02
18	0.001	-0.001	-0.150	0.150	0.75	1.00	0.719E-03	0.371E-02
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.748E+06	0.606E+06
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.206E-05	0.256E-04
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	0.190E-05	0.330E-04
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.870E-06	0.148E-04
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	0.582E-05	0.568E-04
24	-0.001	-0.001	0.150	0.150	0.75	1.00	0.353E-03	0.202E-02
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	0.295E-03	0.259E-02
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	0.632E-03	0.387E-02
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	0.759E-03	0.384E-02
28	-0.001	0.001	0.000	0.000	0.75	1.00	0.588E+07	0.759E+07
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.381E-05	0.432E-04
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.240E-05	0.409E-04
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.187E-05	0.259E-04
32	-0.001	0.001	-0.050	0.050	0.75	1.00	0.673E-05	0.729E-04
33	-0.001	0.001	0.150	0.150	0.75	1.00	0.286E-03	0.174E-02
34	-0.001	0.001	0.150	-0.150	0.75	1.00	0.267E-03	0.244E-02
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	0.530E-03	0.340E-02
36	-0.001	0.001	-0.150	0.150	0.75	1.00	0.741E-03	0.379E-02

Table 5.1.1 Simulation Table for Linear Model versus Second Degree Reduced Model



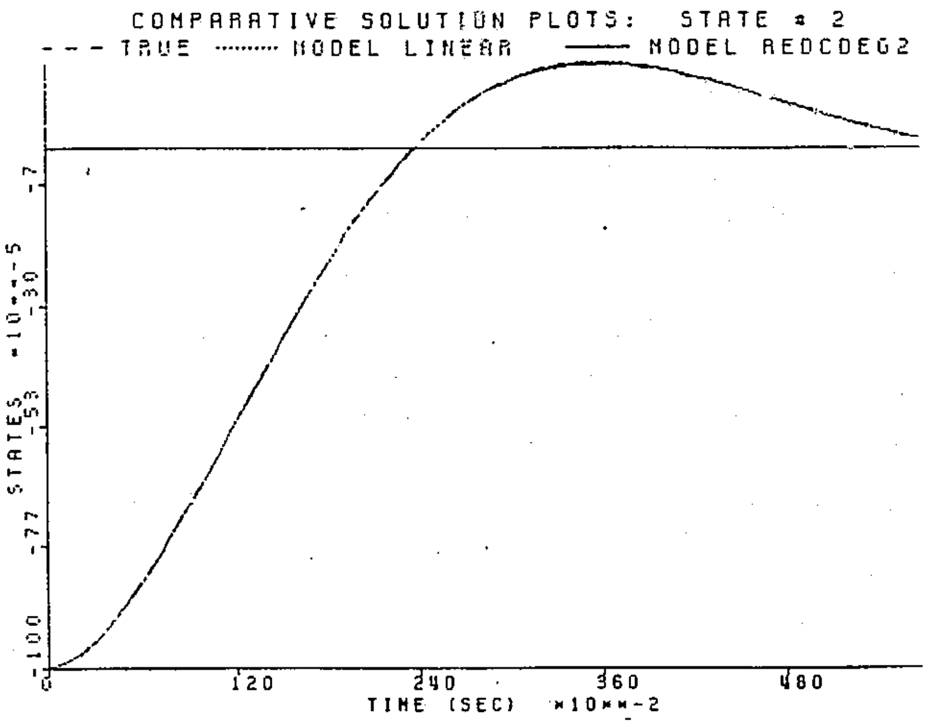
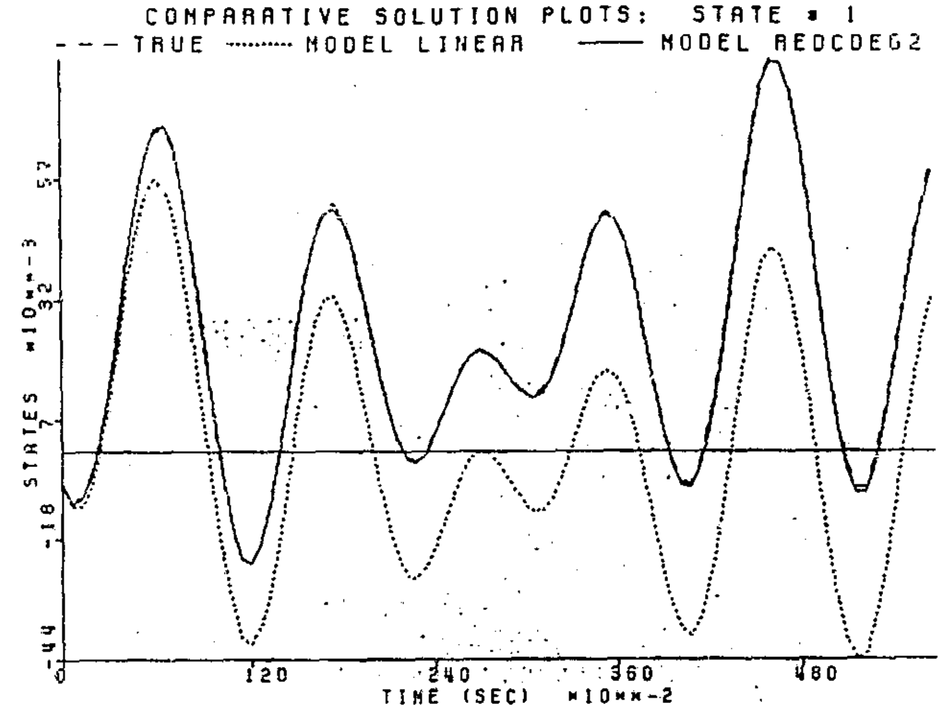


Figure 5.1.3 Simulation Number 10 of Table 5.1.1

ORIGINAL PAGE IS
OF POOR QUALITY



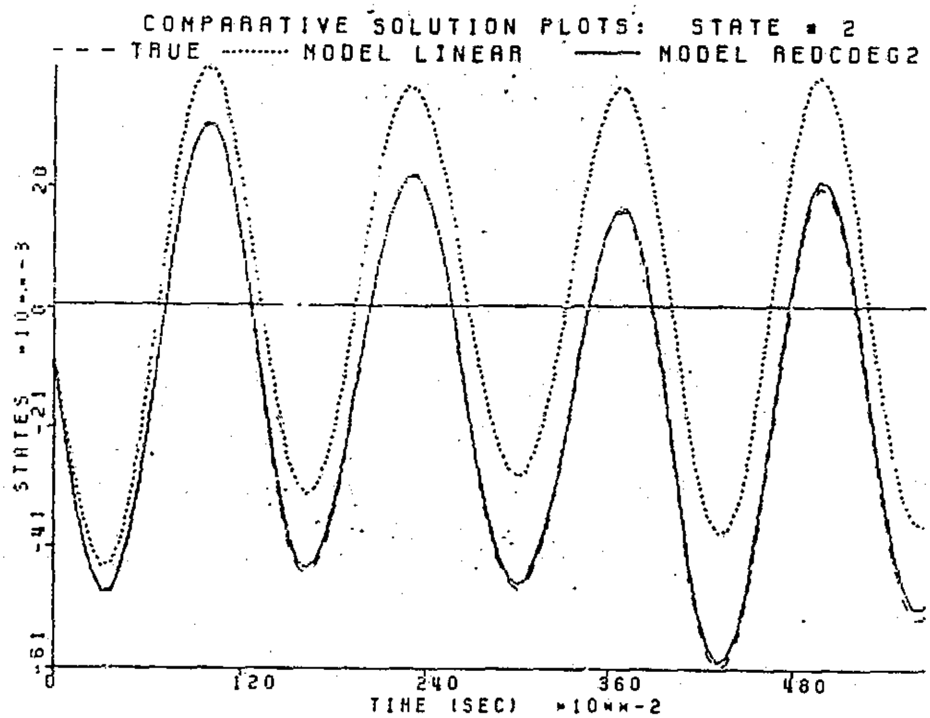
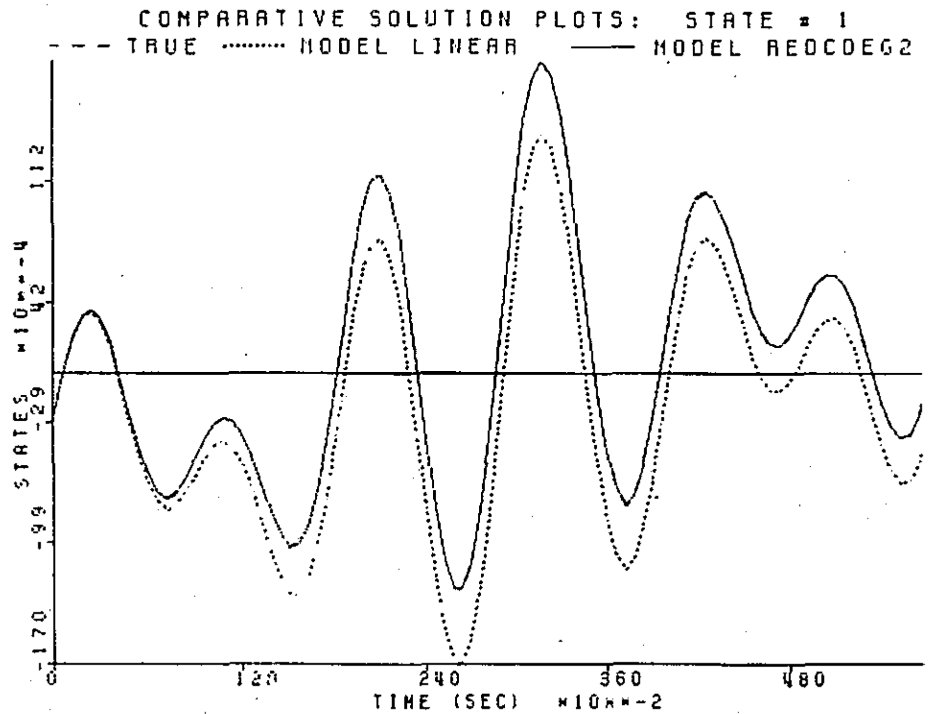


Figure 5.1.4 Simulation Number 25 of Table 5.1.1

```
****************************************
            PROBLEM SUMMARY
  CONFIGURATION:  TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:   6
     DEGREE OF APPROXIMATION: 2
       SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.005	0.005	0.000	0.000	0.75	1.00	805.	891.
2	0.005	0.005	0.050	0.050	0.75	1.00	0.162E-04	0.170E-03
3	0.005	0.005	0.050	-0.050	0.75	1.00	0.510E-05	0.811E-04
4	0.005	0.005	-0.050	-0.050	0.75	1.00	0.118E-04	0.159E-03
5	0.005	0.005	-0.050	0.050	0.75	1.00	0.826E-05	0.116E-03
6	0.005	0.005	0.150	0.150	0.75	1.00	0.133E-03	0.103E-02
7	0.005	0.005	0.150	-0.150	0.75	1.00	0.205E-03	0.203E-02
8	0.005	0.005	-0.150	-0.150	0.75	1.00	0.269E-03	0.208E-02
9	0.005	0.005	-0.150	0.150	0.75	1.00	0.619E-03	0.344E-02
10	0.005	-0.005	0.000	0.000	0.75	1.00	0.704E+04	0.818E+04
11	0.005	-0.005	0.050	0.050	0.75	1.00	0.159E-05	0.217E-04
12	0.005	-0.005	0.050	-0.050	0.75	1.00	0.222E-05	0.396E-04
13	0.005	-0.005	-0.050	-0.050	0.75	1.00	0.856E-06	0.172E-04
14	0.005	-0.005	-0.050	0.050	0.75	1.00	0.342E-05	0.327E-04
15	0.005	-0.005	0.150	0.150	0.75	1.00	0.404E-03	0.219E-02
16	0.005	-0.005	0.150	-0.150	0.75	1.00	0.331E-03	0.270E-02
17	0.005	-0.005	-0.150	-0.150	0.75	1.00	0.698E-03	0.413E-02
18	0.005	-0.005	-0.150	0.150	0.75	1.00	0.688E-03	0.359E-02
19	-0.005	-0.005	0.000	0.000	0.75	1.00	805.	891.
20	-0.005	-0.005	0.050	0.050	0.75	1.00	0.191E-05	0.225E-04
21	-0.005	-0.005	0.050	-0.050	0.75	1.00	0.102E-05	0.181E-04
22	-0.005	-0.005	-0.050	-0.050	0.75	1.00	0.338E-05	0.531E-04
23	-0.005	-0.005	-0.050	0.050	0.75	1.00	0.549E-05	0.332E-04
24	-0.005	-0.005	0.150	0.150	0.75	1.00	0.615E-03	0.305E-02
25	-0.005	-0.005	0.150	-0.150	0.75	1.00	0.394E-03	0.314E-02
26	-0.005	-0.005	-0.150	-0.150	0.75	1.00	0.101E-02	0.553E-02
27	-0.005	-0.005	-0.150	0.150	0.75	1.00	0.905E-03	0.429E-02
28	-0.005	0.005	0.000	0.000	0.75	1.00	0.704E+04	0.818E+04
29	-0.005	0.005	0.050	0.050	0.75	1.00	0.685E-05	0.703E-04
30	-0.005	0.005	0.050	-0.050	0.75	1.00	0.333E-05	0.574E-04
31	-0.005	0.005	-0.050	-0.050	0.75	1.00	0.416E-05	0.499E-04
32	-0.005	0.005	-0.050	0.050	0.75	1.00	0.996E-05	0.115E-03
33	-0.005	0.005	0.150	0.150	0.75	1.00	0.233E-03	0.152E-02
34	-0.005	0.005	0.150	-0.150	0.75	1.00	0.244E-03	0.237E-02
35	-0.005	0.005	-0.150	-0.150	0.75	1.00	0.444E-03	0.299E-02
36	-0.005	0.005	-0.150	0.150	0.75	1.00	0.794E-03	0.401E-02

Table 5.1.2 Simulation Table for Linear Model versus Second Degree Reduced Model



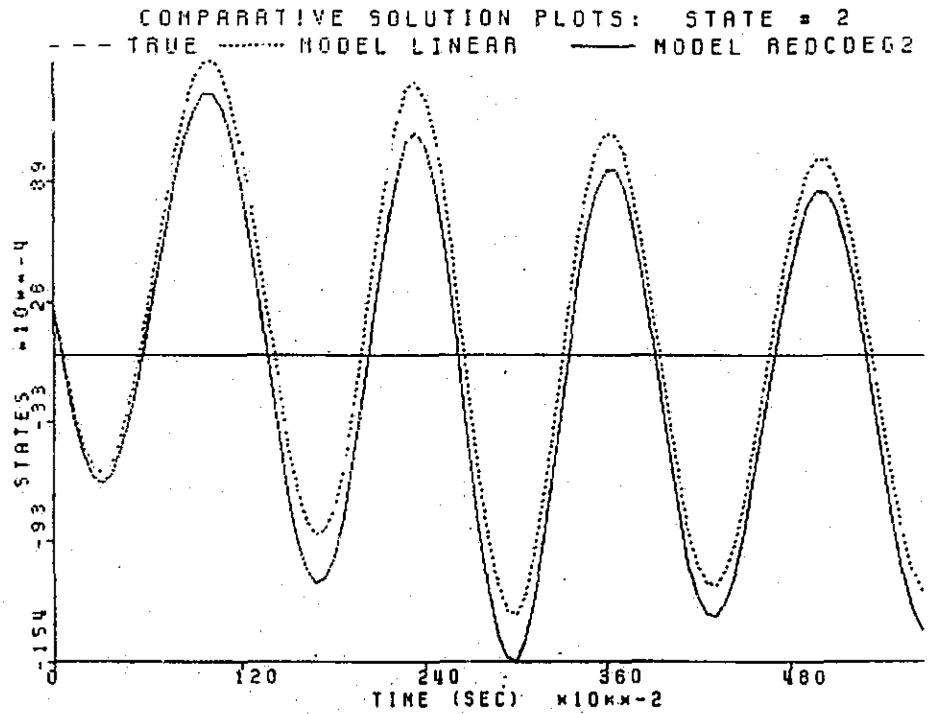
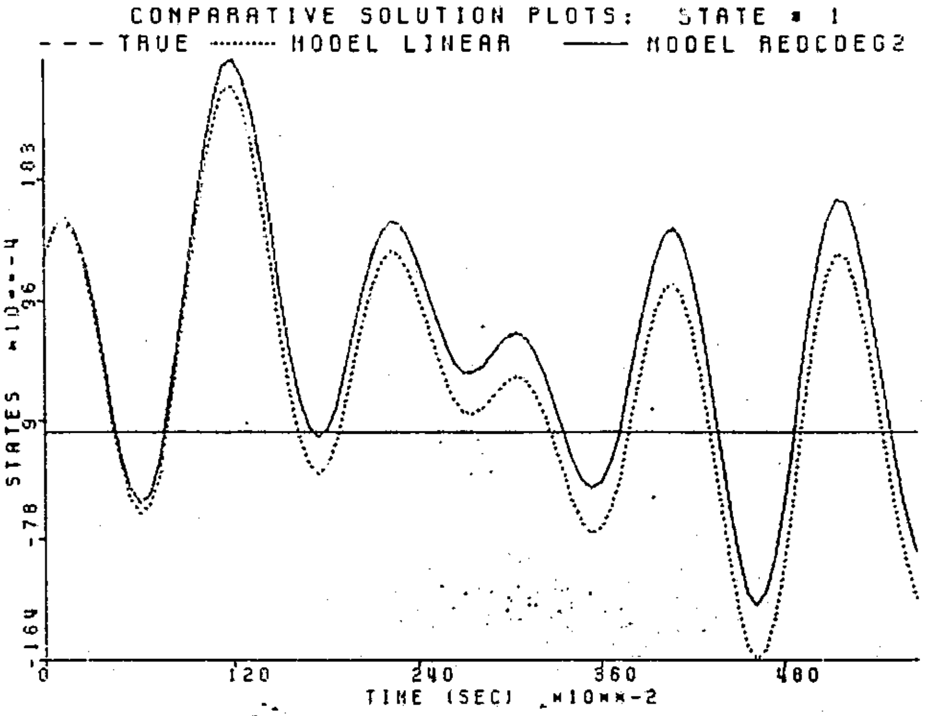


Figure 5.1.5 Simulation Number 29 of Table 5.1.2

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.010	0.010	0.000	0.000	0.75	1.00	39.0	42.4
2	0.010	0.010	0.050	0.050	0.75	1.00	0.439E-04	0.446E-03
3	0.010	0.010	0.050	-0.050	0.75	1.00	0.110E-04	0.166E-03
4	0.010	0.010	-0.050	-0.050	0.75	1.00	0.364E-04	0.507E-03
5	0.010	0.010	-0.050	0.050	0.75	1.00	0.133E-04	0.208E-03
6	0.010	0.010	0.150	0.150	0.75	1.00	0.104E-03	0.692E-03
7	0.010	0.010	0.150	-0.150	0.75	1.00	0.180E-03	0.178E-02
8	0.010	0.010	-0.150	-0.150	0.75	1.00	0.150E-03	0.126E-02
9	0.010	0.010	-0.150	0.150	0.75	1.00	0.570E-03	0.334E-02
10	0.010	-0.010	0.000	0.000	0.75	1.00	447.	514.
11	0.010	-0.010	0.050	0.050	0.75	1.00	0.203E-05	0.229E-04
12	0.010	-0.010	0.050	-0.050	0.75	1.00	0.341E-05	0.635E-04
13	0.010	-0.010	-0.050	-0.050	0.75	1.00	0.239E-05	0.365E-04
14	0.010	-0.010	-0.050	0.050	0.75	1.00	0.180E-05	0.173E-04
15	0.010	-0.010	0.150	0.150	0.75	1.00	0.539E-03	0.267E-02
16	0.010	-0.010	0.150	-0.150	0.75	1.00	0.412E-03	0.306E-02
17	0.010	-0.010	-0.150	-0.150	0.75	1.00	0.873E-03	0.486E-02
18	0.010	-0.010	-0.150	0.150	0.75	1.00	0.669E-03	0.354E-02
19	-0.010	-0.010	0.000	0.000	0.75	1.00	39.0	42.4
20	-0.010	-0.010	0.050	0.050	0.75	1.00	0.143E-04	0.148E-03
21	-0.010	-0.010	0.050	-0.050	0.75	1.00	0.184E-05	0.265E-04
22	-0.010	-0.010	-0.050	-0.050	0.75	1.00	0.211E-04	0.289E-03
23	-0.010	-0.010	-0.050	0.050	0.75	1.00	0.760E-05	0.283E-04
24	-0.010	-0.010	0.150	0.150	0.75	1.00	0.108E-02	0.476E-02
25	-0.010	-0.010	0.150	-0.150	0.75	1.00	0.566E-03	0.403E-02
26	-0.010	-0.010	-0.150	-0.150	0.75	1.00	0.165E-02	0.816E-02
27	-0.010	-0.010	-0.150	0.150	0.75	1.00	0.115E-02	0.507E-02
28	-0.010	0.010	0.000	0.000	0.75	1.00	447.	514.
29	-0.010	0.010	0.050	0.050	0.75	1.00	0.125E-04	0.120E-03
30	-0.010	0.010	0.050	-0.050	0.75	1.00	0.549E-05	0.972E-04
31	-0.010	0.010	-0.050	-0.050	0.75	1.00	0.914E-05	0.105E-03
32	-0.010	0.010	-0.050	0.050	0.75	1.00	0.156E-04	0.192E-03
33	-0.010	0.010	0.150	0.150	0.75	1.00	0.196E-03	0.132E-02
34	-0.010	0.010	0.150	-0.150	0.75	1.00	0.237E-03	0.237E-02
35	-0.010	0.010	-0.150	-0.150	0.75	1.00	0.365E-03	0.257E-02
36	-0.010	0.010	-0.150	0.150	0.75	1.00	0.883E-03	0.437E-02

Table 5.1.3 Simulation Table for Linear Model versus Second Degree Reduced Model



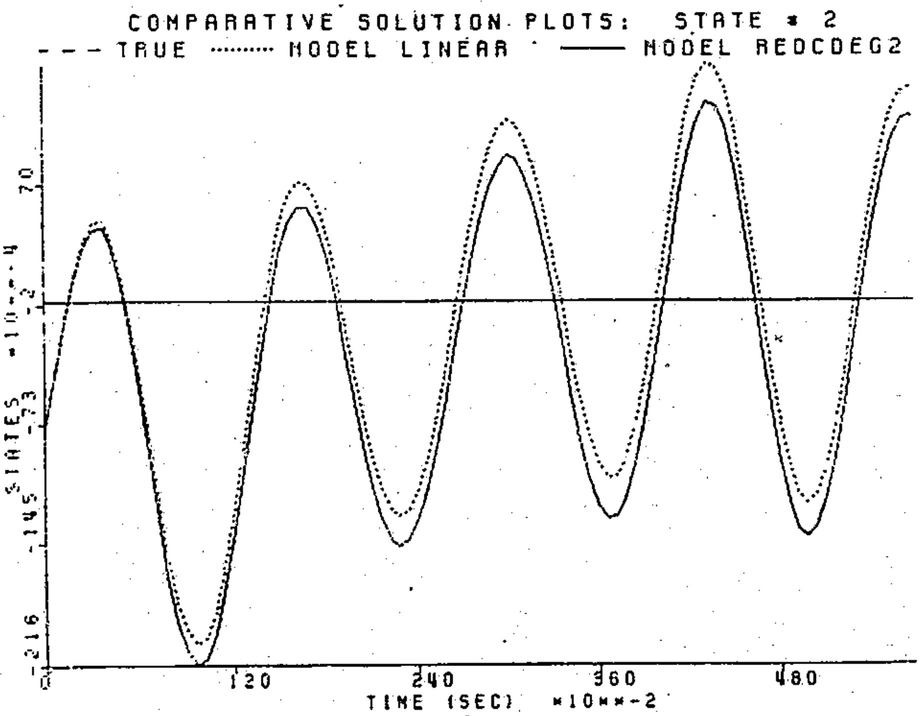
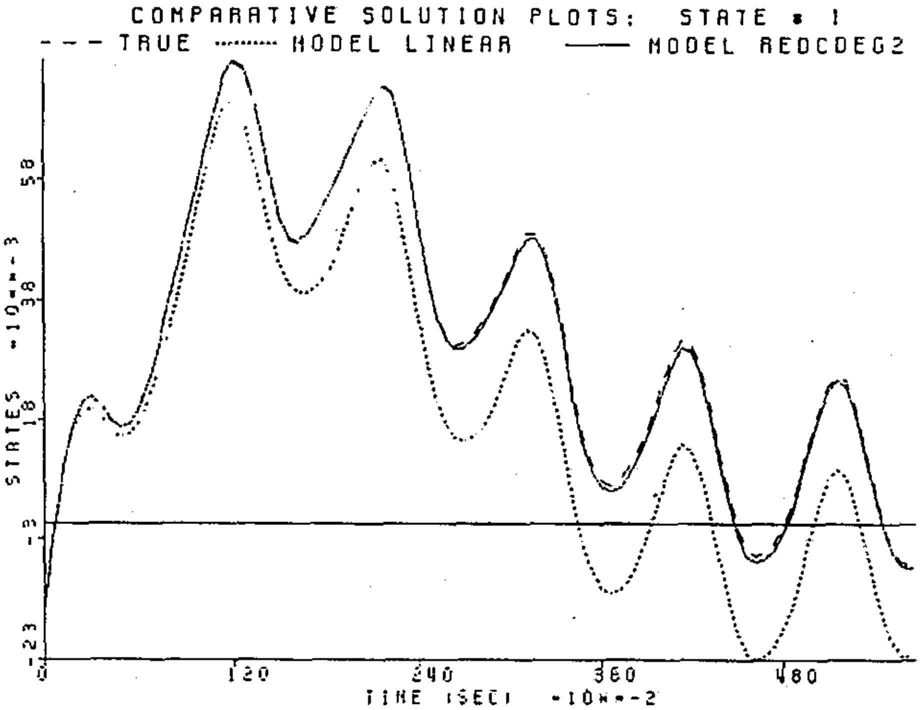


Figure 5.1.6 Simulation Number 14 of Table 5.1.3

```
***************************************
          PROBLEM SUMMARY
  CONFIGURATION:  TRUE,LINEAR,MODEL1
          # OF STATES: 2
          # OF INPUTS: 2
      # OF TERMS IN MODEL 1:   6
      DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
***************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	0.566E-03	0.231E-02
2	0.025	0.025	0.100	-0.100	2.00	1.00	0.497E-03	0.240E-02
3	0.025	0.025	-0.100	0.100	2.00	1.00	0.469E-03	0.195E-02
4	0.025	0.025	-0.100	-0.100	2.00	1.00	0.414E-03	0.224E-02
5	0.025	-0.025	0.100	0.100	2.00	1.00	0.181E-03	0.866E-03
6	0.025	-0.025	0.100	-0.100	2.00	1.00	0.204E-03	0.111E-02
7	0.025	-0.025	-0.100	0.100	2.00	1.00	0.255E-03	0.115E-02
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	0.275E-03	0.148E-02
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.205E-03	0.862E-03
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.151E-03	0.792E-03
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.170E-03	0.806E-03
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	0.136E-03	0.850E-03
13	-0.025	-0.025	0.100	0.100	2.00	1.00	0.868E-03	0.359E-02
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	0.906E-03	0.442E-02
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	0.107E-02	0.424E-02
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	0.104E-02	0.514E-02
17	0.025	0.025	0.200	0.200	2.00	1.00	0.112E-02	0.513E-02
18	0.025	0.025	0.200	-0.200	2.00	1.00	0.778E-03	0.474E-02
19	0.025	0.025	-0.200	0.200	2.00	1.00	0.124E-02	0.610E-02
20	0.025	0.025	-0.200	-0.200	2.00	1.00	0.912E-03	0.616E-02
21	0.025	-0.025	0.200	0.200	2.00	1.00	0.295E-02	0.101E-01
22	0.025	-0.025	0.200	-0.200	2.00	1.00	0.297E-02	0.109E-01
23	0.025	-0.025	-0.200	0.200	2.00	1.00	0.402E-02	0.126E-01
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	0.380E-02	0.135E-01
25	-0.025	0.025	0.200	0.200	2.00	1.00	0.188E-02	0.759E-02
26	-0.025	0.025	0.200	-0.200	2.00	1.00	0.155E-02	0.755E-02
27	-0.025	0.025	-0.200	0.200	2.00	1.00	0.237E-02	0.931E-02
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	0.195E-02	0.950E-02
29	-0.025	-0.025	0.200	0.200	2.00	1.00	0.618E-02	0.170E-01
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	0.620E-02	0.186E-01
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	0.782E-02	0.204E-01
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	0.734E-02	0.219E-01
33	0.025	0.025	0.300	0.300	2.00	1.00	0.873E-02	0.347E-01
34	0.025	0.025	0.300	-0.300	2.00	1.00	0.776E-02	0.348E-01
35	0.025	0.025	-0.300	0.300	2.00	1.00	0.111E-01	0.411E-01
36	0.025	0.025	-0.300	-0.300	2.00	1.00	0.940E-02	0.415E-01
37	0.025	-0.025	0.300	0.300	2.00	1.00	0.173E-01	0.500E-01
38	0.025	-0.025	0.300	-0.300	2.00	1.00	0.169E-01	0.521E-01
39	0.025	-0.025	-0.300	0.300	2.00	1.00	0.218E-01	0.589E-01
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	0.200E-01	0.603E-01
41	-0.025	0.025	0.300	0.300	2.00	1.00	0.131E-01	0.433E-01
42	-0.025	0.025	0.300	-0.300	2.00	1.00	0.121E-01	0.441E-01
43	-0.025	0.025	-0.300	0.300	2.00	1.00	0.164E-01	0.510E-01
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	0.143E-01	0.515E-01
45	-0.025	-0.025	0.300	0.300	2.00	1.00	0.249E-01	0.627E-01
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	0.244E-01	0.654E-01
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	0.305E-01	0.729E-01
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	0.281E-01	0.746E-01

Table 5.1.4 Simulation Table for Linear Model versus Second Degree Reduced Model



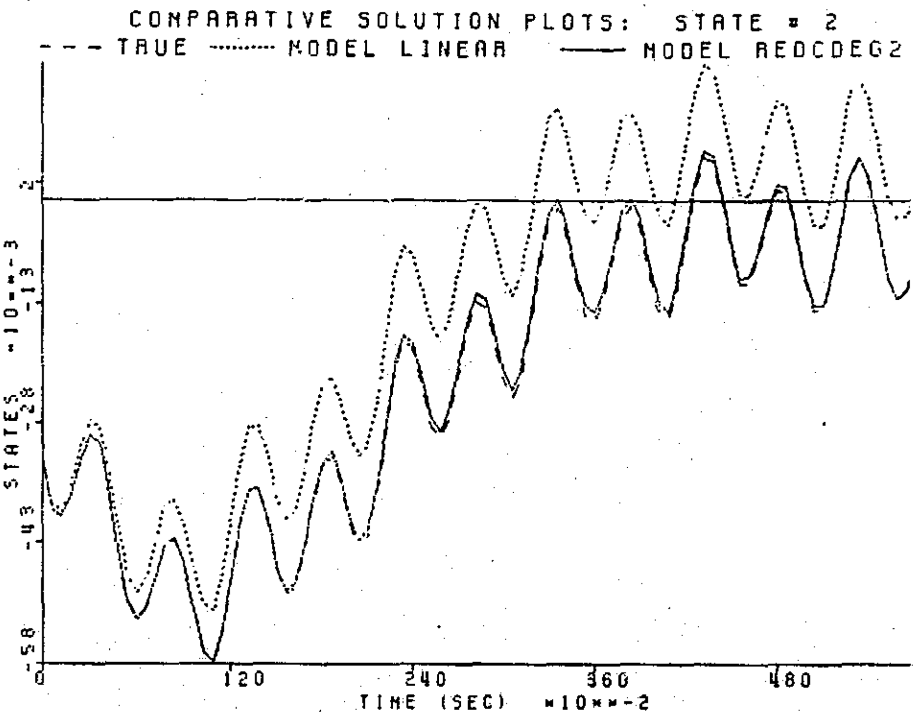


Figure 5.1.7 Simulation Number 13 of Table 5.1.4

```
***************************************
            PROBLEM SUMMARY
   CONFIGURATION:  TRUE,LINEAR,MODEL1
            # OF STATES: 2
            # OF INPUTS: 2
      # OF TERMS IN MODEL 1:    6
      DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
***************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.010	-0.010	0.063	0.055	4.69	4.66	0.552E-04	0.315E-04
2	-0.010	-0.010	0.043	-0.012	3.36	2.29	0.273E-04	0.417E-04
3	-0.010	-0.010	0.059	0.081	3.58	4.77	0.103E-03	0.215E-03
4	-0.010	-0.010	0.067	0.082	4.97	5.24	0.362E-03	0.648E-03
5	-0.010	-0.010	-0.042	0.173	5.87	4.31	0.265E-05	0.408E-04
6	-0.010	-0.010	0.049	-0.198	2.12	5.51	0.113E-04	0.970E-04
7	-0.010	-0.010	-0.005	-0.040	2.48	3.10	0.997E-04	0.216E-01
8	-0.010	-0.010	0.103	0.079	4.07	3.78	0.721E-03	0.137E-02
9	-0.010	-0.010	-0.187	0.065	5.84	3.48	0.458E-02	0.974E-02
10	-0.010	-0.010	-0.103	-0.082	2.89	5.00	0.702E-03	0.125E-02
11	-0.010	-0.010	-0.118	-0.043	5.59	4.14	0.107E-02	0.193E-02
12	-0.010	-0.010	-0.125	-0.078	3.19	5.85	0.125E-02	0.233E-02
13	-0.010	-0.010	0.115	0.147	2.82	2.01	0.552E-03	0.170E-02
14	-0.010	-0.010	0.130	0.130	5.05	4.39	0.129E-02	0.263E-02
15	-0.010	-0.010	0.057	0.023	4.98	5.54	0.142E-03	0.193E-03
16	-0.010	-0.010	-0.166	0.114	5.58	5.28	0.335E-02	0.644E-02
17	-0.010	-0.010	-0.040	0.139	2.05	3.28	0.132E-05	0.265E-04
18	-0.010	-0.010	-0.168	0.194	3.39	3.24	0.641E-03	0.296E-02
19	-0.010	-0.010	-0.027	0.115	5.56	2.10	0.119E-05	0.733E-04
20	-0.010	-0.010	0.034	0.006	4.25	3.58	0.105E-04	0.322E-04
21	-0.010	-0.010	0.116	-0.065	4.40	4.95	0.102E-02	0.182E-02
22	-0.010	-0.010	-0.023	0.051	4.27	5.32	0.662E-04	0.900E-03
23	-0.010	-0.010	-0.063	-0.001	5.37	5.00	0.207E-03	0.274E-03
24	-0.010	-0.010	-0.030	-0.147	2.15	3.09	0.123E-05	0.115E-03
25	-0.010	-0.010	0.020	-0.065	3.54	4.47	0.373E-04	0.177E-02
26	-0.010	-0.010	-0.097	0.032	3.19	4.95	0.623E-03	0.101E-02
27	-0.010	-0.010	-0.058	0.180	5.16	3.63	0.250E-04	0.201E-03
28	-0.010	-0.010	0.133	0.065	3.05	5.04	0.147E-02	0.281E-02
29	-0.010	-0.010	-0.080	-0.162	4.83	4.07	0.197E-03	0.590E-03
30	-0.010	-0.010	-0.117	-0.120	4.92	4.56	0.959E-03	0.193E-02
31	-0.010	-0.010	-0.173	0.055	5.83	4.55	0.355E-02	0.738E-02
32	-0.010	-0.010	-0.099	-0.175	5.73	3.93	0.449E-03	0.111E-02
33	-0.010	-0.010	-0.170	-0.059	5.33	5.72	0.332E-02	0.687E-02
34	-0.010	-0.010	-0.156	-0.110	2.92	5.00	0.250E-02	0.511E-02
35	-0.010	-0.010	0.195	-0.107	3.34	4.82	0.516E-02	0.112E-01
36	-0.010	-0.010	0.144	0.068	6.00	4.36	0.194E-02	0.380E-02
37	-0.010	-0.010	-0.156	0.093	2.65	4.60	0.246E-02	0.503E-02
38	-0.010	-0.010	0.100	-0.092	3.88	2.31	0.490E-03	0.114E-02
39	-0.010	-0.010	-0.079	0.075	5.94	4.18	0.308E-03	0.536E-03
40	-0.010	-0.010	-0.073	-0.002	2.60	5.73	0.312E-03	0.441E-03
41	-0.010	-0.010	0.077	0.133	4.96	4.84	0.474E-03	0.848E-03
42	-0.010	-0.010	0.013	0.005	2.41	5.93	0.672E-02	0.153E-01
43	-0.010	-0.010	0.150	-0.060	4.77	2.51	0.214E-02	0.434E-02
44	-0.010	-0.010	0.088	-0.194	2.34	2.22	0.110E-03	0.754E-03
45	-0.010	-0.010	0.086	0.174	5.03	3.62	0.227E-03	0.706E-03
46	-0.010	-0.010	0.098	-0.073	3.39	5.24	0.620E-03	0.106E-02
47	-0.010	-0.010	-0.007	-0.163	4.80	2.32	0.359E-05	0.427E-03
48	-0.010	-0.010	0.034	-0.121	5.38	3.06	0.590E-06	0.268E-04
49	-0.010	-0.010	0.138	-0.062	4.12	5.88	0.168E-02	0.323E-02
50	-0.010	-0.010	-0.019	-0.159	5.45	3.55	0.455E-05	0.996E-03

Table 5.1.5 Simulation Table for Linear Model versus Second Degree Reduced Model

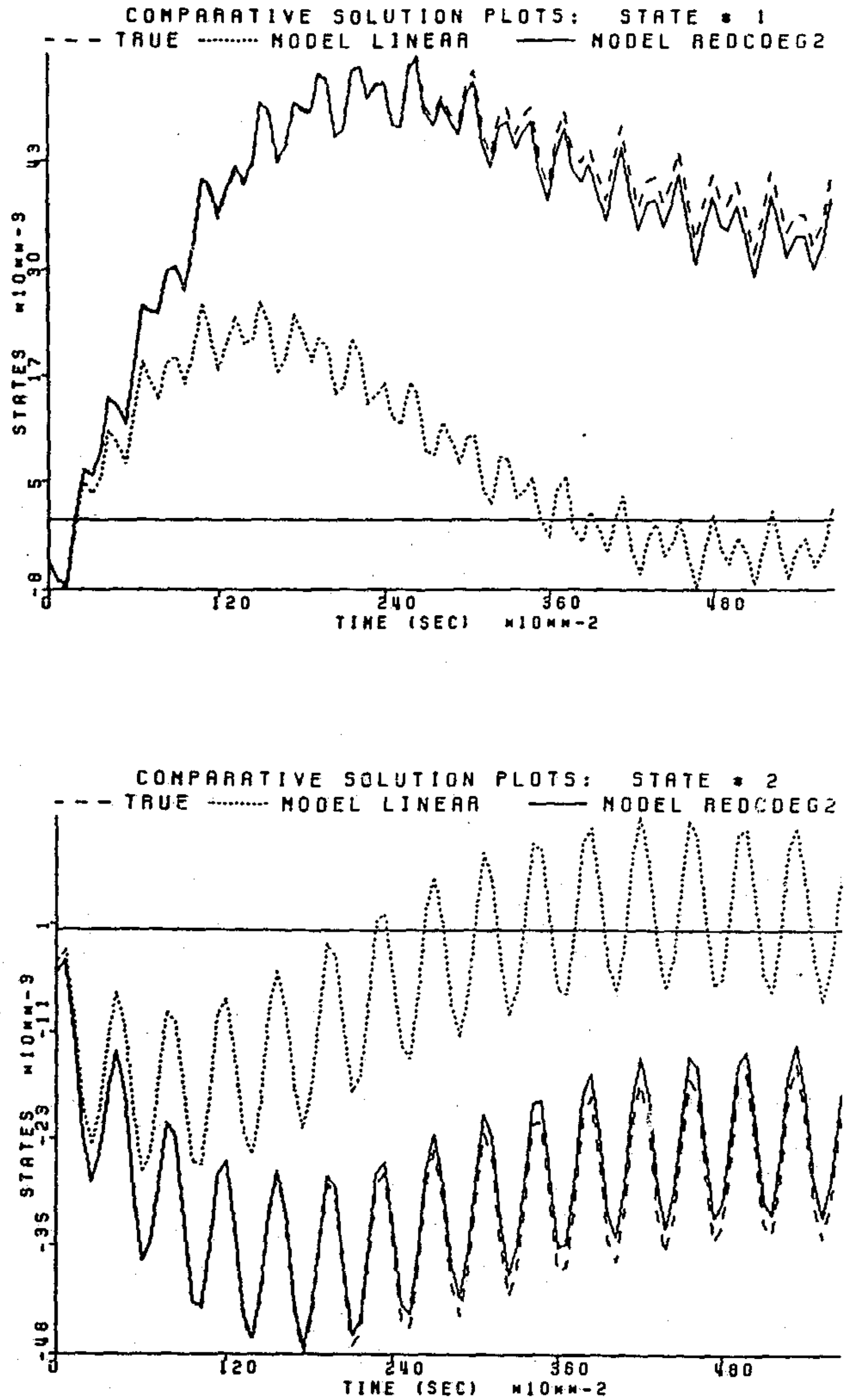
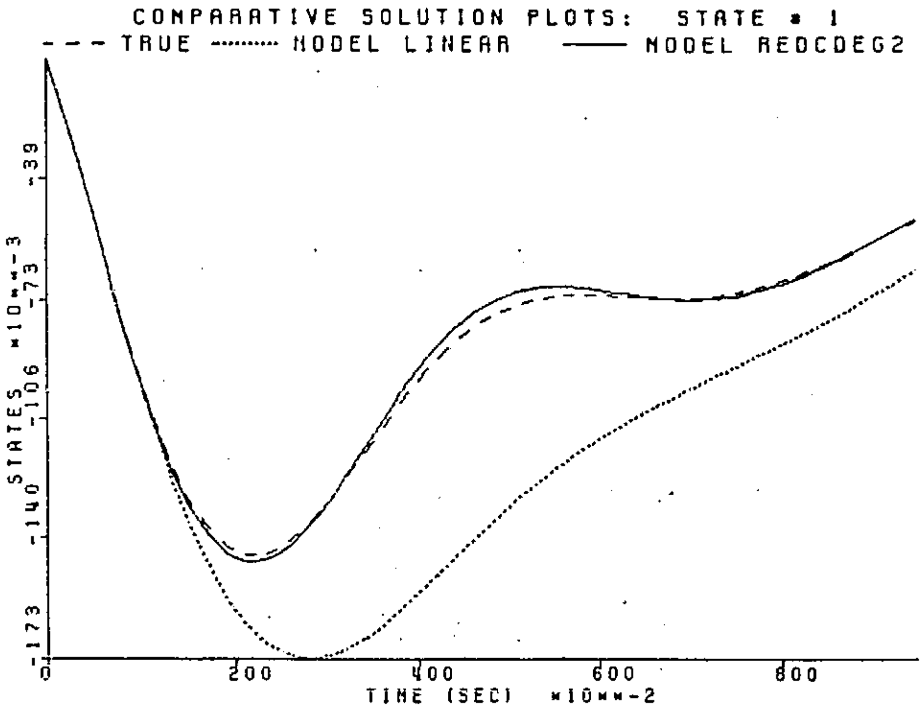


Figure 5.1.8 Simulation Number 37 of Table 5.1.5

```
**************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
       # OF TERMS IN MODEL 1:   6
     DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.827E-04	0.769E-04
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.754E-04	0.695E-04
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.545E-04	0.527E-04
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.218E-04	0.180E-04
5	0.001	-0.001	-0.050	0.050	0.00	0.00	0.683E-02	0.594E-02
6	0.001	-0.001	-0.050	0.050	0.01	0.01	0.644E-02	0.548E-02
7	0.001	-0.001	-0.050	0.050	0.02	0.02	0.543E-02	0.486E-02
8	0.001	-0.001	-0.050	0.050	0.05	0.05	0.197E-02	0.165E-02
9	0.001	-0.001	-0.075	0.010	0.00	0.00	0.359E-01	0.459E-01
10	0.001	-0.001	-0.075	0.010	0.01	0.01	0.394E-01	0.487E-01
11	0.001	-0.001	-0.075	0.010	0.02	0.02	0.564E-01	0.663E-01
12	0.001	-0.001	-0.075	0.010	0.05	0.05	0.412E-01	0.216E-01
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.103E-03	0.115E-03
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.109E-03	0.111E-03
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.141E-03	0.124E-03
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.129E-03	0.181E-03
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.796E-04	0.747E-04
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.724E-04	0.673E-04
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.521E-04	0.507E-04
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.213E-04	0.177E-04
21	0.010	-0.010	-0.050	0.050	0.00	0.00	0.709E-02	0.608E-02
22	0.010	-0.010	-0.050	0.050	0.01	0.01	0.669E-02	0.563E-02
23	0.010	-0.010	-0.050	0.050	0.02	0.02	0.564E-02	0.500E-02
24	0.010	-0.010	-0.050	0.050	0.05	0.05	0.201E-02	0.167E-02
25	0.010	-0.010	-0.075	0.010	0.00	0.00	0.379E-01	0.474E-01
26	0.010	-0.010	-0.075	0.010	0.01	0.01	0.416E-01	0.506E-01
27	0.010	-0.010	-0.075	0.010	0.02	0.02	0.599E-01	0.699E-01
28	0.010	-0.010	-0.075	0.010	0.05	0.05	0.434E-01	0.220E-01
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.103E-03	0.114E-03
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.109E-03	0.109E-03
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.138E-03	0.121E-03
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.116E-03	0.173E-03

Table 5.1.6 Low Frequency Table for Linear Model versus Second Degree Reduced Model



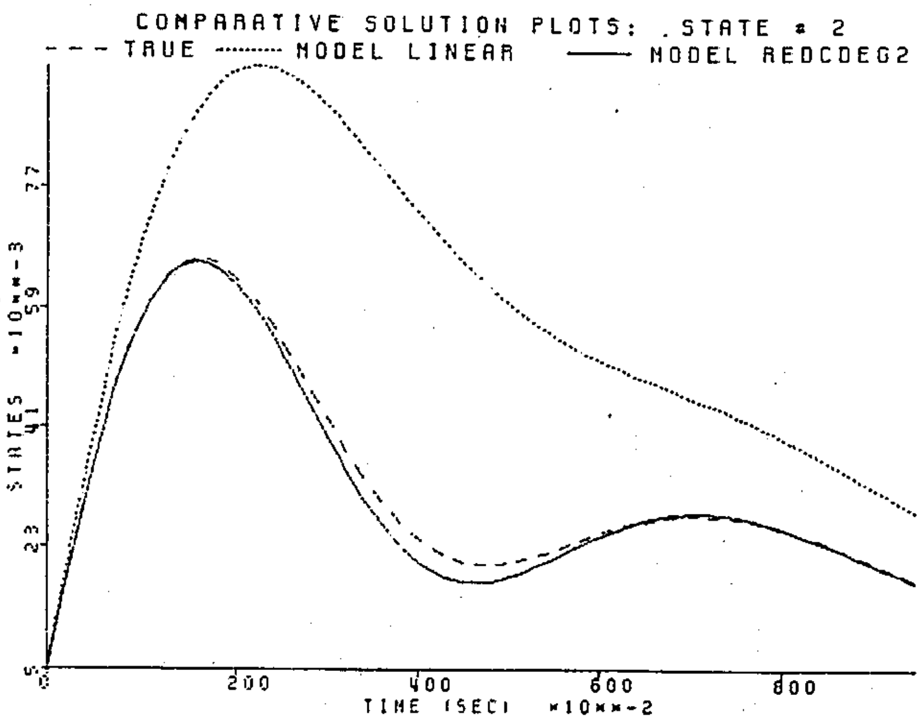


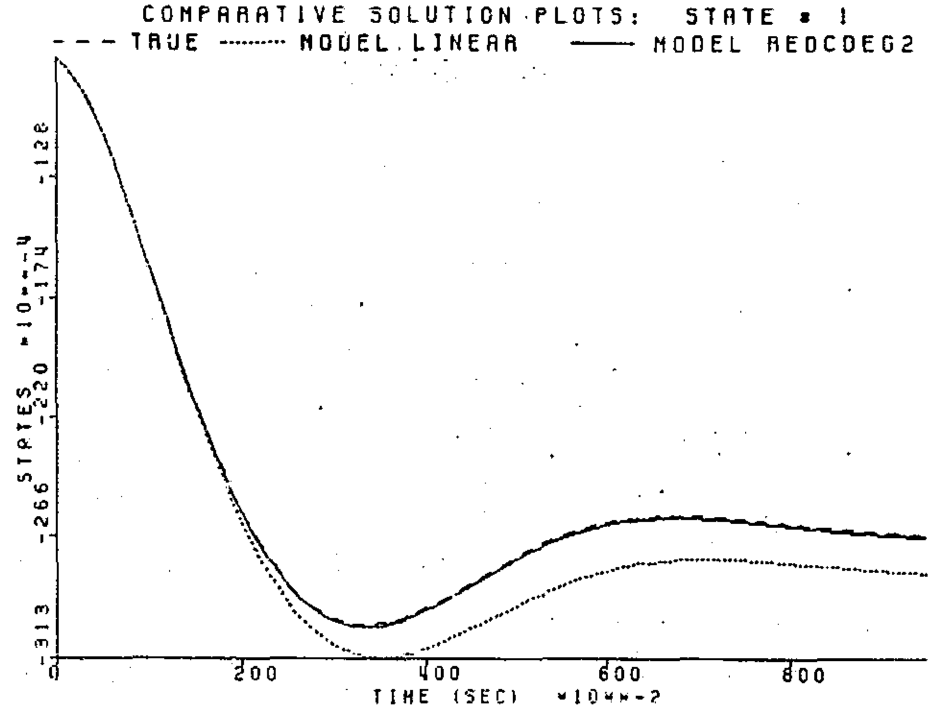
Figure 5.1.9 Simulation Number 15 of Table 5.1.6

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.003	0.010	0.010	-0.021	0.00	0.00	0.506E-05	0.488E-05
2	0.001	-0.002	-0.006	-0.015	0.00	0.00	0.144E-05	0.112E-05
3	0.008	0.003	0.002	0.023	0.00	0.00	0.207E-06	0.123E-06
4	0.010	-0.003	0.015	-0.022	0.00	0.00	0.465E-04	0.464E-04
5	-0.009	0.002	0.010	0.003	0.00	0.00	0.987E-04	0.687E-04
6	0.008	0.002	-0.006	-0.013	0.00	0.00	0.336E-05	0.249E-05
7	0.005	0.005	0.012	-0.002	0.00	0.00	0.677E-05	0.252E-05
8	0.007	0.009	0.016	0.011	0.00	0.00	0.441E-05	0.158E-05
9	-0.005	-0.004	-0.002	0.002	0.00	0.00	0.242E-04	0.177E-04
10	0.009	-0.000	-0.008	-0.018	0.00	0.00	0.224E-05	0.166E-05
11	0.006	-0.005	-0.012	0.011	0.00	0.00	0.240E-05	0.892E-06
12	-0.007	-0.003	-0.004	-0.014	0.00	0.00	0.294E-06	0.229E-06
13	0.000	-0.001	-0.008	-0.007	0.00	0.00	0.153E-04	0.110E-04
14	-0.010	0.008	-0.015	0.003	0.00	0.00	0.431E-03	0.603E-04
15	0.009	-0.005	-0.002	0.004	0.00	0.00	0.856E-05	0.615E-05
16	-0.004	-0.001	-0.001	0.006	0.00	0.00	0.151E-05	0.105E-05
17	0.001	-0.007	-0.005	-0.008	0.00	0.00	0.318E-05	0.251E-05
18	0.001	0.003	0.009	0.008	0.00	0.00	0.127E-04	0.103E-04
19	0.005	-0.004	0.009	-0.024	0.00	0.00	0.296E-05	0.291E-05
20	0.009	0.009	0.002	0.016	0.00	0.00	0.167E-06	0.891E-07
21	-0.008	0.003	0.006	-0.003	0.00	0.00	0.262E-04	0.213E-04
22	-0.009	0.008	0.021	0.023	0.00	0.00	0.155E-05	0.550E-06
23	-0.003	-0.007	-0.002	-0.021	0.00	0.00	0.110E-06	0.635E-07
24	0.008	-0.002	-0.011	0.016	0.00	0.00	0.983E-06	0.911E-06
25	0.003	0.000	-0.023	0.001	0.00	0.00	0.877E-03	0.130E-02
26	0.000	-0.005	0.014	-0.024	0.00	0.00	0.413E-04	0.407E-04
27	-0.004	0.008	0.014	-0.024	0.00	0.00	0.518E-04	0.505E-04
28	0.006	0.008	-0.011	0.024	0.00	0.00	0.416E-05	0.472E-05
29	-0.006	-0.003	0.016	0.014	0.00	0.00	0.363E-05	0.14[illegible]E-05
30	0.001	0.009	-0.019	0.010	0.00	0.00	0.104E-03	0.119E-03
31	0.003	0.008	-0.025	0.022	0.00	0.00	0.343E-03	0.348E-03
32	-0.010	0.004	0.009	-0.024	0.00	0.00	0.446E-05	0.427E-05
33	-0.002	0.002	0.020	0.021	0.00	0.00	0.161E-05	0.394E-06
34	-0.009	0.006	-0.021	-0.009	0.00	0.00	0.922E-05	0.173E-05
35	-0.008	0.004	-0.011	-0.005	0.00	0.00	0.435E-04	0.261E-04
36	0.005	-0.008	-0.017	0.003	0.00	0.00	0.280E-03	0.224E-03
37	-0.006	0.008	0.007	0.006	0.00	0.00	0.167E-04	0.139E-04
38	-0.008	0.008	-0.017	0.018	0.00	0.00	0.313E-04	0.365E-04
39	0.000	0.004	-0.016	0.018	0.00	0.00	0.225E-04	0.257E-04
40	-0.005	0.006	0.019	0.012	0.00	0.00	0.663E-05	0.686E-05
41	-0.008	-0.004	0.020	0.017	0.00	0.00	0.532E-05	0.545E-05
42	-0.007	-0.009	0.014	-0.012	0.00	0.00	0.320E-04	0.325E-04
43	-0.006	-0.001	0.022	0.006	0.00	0.00	0.332E-02	0.381E-02
44	-0.008	0.003	-0.017	-0.012	0.00	0.00	0.980E-05	0.342E-05
45	-0.005	0.005	0.007	-0.007	0.00	0.00	0.417E-05	0.314E-05
46	-0.004	0.002	-0.005	0.003	0.00	0.00	0.666E-04	0.483E-04
47	-0.004	-0.005	0.020	-0.022	0.00	0.00	0.281E-03	0.286E-03
48	0.003	-0.009	0.015	-0.004	0.00	0.00	0.694E-04	0.788E-04
49	-0.007	-0.010	-0.001	0.011	0.00	0.00	0.251E-05	0.163E-05
50	0.007	-0.003	-0.006	0.001	0.00	0.00	0.430E-03	0.303E-03

Table 5.1.7 Step Response Table for Linear Model versus Second Degree Reduced Model

ORIGINAL PAGE IS
OF POOR QUALITY



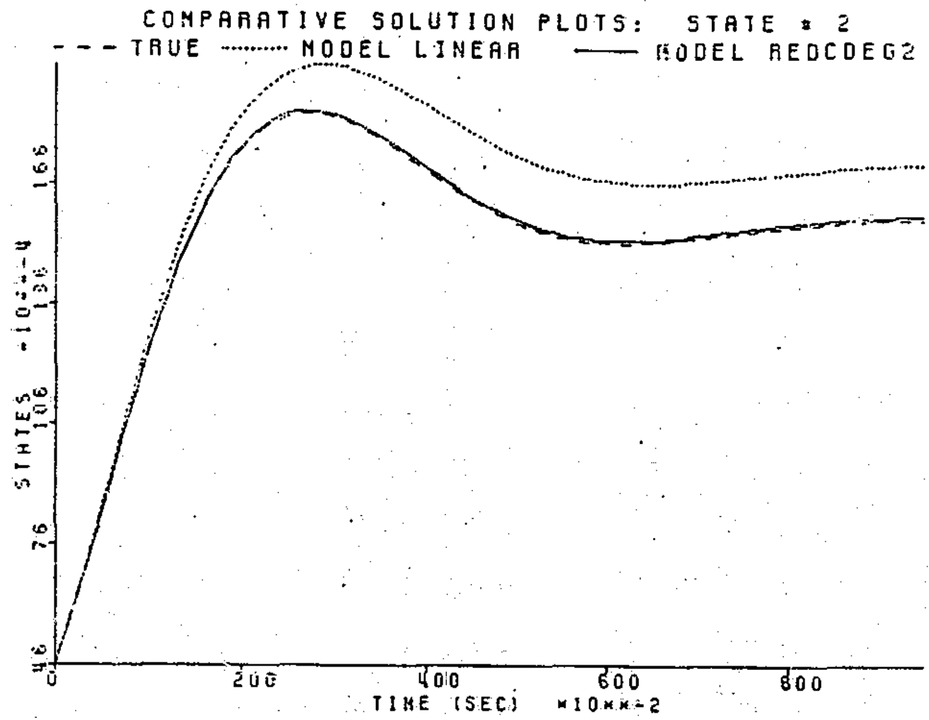


Figure 5.1.10 Simulation Number 35 of Table 5.1.7

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.022	0.031	-0.062	-0.034	0.00	0.00	0.451E-03	0.520E-03
2	0.010	0.011	0.005	-0.050	0.00	0.00	0.617E-05	0.561E-05
3	-0.044	-0.004	0.055	0.063	0.00	0.00	0.339E-02	0.426E-02
4	0.015	-0.006	-0.007	0.042	0.00	0.00	0.350E-05	0.344E-05
5	0.019	0.042	0.038	-0.049	0.00	0.00	0.581E-02	0.636E-02
6	0.023	-0.029	0.068	0.027	0.00	0.00	0.260	0.346
7	0.048	0.035	-0.027	-0.052	0.00	0.00	0.391E-04	[illegible]
8	-0.006	0.016	0.024	-0.027	0.00	0.00	0.724E-03	[illegible]
9	0.026	-0.043	0.049	0.044	0.00	0.00	0.622E-02	0.735E-02
10	0.018	-0.016	0.054	0.071	0.00	0.00	0.648E-03	0.943E-03
11	0.029	0.042	-0.018	0.066	0.00	0.00	0.240E-03	0.203E-03
12	0.011	-0.001	0.049	-0.064	0.00	0.00	0.223E-01	0.262E-01
13	-0.027	0.022	-0.003	-0.045	0.00	0.00	0.400E-06	0.269E-06
14	-0.028	0.030	0.033	0.029	0.00	0.00	0.190E-03	0.245E-03
15	-0.025	-0.002	0.068	0.056	0.00	0.00	0.143	0.240
16	0.006	0.010	-0.015	-0.014	0.00	0.00	0.862E-05	0.405E-05
17	0.011	-0.001	-0.036	-0.057	0.00	0.00	0.357E-05	0.451E-06
18	0.011	-0.005	0.070	0.004	0.00	0.00	0.140	0.180
19	0.021	-0.044	0.010	-0.010	0.00	0.00	0.309E-04	0.217E-04
20	-0.026	0.030	-0.013	0.053	0.00	0.00	0.277E-04	0.277E-04
21	0.035	-0.021	0.008	0.059	0.00	0.00	0.356E-06	0.253E-06
22	-0.024	-0.040	-0.018	-0.046	0.00	0.00	0.106E-05	0.654E-06
23	0.047	0.036	-0.022	0.064	0.00	0.00	0.482E-03	0.395E-03
24	-0.027	-0.018	-0.018	0.047	0.00	0.00	0.109E-03	0.972E-04
25	-0.038	0.015	-0.014	0.012	0.00	0.00	0.526E-05	0.588E-05
26	-0.004	0.050	0.070	-0.068	0.00	0.00	0.130	0.184
27	-0.008	0.026	0.050	-0.020	0.00	0.00	0.261E-01	[illegible]
28	-0.036	0.020	-0.064	0.041	0.00	0.00	0.406E-01	0.342E-01
29	-0.000	0.047	0.021	0.070	0.00	0.00	0.279E-06	0.276E-06
30	-0.027	-0.004	-0.033	-0.005	0.00	0.00	0.393E-03	0.529E-03
31	-0.025	0.035	0.047	-0.012	0.00	0.00	0.225E-01	0.266E-01
32	0.029	-0.024	0.062	-0.020	0.00	0.00	0.709E-01	0.885E-01
33	0.041	-0.038	-0.014	-0.053	0.00	0.00	0.179E-06	0.149E-06
34	-0.042	0.018	-0.016	-0.069	0.00	0.00	0.611E-06	0.347E-06
35	0.038	-0.006	0.022	0.063	0.00	0.00	0.124E-05	0.995E-06
36	0.049	-0.015	0.075	0.005	0.00	0.00	0.173	0.227
37	0.012	-0.036	-0.033	0.053	0.00	0.00	0.112E-02	0.950E-03
38	-0.017	-0.006	0.023	-0.033	0.00	0.00	0.591E-03	0.595E-03
39	-0.039	[illegible]	-0.049	0.037	0.00	0.00	0.942E-02	0.787E-02
40	-0.006	0.007	-0.055	0.067	0.00	0.00	0.785E-02	0.673E-02
41	0.022	0.016	0.040	-0.011	0.00	0.00	0.106E-01	0.123E-01
42	0.030	-0.011	0.030	0.016	0.00	0.00	0.936E-03	0.117E-02
43	0.009	-0.018	0.043	-0.026	0.00	0.00	0.120E-01	0.136E-01
44	0.015	0.001	-0.015	-0.009	0.00	0.00	0.135E-04	0.561E-05
45	0.034	0.042	0.063	0.003	0.00	0.00	0.873E-01	0.114
46	0.007	-0.047	0.007	0.032	0.00	0.00	0.290E-05	0.953E-06
47	0.008	-0.038	0.042	0.071	0.00	0.00	0.809E-04	0.762E-04
48	-0.036	0.018	-0.074	0.031	0.00	0.00	0.474	0.514
49	0.026	0.017	0.061	0.006	0.00	0.00	0.920E-01	0.119
50	-0.029	0.038	-0.023	-0.055	0.00	0.00	0.254E-05	0.593E-06

Table 5.1.8 Step Response Table for Linear Model versus Second Degree Reduced Model

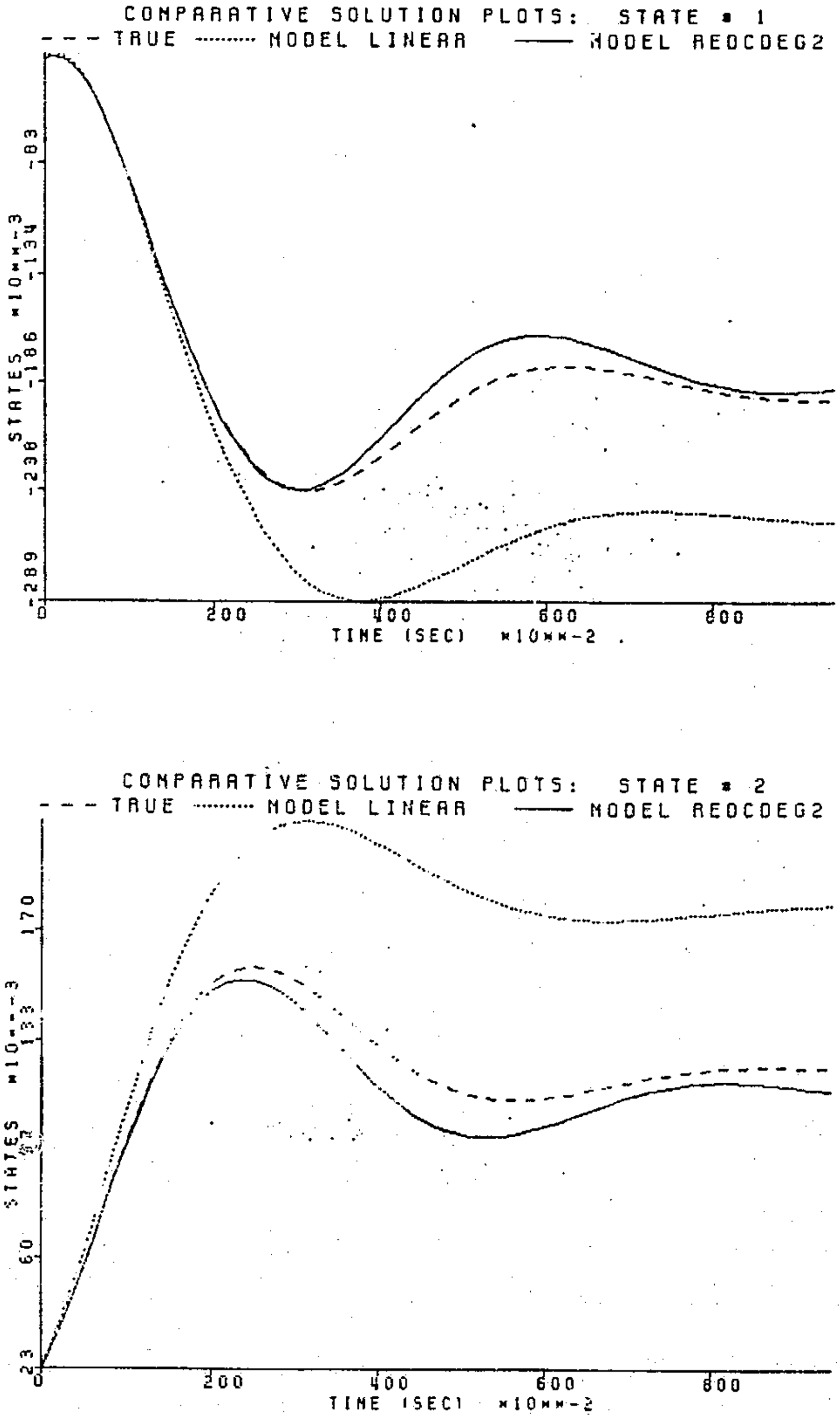


Figure 5.1.11 Simulation Number 48 of Table 5.1.8

From the tables and plots it is obvious that the second degree reduced model performs very well with respect to the linear model. Clearly, the loss of the eight terms in the model has not caused model performance to deteriorate significantly. In order to better check the reduced model's degradation of tracking ability, it is compared against the full second degree model. Tables 5.1.9 through 5.1.13 and the corresponding plots will show that good model behavior is preserved despite the loss of more than half the full model's terms. In each case now, we do not expect to outperform the full model, rather we just hope to approximate it. The error ratio, now

$$R_i = \frac{MSE(MODEL2_i)}{MSE(MODEL1_i)} \quad , \quad i = 1,2,\ldots,n,$$

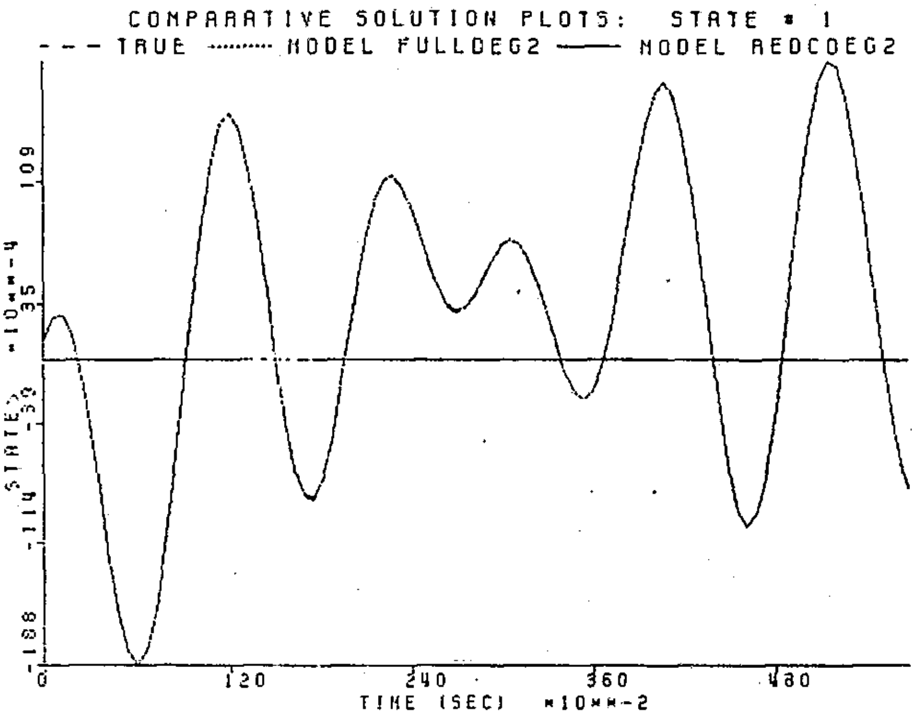
(where MODEL1 and MODEL2 refer to the full and reduced models respectively) is expected to be around unity. This may or may not be true, again depending on the relative closeness of a model's solution to the true solution, so we rely on the graphical data. Table 5.1.9 is a table of simulations close to the origin. The error ratios meet their expectation and the plots in Figures 5.1.12 and 5.1.13 present typical curves. Table 5.1.10 moves the excitation parameters farther away from the origin. Figure 5.1.14 gives a representative set of plots, and Figure 5.1.15 is a blow up of Figure 5.1.14 to give a better look at the reduced model performance. Tables 5.1.11 and 5.1.2 are tables with constant initial state conditions and randomly chosen frequency and amplitude pairs. Table 5.1.11 has amplitudes in the range (-0.05, 0.05), and Table 5.1.12 has amplitudes in the range (-0.25, 0.25). Both have frequencies chosen from the range (2,6). Finally, Table 5.1.13 tests the low frequency and d.c. behavior of the full and reduced second degree models.

From all of the results presented so far it appears the model reduction technique is working quite well as both criteria for judging reduced models

```
****************************************
            PROBLEM SUMMARY
   CONFIGURATION:  TRUE,MODEL1,MODEL2
            # OF STATES: 2
            # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
       DEGREE OF APPROXIMATION: 2
       # OF TERMS IN MODEL 2:   6
       DEGREE OF APPROXIMATION: 2
         SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.713	0.722
2	0.001	0.001	0.050	0.050	0.75	1.00	0.765	0.834
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.278	0.475
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.486	0.641
5	0.001	0.001	-0.050	0.050	0.75	1.00	2.40	2.95
6	0.001	0.001	0.150	0.150	0.75	1.00	0.523	0.686
7	0.001	0.001	0.150	-0.150	0.75	1.00	0.661	0.762
8	0.001	0.001	-0.150	-0.150	0.75	1.00	0.582	0.705
9	0.001	0.001	-0.150	0.150	0.75	1.00	1.15	0.944
10	0.001	-0.001	0.000	0.000	0.75	1.00	1.18	1.26
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.554	0.738
12	0.001	-0.001	0.050	-0.050	0.75	1.00	0.219	0.369
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.246	0.426
14	0.001	-0.001	-0.050	0.050	0.75	1.00	2.52	3.07
15	0.001	-0.001	0.150	0.150	0.75	1.00	0.561	0.706
16	0.001	-0.001	0.150	-0.150	0.75	1.00	0.654	0.754
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	0.608	0.720
18	0.001	-0.001	-0.150	0.150	0.75	1.00	1.11	0.928
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.899	0.910
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.397	0.596
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	0.219	0.369
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.159	0.308
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	3.30	3.71
24	-0.001	-0.001	0.150	0.150	0.75	1.00	0.572	0.711
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	0.667	0.758
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	0.616	0.724
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	1.10	0.926
28	-0.001	0.001	0.000	0.000	0.75	1.00	1.03	1.10
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.605	0.715
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.286	0.489
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.345	0.493
32	-0.001	0.001	-0.050	0.050	0.75	1.00	3.19	3.47
33	-0.001	0.001	0.150	0.150	0.75	1.00	0.536	0.692
34	-0.001	0.001	0.150	-0.150	0.75	1.00	0.675	0.766
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	0.591	0.710
36	-0.001	0.001	-0.150	0.150	0.75	1.00	1.14	0.942

Table 5.1.9 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



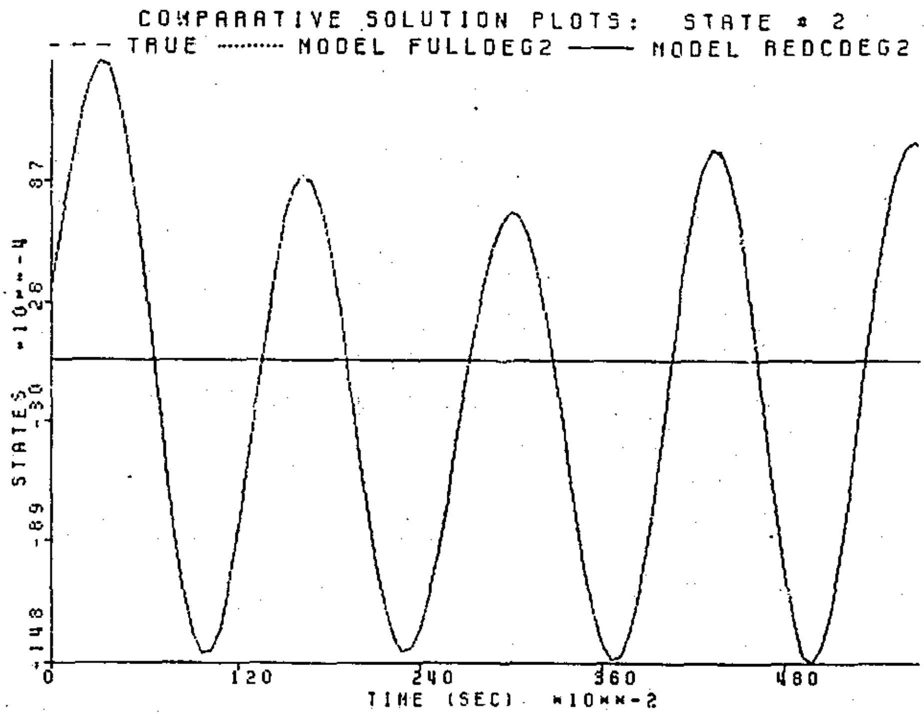
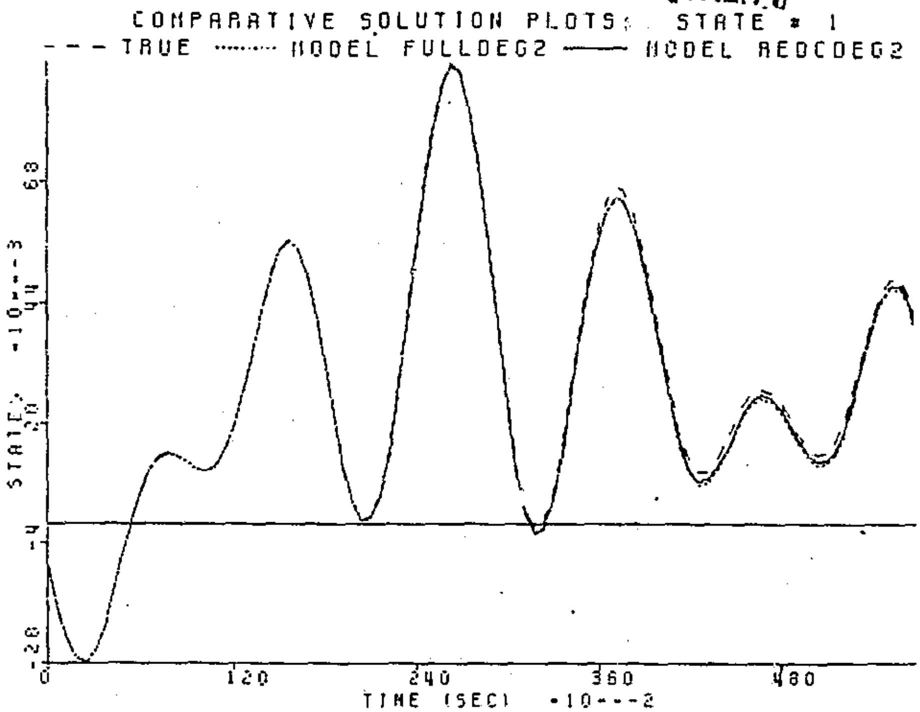


Figure 5.1.12 Simulation Number 32 of Table 5.1.9



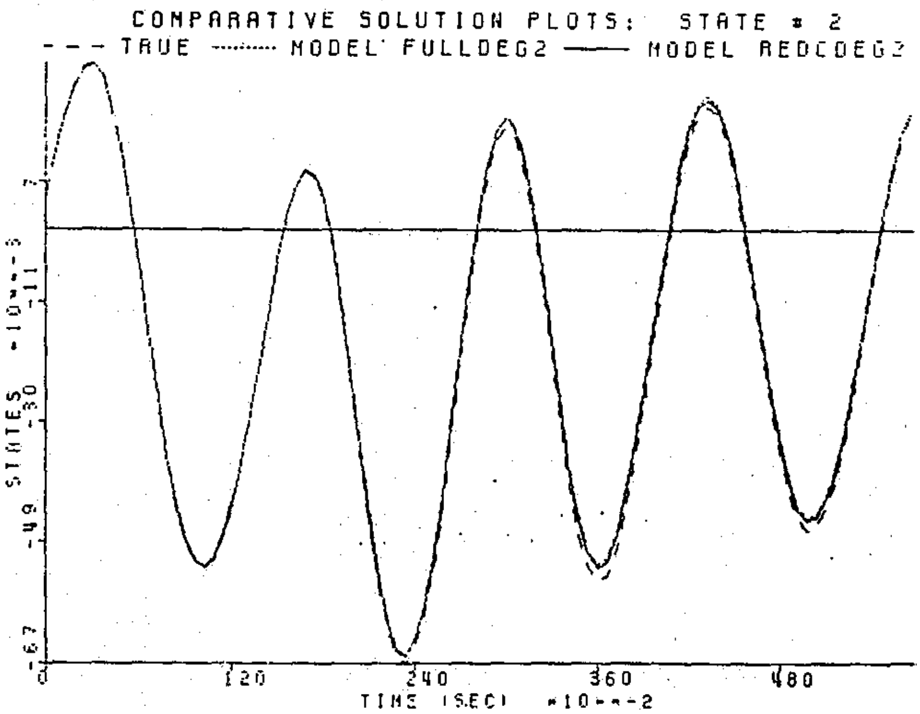
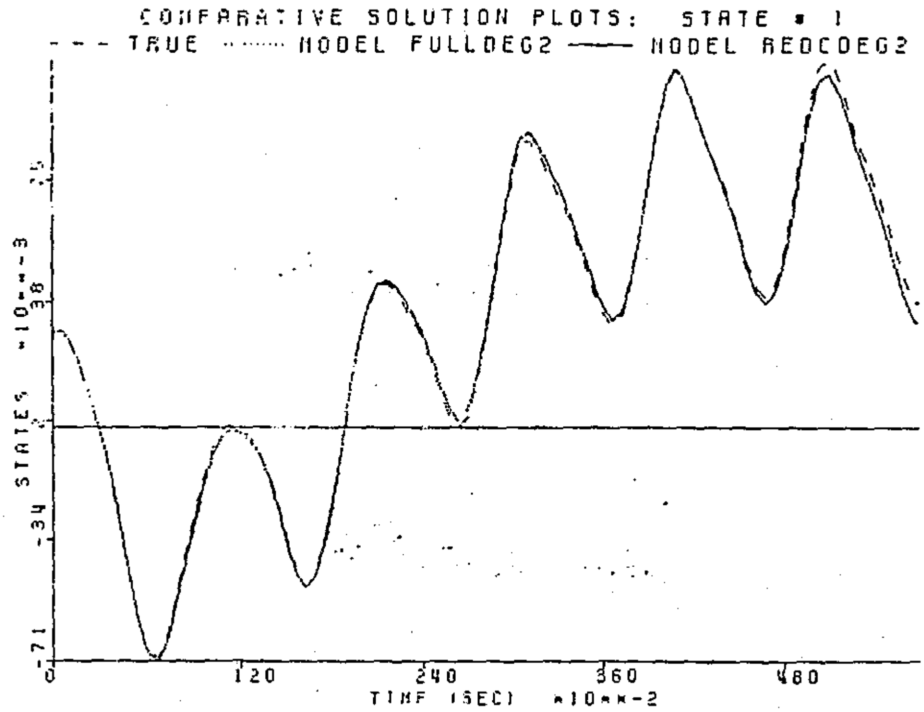


Figure 5.1.13 Simulation Number 26 of Table 5.1.9

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	1.21	1.21
2	0.025	0.025	0.100	-0.100	2.00	1.00	1.18	1.17
3	0.025	0.025	-0.100	0.100	2.00	1.00	1.22	1.22
4	0.025	0.025	-0.100	-0.100	2.00	1.00	1.20	1.22
5	0.025	-0.025	0.100	0.100	2.00	1.00	0.876	0.936
6	0.025	-0.025	0.100	-0.100	2.00	1.00	0.855	0.920
7	0.025	-0.025	-0.100	0.100	2.00	1.00	0.924	0.979
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	0.889	0.939
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.978	0.987
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.924	0.930
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.992	1.00
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	1.00	1.03
13	-0.025	-0.025	0.100	0.100	2.00	1.00	0.717	0.743
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	0.710	0.736
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	0.729	0.756
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	0.724	0.749
17	0.025	0.025	0.200	0.200	2.00	1.00	0.913	0.932
18	0.025	0.025	0.200	-0.200	2.00	1.00	0.900	0.929
19	0.025	0.025	-0.200	0.200	2.00	1.00	0.924	0.940
20	0.025	0.025	-0.200	-0.200	2.00	1.00	0.933	0.951
21	0.025	-0.025	0.200	0.200	2.00	1.00	0.929	0.949
22	0.025	-0.025	0.200	-0.200	2.00	1.00	0.932	0.952
23	0.025	-0.025	-0.200	0.200	2.00	1.00	0.936	0.953
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	0.946	0.959
25	-0.025	0.025	0.200	0.200	2.00	1.00	0.907	0.931
26	-0.025	0.025	0.200	-0.200	2.00	1.00	0.909	0.934
27	-0.025	0.025	-0.200	0.200	2.00	1.00	0.916	0.936
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	0.937	0.950
29	-0.025	-0.025	0.200	0.200	2.00	1.00	0.880	0.907
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	0.880	0.906
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	0.885	0.910
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	0.892	0.914
33	0.025	0.025	0.300	0.300	2.00	1.00	0.892	0.915
34	0.025	0.025	0.300	-0.300	2.00	1.00	0.898	0.918
35	0.025	0.025	-0.300	0.300	2.00	1.00	0.905	0.923
36	0.025	0.025	-0.300	-0.300	2.00	1.00	0.916	0.929
37	0.025	-0.025	0.300	0.300	2.00	1.00	0.907	0.925
38	0.025	-0.025	0.300	-0.300	2.00	1.00	0.911	0.927
39	0.025	-0.025	-0.300	0.300	2.00	1.00	0.915	0.930
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	0.923	0.935
41	-0.025	0.025	0.300	0.300	2.00	1.00	0.892	0.915
42	-0.025	0.025	0.300	-0.300	2.00	1.00	0.896	0.917
43	-0.025	0.025	-0.300	0.300	2.00	1.00	0.902	0.921
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	0.912	0.927
45	-0.025	-0.025	0.300	0.300	2.00	1.00	0.896	0.917
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	0.898	0.918
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	0.901	0.921
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	0.908	0.926

Table 5.1.10 Simulation Tables for Second Degree Full Model versus Second Degree Reduced Model



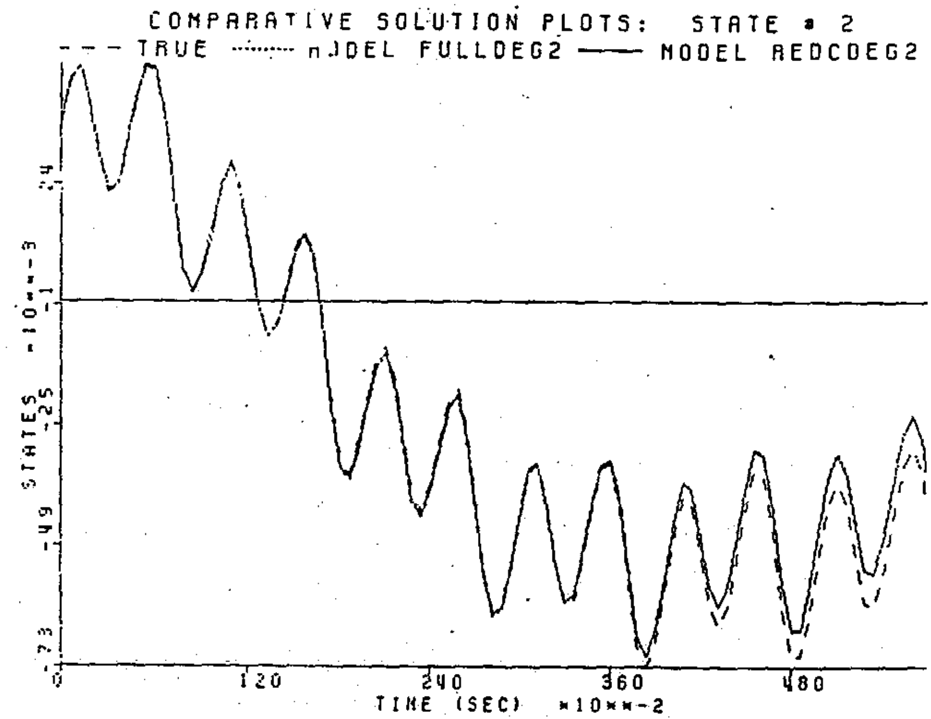
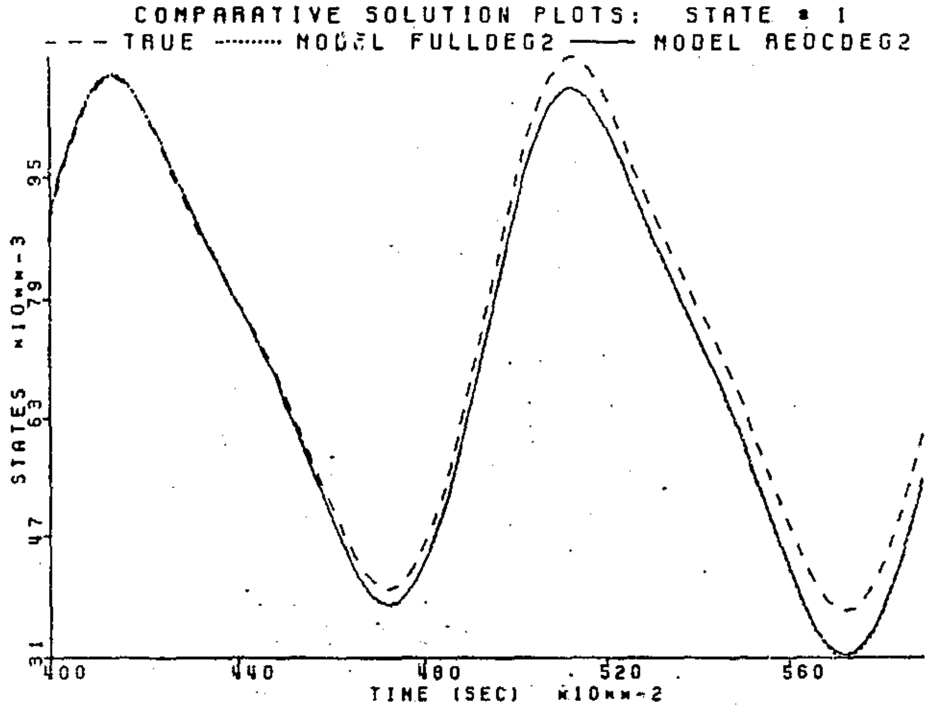


Figure 5.1.14 Simulation Number 19 of Table 5.1.10



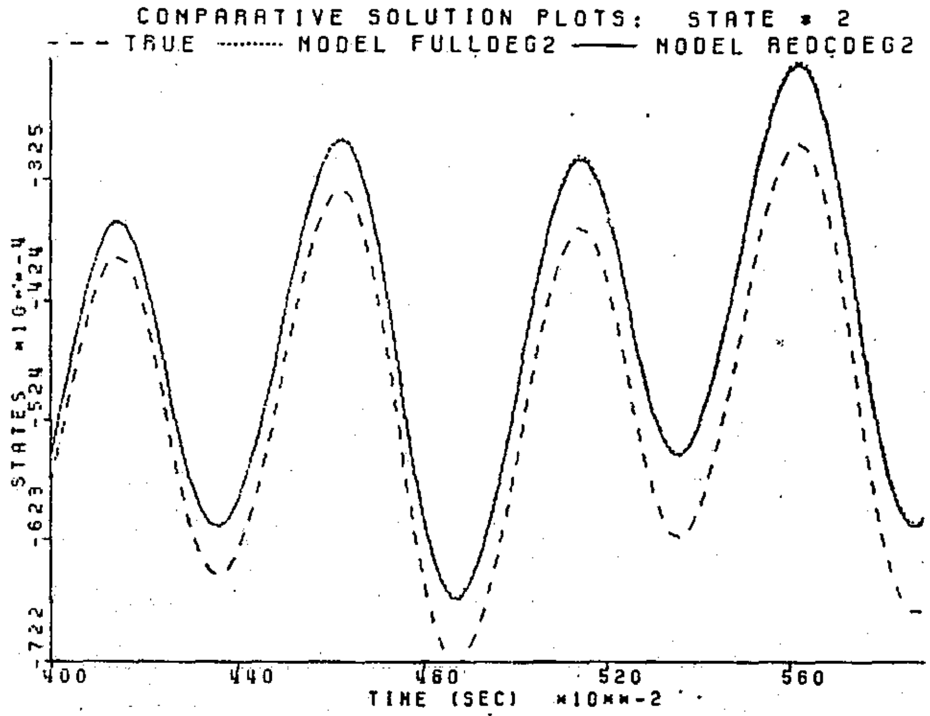
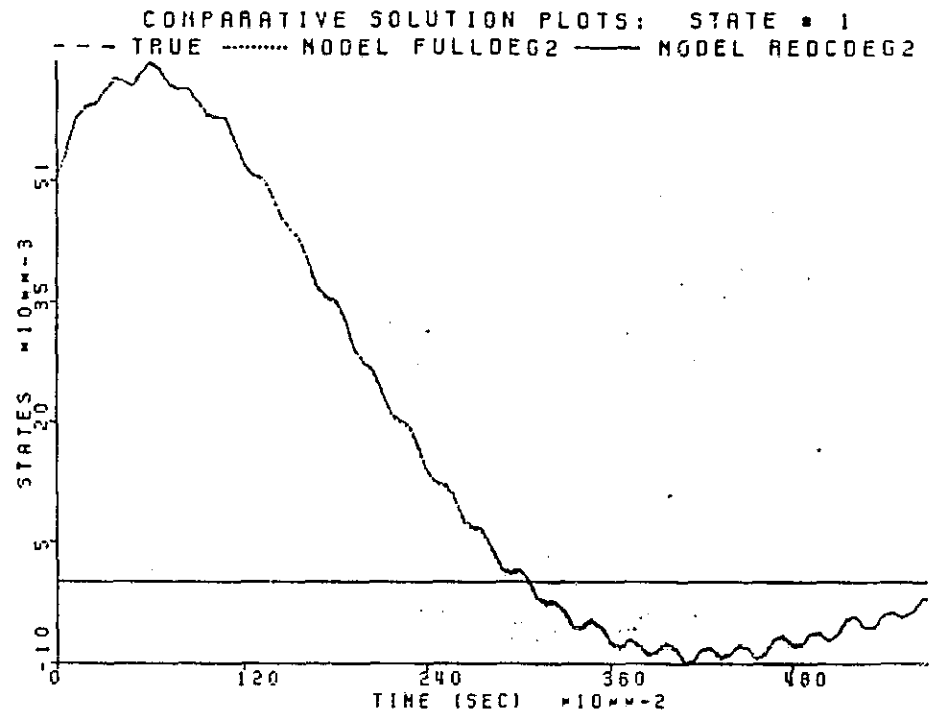


Figure 5.1.15 Expanded Section of Figure 5.1.14

```
**************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.050	-0.050	-0.032	-0.049	5.26	3.68	0.140	0.120
2	0.050	-0.050	0.004	0.046	2.40	4.77	0.320E-01	0.898E-02
3	0.050	-0.050	-0.049	0.018	4.96	3.33	0.325	0.315
4	0.050	-0.050	0.043	-0.045	3.45	2.92	0.238	0.224
5	0.050	-0.050	-0.034	-0.019	4.41	3.60	0.165	0.146
6	0.050	-0.050	0.027	-0.021	2.51	3.71	0.104	0.818E-01
7	0.050	-0.050	0.020	0.028	5.60	2.08	0.647E-01	0.409E-01
8	0.050	-0.050	-0.027	-0.033	5.71	4.73	0.104	0.819E-01
9	0.050	-0.050	-0.017	-0.019	5.59	3.19	0.527E-01	0.284E-01
10	0.050	-0.050	0.024	0.015	2.76	4.47	0.887E-01	0.657E-01
11	0.050	-0.050	0.027	-0.022	2.53	2.19	0.924E-01	0.766E-01
12	0.050	-0.050	0.010	0.011	3.50	3.39	0.374E-01	0.103E-01
13	0.050	-0.050	0.027	-0.025	2.38	4.15	0.103	0.801E-01
14	0.050	-0.050	-0.041	0.024	3.52	2.87	0.227	0.211
15	0.050	-0.050	0.045	0.049	3.51	2.91	0.279	0.267
16	0.050	-0.050	-0.006	0.046	3.72	5.72	0.325E-01	0.934E-02
17	0.050	-0.050	-0.006	0.026	5.79	5.32	0.322E-01	0.898E-02
18	0.050	-0.050	-0.025	-0.027	4.14	5.40	0.942E-01	0.717E-01
19	0.050	-0.050	-0.021	-0.010	2.01	4.65	0.671E-01	0.432E-01
20	0.050	-0.050	-0.043	-0.017	3.74	2.51	0.257	0.243
21	0.050	-0.050	-0.011	-0.022	2.57	3.82	0.387E-01	0.145E-01
22	0.050	-0.050	0.037	0.035	3.59	4.53	0.195	0.179
23	0.050	-0.050	0.002	0.002	3.50	2.78	0.327E-01	0.890E-02
24	0.050	-0.050	0.028	-0.000	2.30	3.67	0.113	0.907E-01
25	0.050	-0.050	0.019	-0.044	3.94	4.38	0.463E-01	0.258E-01
26	0.050	-0.050	0.011	0.041	4.97	2.41	0.378E-01	0.140E-01
27	0.050	-0.050	-0.011	-0.036	2.82	3.91	0.385E-01	0.147E-01
28	0.050	-0.050	-0.044	-0.040	2.59	4.69	0.258	0.244
29	0.050	-0.050	0.020	0.030	5.93	2.33	0.626E-01	0.387E-01
30	0.050	-0.050	-0.012	0.010	5.89	4.09	0.404E-01	0.160E-01
31	0.050	-0.050	0.019	-0.006	2.21	5.51	0.608E-01	0.365E-01
32	0.050	-0.050	-0.039	0.021	4.14	5.40	0.215	0.199
33	0.050	-0.050	-0.038	-0.008	3.60	4.82	0.205	0.188
34	0.050	-0.050	0.040	-0.026	3.98	4.82	0.224	0.207
35	0.050	-0.050	-0.002	-0.008	3.32	5.85	0.325E-01	0.875E-02
36	0.050	-0.050	-0.044	0.026	4.93	2.49	0.264	0.250
37	0.050	-0.050	-0.041	-0.015	3.00	2.84	0.230	0.204
38	0.050	-0.050	-0.023	0.021	3.49	2.71	0.770E-01	0.533E-01
39	0.050	-0.050	0.007	-0.038	3.21	3.53	0.298E-01	0.968E-02
40	0.050	-0.050	0.045	0.000	4.74	3.60	0.276	0.263
41	0.050	-0.050	0.046	-0.007	3.70	4.53	0.293	0.280
42	0.050	-0.050	0.017	0.019	2.83	3.44	0.538E-01	0.297E-01
43	0.050	-0.050	0.045	0.030	4.78	2.79	0.278	0.265
44	0.050	-0.050	0.009	-0.017	5.55	5.42	0.370E-01	0.161E-01
45	0.050	-0.050	-0.020	-0.047	5.28	3.08	0.633E-01	0.396E-01
46	0.050	-0.050	-0.038	0.048	4.37	4.33	0.321	0.308
47	0.050	-0.050	0.033	-0.009	2.36	3.31	0.153	0.134
48	0.050	-0.050	0.015	-0.007	5.22	4.14	0.486E-01	0.240E-01
49	0.050	-0.050	-0.033	0.020	5.43	2.44	0.156	0.136
50	0.050	-0.050	-0.028	0.029	3.16	3.97	0.111	0.889E-01

Table 5.1.11 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



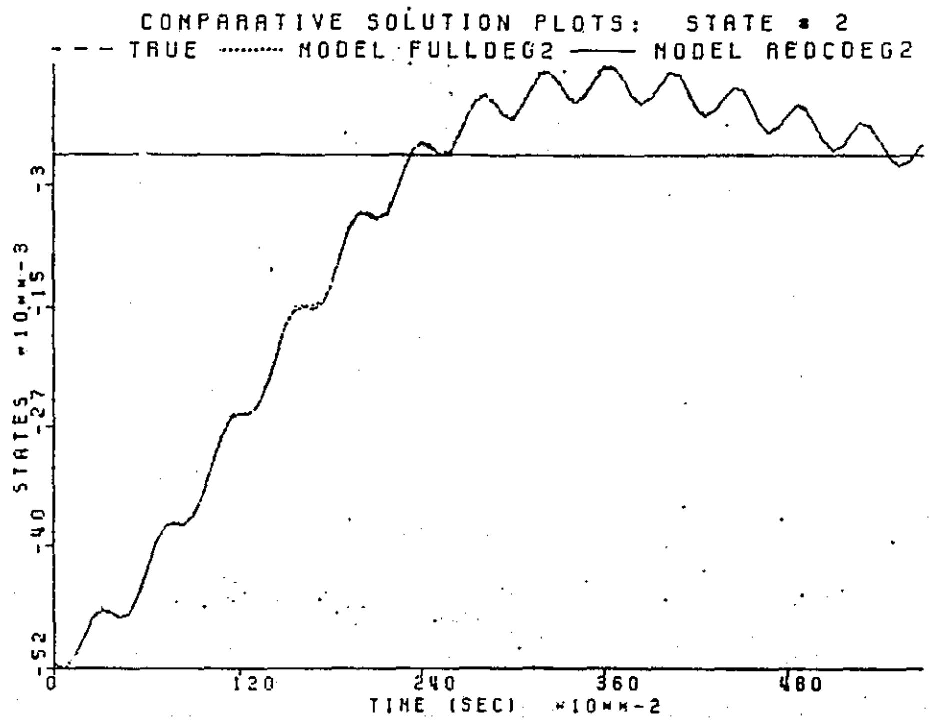
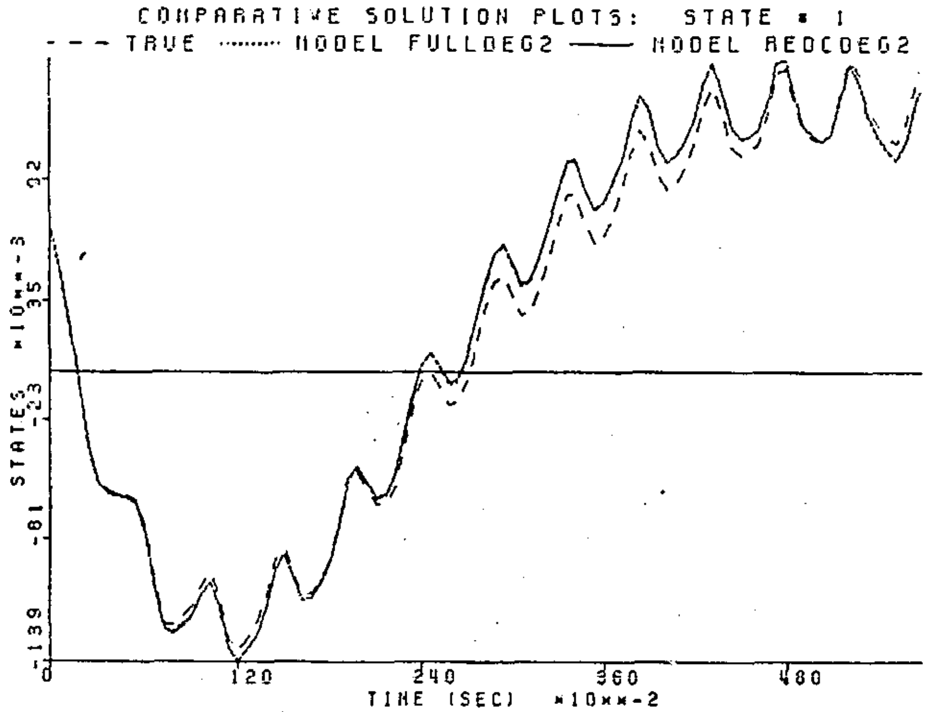


Figure 5.1.16 Simulation Number 13 of Table 5.1.11

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.075	0.075	-0.249	0.243	4.40	2.12	1.03	0.983
2	0.075	0.075	-0.041	0.179	5.70	5.30	2.45	3.91
3	0.075	0.075	-0.147	0.192	3.09	3.27	1.69	1.61
4	0.075	0.075	-0.205	-0.164	2.82	3.50	1.13	1.11
5	0.075	0.075	-0.090	0.082	3.86	5.54	2.03	2.22
6	0.075	0.075	-0.171	0.070	3.41	5.45	1.23	1.24
7	0.075	0.075	-0.155	0.162	4.59	2.50	1.34	1.36
8	0.075	0.075	-0.108	-0.171	2.41	2.67	1.62	1.63
9	0.075	0.075	0.017	0.066	5.82	2.42	3.33	6.46
10	0.075	0.075	-0.186	0.180	4.91	5.17	1.29	1.27
11	0.075	0.075	0.140	0.188	3.55	4.25	1.46	1.49
12	0.075	0.075	-0.035	-0.145	2.02	4.44	3.36	5.78
13	0.075	0.075	-0.055	-0.145	5.69	4.90	3.01	4.06
14	0.075	0.075	-0.024	0.070	4.27	2.63	3.31	6.05
15	0.075	0.075	-0.114	0.154	3.72	5.06	1.70	1.79
16	0.075	0.075	-0.085	0.086	4.57	3.16	2.13	2.37
17	0.075	0.075	0.129	-0.134	3.34	3.57	1.75	1.84
18	0.075	0.075	0.120	-0.122	3.43	5.18	1.61	1.67
19	0.075	0.075	-0.069	0.096	5.64	4.58	2.52	3.03
20	0.075	0.075	0.217	0.223	5.88	2.14	1.11	1.07
21	0.075	0.075	-0.006	-0.234	5.06	3.57	3.58	8.91
22	0.075	0.075	-0.175	0.074	4.13	3.97	1.24	1.22
23	0.075	0.075	0.160	0.179	2.63	4.95	1.33	1.34
24	0.075	0.075	-0.240	0.035	5.84	2.90	1.03	0.997
25	0.075	0.075	-0.117	-0.124	4.33	2.47	1.65	1.72
26	0.075	0.075	-0.179	-0.058	4.63	4.11	1.20	1.20
27	0.075	0.075	-0.091	-0.241	4.40	5.94	2.23	2.52
28	0.075	0.075	0.184	0.223	3.83	4.44	1.22	1.20
29	0.075	0.075	-0.056	0.249	5.96	5.37	2.84	4.15
30	0.075	0.075	0.225	0.106	3.39	4.23	1.07	1.04
31	0.075	0.075	-0.054	0.023	2.73	2.81	2.93	3.66
32	0.075	0.075	0.036	-0.195	3.44	3.88	2.65	4.45
33	0.075	0.075	-0.034	0.220	2.20	5.13	3.52	6.72
34	0.075	0.075	-0.239	-0.235	3.23	2.52	1.06	1.01
35	0.075	0.075	-0.182	-0.139	5.47	2.94	1.20	1.20
36	0.075	0.075	-0.129	-0.062	4.81	3.67	1.50	1.54
37	0.075	0.075	0.131	0.013	4.82	4.70	1.47	1.52
38	0.075	0.075	-0.163	0.170	5.15	2.38	1.29	1.31
39	0.075	0.075	0.158	-0.186	5.27	2.75	1.33	1.35
40	0.075	0.075	-0.031	-0.045	5.50	2.12	3.26	5.50
41	0.075	0.075	-0.193	0.187	3.34	5.53	1.17	1.16
42	0.075	0.075	-0.037	-0.024	3.35	4.16	3.20	5.06
43	0.075	0.075	0.220	-0.098	3.94	4.60	1.08	1.05
44	0.075	0.075	0.144	-0.205	3.16	3.83	1.41	1.46
45	0.075	0.075	0.157	0.196	3.77	4.01	1.22	1.24
46	0.075	0.075	0.126	0.131	4.21	2.40	1.55	1.60
47	0.075	0.075	-0.226	0.020	2.30	5.98	1.07	1.04
48	0.075	0.075	-0.170	-0.087	5.25	4.87	1.25	1.25
49	0.075	0.075	0.216	0.148	2.21	5.99	1.10	1.07
50	0.075	0.075	-0.197	-0.218	4.06	2.30	1.17	1.15

Table 5.1.12 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



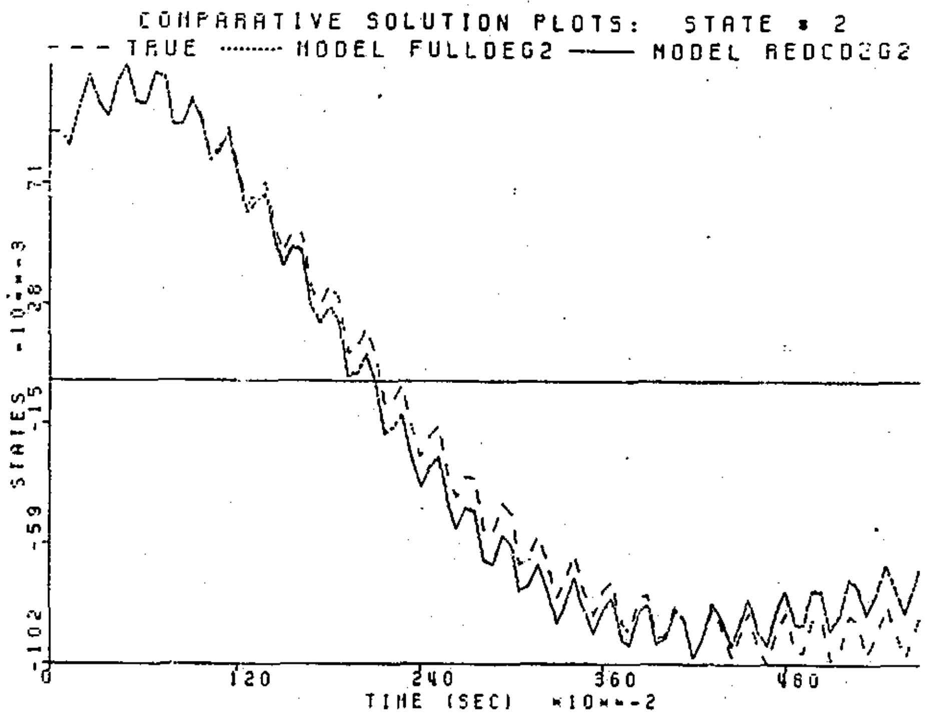


Figure 5.1.17 Simulation Number 1 of Table 5.1.1

```
***************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:   6
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
***************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.312	0.318
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.300	0.307
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.260	0.27[illegible]
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.210	0.232
5	0.001	-0.001	-0.050	0.050	0.00	0.00	1.67	1.74
6	0.001	-0.001	-0.050	0.050	0.01	0.01	1.73	1.80
7	0.001	-0.001	-0.050	0.050	0.02	0.02	1.99	2.03
8	0.001	-0.001	-0.050	0.050	0.05	0.05	3.13	1.52
9	0.001	-0.001	-0.075	0.010	0.00	0.00	1.20	1.23
10	0.001	-0.001	-0.075	0.010	0.01	0.01	1.23	1.26
11	0.001	-0.001	-0.075	0.010	0.02	0.02	1.34	1.35
12	0.001	-0.001	-0.075	0.010	0.05	0.05	1.81	1.07
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.694	0.682
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.713	0.695
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.783	0.754
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.907	0.774
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.310	0.316
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.298	0.305
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.258	0.271
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.208	0.229
21	0.010	-0.010	-0.050	0.050	0.00	0.00	1.67	1.73
22	0.010	-0.010	-0.050	0.050	0.01	0.01	1.72	1.79
23	0.010	-0.010	-0.050	0.050	0.02	0.02	1.98	2.02
24	0.010	-0.010	-0.050	0.050	0.05	0.05	3.08	1.53
25	0.010	-0.010	-0.075	0.010	0.00	0.00	1.21	1.24
26	0.010	-0.010	-0.075	0.010	0.01	0.01	1.23	1.26
27	0.010	-0.010	-0.075	0.010	0.02	0.02	1.34	1.36
28	0.010	-0.010	-0.075	0.010	0.05	0.05	1.83	1.08
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.697	0.683
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.716	0.696
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.787	0.756
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.915	0.772

Table 5.1.13 Low Frequency Table for Second Degree Full Model versus Second Degree Reduced Model

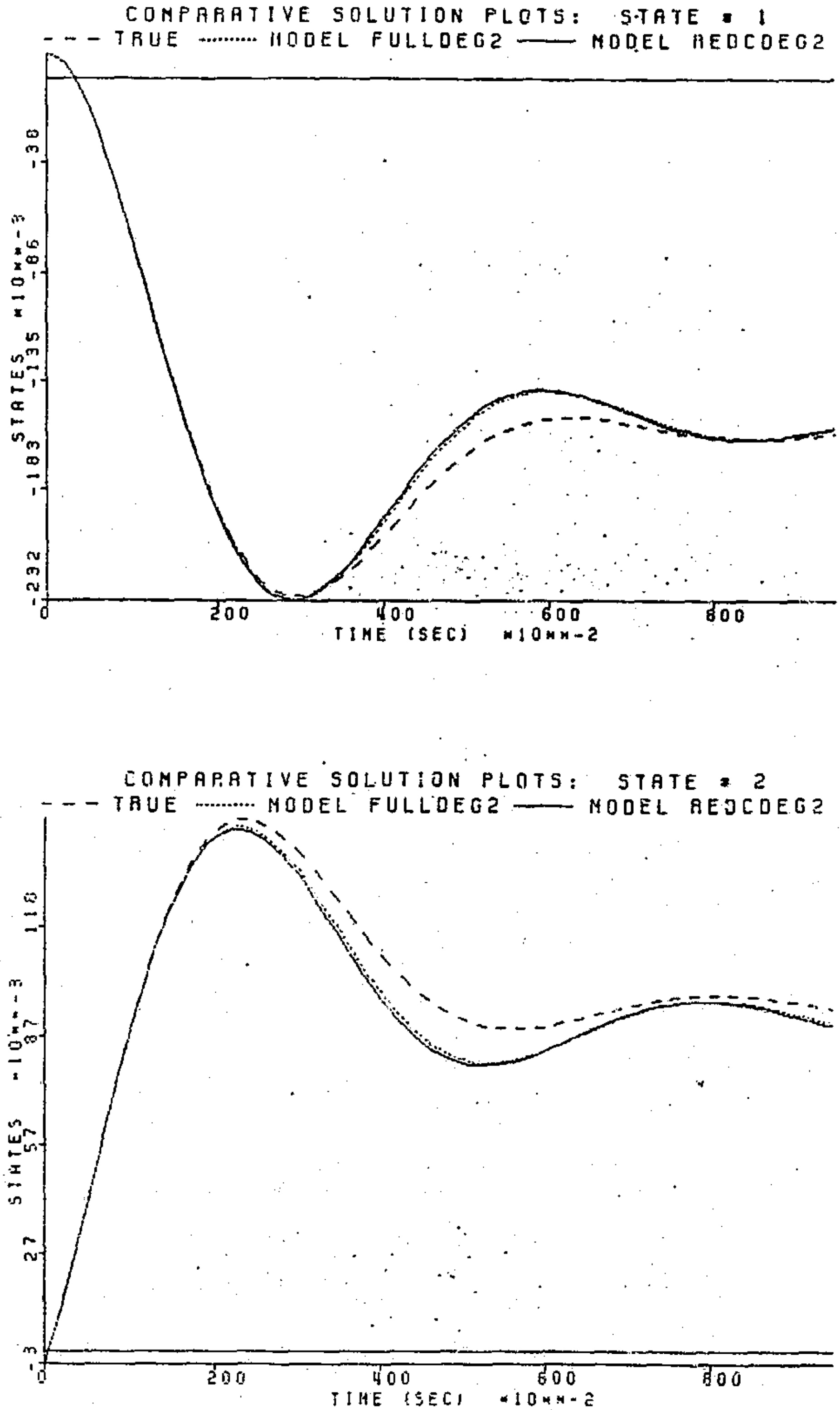


Figure 5.1.18 Simulation Number 26 of Table 5.1.13

were met. But, in this case the full model has only fourteen terms which is not unreasonably large. The true usefulness of the reduction method will appear when used on models with many nonlinear terms. With this in mind, the third degree approximation of this same two state system is studied.

5.1.2 DEGREE THREE APPROXIMATION

In this example, the third degree approximation now has 34 terms corresponding to the products

$$\begin{gathered}x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1x_2,\ x_2x_2,\\ x_1u_1,\ x_1u_2,\ x_2u_1,\ x_2u_2,\ u_1u_1,\ u_1u_2,\\ u_2u_2,\ x_1x_1x_1,\ x_1x_1x_2,\ x_1x_2x_2,\\ x_2x_2x_2,\ x_1x_1u_1,\ x_1x_1u_2,\ x_1x_2u_1,\ x_1x_2u_2,\\ x_2x_2u_1,\ x_2x_2u_2,\ x_1u_1u_1,\ x_1u_1u_2,\ x_1u_2u_2,\\ x_2u_1u_1,\ x_2u_1u_2,\ x_2u_2u_2,\ u_1u_1u_1,\ u_1u_1u_2,\\ u_1u_2u_2,\ u_2u_2u_2.\end{gathered}$$

The appendix contains the locally optimum, third degree model to be used as the starting model for the reduction. See Figure 5.1.19 for the original third degree model and the "first pass" reduction test. Note the model is identified using initial state conditions (0.001, -0.001). It also uses as control input excitations the same sum of two sinusoids, although with different amplitudes and frequencies,

$$u_1(t) = 0.055\sin(2\pi t\cdot\phi_1) + 0.055\sin(2\pi t\cdot 2\phi_1)$$

$$u_2(t) = 0.0325\sin(2\pi t\cdot\phi_2) + 0.0325\sin(2\pi t\cdot 2\phi_2)$$

where $\phi_1 = 0.5090$ and $\phi_2 = 2.0098$.

Directing our attention to the reduction test and the terms with changes in errors of magnitude less than 4, it is found that 18 terms can be omitted. The remaining 16 terms are

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010     -0.0010

ENTER SAMPLING PARAMETERS:
           APPROXIMATION DEGREE
           # SAMPLE POINTS
           INTERVAL BETWEEN SAMPLES
           INTEGRATION STEPSIZE
   3  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2            2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.5090      2.0098

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N


ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
           1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  34 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



TSO SPEAKEASY III PI+  9:57 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
```

Figure 5.1.19a Full Third Degree Model and Reduction Test (First Pass)

```
 -1.9977 -3.0008
  .99967  1.0002

 ...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.49874+.867671 -.49874-.867671


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  .0019891    .99779
 -1.0001     -6.2231E-7


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
 -.19024     -.012365    -.0034771
  .036033     .0022184    1.5017E-4


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0692      .026129     .027909   -.0078805
  .021083     .002591     .0026825   .0035398


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.0039675    .0071601   -.012386
 -.99787      .0010537    7.4681E-4


PARTITION NUMBER 6.
  PRTITION (A 2 BY 4 ARRAY)
  .70548  1.6627  3.3163  .78087
 -.2223  -.6684  -1.2624 -.3395


PARTITION NUMBER 7.
  PRTITION (A 2 BY 6 ARRAY)
  .62148     -.61366    -.25223    -.76455     .44724     -.67544
 -.28909     -.025264   -.05473    -.058202   -.048646   -.0056896


PARTITION NUMBER 8.
  PRTITION (A 2 BY 6 ARRAY)
 -3.5488   -.10235    .41411     .16518    .036896   .073615
 -.17155    .96002   -.044252  -.055923  -.035393  -.013022


PARTITION NUMBER 9.
  PRTITION (A 2 BY 4 ARRAY)
 -.18738     -.032069   -.0251      -.019747
 -.51749     -.016328   -.019048    -.0011802


  S (A 34 COMPONENT ARRAY)
  12.581    8.5987    7.997     6.0242    4.3769    4.1246    3.1931    2.9288
  2.6574    2.5454    2.1057    1.8416    1.5116    1.4722    1.3897    1.3138
  1.1668    .99995    .92334    .82698    .76732    .71888    .5978     .53451
  .40492    .35004    .32218    .20171    .18144    .11826    .023934   .045849
  .084429   .063016
 THE MAXIMUM SINGULAR VALUE:
MAX =  12.581
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .023934

 ...AND THEIR RATIO:
RTIO =  525.67
```

Figure 5.1.19b Full Third Degree Model and Reduction Test (First Pass)

```
WANT TO TRY REDUCTION TEST?[Y/N]: Y

FULL SYSTEM ERROR:     208.5798
FULL STATE ERRORS:     152.0251        56.5547

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     273.2820
CHANGE IN STATE ERRORS:     226.0011         47.2809

TERM:          X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:     -18.8934
CHANGE IN STATE ERRORS:     -16.9980         -1.8954

TERM:          U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:     -25.4317
CHANGE IN STATE ERRORS:       0.0450        -25.4768

TERM:          U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:      14.0826
CHANGE IN STATE ERRORS:      14.0826         -0.0000

TERM:       X1.X1.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:      12.2573
CHANGE IN STATE ERRORS:      11.6348          0.6225

TERM:       X1.X2.  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:      -0.9152
CHANGE IN STATE ERRORS:      -0.8697         -0.0455

TERM:       X2.X2.  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:       0.2715
CHANGE IN STATE ERRORS:       0.2671          0.0044

TERM:       X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:      48.0619
CHANGE IN STATE ERRORS:      48.0063          0.0556

TERM:       X1.U2.  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:      -0.0250
CHANGE IN STATE ERRORS:      -0.0373          0.0123

TERM:       X2.U1.  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:      -0.3615
CHANGE IN STATE ERRORS:      -0.3495         -0.0120

TERM:       X2.U2.  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:      -0.0061
CHANGE IN STATE ERRORS:      -0.0176          0.0114

TERM:       U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -30.3638
CHANGE IN STATE ERRORS:      -0.0348        -30.3290

TERM:       U1.U2.  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:      -0.0164
CHANGE IN STATE ERRORS:      -0.0045         -0.0119

TERM:       U2.U2.  COLUMN #:  14.
CHANGE IN SYSTEM ERROR:       0.3772
CHANGE IN STATE ERRORS:       0.3599          0.0173

TERM:    X1.X1.X1.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:     -29.9081
CHANGE IN STATE ERRORS:     -27.9561         -1.9520
```

Figure 5.1.19c Full Third Degree Model and Reduction Test (First Pass)

```
TERM:    X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:     136.7215
CHANGE IN STATE ERRORS:     122.6539       14.0676

TERM:    X1.X2.X2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:     -77.4448
CHANGE IN STATE ERRORS:     -75.6628       -1.7820

TERM:    X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:      79.0131
CHANGE IN STATE ERRORS:      69.5179        9.4952

TERM:    X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:       7.9639
CHANGE IN STATE ERRORS:       7.8924        0.0715

TERM:    X1.X1.U2.  COLUMN #:  20.
CHANGE IN SYSTEM ERROR:       1.3980
CHANGE IN STATE ERRORS:       1.5089       -0.1108

TERM:    X1.X2.U1.  COLUMN #:  21.
CHANGE IN SYSTEM ERROR:       1.7702
CHANGE IN STATE ERRORS:       1.6826        0.0876

TERM:    X1.X2.U2.  COLUMN #:  22.
CHANGE IN SYSTEM ERROR:       2.6437
CHANGE IN STATE ERRORS:       2.5047        0.1390

TERM:    X2.X2.U1.  COLUMN #:  23.
CHANGE IN SYSTEM ERROR:       2.3302
CHANGE IN STATE ERRORS:       2.7091       -0.3789

TERM:    X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:       5.6993
CHANGE IN STATE ERRORS:       5.7065       -0.0071

TERM:    X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:      57.3052
CHANGE IN STATE ERRORS:      58.0436       -0.7384

TERM:    X1.U1.U2.  COLUMN #:  26.
CHANGE IN SYSTEM ERROR:      -2.3094
CHANGE IN STATE ERRORS:       0.0114       -2.3208

TERM:    X1.U2.U2.  COLUMN #:  27.
CHANGE IN SYSTEM ERROR:      -8.9582
CHANGE IN STATE ERRORS:      -8.6700       -0.2882

TERM:    X2.U1.U1.  COLUMN #:  28.
CHANGE IN SYSTEM ERROR:       1.1869
CHANGE IN STATE ERRORS:       0.7543        0.4325

TERM:    X2.U1.U2.  COLUMN #:  29.
CHANGE IN SYSTEM ERROR:      -0.0037
CHANGE IN STATE ERRORS:       0.2052       -0.2090

TERM:    X2.U2.U2.  COLUMN #:  30.
CHANGE IN SYSTEM ERROR:       2.7262
CHANGE IN STATE ERRORS:       2.5737        0.1524

TERM:    U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:     -20.7670
CHANGE IN STATE ERRORS:      -2.2527      -18.5142

TERM:    U1.U1.U2.  COLUMN #:  32.
CHANGE IN SYSTEM ERROR:       0.2494
CHANGE IN STATE ERRORS:       0.2183        0.0311
```

Figure 5.1.19d Full Third Degree Model and Reduction Test (First Pass)

```
TERM:   U1.U2.U2.  COLUMN #:  33.
CHANGE IN SYSTEM ERROR:      -0.3957
CHANGE IN STATE ERRORS:      -0.2063       -0.1894

TERM:   U2.U2.U2.  COLUMN #:  34.
CHANGE IN SYSTEM ERROR:       0.2591
CHANGE IN STATE ERRORS:       0.2608       -0.0016

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 16

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(10:1,2,3,4,5,8,12,15,16,17)

A = ARRAY(6:18,19,24,25,27,31)

MANUAL MODE
 SPACE USED  113 K NOW,  138 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.19e Full Third Degree Model and Reduction Test (First Pass)

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1x_1,\ x_1u_1,\ u_1u_1,$$
$$x_1x_1x_1,\ x_1x_1x_2,\ x_1x_2x_2,\ x_2x_2x_2,$$
$$x_1x_1u_1,\ x_2x_2u_2,\ x_1u_1u_1,\ x_1u_2u_2,$$
$$u_1u_1u_1.$$

These terms are kept and a new local optimum is found. Optimization over frequencies yielded $\phi_1 = 0.3950$ and $\phi_2 = 1.3506$. See Figure 5.1.20 for the "second pass". Note that the IMSL optimization routine ended with a "terminal error". This error (IER = 130) is an IMSL provided error check to insure that roundoff error does not become excessive. Note further that a FORTRAN error message summary is provided. In this instance error 208 was encountered 34 times. Error 208 is an underflow error and indicates that an internal variable was less than 10^{-78}. The standard fixup is to set that variable to zero, which is perfectly acceptable.

The "second pass" reduction test indicates that another 3 terms are fairly insignificant. This brings the total number of terms to 13. They are

$$x_1,\ x_2,\ u_1,\ u_2,\ x_1u_1,\ u_1u_1,$$
$$x_1x_1x_1,\ x_1x_1x_2,\ x_2x_2x_2,\ x_1x_1u_1,$$
$$x_2x_2u_2,\ x_1u_1u_1,\ u_1u_1u_1.$$

Again, only these 13 terms are kept and a locally optimum frequency set is determined. The model is identified and a reduction test performed. (See Figure 5.1.21 for this "third pass".) The reduction test indicates that term 19 ($x_1x_1u_1$) could be removed, but after execution of the "fourth pass", the model simulations showed that the 12 term model was not acceptable. So the 13 term model is the final model, and the final reduction is 21 terms removed out of a possible 34; a 62% reduction.

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010      -0.0010

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   3  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2          2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.5090        2.0098

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2     200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.5253147D+00  0.2009603D+01
THE CONDITION NUMBER:  0.152D+10
OBJECTIVE FUNCTION VALUE: 0.5532D+04

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.5254612D+00  0.2008699D+01
THE CONDITION NUMBER:  0.152D+10
OBJECTIVE FUNCTION VALUE: 0.5516D+04

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.5199821D+00  0.1989839D+01
THE CONDITION NUMBER:  0.145D+10
OBJECTIVE FUNCTION VALUE: 0.5316D+04
```

Figure 5.1.20a Reduced Third Degree Model and Reduction Test (Second Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.5039734D+00  0.1890305D+01
THE CONDITION NUMBER:   0.123D+10
OBJECTIVE FUNCTION VALUE: 0.4721D+04

 50 CALLS OF FCN.....
THE PARAMETERS:
 0.4076332D+00  0.1414816D+01
THE CONDITION NUMBER:   0.470D+09
OBJECTIVE FUNCTION VALUE: 0.2133D+04
*** TERMINAL ERROR -        (IER = 130) FROM IMSL ROUTINE ZXMIN

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  16 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:       52

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.395      FREQ(2)=  1.351      FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



 MESSAGE SUMMARY: MESSAGE NUMBER - COUNT
                              208      34

TSO SPEAKEASY III PI+ 10:59 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 16 COMPONENT ARRAY)
  1   2   3   4   5   8   12  15  16  17  18  19  24  25  27  31


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -1.9999 -3.0002
  1.0011  .99978

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50005+.86841i -.50005-.86841i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.5116E-4   .99902
 -.99844     -.0013877


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
 -.028226  0         0
 -.093347  0         0
```

Figure 5.1.20b Reduced Third Degree Model and Reduction Test (Second Pass)

```
PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0367   0          0          0
 -.062668  0          0          0


PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -.0022205  0          0
 -1.0093    0          0


PARTITION NUMBER 6.
  PRTITION (A 2 BY 4 ARRAY)
 -.91373  .39023  .40494 -.40803
  .48991 -1.1593 -.11776  .55633


PARTITION NUMBER 7.
  PRTITION (A 2 BY 6 ARRAY)
  .11824   0          0          0          0          .049393
  .58555   0          0          0          0          .62134


PARTITION NUMBER 8.
  PRTITION (A 2 BY 6 ARRAY)
 -3.9208   0          .028512    0          0          0
  .48114   0          .12065     0          0          0


PARTITION NUMBER 9.
  PRTITION (A 2 BY 4 ARRAY)
 -.058225  0          0          0
 -.57913   0          0          0


  S (A 16 COMPONENT ARRAY)
  11.449    7.1423    6.0617    5.0361    2.7425    2.088     1.9426    1.3796
  1.1688    1.1436    .8219     .56891    .31614    .17437    .084745   .034909
 THE MAXIMUM SINGULAR VALUE:
MAX =  11.449
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .034909

 ...AND THEIR RATIO:
RTIO =  327.96


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:     138.4610
FULL STATE ERRORS:      70.3943      68.0666

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:    105.5239
CHANGE IN STATE ERRORS:    109.8157       -4.2918

TERM:          X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:    234.7848
CHANGE IN STATE ERRORS:    194.0638       40.7210

TERM:          U1.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:    -41.7820
CHANGE IN STATE ERRORS:     -0.0060      -41.7761

TERM:          U2.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:    -16.5398
```

Figure 5.1.20c Reduced Third Degree Model and Reduction Test (Second Pass)

```
CHANGE IN STATE ERRORS:    -16.5602        0.0204

TERM:       X1.X1.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:       3.0913
CHANGE IN STATE ERRORS:       0.0624       3.0289

TERM:       X1.U1.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:     113.9193
CHANGE IN STATE ERRORS:     114.6507      -0.7314

TERM:       U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -19.9546
CHANGE IN STATE ERRORS:      -0.0270     -19.9276

TERM:    X1.X1.X1.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:       4.4767
CHANGE IN STATE ERRORS:      15.1272     -10.6505

TERM:    X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:      -2.5463
CHANGE IN STATE ERRORS:       5.6283      -8.1746

TERM:    X1.X2.X2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:      -1.4812
CHANGE IN STATE ERRORS:      -3.6453       2.1641

TERM:    X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:       2.7722
CHANGE IN STATE ERRORS:      -7.2330      10.0051

TERM:    X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:       7.1812
CHANGE IN STATE ERRORS:      -0.6173       7.7985

TERM:    X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:      -6.0369
CHANGE IN STATE ERRORS:      -0.9311      -5.1057

TERM:    X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:       9.4809
CHANGE IN STATE ERRORS:      11.9710      -2.4901

TERM:    X1.U2.U2.  COLUMN #:  27.
CHANGE IN SYSTEM ERROR:      -1.2075
CHANGE IN STATE ERRORS:       0.1620      -1.3695

TERM:    U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:     -24.7955
CHANGE IN STATE ERRORS:       0.7590     -25.5545

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 13

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(1,2,3,4,8,12,15,16,18,19)

A = ARRAY(24,25,31)

MANUAL MODE
 SPACE USED  70 K NOW,  81 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.20d Reduced Third Degree Model and Reduction Test (Second Pass)

```
ENTER INITIAL CONDITIONS FOR THE 2 STATES:
     0.0010     -0.0010

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   3  100     0.0400     0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2            2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.55000000000000002E-01  0.32500000000000002E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     0.3950       1.3506

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y


ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
           3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2     200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.3986263D+00  0.1343124D+01
THE CONDITION NUMBER:  0.400D+08
OBJECTIVE FUNCTION VALUE: 0.1685D+03

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.3991838D+00  0.1352194D+01
THE CONDITION NUMBER:  0.397D+08
OBJECTIVE FUNCTION VALUE: 0.1681D+03

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.3991642D+00  0.1352155D+01
THE CONDITION NUMBER:  0.397D+08
OBJECTIVE FUNCTION VALUE: 0.1681D+03
```

Figure 5.1.21a Reduced Third Degree Model and Reduction Test (Third Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.3991505D+00  0.1352110D+01
THE CONDITION NUMBER:  0.397D+08
OBJECTIVE FUNCTION VALUE: 0.1681D+03

CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  13 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        48 .

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.399     FREQ(2)=  1.352     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



 MESSAGE SUMMARY: MESSAGE NUMBER - COUNT
                     208         55

TSO SPEAKEASY III PI+ 11:30 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 13 COMPONENT ARRAY)
  1   2   3   4   8   12  15  16  18  19  24  25  31


PARTITION NUMBER 1.
  PRTITION (A 2 BY 2 ARRAY)
 -1.9999 -2.9999
  .99957  .99945

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 2 COMPONENTS)
 -.50021+.86582i -.50021-.86582i


PARTITION NUMBER 2.
  PRTITION (A 2 BY 2 ARRAY)
  6.3827E-4  .99898
 -.99849    -.0014354


PARTITION NUMBER 3.
  PRTITION (A 2 BY 3 ARRAY)
  0  0  0
  0  0  0


PARTITION NUMBER 4.
  PRTITION (A 2 BY 4 ARRAY)
 -4.0226    0         0         0
 -.026954   0         0         0
```

Figure 5.1.21b Reduced Third Degree Model and Reduction Test (Third Pass)

C - 2

```
PARTITION NUMBER 5.
  PRTITION (A 2 BY 3 ARRAY)
 -6.5226E-4  0          0
 -1.0026     0          0


PARTITION NUMBER 6.
  PRTITION (A 2 BY 4 ARRAY)
 -1.2411   .15993  0      -.5407
 -.26768 -.91146   0       .77552


PARTITION NUMBER 7.
  PRTITION (A 2 BY 6 ARRAY)
 -.063216  0        0        0        0        .073678
  .20884   0        0        0        0        .68143


PARTITION NUMBER 8.
  PRTITION (A 2 BY 6 ARRAY)
 -4.0074  0        0        0        0        0
  .27416  0        0        0        0        0


PARTITION NUMBER 9.
  PRTITION (A 2 BY 4 ARRAY)
 -.078589  0        0        0
 -.66691   0        0        0


   S (A 13 COMPONENT ARRAY)
   9.8648  7.1115  6.0803  4.9378  2.4718  1.9875  1.3949  1.2823  1.1637
   .95869  .56438  .26112  .16092
  THE MAXIMUM SINGULAR VALUE:
 MAX =   9.8648
  THE MINIMUM NONZERO SINGULAR VALUE:
 MIN =   .16092

  ...AND THEIR RATIO:
 RTIO =  61.303


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:      133.9531
FULL STATE ERRORS:       70.3439        63.6092

TERM:             X1.  COLUMN #:    1.
CHANGE IN SYSTEM ERROR:     121.0994
CHANGE IN STATE ERRORS:      85.4754        35.6240

TERM:             X2.  COLUMN #:    2.
CHANGE IN SYSTEM ERROR:     222.9544
CHANGE IN STATE ERRORS:     220.0404         2.9140

TERM:             U1.  COLUMN #:    3.
CHANGE IN SYSTEM ERROR:     -37.8873
CHANGE IN STATE ERRORS:      -0.0055       -37.8818

TERM:             U2.  COLUMN #:    4.
CHANGE IN SYSTEM ERROR:     -15.6390
CHANGE IN STATE ERRORS:     -15.6634         0.0243

TERM:          X1.U1.  COLUMN #:    8.
CHANGE IN SYSTEM ERROR:     123.9281
```

Figure 5.1.21c Reduced Third Degree Model and Reduction Test (Third Pass)

```
CHANGE IN STATE ERRORS:     124.1291        -0.2010

TERM:      U1.U1.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:     -27.5552
CHANGE IN STATE ERRORS:      -0.0115       -27.5438

TERM:   X1.X1.X1.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:      15.9879
CHANGE IN STATE ERRORS:      15.2055         0.7824

TERM:   X1.X1.X2.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:      16.6559
CHANGE IN STATE ERRORS:       0.5196        16.1362

TERM:   X2.X2.X2.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:      -8.7379
CHANGE IN STATE ERRORS:      -4.3799        -4.3580

TERM:   X1.X1.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:       1.9149
CHANGE IN STATE ERRORS:       0.4854         1.4295

TERM:   X2.X2.U2.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:      -5.8982
CHANGE IN STATE ERRORS:      -1.3748        -4.5234

TERM:   X1.U1.U1.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:       7.7140
CHANGE IN STATE ERRORS:       7.7567        -0.0427

TERM:   U1.U1.U1.  COLUMN #:  31.
CHANGE IN SYSTEM ERROR:     -25.8595
CHANGE IN STATE ERRORS:       0.9593       -26.8188

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N

MANUAL MODE
 SPACE USED  62 K NOW,  71 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.1.21d Reduced Third Degree Model and Reduction Test (Third Pass)

Now all that remains is simulation to bear out the model's validity. First we test against the full second degree model and the error ratio for each state is the mean square of the reduced third degree over the mean square error of the full second degree. In each of the following tables MODEL1 is the second degree model and MODEL2 is the reduced third degree. The first two tables (Table 5.1.14 and Table 5.1.15) and their corresponding plots display the behavior when excitation parameters are chosen close to the origin. The following three tables stretch the initial conditions and amplitudes farther from the origin. From this set of data, it is clear that the reduced third degree model outperforms the full second degree. Table 5.1.19 is a table including randomly chosen amplitudes and frequencies. The corresponding figures (Figures 5.1.29 and 5.1.30) show a worst case and a typical plot of model behavior. The final two tables show the low frequency and d.c. behavior of the two models. Again a definite improvement is observed.

Now, comparison against the full degree three model is in order. The error ratios for each state are now the mean square error of the reduced model over the mean square error of the full third degree approximation. The first table (Table 5.1.22) and its corresponding set of plots (Figure 5.1.34) show the typical comparison of model behavior near the origin. The next two tables move the control amplitudes farther out. Throughout these tables, the reduced model behavior is quite good and sometimes even better than that of the full model, but the reduced model went unstable as the amplitudes reached ±0.6, while the full model remained stable and did well (see the appendix). Table 5.1.25 is another random table with amplitudes in the range (-0.4, 0.4) and frequencies in the range (2,6). Table 5.1.26 is a low frequency table and Table 5.1.27 is a table of random steps. In these cases, we note that re-

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.148E-01	0.133E-01
2	0.001	0.001	0.050	0.050	0.75	1.00	0.226	0.394
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.197	0.419
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.279	0.639
5	0.001	0.001	-0.050	0.050	0.75	1.00	0.614	1.21
6	0.001	0.001	0.150	0.150	0.75	1.00	0.149	0.947E-01
7	0.001	0.001	0.150	-0.150	0.75	1.00	0.205	0.114
8	0.001	0.001	-0.150	-0.150	0.75	1.00	0.141	0.879E-01
9	0.001	0.001	-0.150	0.150	0.75	1.00	0.181	0.105
10	0.001	-0.001	0.000	0.000	0.75	1.00	0.149E-01	0.107E-01
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.321	0.599
12	0.001	-0.001	0.050	-0.050	0.75	1.00	0.208	0.415
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.385	0.886
14	0.001	-0.001	-0.050	0.050	0.75	1.00	0.705	1.57
15	0.001	-0.001	0.150	0.150	0.75	1.00	0.144	0.929E-01
16	0.001	-0.001	0.150	-0.150	0.75	1.00	0.185	0.108
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	0.134	0.852E-01
18	0.001	-0.001	-0.150	0.150	0.75	1.00	0.168	0.101
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.187E-01	0.168E-01
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.323	0.647
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	0.238	0.472
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.401	0.907
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	0.850	1.90
24	-0.001	-0.001	0.150	0.150	0.75	1.00	0.141	0.919E-01
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	0.180	0.105
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	0.132	0.844E-01
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	0.161	0.992E-01
28	-0.001	0.001	0.000	0.000	0.75	1.00	0.131E-01	0.943E-02
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.229	0.428
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.222	0.475
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.319	0.737
32	-0.001	0.001	-0.050	0.050	0.75	1.00	0.753	1.42
33	-0.001	0.001	0.150	0.150	0.75	1.00	0.147	0.939E-01
34	-0.001	0.001	0.150	-0.150	0.75	1.00	0.200	0.111
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	0.139	0.871E-01
36	-0.001	0.001	-0.150	0.150	0.75	1.00	0.173	0.103

Table 5.1.14 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

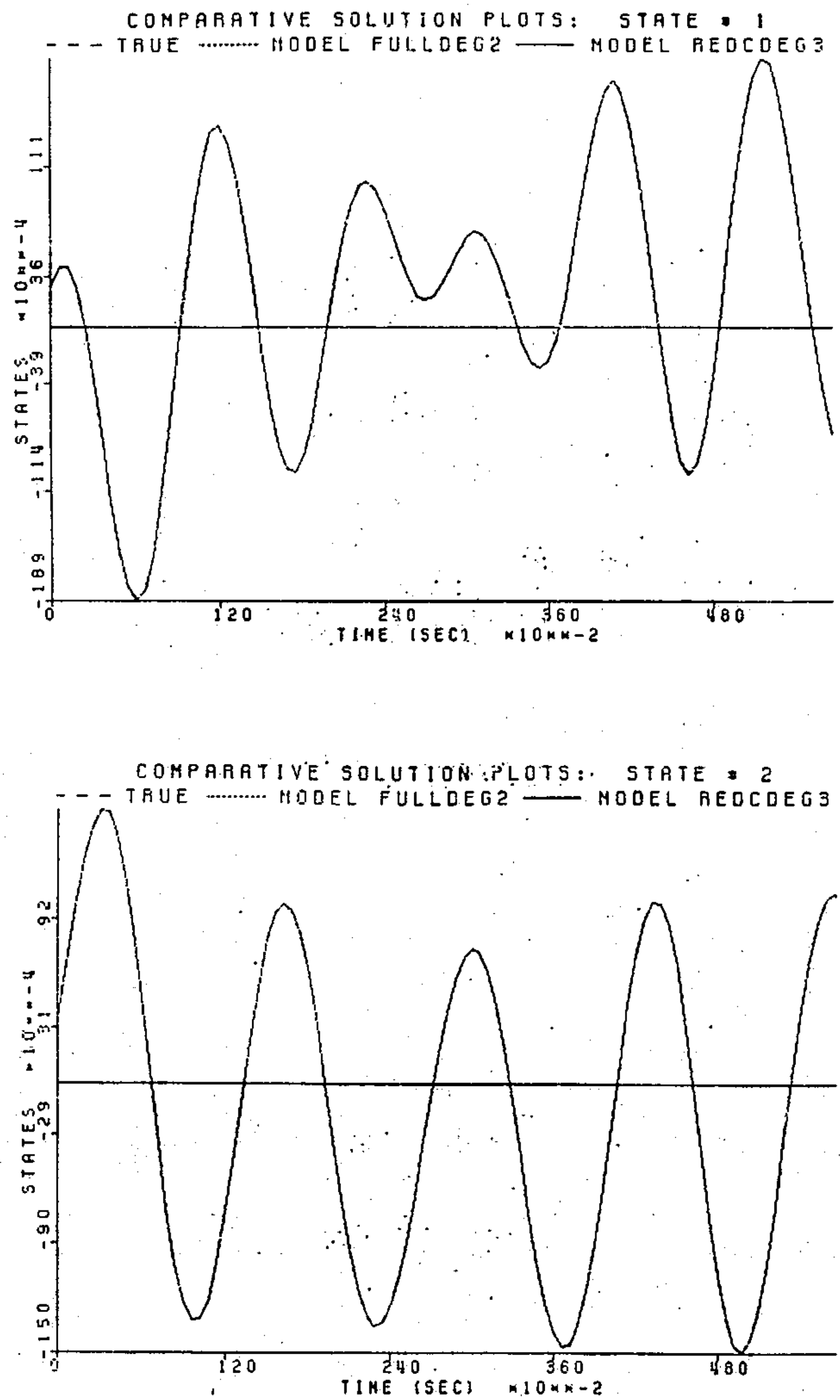


Figure 5.1.22 Simulation Number 5 of Table 5.1.14

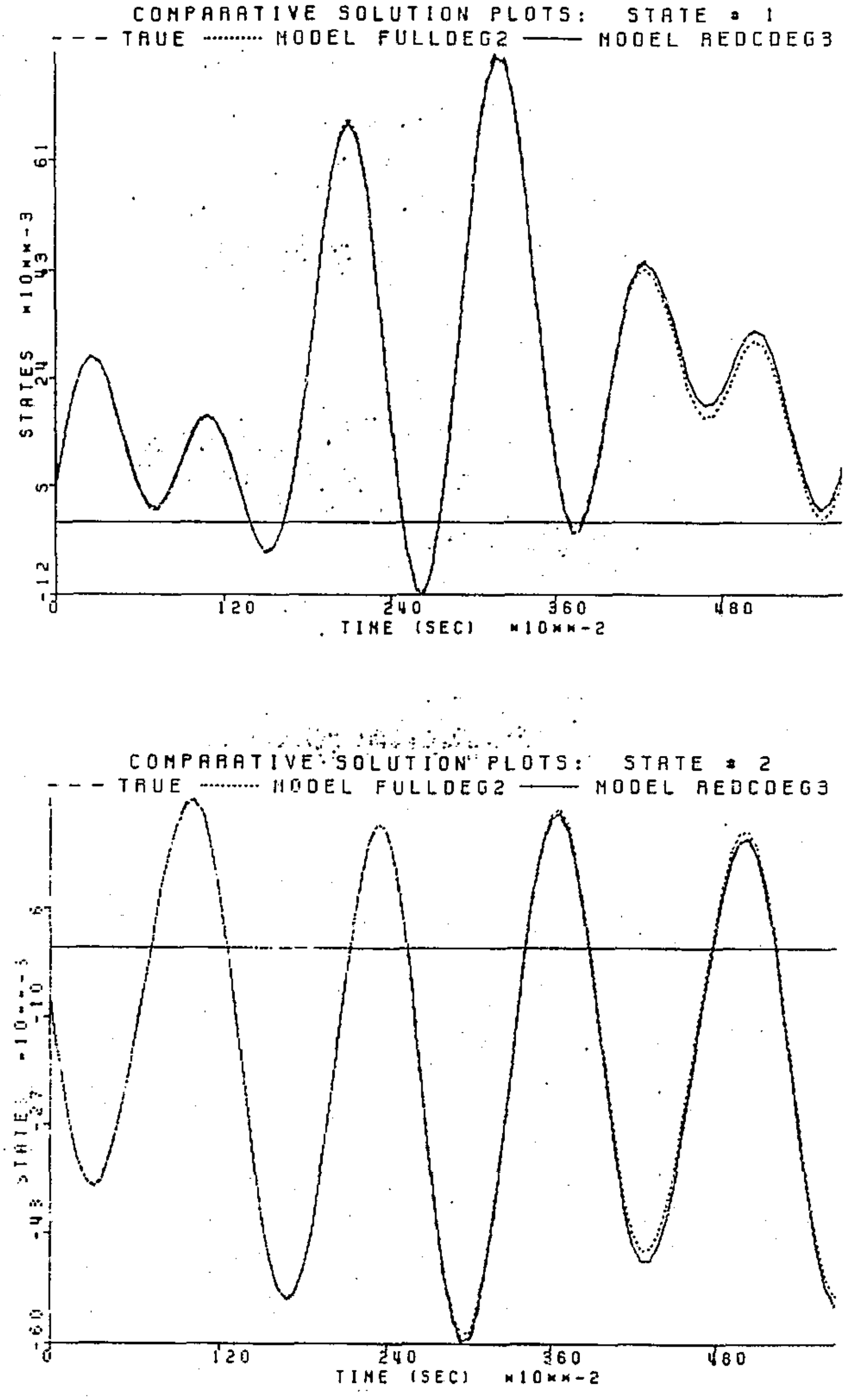


Figure 5.1.23 Simulation Number 33 of Table 5.1.14

```
**************************************
           PROBLEM SUMMARY
  CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
       DEGREE OF APPROXIMATION: 2
       # OF TERMS IN MODEL 2:  13
       DEGREE OF APPROXIMATION: 3
         SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.005	0.005	0.000	0.000	0.75	1.00	0.101E-01	0.899E-02
2	0.005	0.005	0.050	0.050	0.75	1.00	0.164	0.221
3	0.005	0.005	0.050	-0.050	0.75	1.00	0.154	0.345
4	0.005	0.005	-0.050	-0.050	0.75	1.00	0.966E-01	0.226
5	0.005	0.005	-0.050	0.050	0.75	1.00	0.344	0.681
6	0.005	0.005	0.150	0.150	0.75	1.00	0.155	0.954E-01
7	0.005	0.005	0.150	-0.150	0.75	1.00	0.260	0.133
8	0.005	0.005	-0.150	-0.150	0.75	1.00	0.153	0.929E-01
9	0.005	0.005	-0.150	0.150	0.75	1.00	0.227	0.118
10	0.005	-0.005	0.000	0.000	0.75	1.00	0.206E-01	0.148E-01
11	0.005	-0.005	0.050	0.050	0.75	1.00	0.564	1.19
12	0.005	-0.005	0.050	-0.050	0.75	1.00	0.176	0.296
13	0.005	-0.005	-0.050	-0.050	0.75	1.00	0.446	0.980
14	0.005	-0.005	-0.050	0.050	0.75	1.00	0.515	1.28
15	0.005	-0.005	0.150	0.150	0.75	1.00	0.136	0.898E-01
16	0.005	-0.005	0.150	-0.150	0.75	1.00	0.158	0.100
17	0.005	-0.005	-0.150	-0.150	0.75	1.00	0.126	0.813E-01
18	0.005	-0.005	-0.150	0.150	0.75	1.00	0.158	0.966E-01
19	-0.005	-0.005	0.000	0.000	0.75	1.00	0.344E-01	0.306E-01
20	-0.005	-0.005	0.050	0.050	0.75	1.00	0.316	0.595
21	-0.005	-0.005	0.050	-0.050	0.75	1.00	0.300	0.458
22	-0.005	-0.005	-0.050	-0.050	0.75	1.00	0.272	0.415
23	-0.005	-0.005	-0.050	0.050	0.75	1.00	0.447	1.26
24	-0.005	-0.005	0.150	0.150	0.75	1.00	0.124	0.849E-01
25	-0.005	-0.005	0.150	-0.150	0.75	1.00	0.139	0.895E-01
26	-0.005	-0.005	-0.150	-0.150	0.75	1.00	0.116	0.770E-01
27	-0.005	-0.005	-0.150	0.150	0.75	1.00	0.128	0.876E-01
28	-0.005	0.005	0.000	0.000	0.75	1.00	0.106E-01	0.761E-02
29	-0.005	0.005	0.050	0.050	0.75	1.00	0.127	0.251
30	-0.005	0.005	0.050	-0.050	0.75	1.00	0.246	0.560
31	-0.005	0.005	-0.050	-0.050	0.75	1.00	0.202	0.476
32	-0.005	0.005	-0.050	0.050	0.75	1.00	0.777	1.05
33	-0.005	0.005	0.150	0.150	0.75	1.00	0.149	0.946E-01
34	-0.005	0.005	0.150	-0.150	0.75	1.00	0.234	0.117
35	-0.005	0.005	-0.150	-0.150	0.75	1.00	0.146	0.907E-01
36	-0.005	0.005	-0.150	0.150	0.75	1.00	0.180	0.107

Table 5.1.15 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

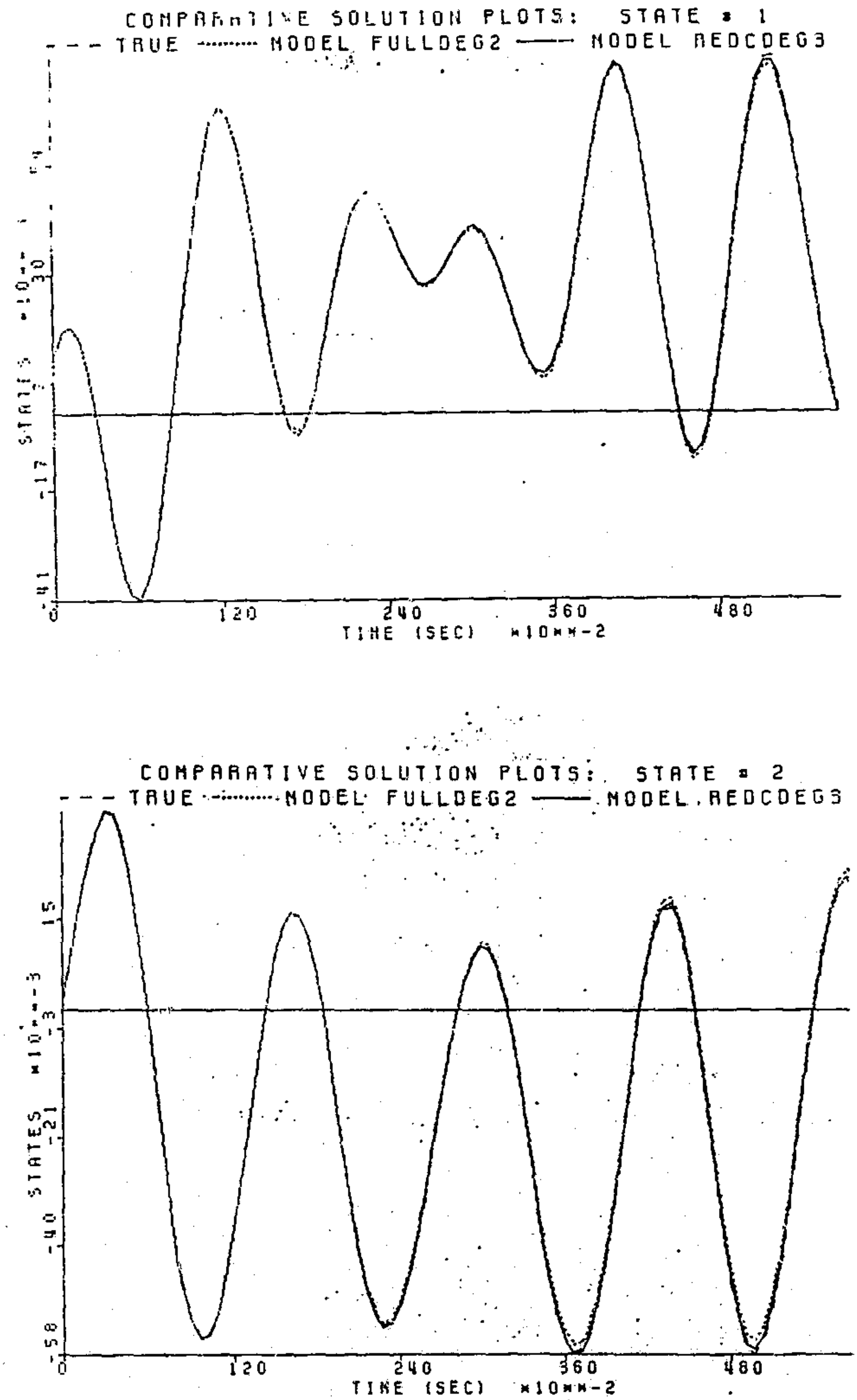


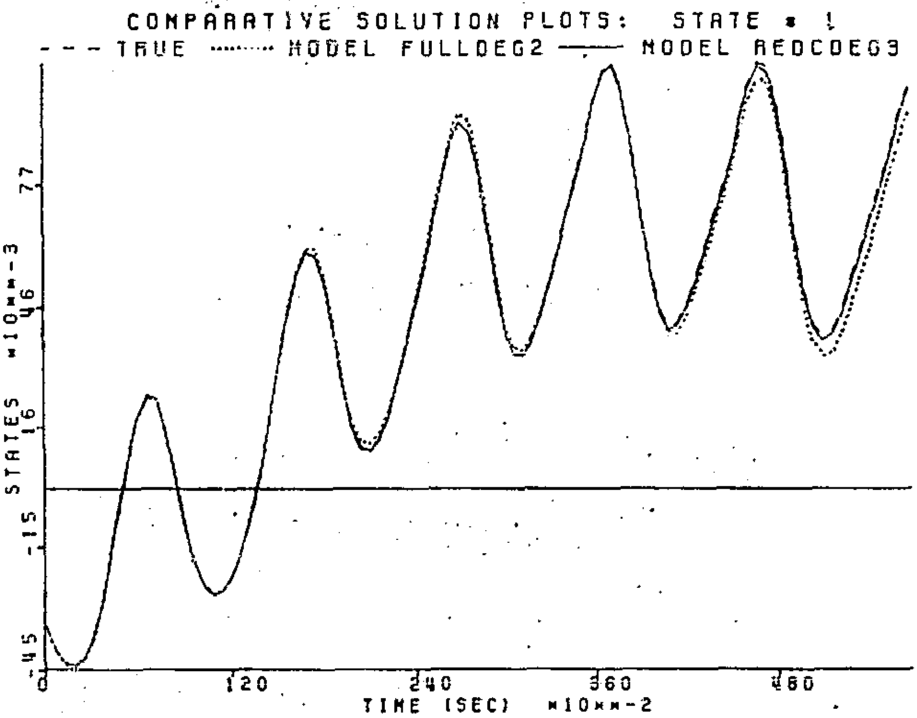
Figure 5.1.24 Simulation Number 18 of Table 5.1.15

```
***************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
        # OF TERMS IN MODEL 1:  14
       DEGREE OF APPROXIMATION: 2
        # OF TERMS IN MODEL 2:  13
       DEGREE OF APPROXIMATION: 3
         SIMULATION WITH COSINE
***************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	0.518E-01	0.453E-01
2	0.025	0.025	0.100	-0.100	2.00	1.00	0.384E-01	0.351E-01
3	0.025	0.025	-0.100	0.100	2.00	1.00	0.181E-01	0.231E-01
4	0.025	0.025	-0.100	-0.100	2.00	1.00	0.234E-01	0.260E-01
5	0.025	-0.025	0.100	0.100	2.00	1.00	0.143E-01	0.150E-01
6	0.025	-0.025	0.100	-0.100	2.00	1.00	0.682E-02	0.101E-01
7	0.025	-0.025	-0.100	0.100	2.00	1.00	0.576E-01	0.391E-01
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	0.569E-01	0.394E-01
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.391E-01	0.364E-01
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.385E-01	0.341E-01
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.432E-01	0.373E-01
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	0.541E-01	0.437E-01
13	-0.025	-0.025	0.100	0.100	2.00	1.00	0.836E-02	0.727E-02
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	0.644E-02	0.644E-02
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	0.217E-01	0.169E-01
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	0.256E-01	0.187E-01
17	0.025	0.025	0.200	0.200	2.00	1.00	0.476E-01	0.255E-01
18	0.025	0.025	0.200	-0.200	2.00	1.00	0.446E-01	0.201E-01
19	0.025	0.025	-0.200	0.200	2.00	1.00	0.143	0.584E-01
20	0.025	0.025	-0.200	-0.200	2.00	1.00	0.179	0.659E-01
21	0.025	-0.025	0.200	0.200	2.00	1.00	0.348E-01	0.178E-01
22	0.025	-0.025	0.200	-0.200	2.00	1.00	0.326E-01	0.166E-01
23	0.025	-0.025	-0.200	0.200	2.00	1.00	0.900E-01	0.471E-01
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	0.945E-01	0.501E-01
25	-0.025	0.025	0.200	0.200	2.00	1.00	0.363E-01	0.195E-01
26	-0.025	0.025	0.200	-0.200	2.00	1.00	0.389E-01	0.181E-01
27	-0.025	0.025	-0.200	0.200	2.00	1.00	0.109	0.506E-01
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	0.123	0.550E-01
29	-0.025	-0.025	0.200	0.200	2.00	1.00	0.252E-01	0.149E-01
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	0.238E-01	0.143E-01
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	0.567E-01	0.351E-01
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	0.615E-01	0.377E-01
33	0.025	0.025	0.300	0.300	2.00	1.00	0.552E-01	0.243E-01
34	0.025	0.025	0.300	-0.300	2.00	1.00	0.606E-01	0.241E-01
35	0.025	0.025	-0.300	0.300	2.00	1.00	0.126	0.545E-01
36	0.025	0.025	-0.300	-0.300	2.00	1.00	0.137	0.588E-01
37	0.025	-0.025	0.300	0.300	2.00	1.00	0.467E-01	0.242E-01
38	0.025	-0.025	0.300	-0.300	2.00	1.00	0.467E-01	0.238E-01
39	0.025	-0.025	-0.300	0.300	2.00	1.00	0.907E-01	0.495E-01
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	0.964E-01	0.531E-01
41	-0.025	0.025	0.300	0.300	2.00	1.00	0.467E-01	0.232E-01
42	-0.025	0.025	0.300	-0.300	2.00	1.00	0.496E-01	0.231E-01
43	-0.025	0.025	-0.300	0.300	2.00	1.00	0.999E-01	0.501E-01
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	0.107	0.537E-01
45	-0.025	-0.025	0.300	0.300	2.00	1.00	0.391E-01	0.226E-01
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	0.385E-01	0.221E-01
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	0.723E-01	0.443E-01
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	0.781E-01	0.478E-01

Table 5.1.16 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

ORIGINAL PAGE IS
OF POOR QUALITY



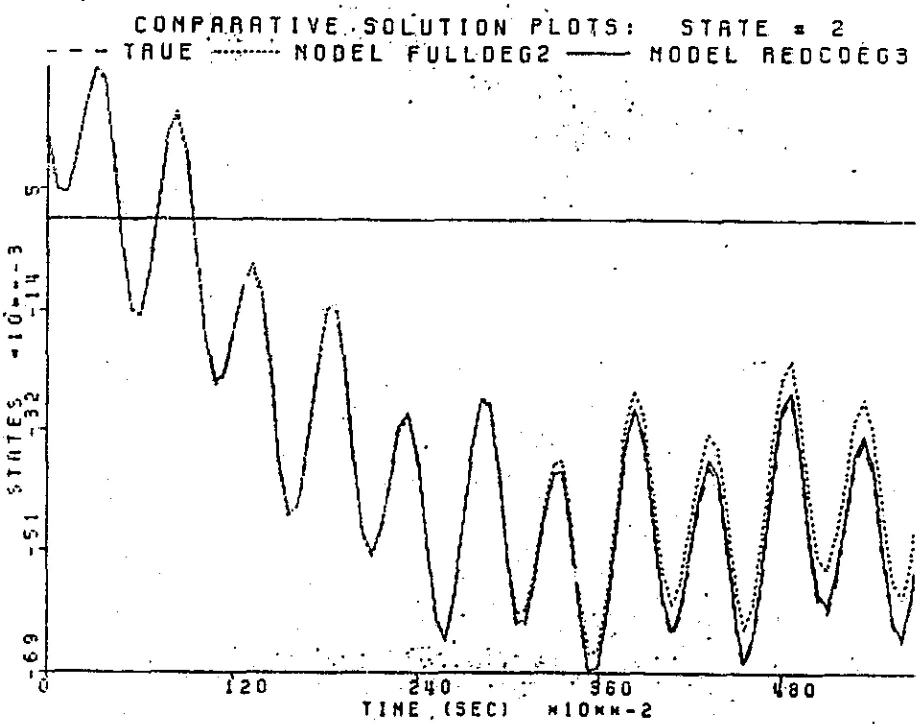


Figure 5.1.25 Simulation Number 26 of Table 5.1.16

```
*************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,MODEL1,MODEL2
            # OF STATES: 2
            # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
      DEGREE OF APPROXIMATION: 2
       # OF TERMS IN MODEL 2:  13
      DEGREE OF APPROXIMATION: 3
        SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.400	0.400	2.00	1.00	0.518E-01	0.282E-01
2	0.025	0.025	0.400	-0.400	2.00	1.00	0.566E-01	0.295E-01
3	0.025	0.025	-0.400	0.400	2.00	1.00	0.103	0.583E-01
4	0.025	0.025	-0.400	-0.400	2.00	1.00	0.109	0.631E-01
5	0.025	-0.025	0.400	0.400	2.00	1.00	0.485E-01	0.303E-01
6	0.025	-0.025	0.400	-0.400	2.00	1.00	0.497E-01	0.309E-01
7	0.025	-0.025	-0.400	0.400	2.00	1.00	0.865E-01	0.578E-01
8	0.025	-0.025	-0.400	-0.400	2.00	1.00	0.922E-01	0.630E-01
9	-0.025	0.025	0.400	0.400	2.00	1.00	0.472E-01	0.280E-01
10	-0.025	0.025	0.400	-0.400	2.00	1.00	0.501E-01	0.290E-01
11	-0.025	0.025	-0.400	0.400	2.00	1.00	0.902E-01	0.563E-01
12	-0.025	0.025	-0.400	-0.400	2.00	1.00	0.957E-01	0.612E-01
13	-0.025	-0.025	0.400	0.400	2.00	1.00	0.439E-01	0.298E-01
14	-0.025	-0.025	0.400	-0.400	2.00	1.00	0.442E-01	0.300E-01
15	-0.025	-0.025	-0.400	0.400	2.00	1.00	0.760E-01	0.553E-01
16	-0.025	-0.025	-0.400	-0.400	2.00	1.00	0.822E-01	0.609E-01
17	0.025	0.025	0.500	0.500	2.00	1.00	0.268E-01	0.158E-01
18	0.025	0.025	0.500	-0.500	2.00	1.00	0.333E-01	0.198E-01
19	0.025	0.025	-0.500	0.500	2.00	1.00	0.608E-01	0.450E-01
20	0.025	0.025	-0.500	-0.500	2.00	1.00	0.676E-01	0.536E-01
21	0.025	-0.025	0.500	0.500	2.00	1.00	0.231E-01	0.147E-01
22	0.025	-0.025	0.500	-0.500	2.00	1.00	0.271E-01	0.175E-01
23	0.025	-0.025	-0.500	0.500	2.00	1.00	0.483E-01	0.399E-01
24	0.025	-0.025	-0.500	-0.500	2.00	1.00	0.572E-01	0.504E-01
25	-0.025	0.025	0.500	0.500	2.00	1.00	0.235E-01	0.143E-01
26	-0.025	0.025	0.500	-0.500	2.00	1.00	0.285E-01	0.175E-01
27	-0.025	0.025	-0.500	0.500	2.00	1.00	0.522E-01	0.411E-01
28	-0.025	0.025	-0.500	-0.500	2.00	1.00	0.600E-01	0.504E-01
29	-0.025	-0.025	0.500	0.500	2.00	1.00	0.194E-01	0.122E-01
30	-0.025	-0.025	0.500	-0.500	2.00	1.00	0.223E-01	0.140E-01
31	-0.025	-0.025	-0.500	0.500	2.00	1.00	0.394E-01	0.336E-01
32	-0.025	-0.025	-0.500	-0.500	2.00	1.00	0.493E-01	0.451E-01

Table 5.1.17 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

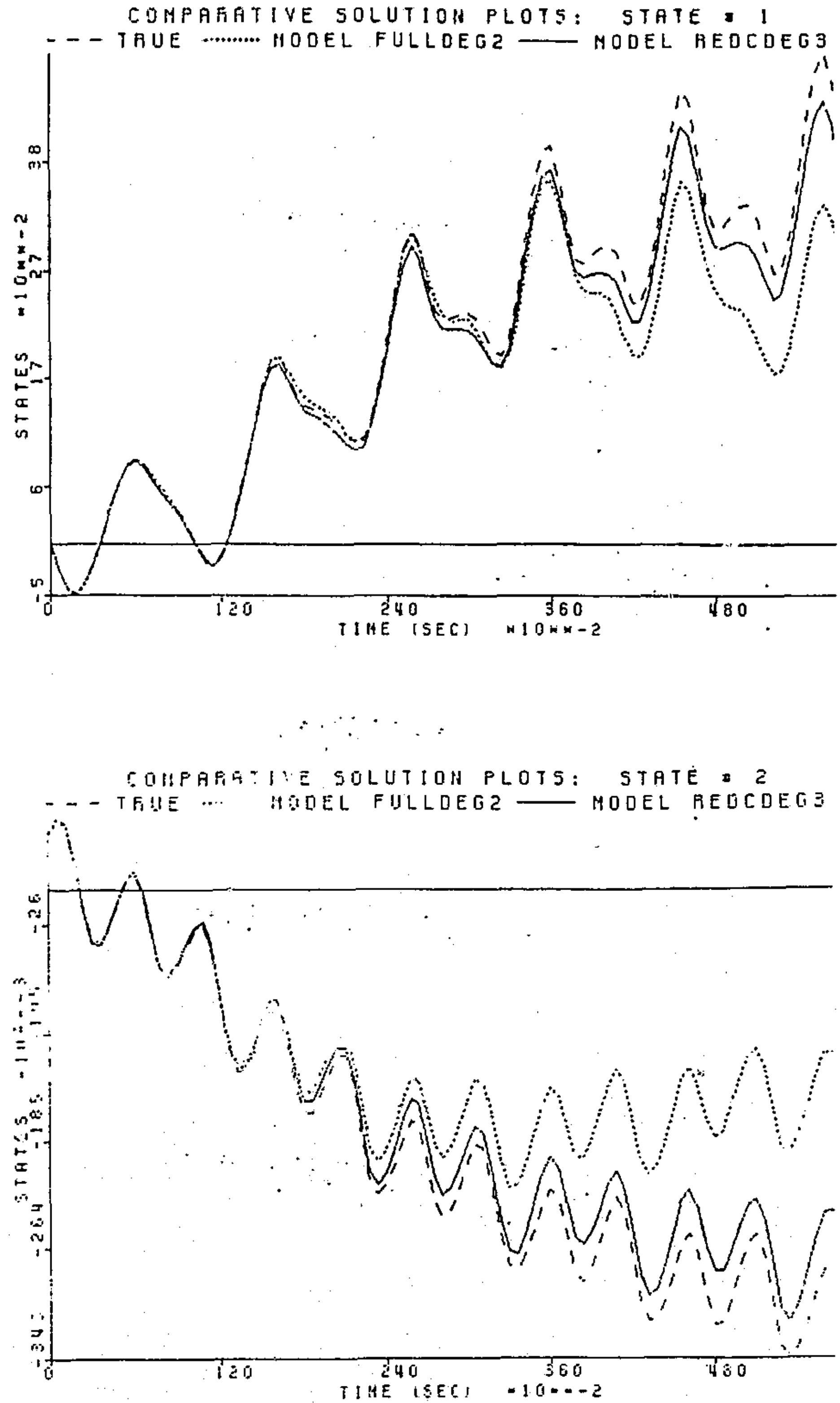
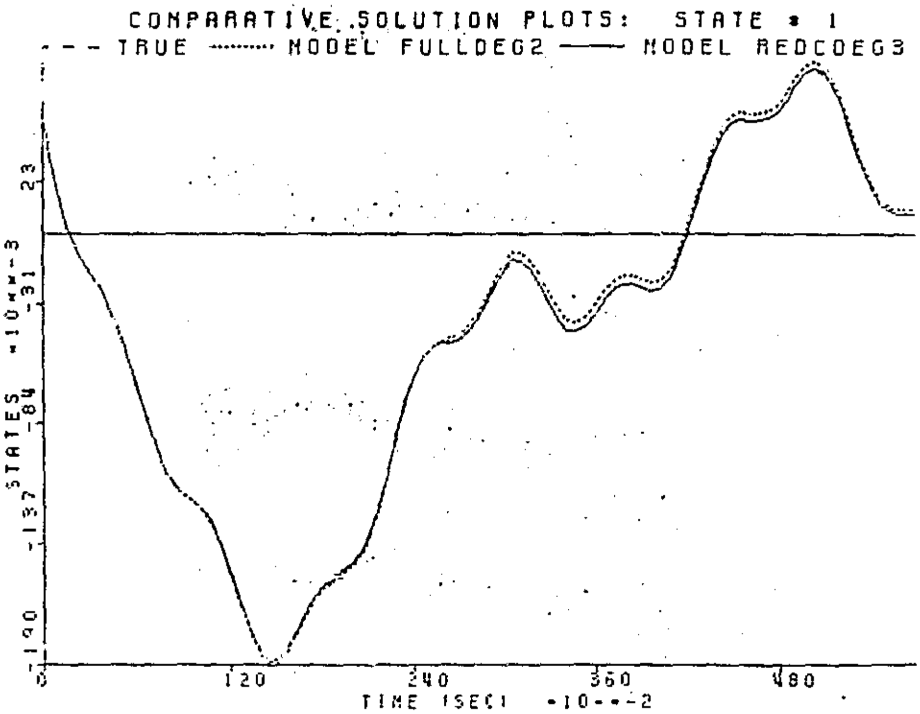


Figure 5.1.26 Simulation Number 4 of Table 5.1.17

```
**************************************
           PROBLEM SUMMARY
   CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
      DEGREE OF APPROXIMATION: 2
       # OF TERMS IN MODEL 2:  13
      DEGREE OF APPROXIMATION: 3
        SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.075	0.075	0.100	0.100	0.50	1.50	0.473E-02	0.661E-02
2	0.075	0.075	0.100	-0.100	0.50	1.50	0.444E-02	0.617E-02
3	0.075	0.075	-0.100	0.100	0.50	1.50	0.103	0.996E-01
4	0.075	0.075	-0.100	-0.100	0.50	1.50	0.994E-01	0.919E-01
5	0.075	-0.075	0.100	0.100	0.50	1.50	0.665E-02	0.692E-02
6	0.075	-0.075	0.100	-0.100	0.50	1.50	0.635E-02	0.695E-02
7	0.075	-0.075	-0.100	0.100	0.50	1.50	0.172E-02	0.224E-02
8	0.075	-0.075	-0.100	-0.100	0.50	1.50	0.165E-02	0.173E-02
9	-0.075	0.075	0.100	0.100	0.50	1.50	0.231E-01	0.286E-01
10	-0.075	0.075	0.100	-0.100	0.50	1.50	0.242E-01	0.296E-01
11	-0.075	0.075	-0.100	0.100	0.50	1.50	0.854E-02	0.107E-01
12	-0.075	0.075	-0.100	-0.100	0.50	1.50	0.465E-02	0.103E-01
13	-0.075	-0.075	0.100	0.100	0.50	1.50	0.428E-01	0.405E-01
14	-0.075	-0.075	0.100	-0.100	0.50	1.50	0.429E-01	0.416E-01
15	-0.075	-0.075	-0.100	0.100	0.50	1.50	0.302E-01	0.290E-01
16	-0.075	-0.075	-0.100	-0.100	0.50	1.50	0.290E-01	0.279E-01
17	0.075	0.075	0.200	0.200	0.50	1.50	0.655E-02	0.752E-02
18	0.075	0.075	0.200	-0.200	0.50	1.50	0.787E-02	0.110E-01
19	0.075	0.075	-0.200	0.200	0.50	1.50	0.344E-01	0.371E-01
20	0.075	0.075	-0.200	-0.200	0.50	1.50	0.231E-01	0.216E-01
21	0.075	-0.075	0.200	0.200	0.50	1.50	0.168E-01	0.182E-01
22	0.075	-0.075	0.200	-0.200	0.50	1.50	0.150E-01	0.179E-01
23	0.075	-0.075	-0.200	0.200	0.50	1.50	0.130E-01	0.148E-01
24	0.075	-0.075	-0.200	-0.200	0.50	1.50	0.154E-01	0.151E-01
25	-0.075	0.075	0.200	0.200	0.50	1.50	0.263E-02	0.599E-02
26	-0.075	0.075	0.200	-0.200	0.50	1.50	0.168E-01	0.237E-01
27	-0.075	0.075	-0.200	0.200	0.50	1.50	0.836E-02	0.152E-01
28	-0.075	0.075	-0.200	-0.200	0.50	1.50	0.722E-02	0.114E-01
29	-0.075	-0.075	0.200	0.200	0.50	1.50	0.575E-01	0.606E-01
30	-0.075	-0.075	0.200	-0.200	0.50	1.50	0.566E-01	0.628E-01
31	-0.075	-0.075	-0.200	0.200	0.50	1.50	0.751E-01	0.795E-01
32	-0.075	-0.075	-0.200	-0.200	0.50	1.50	0.781E-01	0.829E-01
33	0.075	0.075	0.300	0.300	0.50	1.50	0.201E-01	0.343E-01
34	0.075	0.075	0.300	-0.300	0.50	1.50	0.528E-01	0.469E-01
35	0.075	0.075	-0.300	0.300	0.50	1.50	0.385E-01	0.479E-01
36	0.075	0.075	-0.300	-0.300	0.50	1.50	0.267E-01	0.351E-01
37	0.075	-0.075	0.300	0.300	0.50	1.50	0.498E-01	0.573E-01
38	0.075	-0.075	0.300	-0.300	0.50	1.50	0.379E-01	0.445E-01
39	0.075	-0.075	-0.300	0.300	0.50	1.50	0.697E-01	0.751E-01
40	0.075	-0.075	-0.300	-0.300	0.50	1.50	0.830E-01	0.895E-01
41	-0.075	0.075	0.300	0.300	0.50	1.50	0.272E-01	0.362E-01
42	-0.075	0.075	0.300	-0.300	0.50	1.50	0.298E-01	0.330E-01
43	-0.075	0.075	-0.300	0.300	0.50	1.50	0.197E-01	0.356E-01
44	-0.075	0.075	-0.300	-0.300	0.50	1.50	0.254E-01	0.426E-01
45	-0.075	-0.075	0.300	0.300	0.50	1.50	0.113	0.128
46	-0.075	-0.075	0.300	-0.300	0.50	1.50	0.969E-01	0.115
47	-0.075	-0.075	-0.300	0.300	0.50	1.50	0.168	0.204
48	-0.075	-0.075	-0.300	-0.300	0.50	1.50	0.180	0.223

Table 5.1.18 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model



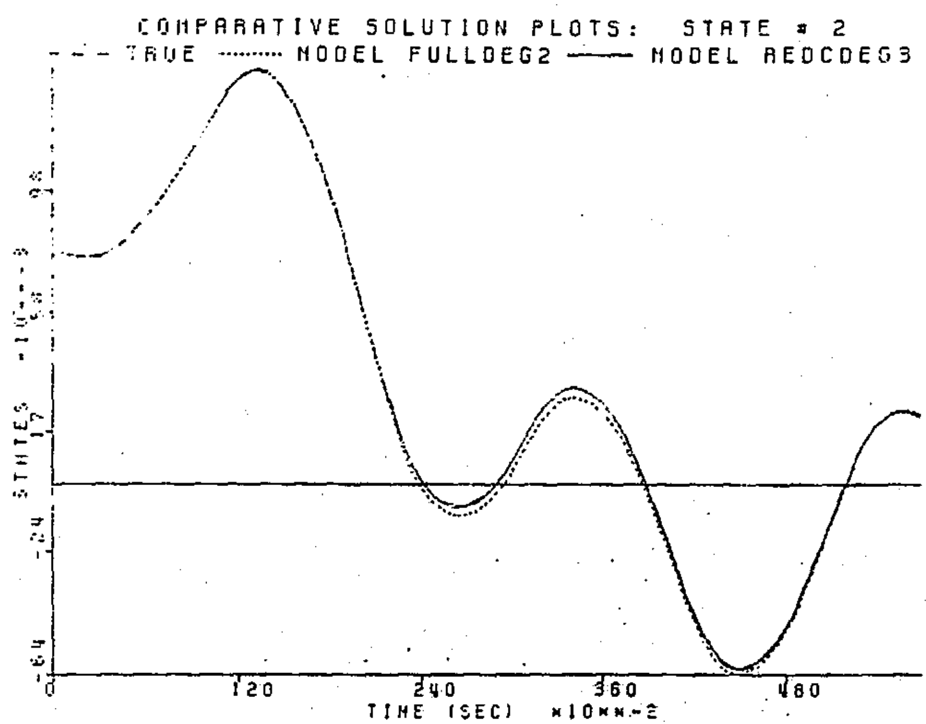
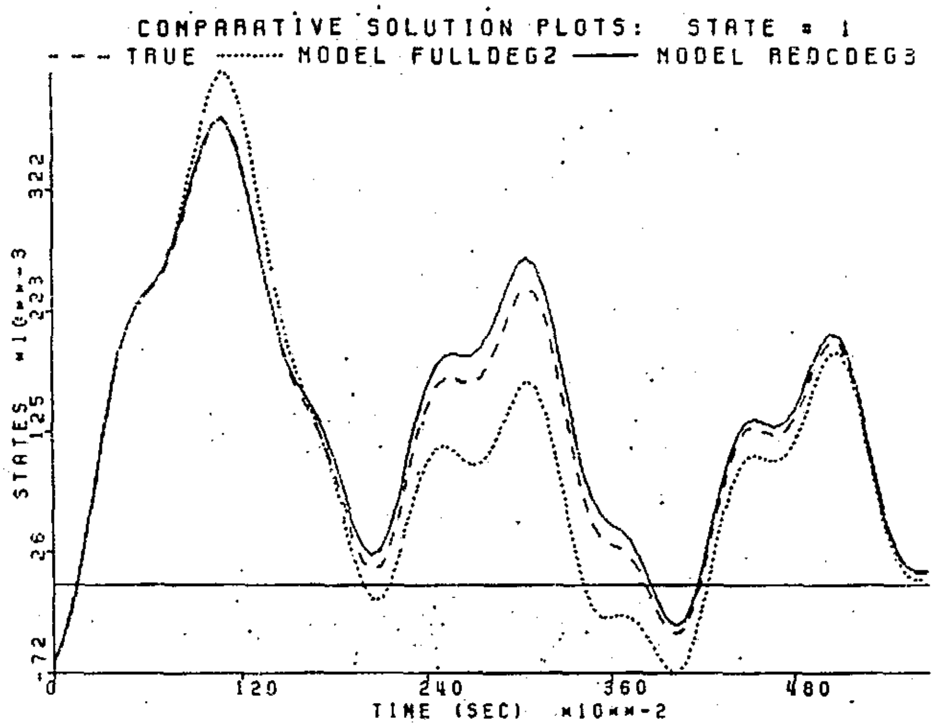


Figure 5.1.27 Simulation Number 2 of Table 5.1.18

ORIGINAL PAGE IS
OF POOR QUALITY



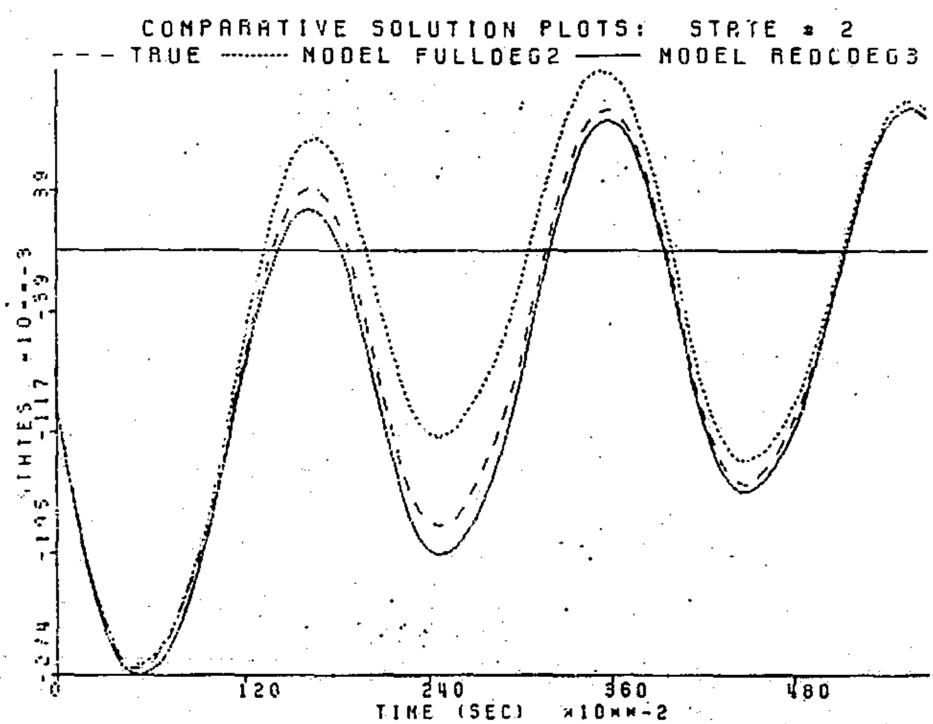


Figure 5.1.28 Simulation Number 46 of Table 5.1.18

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.050	-0.050	-0.107	0.251	3.43	3.75	0.127	0.111
2	0.050	-0.050	-0.015	-0.088	2.63	2.13	0.661E-02	0.642E-02
3	0.050	-0.050	-0.020	0.172	4.05	4.91	0.705E-02	0.623E-02
4	0.050	-0.050	0.217	0.223	4.82	2.08	0.558E-01	0.309E-01
5	0.050	-0.050	0.364	0.141	4.84	3.37	0.863E-01	0.576E-01
6	0.050	-0.050	-0.304	-0.014	4.12	4.32	0.794E-01	0.454E-01
7	0.050	-0.050	0.127	0.265	3.32	5.34	0.177E-01	0.125E-01
8	0.050	-0.030	0.152	-0.298	3.54	4.50	0.224E-01	0.167E-01
9	0.050	-0.050	-0.256	0.238	2.49	2.46	0.289	0.181
10	0.050	-0.050	-0.335	-0.342	4.35	4.70	0.95RE-01	0.606E-01
11	0.050	-0.050	-0.032	-0.094	4.14	4.81	0.271E-02	0.433E-02
12	0.050	-0.050	0.072	0.193	2.90	2.83	1.79	2.44
13	0.050	-0.050	-0.281	0.078	4.69	2.89	0.727E-01	0.410E-01
14	0.050	-0.050	-0.196	0.292	4.07	2.98	0.442E-01	0.267E-01
15	0.050	-0.050	0.382	-0.203	4.92	3.21	0.865E-01	0.616E-01
16	0.050	-0.050	0.060	0.090	4.93	5.41	0.128E-01	0.150E-01
17	0.050	-0.050	0.022	-0.083	3.92	5.43	0.627E-02	0.462E-02
18	0.050	-0.050	0.376	-0.391	4.78	5.05	0.108	0.748E-01
19	0.050	-0.050	-0.191	-0.309	4.58	5.53	0.405E-01	0.251E-01
20	0.050	-0.050	-0.316	-0.241	2.86	2.39	0.737E-01	0.454E-01
21	0.050	-0.050	-0.317	-0.052	3.77	2.47	0.781E-01	0.462E-01
22	0.050	-0.050	0.161	-0.124	4.16	4.24	0.178	0.117
23	0.050	-0.050	0.140	-0.246	5.05	2.81	0.225E-01	0.152E-01
24	0.050	-0.050	-0.125	-0.047	5.44	3.97	0.247E-01	0.169E-01
25	0.050	-0.050	0.335	0.397	3.21	4.90	0.644E-01	0.395E-01
26	0.050	-0.050	-0.054	-0.218	5.31	3.38	0.238E-02	0.361E-02
27	0.050	-0.050	-0.319	0.077	5.57	4.00	0.805E-01	0.482E-01
28	0.050	-0.050	-0.139	-0.229	2.30	3.57	0.231E-01	0.161E-01
29	0.050	-0.050	0.280	0.128	2.23	2.41	0.116	0.637E-01
30	0.050	-0.050	0.055	0.124	5.93	3.36	0.179E-02	0.228E-02
31	0.050	-0.050	0.141	-0.087	2.91	3.16	0.382E-01	0.284E-01
32	0.050	-0.050	-0.037	0.369	2.31	2.37	0.325	0.232
33	0.050	-0.050	0.143	-0.014	4.99	3.11	0.317E-01	0.205E-01
34	0.050	-0.050	-0.350	0.139	5.40	3.31	0.846E-01	0.545E-01
35	0.050	-0.050	0.084	0.307	4.11	4.78	0.167E-01	0.238E-01
36	0.050	-0.050	-0.361	-0.371	3.00	4.89	0.680E-01	0.442E-01
37	0.050	-0.050	-0.201	0.400	4.80	2.85	0.416E-01	0.241E-01
38	0.050	-0.050	-0.133	-0.185	5.11	3.50	0.259E-01	0.176E-01
39	0.050	-0.050	0.017	0.260	2.38	3.32	0.378E-02	0.430E-02
40	0.050	-0.050	0.123	0.256	4.66	2.12	0.260E-01	0.188E-01
41	0.050	-0.050	0.020	-0.138	4.99	5.66	0.101E-01	0.875E-02
42	0.050	-0.050	0.100	-0.381	4.90	3.77	0.456E-02	0.788E-02
43	0.050	-0.050	-0.292	0.080	5.01	3.90	0.750E-01	0.431E-01
44	0.050	-0.050	0.129	-0.155	2.80	2.16	0.190E-01	0.153E-01
45	0.050	-0.050	0.130	-0.330	3.10	5.11	0.143E-01	0.103E-01
46	0.050	-0.050	-0.350	0.296	2.07	3.35	0.626E-01	0.376E-01
47	0.050	-0.050	0.201	-0.091	5.22	3.65	0.527E-01	0.300E-01
48	0.050	-0.050	-0.027	-0.011	4.76	2.88	0.458E-02	0.326E-02
49	0.050	-0.050	0.265	0.019	2.02	3.20	0.618E-01	0.333E-01
50	0.050	-0.050	-0.119	0.171	4.24	2.59	0.197E-01	0.140E-01

Table 5.1.19 Simulation Table for Second Degree Full Model versus Third Degree Reduced Model

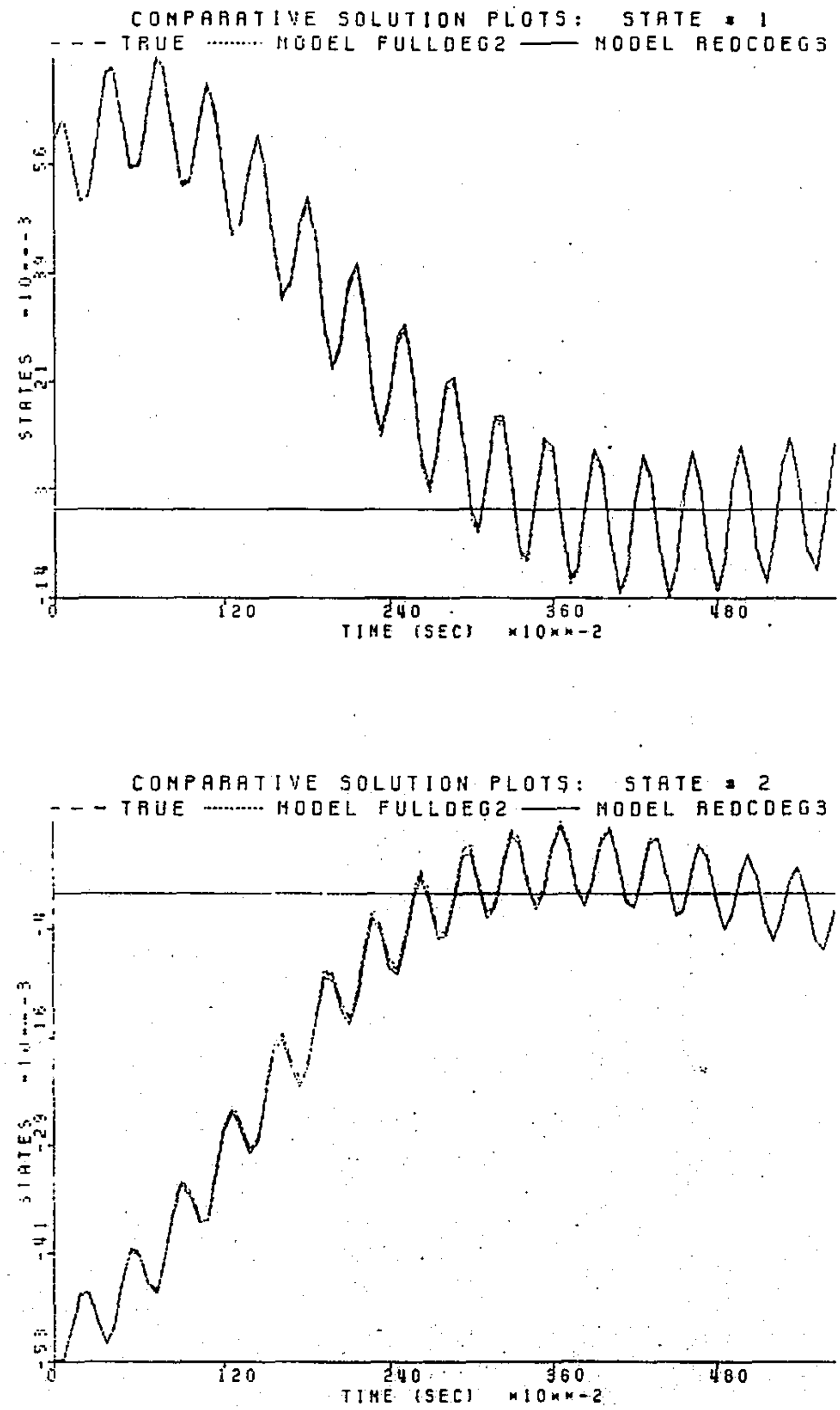


Figure 5.1.29 Simulation Number 12 of Table 5.1.19

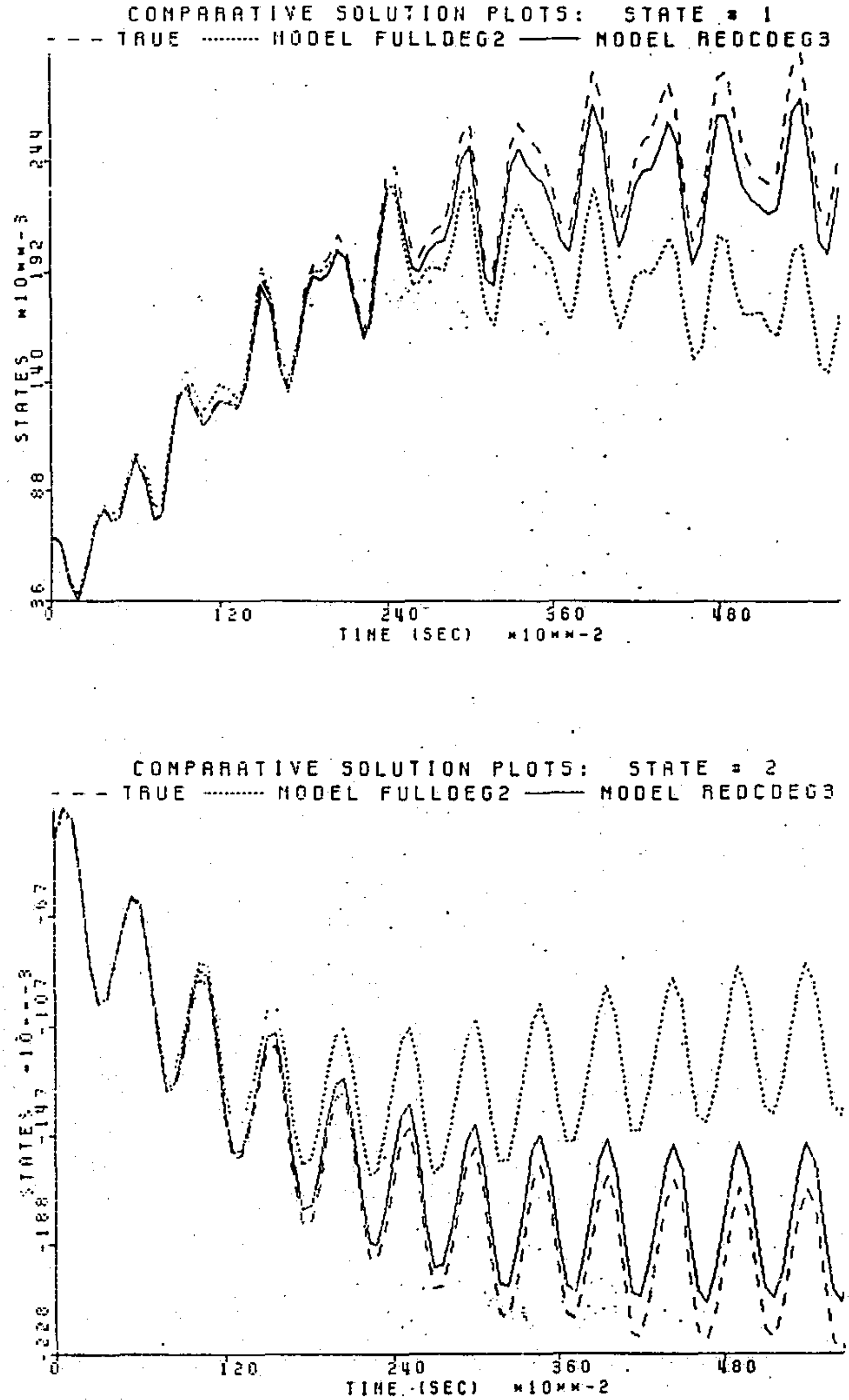


Figure 5.1.30 Simulation Number 46 of Table 5.1.19

****

PROBLEM SUMMARY

CONFIGURATION: TRUE,MODEL1,MODEL2

OF STATES: 2

OF INPUTS: 2

OF TERMS IN MODEL 1: 14

DEGREE OF APPROXIMATION: 2

OF TERMS IN MODEL 2: 13

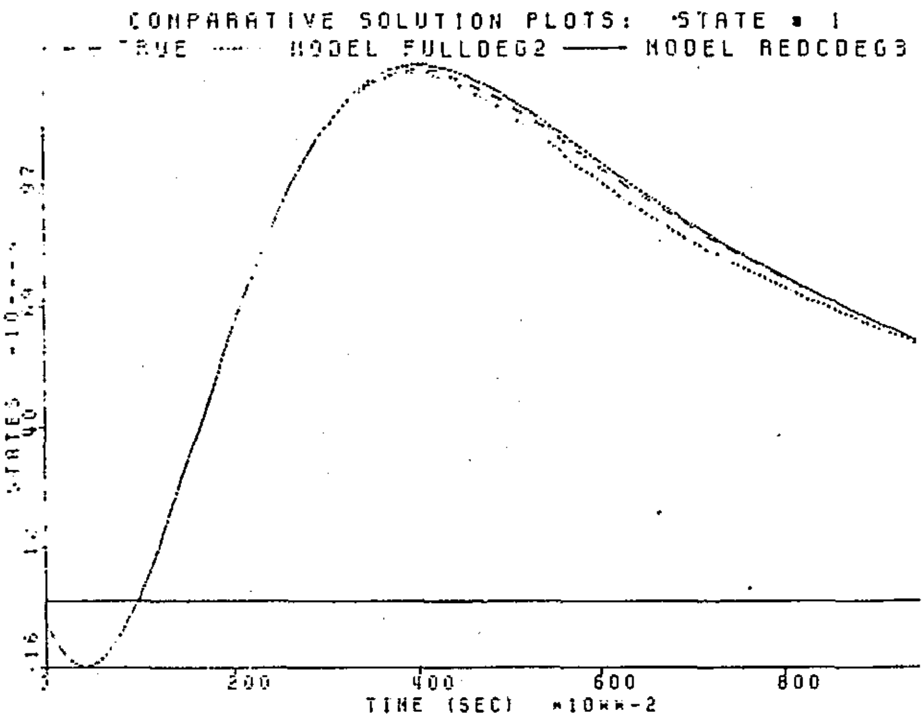
DEGREE OF APPROXIMATION: 3

SIMULATION WITH COSINE

****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.304	0.161
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.259	0.140
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.180	0.111
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.993E-01	0.767E-01
5	0.001	-0.001	-0.050	0.050	0.00	0.00	0.748E-01	0.220E-01
6	0.001	-0.001	-0.050	0.050	0.01	0.01	0.658E-01	0.212E-01
7	0.001	-0.001	-0.050	0.050	0.02	0.02	0.348E-01	0.121E-01
8	0.001	-0.001	-0.050	0.050	0.05	0.05	0.587E-01	0.179E-01
9	0.001	-0.001	-0.075	0.010	0.00	0.00	0.974E-01	0.226E-01
10	0.001	-0.001	-0.075	0.010	0.01	0.01	0.880E-01	0.230E-01
11	0.001	-0.001	-0.075	0.010	0.02	0.02	0.659E-01	0.203E-01
12	0.001	-0.001	-0.075	0.010	0.05	0.05	0.833E-01	0.145E-01
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.118	0.527E-01
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.970E-01	0.469E-01
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.586E-01	0.367E-01
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.345E-01	0.114E-01
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.305	0.160
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.260	0.139
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.180	0.110
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.101	0.760E-01
21	0.010	-0.010	-0.050	0.050	0.00	0.00	0.689E-01	0.207E-01
22	0.010	-0.010	-0.050	0.050	0.01	0.01	0.598E-01	0.198E-01
23	0.010	-0.010	-0.050	0.050	0.02	0.02	0.294E-01	0.105E-01
24	0.010	-0.010	-0.050	0.050	0.05	0.05	0.502E-01	0.169E-01
25	0.010	-0.010	-0.075	0.010	0.00	0.00	0.912E-01	0.215E-01
26	0.010	-0.010	-0.075	0.010	0.01	0.01	0.820E-01	0.217E-01
27	0.010	-0.010	-0.075	0.010	0.02	0.02	0.604E-01	0.186E-01
28	0.010	-0.010	-0.075	0.010	0.05	0.05	0.749E-01	0.134E-01
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.118	0.528E-01
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.966E-01	0.468E-01
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.564E-01	0.355E-01
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.341E-01	0.102E-01

Table 5.1.20 Low Frequency Table for Second Degree Full Model versus Third Degree Reduced Model



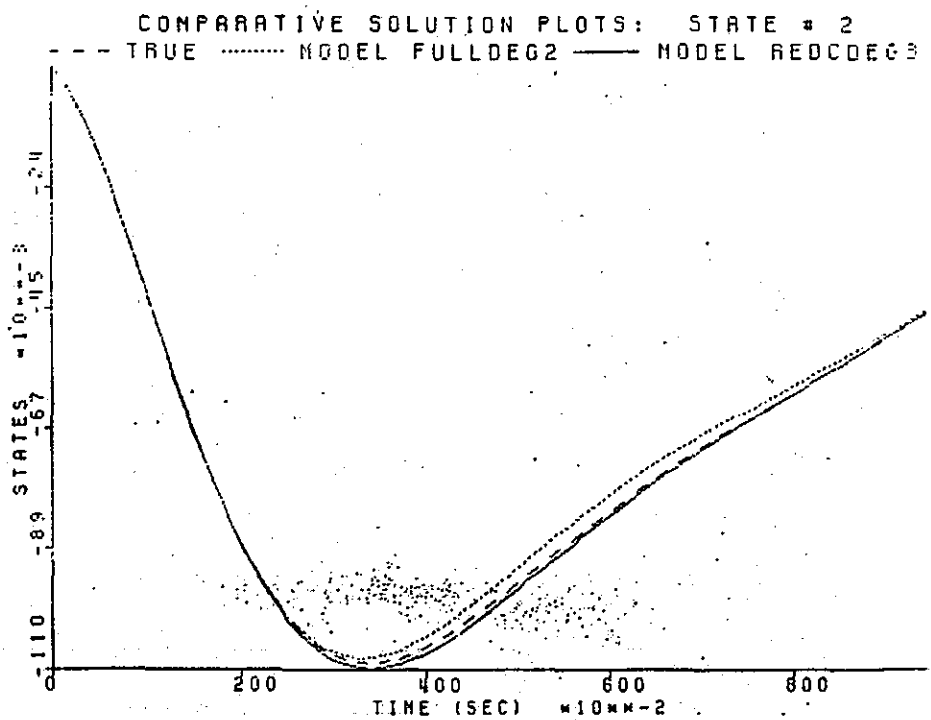


Figure 5.1.31 Simulation Number 3 of Table 5.1.20

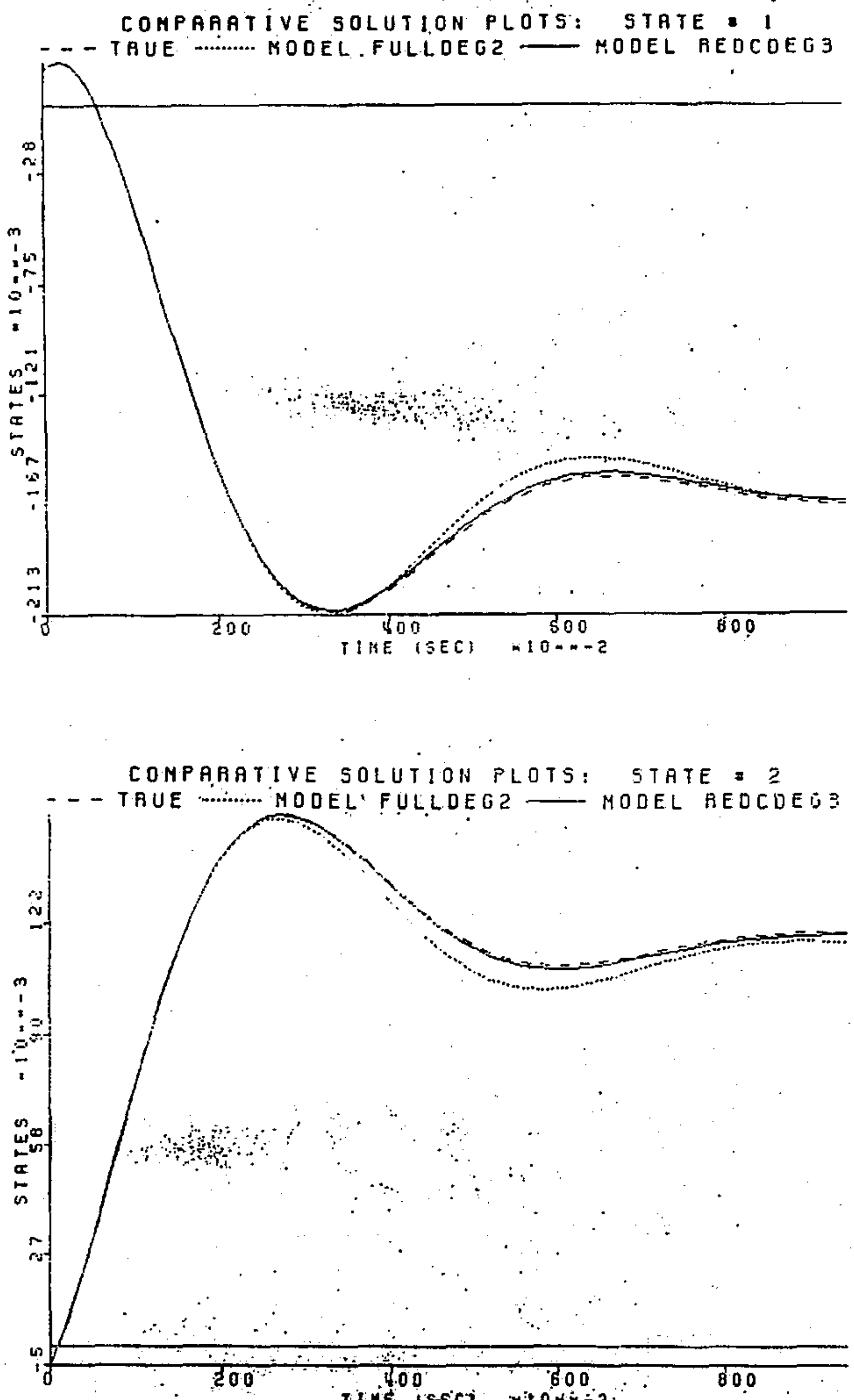


Figure 5.1.32 Simulation Number 21 of Table 5.1.20

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.000	0.010	-0.022	-0.008	0.00	0.00	4.29	7.10
2	-0.006	0.004	0.008	-0.011	0.00	0.00	0.721	0.424
3	0.009	0.006	-0.010	0.017	0.00	0.00	1.15	0.737
4	0.006	0.010	0.022	-0.010	0.00	0.00	0.756E-02	0.405E-02
5	0.006	-0.007	-0.018	-0.003	0.00	0.00	3.59	4.02
6	-0.004	0.001	-0.021	-0.005	0.00	0.00	6.46	10.7
7	0.006	0.003	0.012	0.018	0.00	0.00	1.96	1.79
8	-0.009	-0.003	-0.021	0.024	0.00	0.00	4.47	4.25
9	-0.000	0.003	0.004	0.003	0.00	0.00	0.340E-01	0.194E-01
10	-0.000	0.002	-0.009	-0.013	0.00	0.00	0.432	0.334
11	0.009	0.001	0.011	0.011	0.00	0.00	0.792	0.796
12	0.002	-0.008	0.017	0.002	0.00	0.00	0.117E-01	0.466E-02
13	0.005	-0.008	-0.019	-0.010	0.00	0.00	2.74	3.75
14	-0.004	-0.005	0.022	-0.018	0.00	0.00	0.237E-02	0.146E-02
15	-0.007	0.007	-0.001	-0.020	0.00	0.00	35.1	41.4
16	-0.010	0.005	-0.005	-0.006	0.00	0.00	0.275	0.189
17	-0.007	0.006	-0.018	0.009	0.00	0.00	5.37	5.41
18	-0.001	-0.004	0.015	0.021	0.00	0.00	4.07	5.92
19	0.008	-0.005	0.007	0.017	0.00	0.00	4.12	3.51
20	0.003	-0.003	-0.016	-0.008	0.00	0.00	1.32	1.15
21	0.004	-0.003	0.019	0.006	0.00	0.00	0.391E-02	0.151E-02
22	-0.003	0.006	-0.010	0.007	0.00	0.00	1.25	0.789
23	0.007	-0.002	0.014	-0.012	0.00	0.00	0.575E-01	0.309E-01
24	-0.003	-0.004	0.010	-0.009	0.00	0.00	0.258	0.142
25	-0.000	-0.009	-0.001	-0.013	0.00	0.00	46.4	21.0
26	-0.009	0.007	0.007	-0.015	0.00	0.00	0.887	0.556
27	-0.002	0.002	-0.006	-0.023	0.00	0.00	40.2	34.2
28	0.009	0.002	-0.012	0.012	0.00	0.00	1.59	1.07
29	-0.006	-0.007	-0.003	0.006	0.00	0.00	0.956	0.616
30	0.004	-0.000	-0.000	-0.009	0.00	0.00	18.9	11.0
31	0.001	-0.007	0.022	0.014	0.00	0.00	0.395E-01	0.193E-01
32	0.008	-0.002	0.004	0.024	0.00	0.00	2.89	1.93
33	0.001	-0.007	-0.023	0.018	0.00	0.00	4.64	3.73
34	0.003	-0.005	-0.024	0.024	0.00	0.00	2.39	1.48
35	-0.001	-0.007	0.017	0.012	0.00	0.00	0.242	0.119
36	-0.010	-0.006	0.015	0.014	0.00	0.00	1.88	1.49
37	-0.009	-0.006	-0.004	0.002	0.00	0.00	0.637	0.415
38	0.001	0.008	-0.005	0.020	0.00	0.00	0.743	0.449
39	0.000	-0.001	-0.004	0.025	0.00	0.00	0.592	0.376
40	0.003	-0.009	-0.011	0.009	0.00	0.00	1.26	0.832
41	-0.000	0.004	0.011	-0.006	0.00	0.00	0.362	0.192
42	0.008	-0.005	-0.010	0.010	0.00	0.00	1.16	0.742
43	-0.009	-0.008	0.012	-0.021	0.00	0.00	0.236E-01	0.148E-01
44	0.003	-0.007	-0.021	0.015	0.00	0.00	6.64	8.19
45	0.001	0.007	-0.016	0.012	0.00	0.00	3.74	3.24
46	-0.008	0.004	0.005	0.015	0.00	0.00	13.7	14.6
47	-0.001	-0.002	0.015	0.010	0.00	0.00	0.892	0.469
48	-0.007	0.009	0.018	-0.002	0.00	0.00	0.176E-01	0.819E-02
49	0.006	-0.005	0.004	0.008	0.00	0.00	2.03	1.60
50	-0.006	0.003	-0.011	-0.021	0.00	0.00	4.09	4.17

Table 5.1.21 *Step Response Table for Second Degree Full Model versus Third Degree Reduced Model*

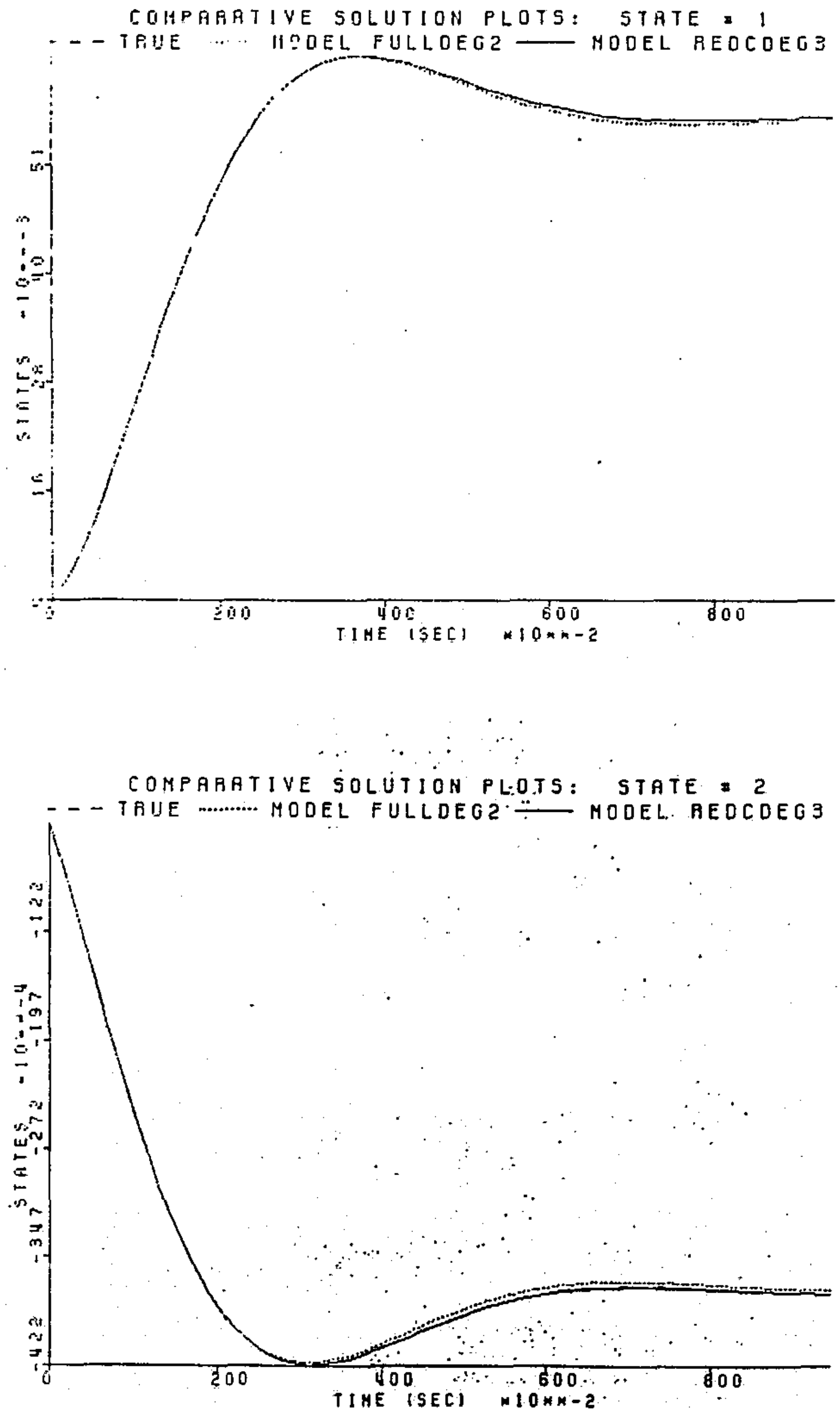


Figure 5.1.33 Simulation Number 15 of Table 5.1.21

```
**************************************
          PROBLEM SUMMARY
 CONFIGURATION:  TRUE,MODEL1,MODEL2
          # OF STATES: 2
          # OF INPUTS: 2
     # OF TERMS IN MODEL 1:  34
    DEGREE OF APPROXIMATION: 3
     # OF TERMS IN MODEL 2:  13
    DEGREE OF APPROXIMATION: 3
      SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.240E-01	0.241E-01
2	0.001	0.001	0.050	0.050	0.75	1.00	0.487	1.85
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.859	2.27
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.497	2.30
5	0.001	0.001	-0.050	0.050	0.75	1.00	0.711	2.00
6	0.001	0.001	0.150	0.150	0.75	1.00	11.1	27.6
7	0.001	0.001	0.150	-0.150	0.75	1.00	18.1	64.9
8	0.001	0.001	-0.150	-0.150	0.75	1.00	16.8	41.7
9	0.001	0.001	-0.150	0.150	0.75	1.00	8.19	16.0
10	0.001	-0.001	0.000	0.000	0.75	1.00	0.239E-01	0.195E-01
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.758	4.16
12	0.001	-0.001	0.050	-0.050	0.75	1.00	1.08	3.70
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.805	5.26
14	0.001	-0.001	-0.050	0.050	0.75	1.00	0.782	3.35
15	0.001	-0.001	0.150	0.150	0.75	1.00	13.4	32.5
16	0.001	-0.001	0.150	-0.150	0.75	1.00	17.5	61.3
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	20.1	48.9
18	0.001	-0.001	-0.150	0.150	0.75	1.00	8.44	17.4
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.378E-01	0.382E-01
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.870	5.98
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	1.14	4.57
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.903	6.05
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	0.726	3.25
24	-0.001	-0.001	0.150	0.150	0.75	1.00	14.6	34.9
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	16.9	57.9
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	21.0	50.9
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	8.19	16.8
28	-0.001	0.001	0.000	0.000	0.75	1.00	0.183E-01	0.149E-01
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.556	2.50
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.938	2.89
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.573	2.74
32	-0.001	0.001	-0.050	0.050	0.75	1.00	0.671	2.05
33	-0.001	0.001	0.150	0.150	0.75	1.00	12.2	30.0
34	-0.001	0.001	0.150	-0.150	0.75	1.00	17.6	61.3
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	17.6	43.6
36	-0.001	0.001	-0.150	0.150	0.75	1.00	7.94	15.5

Table 5.1.22 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model

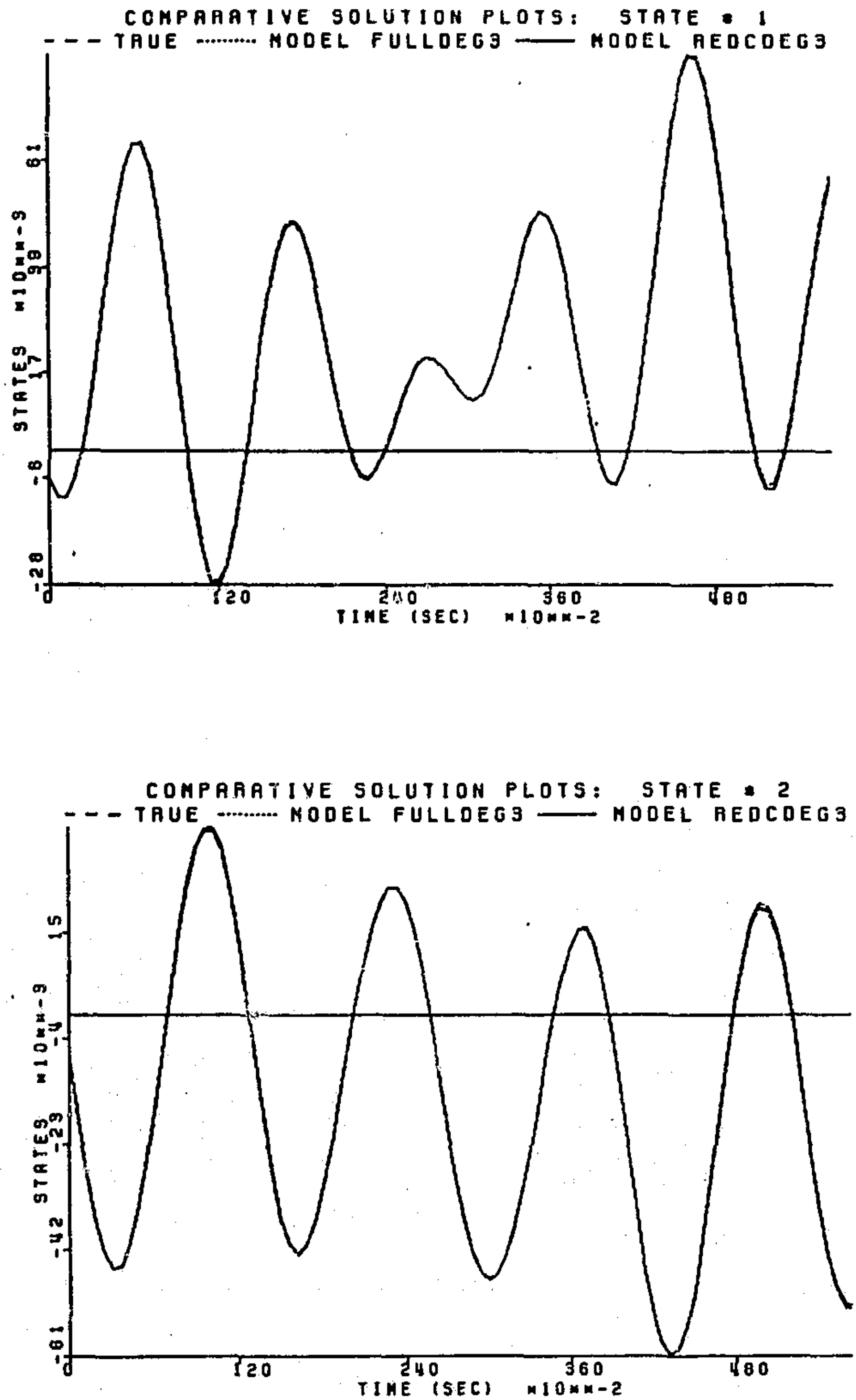
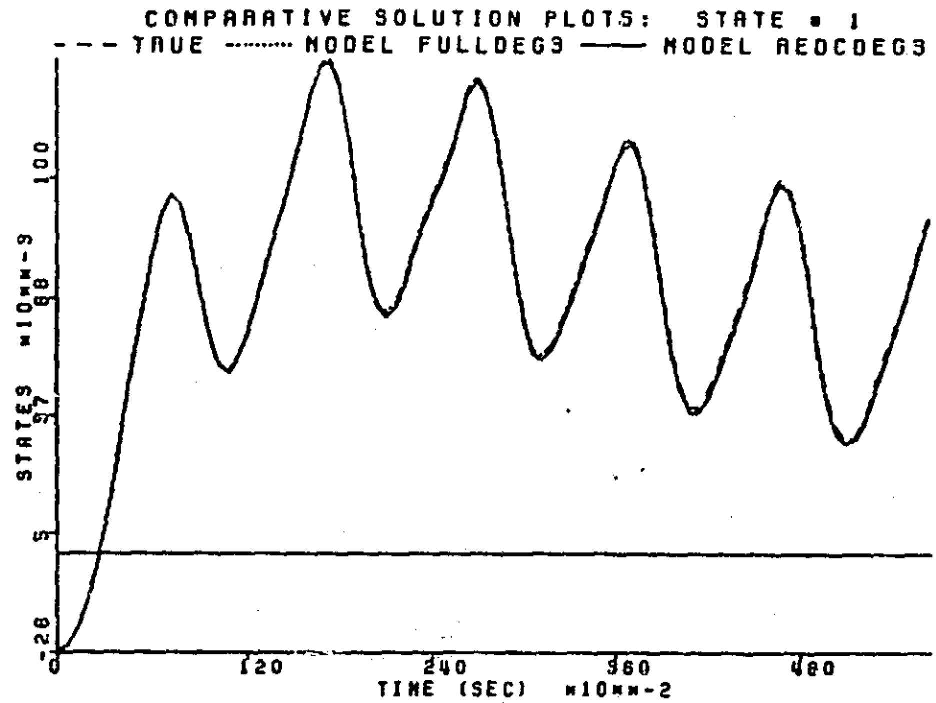


Figure 5.1.34 Simulation Number 7 of Table 5.1.22

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	0.116	0.110
2	0.025	0.025	0.100	-0.100	2.00	1.00	0.955E-01	0.938E-01
3	0.025	0.025	-0.100	0.100	2.00	1.00	0.395E-01	0.592E-01
4	0.025	0.025	-0.100	-0.100	2.00	1.00	0.526E-01	0.726E-01
5	0.025	-0.025	0.100	0.100	2.00	1.00	3.75	5.24
6	0.025	-0.025	0.100	-0.100	2.00	1.00	2.30	3.56
7	0.025	-0.025	-0.100	0.100	2.00	1.00	7.59	15.9
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	8.40	20.1
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.187	0.211
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.190	0.203
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.205	0.277
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	0.283	0.358
13	-0.025	-0.025	0.100	0.100	2.00	1.00	3.41	10.5
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	2.79	11.5
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	4.50	6.26
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	5.34	8.27
17	0.025	0.025	0.200	0.200	2.00	1.00	1.51	3.20
18	0.025	0.025	0.200	-0.200	2.00	1.00	1.55	3.60
19	0.025	0.025	-0.200	0.200	2.00	1.00	7.07	6.06
20	0.025	0.025	-0.200	-0.200	2.00	1.00	8.44	7.37
21	0.025	-0.025	0.200	0.200	2.00	1.00	14.3	4.16
22	0.025	-0.025	0.200	-0.200	2.00	1.00	13.0	3.88
23	0.025	-0.025	-0.200	0.200	2.00	1.00	8.03	3.34
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	8.31	3.71
25	-0.025	0.025	0.200	0.200	2.00	1.00	2.99	3.89
26	-0.025	0.025	0.200	-0.200	2.00	1.00	3.78	4.21
27	-0.025	0.025	-0.200	0.200	2.00	1.00	8.96	4.57
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	10.7	5.07
29	-0.025	-0.025	0.200	0.200	2.00	1.00	3.00	1.77
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	2.92	1.73
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	3.61	1.98
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	3.99	2.26
33	0.025	0.025	0.300	0.300	2.00	1.00	5.33	1.34
34	0.025	0.025	0.300	-0.300	2.00	1.00	7.26	1.35
35	0.025	0.025	-0.300	0.300	2.00	1.00	9.18	1.70
36	0.025	0.025	-0.300	-0.300	2.00	1.00	10.8	1.95
37	0.025	-0.025	0.300	0.300	2.00	1.00	4.30	0.970
38	0.025	-0.025	0.300	-0.300	2.00	1.00	4.58	0.963
39	0.025	-0.025	-0.300	0.300	2.00	1.00	4.54	1.24
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	5.23	1.43
41	-0.025	0.025	0.300	0.300	2.00	1.00	4.19	1.07
42	-0.025	0.025	0.300	-0.300	2.00	1.00	5.08	1.09
43	-0.025	0.025	-0.300	0.300	2.00	1.00	6.03	1.41
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	7.12	1.61
45	-0.025	-0.025	0.300	0.300	2.00	1.00	2.05	0.701
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	2.09	0.692
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	2.49	0.926
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	2.88	1.07

Table 5.1.23 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



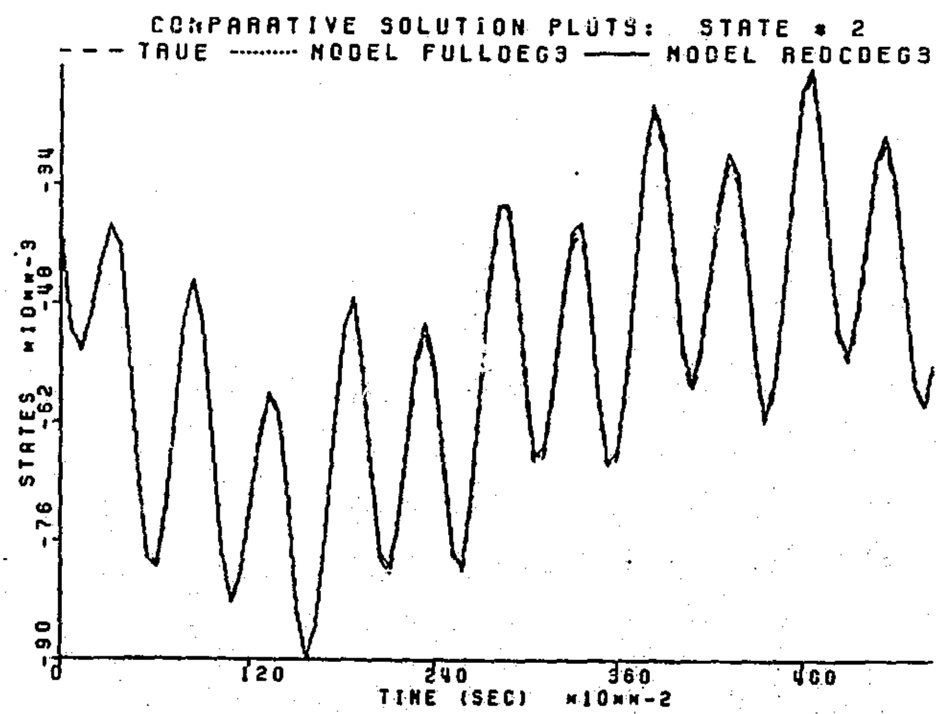


Figure 5.1.35 Simulation Number 30 of Table 5.1.23

ORIGINAL PAGE IS
OF POOR QUALITY

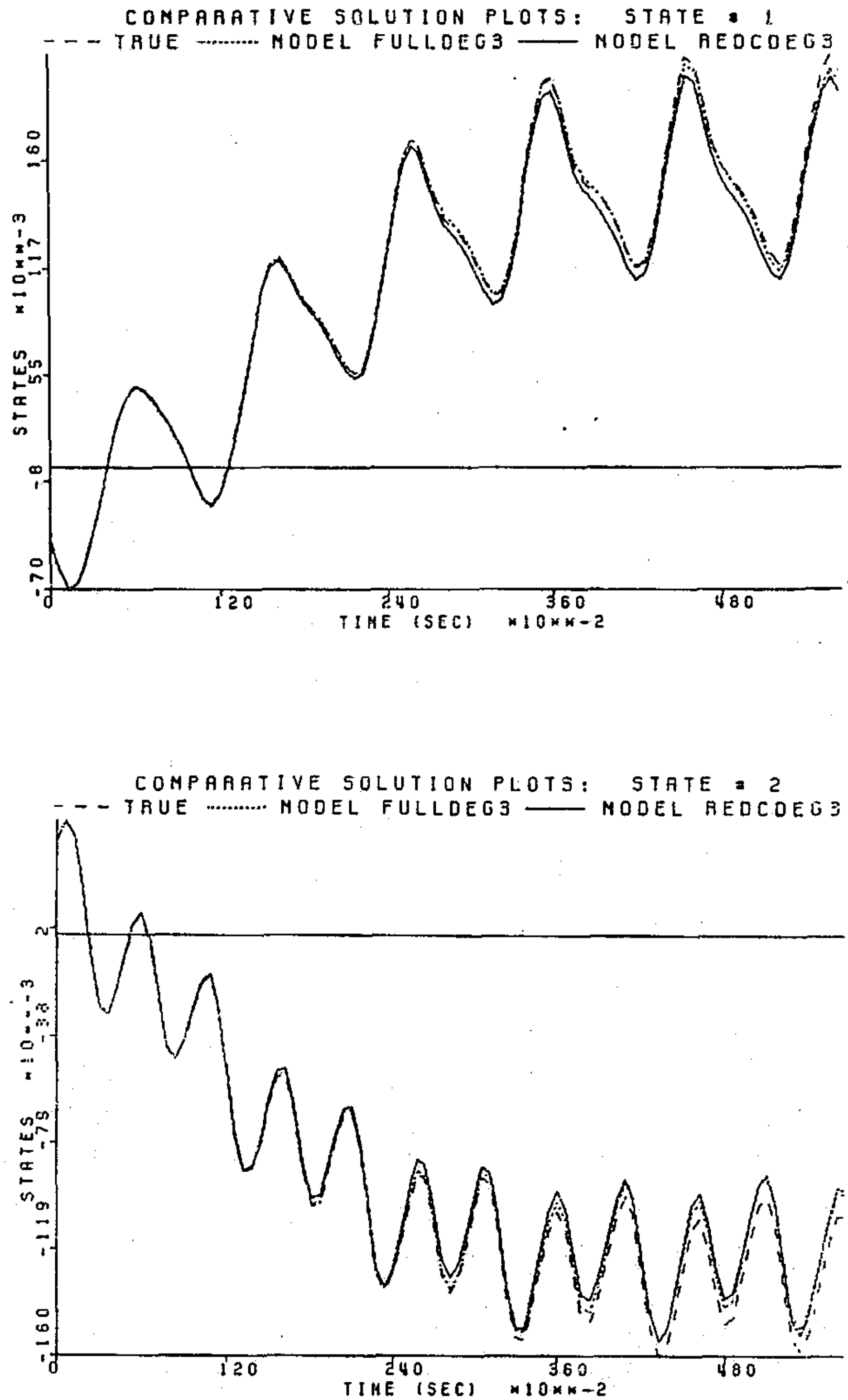
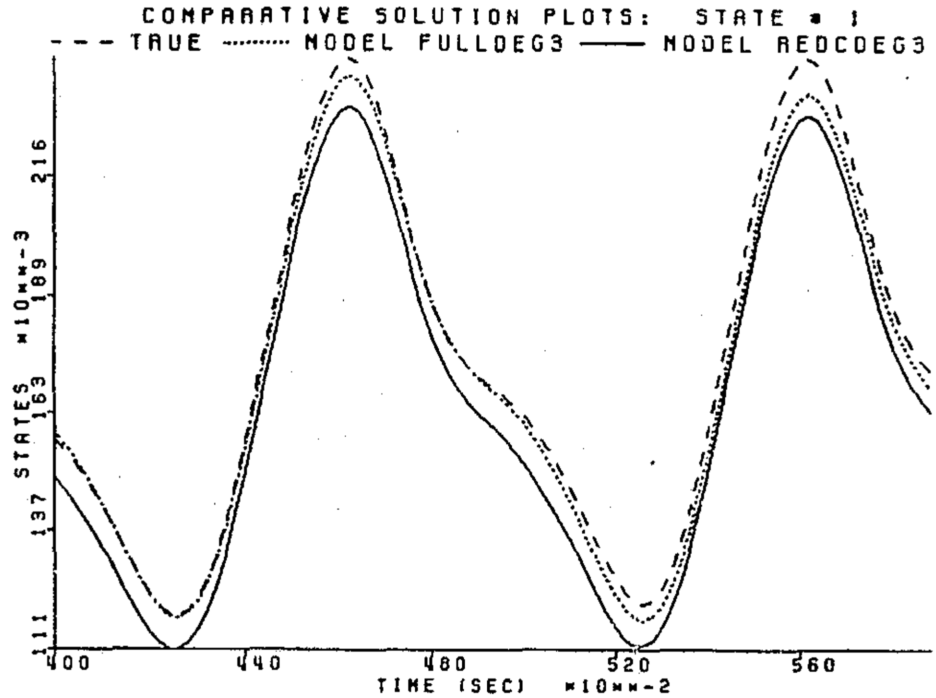


Figure 5.1.36 Simulation Number 44 of Table 5.1.23



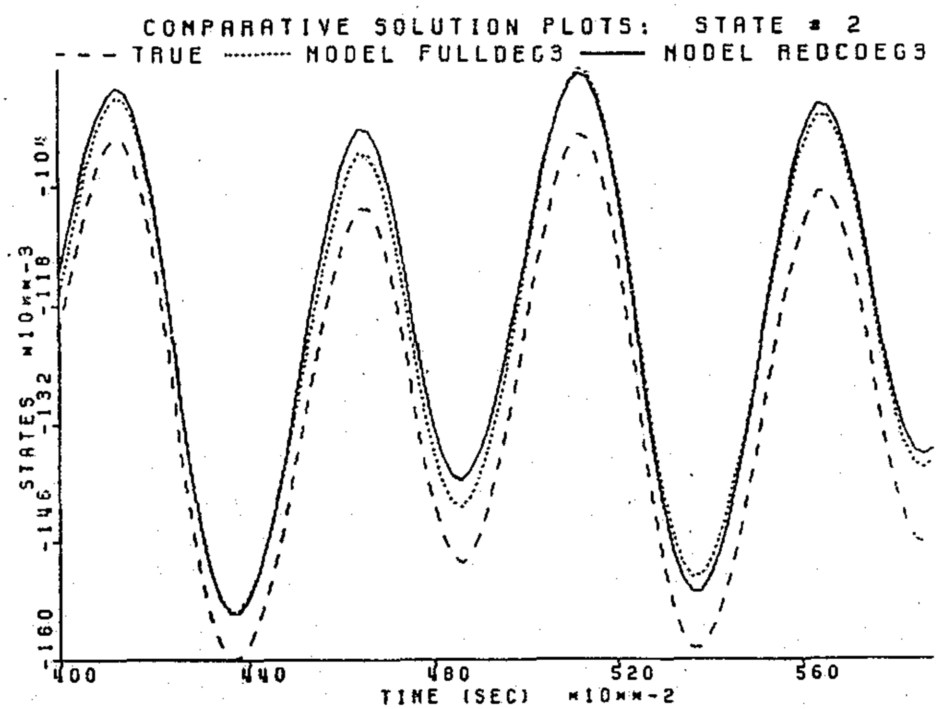
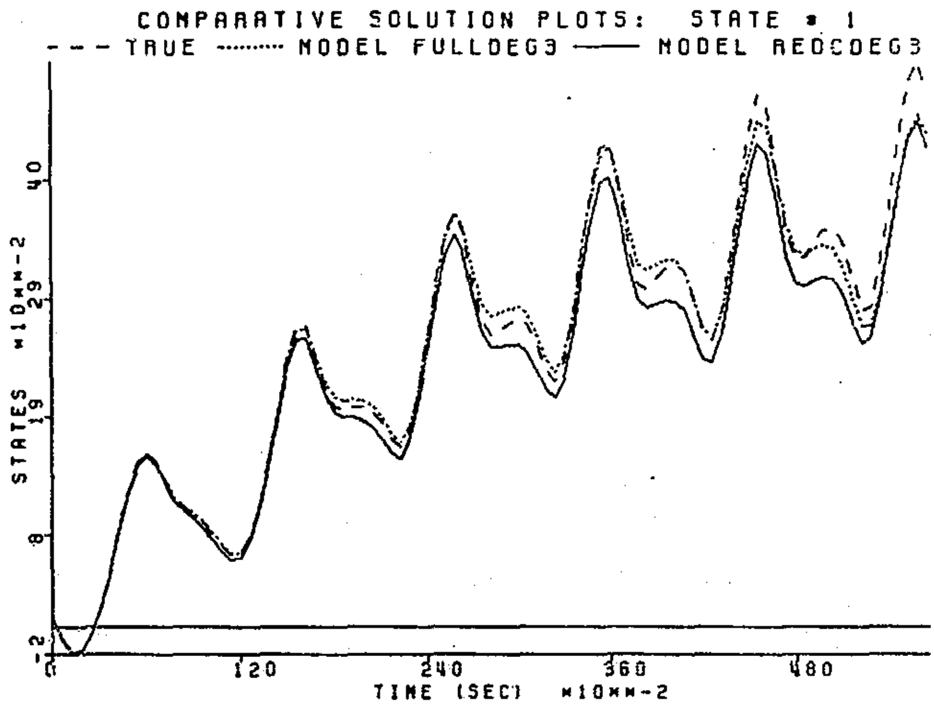


Figure 5.1.37 Expanded Section of Figure 5.1.36

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.400	0.400	2.00	1.00	2.12	0.478
2	0.025	0.025	0.400	-0.400	2.00	1.00	2.71	0.497
3	0.025	0.025	-0.400	0.400	2.00	1.00	4.32	0.698
4	0.025	0.025	-0.400	-0.400	2.00	1.00	4.94	0.816
5	0.025	-0.025	0.400	0.400	2.00	1.00	1.84	0.395
6	0.025	-0.025	0.400	-0.400	2.00	1.00	2.07	0.399
7	0.025	-0.025	-0.400	0.400	2.00	1.00	2.57	0.555
8	0.025	-0.025	-0.400	-0.400	2.00	1.00	3.06	0.648
9	-0.025	0.025	0.400	0.400	2.00	1.00	1.82	0.417
10	-0.025	0.025	0.400	-0.400	2.00	1.00	2.17	0.429
11	-0.025	0.025	-0.400	0.400	2.00	1.00	3.17	0.606
12	-0.025	0.025	-0.400	-0.400	2.00	1.00	3.71	0.709
13	-0.025	-0.025	0.400	0.400	2.00	1.00	1.26	0.334
14	-0.025	-0.025	0.400	-0.400	2.00	1.00	1.34	0.333
15	-0.025	-0.025	-0.400	0.400	2.00	1.00	1.70	0.467
16	-0.025	-0.025	-0.400	-0.400	2.00	1.00	2.03	0.549
17	0.025	0.025	0.500	0.500	2.00	1.00	0.307	0.110
18	0.025	0.025	0.500	-0.500	2.00	1.00	0.418	0.132
19	0.025	0.025	-0.500	0.500	2.00	1.00	0.919	0.231
20	0.025	0.025	-0.500	-0.500	2.00	1.00	0.951	0.299
21	0.025	-0.025	0.500	0.500	2.00	1.00	0.318	0.801E-01
22	0.025	-0.025	0.500	-0.500	2.00	1.00	0.410	0.920E-01
23	0.025	-0.025	-0.500	0.500	2.00	1.00	0.723	0.165
24	0.025	-0.025	-0.500	-0.500	2.00	1.00	0.875	0.224
25	-0.025	0.025	0.500	0.500	2.00	1.00	0.288	0.894E-01
26	-0.025	0.025	0.500	-0.500	2.00	1.00	0.380	0.106
27	-0.025	0.025	-0.500	0.500	2.00	1.00	0.787	0.192
28	-0.025	0.025	-0.500	-0.500	2.00	1.00	0.875	0.255
29	-0.025	-0.025	0.500	0.500	2.00	1.00	0.254	0.596E-01
30	-0.025	-0.025	0.500	-0.500	2.00	1.00	0.317	0.665E-01
31	-0.025	-0.025	-0.500	0.500	2.00	1.00	0.518	0.127
32	-0.025	-0.025	-0.500	-0.500	2.00	1.00	0.678	0.181

Table 5.1.24 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



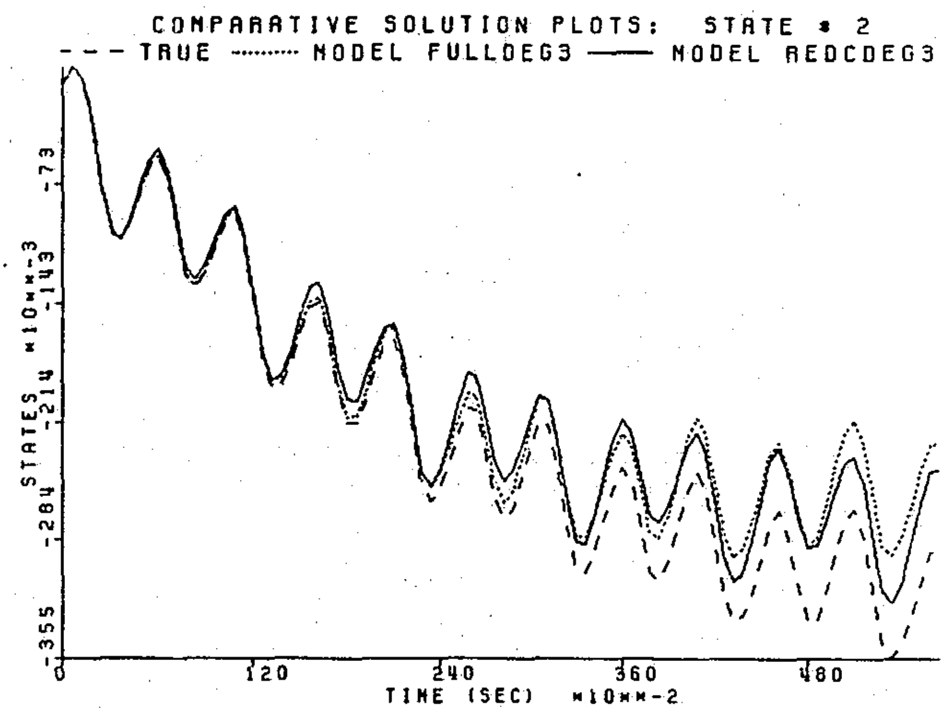
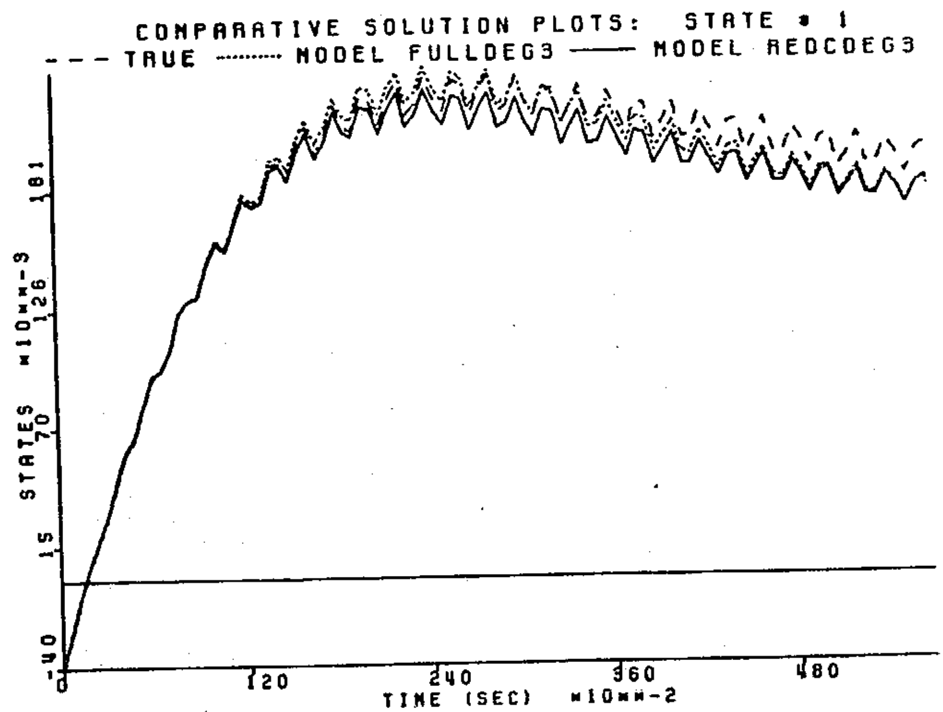


Figure 5.1.38 Simulation Number 8 of Table 5.1.24

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.050	-0.050	-0.401	0.128	4.13	2.26	1.41	0.485
2	-0.050	-0.050	0.160	0.133	4.63	2.84	0.557	0.503
3	-0.050	-0.050	0.442	-0.320	2.12	3.38	0.885	0.273
4	-0.050	-0.050	-0.085	-0.051	3.16	2.36	0.273E-01	0.420E-01
5	-0.050	-0.050	0.182	0.441	4.01	3.72	2.41	2.10
6	-0.050	-0.050	0.027	0.306	2.59	3.58	0.168E-01	0.195E-01
7	-0.050	-0.050	0.027	0.054	2.65	4.01	0.190	0.214
8	-0.050	-0.050	0.179	0.325	4.01	2.25	0.274	0.233
9	-0.050	-0.050	-0.315	-0.431	3.68	4.00	1.09	0.677
10	-0.050	-0.050	0.244	0.131	3.03	2.65	1.57	1.04
11	-0.050	-0.050	0.233	0.208	4.29	5.54	1.01	0.684
12	-0.050	-0.050	-0.308	0.001	4.98	3.72	1.62	0.799
13	-0.050	-0.050	-0.368	0.140	5.56	2.95	1.48	0.572
14	-0.050	-0.050	-0.084	-0.285	5.38	3.84	0.376E-02	0.143E-01
15	-0.050	-0.050	0.299	-0.147	3.14	2.88	1.57	0.807
16	-0.050	-0.050	0.319	-0.159	5.56	5.36	0.821	0.537
17	-0.050	-0.050	-0.318	0.363	4.84	5.60	0.937	0.472
18	-0.050	-0.050	-0.239	-0.343	3.87	4.33	0.765	0.572
19	-0.050	-0.050	-0.173	0.414	4.25	4.06	1.96	1.85
20	-0.050	-0.050	-0.347	-0.064	3.40	4.48	1.51	0.635
21	-0.050	-0.050	0.267	0.438	4.00	2.39	0.548	0.322
22	-0.050	-0.050	0.423	0.153	2.08	3.57	1.22	0.365
23	-0.050	-0.050	0.041	0.365	5.66	3.23	0.858E-02	0.103E-01
24	-0.050	-0.050	-0.262	0.126	2.67	5.00	1.33	0.813
25	-0.050	-0.050	-0.127	-0.433	4.19	2.07	0.576E-01	0.790E-01
26	-0.050	-0.050	-0.225	0.329	4.35	3.19	0.555	0.392
27	-0.050	-0.050	0.020	-0.008	3.14	2.91	0.260	0.294
28	-0.050	-0.050	0.402	0.083	5.20	4.54	1.50	0.509
29	-0.050	-0.050	0.370	-0.295	5.56	2.06	1.27	0.492
30	-0.050	-0.050	-0.105	-0.250	2.30	3.72	0.250E-01	0.414E-01
31	-0.050	-0.050	0.151	-0.268	4.92	4.63	1.34	1.48
32	-0.050	-0.050	-0.078	-0.270	4.08	4.69	0.391E-01	0.887E-01
33	-0.050	-0.050	0.066	-0.307	5.55	2.45	0.127E-02	0.506E-02
34	-0.050	-0.050	0.394	0.139	2.89	5.62	1.38	0.475
35	-0.050	-0.050	0.386	-0.324	2.09	4.98	1.07	0.374
36	-0.050	-0.050	0.251	-0.342	4.71	5.03	1.29	0.857
37	-0.050	-0.050	0.336	0.368	3.66	2.50	1.03	0.467
38	-0.050	-0.050	-0.429	-0.317	3.12	2.35	1.07	0.358
39	-0.050	-0.050	-0.418	0.328	5.15	3.49	1.14	0.393
40	-0.050	-0.050	0.133	-0.319	3.00	3.39	0.719	0.937
41	-0.050	-0.050	-0.096	0.362	4.70	4.08	0.928E-01	0.168
42	-0.050	-0.050	-0.156	-0.370	2.46	4.10	0.870E-01	0.872E-01
43	-0.050	-0.050	0.334	-0.431	3.13	4.73	0.800	0.367
44	-0.050	-0.050	-0.202	0.151	2.74	3.98	0.884	0.696
45	-0.050	-0.050	-0.449	-0.062	2.25	2.36	1.45	0.385

Table 5.1.25 Simulation Table for Third Degree Full Model versus Third Degree Reduced Model



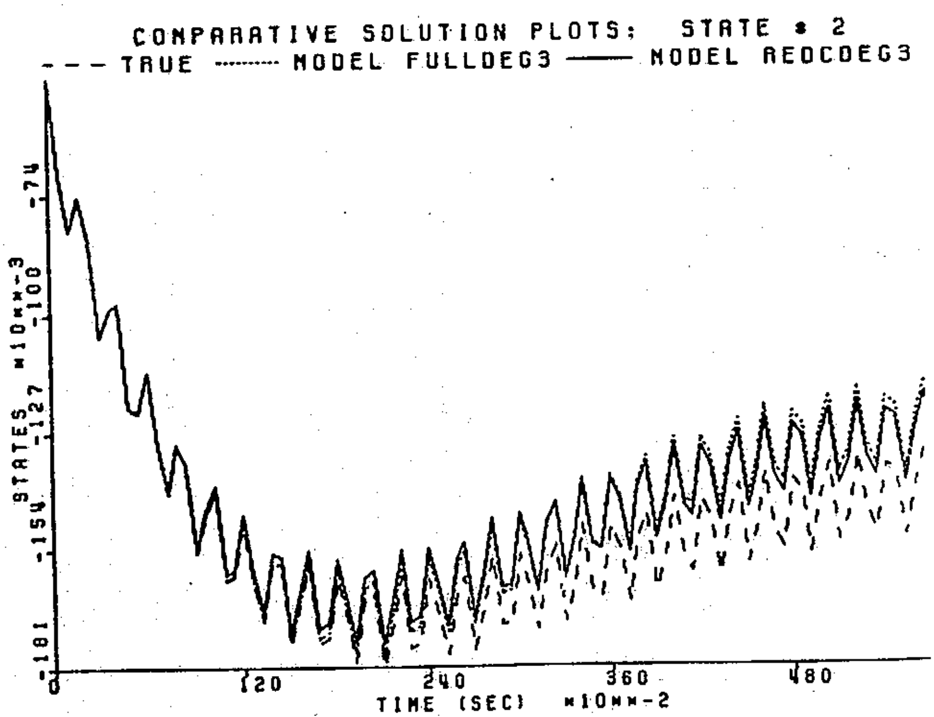
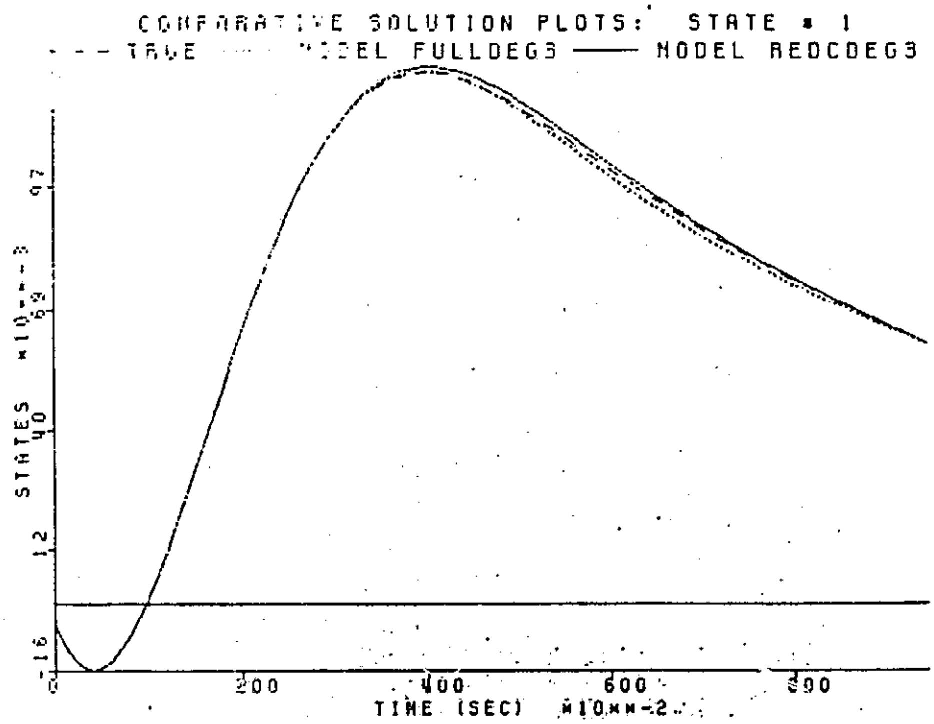


Figure 5.1.39 Simulation Number 3 of Table 5.1.25

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  34
DEGREE OF APPROXIMATION: 3
# OF TERMS IN MODEL 2:  13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	1.42	0.534
2	0.001	-0.001	0.010	-0.075	0.01	0.01	1.16	0.492
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.771	0.467
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.496	0.209
5	0.001	-0.001	-0.050	0.050	0.00	0.00	0.122E-01	0.311E-02
6	0.001	-0.001	-0.050	0.050	0.01	0.01	0.961E-02	0.274E-02
7	0.001	-0.001	-0.050	0.050	0.02	0.02	0.337E-02	0.117E-02
8	0.001	-0.001	-0.050	0.050	0.05	0.05	0.245E-02	0.229E-02
9	0.001	-0.001	-0.075	0.010	0.00	0.00	0.272E-01	0.671E-02
10	0.001	-0.001	-0.075	0.010	0.01	0.01	0.228E-01	0.627E-02
11	0.001	-0.001	-0.075	0.010	0.02	0.02	0.129E-01	0.433E-02
12	0.001	-0.001	-0.075	0.010	0.05	0.05	0.791E-02	0.473E-02
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.541E-01	0.335E-01
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.426E-01	0.267E-01
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.219E-01	0.164E-01
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.839E-02	0.656E-02
17	0.010	-0.010	0.010	-0.075	0.00	0.00	1.48	0.538
18	0.010	-0.010	0.010	-0.075	0.01	0.01	1.20	0.495
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.791	0.468
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.504	0.208
21	0.010	-0.010	-0.050	0.050	0.00	0.00	0.110E-01	0.293E-02
22	0.010	-0.010	-0.050	0.050	0.01	0.01	0.858E-02	0.256E-02
23	0.010	-0.010	-0.050	0.050	0.02	0.02	0.283E-02	0.102E-02
24	0.010	-0.010	-0.050	0.050	0.05	0.05	0.213E-02	0.215E-02
25	0.010	-0.010	-0.075	0.010	0.00	0.00	0.246E-01	0.629E-02
26	0.010	-0.010	-0.075	0.010	0.01	0.01	0.206E-01	0.584E-02
27	0.010	-0.010	-0.075	0.010	0.02	0.02	0.116E-01	0.392E-02
28	0.010	-0.010	-0.075	0.010	0.05	0.05	0.696E-02	0.430E-02
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.530E-01	0.333E-01
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.417E-01	0.264E-01
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.205E-01	0.155E-01
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.779E-02	0.599E-02

Table 5.1.26 Low Frequency Table for Third Degree Full Model versus Third Degree Reduced Model



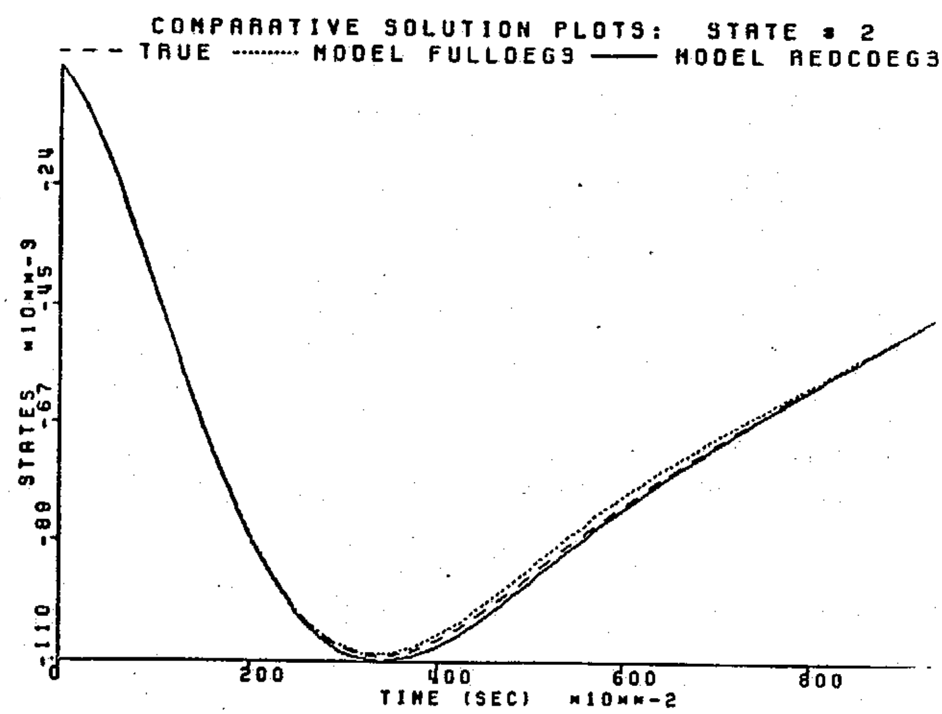
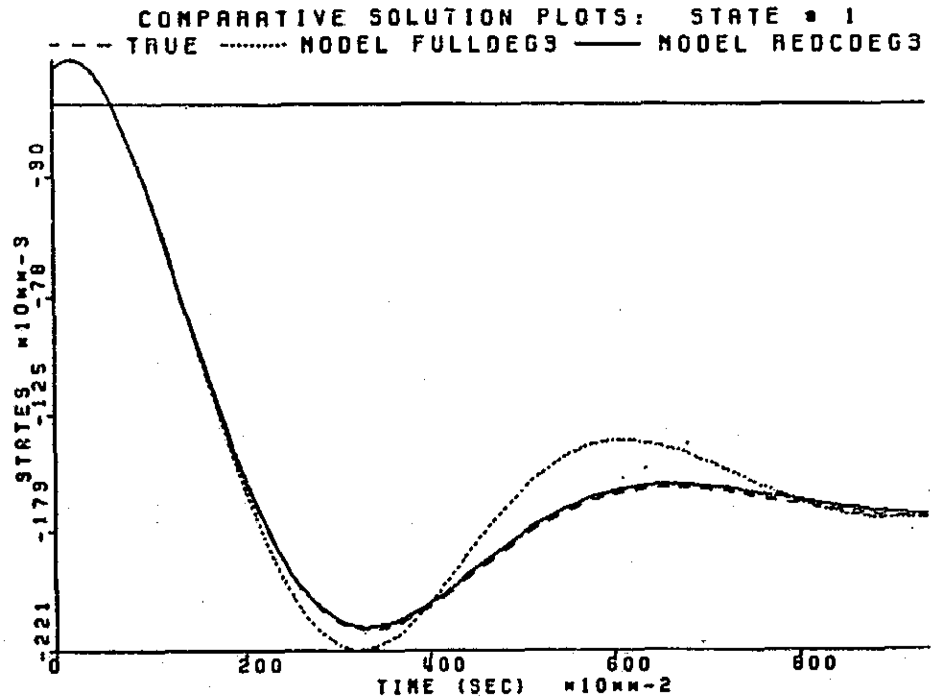


Figure 5.1.40 Simulation Number 3 of Table 5.1.26



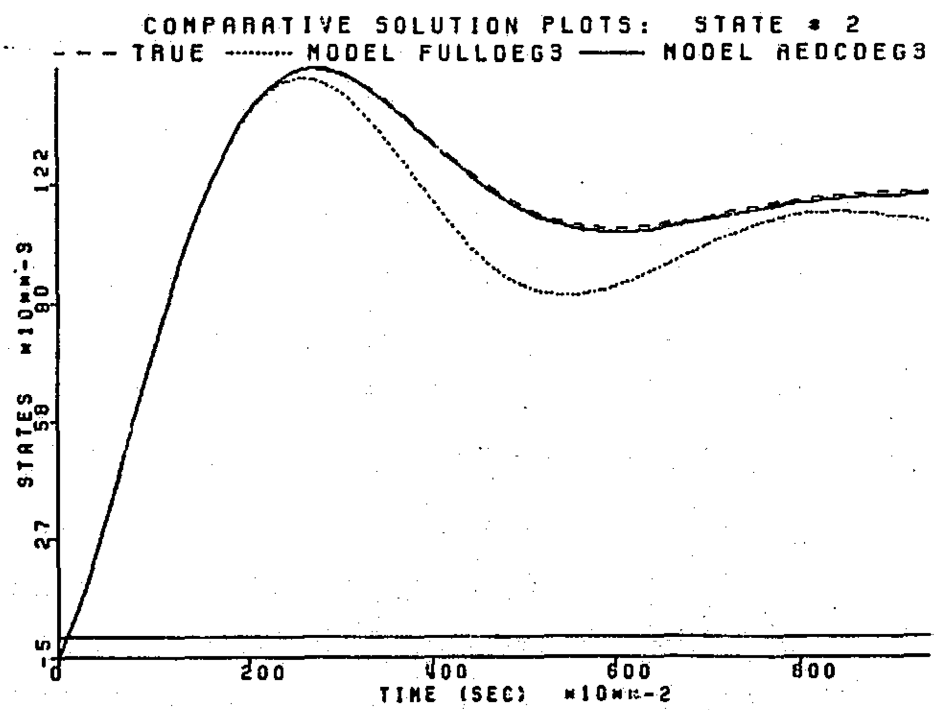
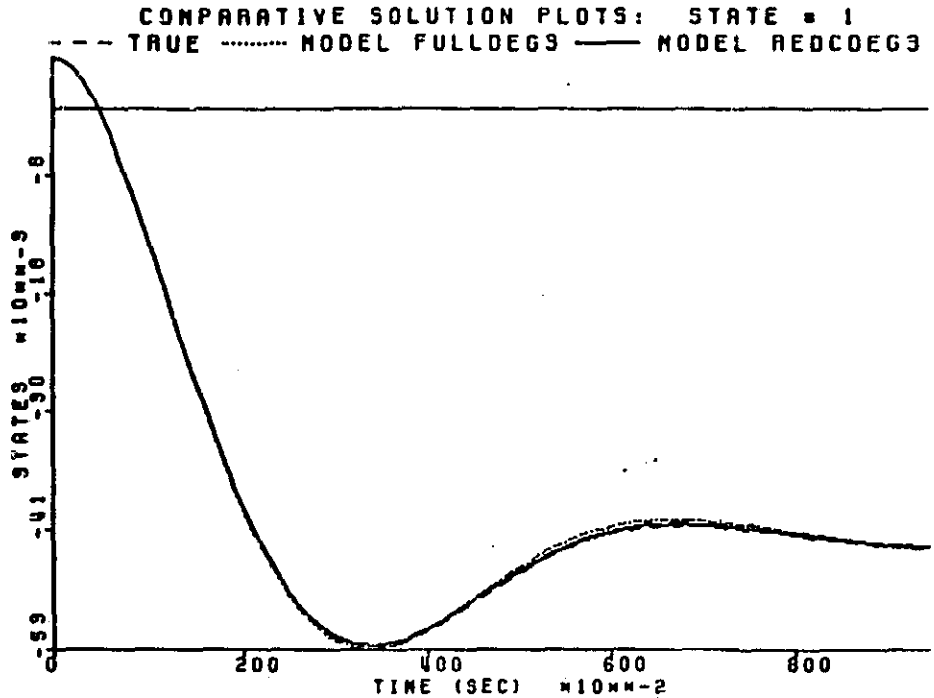


Figure 5.1.41 Simulation Number 21 of Table 5.1.26

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 34
DEGREE OF APPROXIMATION: 3
OF TERMS IN MODEL 2: 13
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.000	0.010	-0.022	-0.008	0.00	0.00	0.105	0.383E-01
2	-0.006	0.004	0.008	-0.011	0.00	0.00	6.67	9.18
3	0.009	0.006	-0.010	0.017	0.00	0.00	0.119	0.446E-01
4	0.006	0.010	0.022	-0.010	0.00	0.00	0.136	0.644E-01
5	0.006	-0.007	-0.018	-0.003	0.00	0.00	0.137	0.478E-01
6	-0.004	0.001	-0.021	-0.005	0.00	0.00	0.144	0.457E-01
7	0.006	0.003	0.012	0.018	0.00	0.00	42.3	10.4
8	-0.009	-0.003	-0.021	0.024	0.00	0.00	0.510E-01	0.179E-01
9	-0.000	0.003	0.004	0.003	0.00	0.00	0.273	0.107
10	-0.000	0.002	-0.009	-0.013	0.00	0.00	0.346	0.153
11	0.009	0.001	0.011	0.011	0.00	0.00	19.5	7.18
12	0.002	-0.008	0.017	0.002	0.00	0.00	1.40	2.19
13	0.005	-0.008	-0.019	-0.010	0.00	0.00	0.115	0.398E-01
14	-0.004	-0.005	0.022	-0.018	0.00	0.00	0.288E-01	0.143E-01
15	-0.007	0.007	-0.001	-0.020	0.00	0.00	8.16	4.11
16	-0.010	0.005	-0.005	-0.006	0.00	0.00	0.740	0.245
17	-0.007	0.006	-0.018	0.009	0.00	0.00	0.137	0.439E-01
18	-0.001	-0.004	0.015	0.021	0.00	0.00	17.3	5.51
19	0.008	-0.005	0.007	0.017	0.00	0.00	15.8	8.09
20	0.003	-0.003	-0.016	-0.008	0.00	0.00	0.137	0.460E-01
21	0.004	-0.003	0.019	0.006	0.00	0.00	0.553	1.47
22	-0.003	0.006	-0.010	0.007	0.00	0.00	0.227	0.749E-01
23	0.007	-0.002	0.014	-0.012	0.00	0.00	2.91	2.87
24	-0.003	-0.004	0.010	-0.009	0.00	0.00	5.69	9.72
25	-0.000	-0.009	-0.001	-0.013	0.00	0.00	5.21	2.40
26	-0.009	0.007	0.007	-0.015	0.00	0.00	7.88	9.36
27	-0.002	0.002	-0.006	-0.023	0.00	0.00	6.99	3.40
28	0.009	0.002	-0.012	0.012	0.00	0.00	0.133	0.486E-01
29	-0.006	-0.007	-0.003	0.006	0.00	0.00	0.592	0.231
30	0.004	-0.000	-0.000	-0.009	0.00	0.00	5.78	2.41
31	0.001	-0.007	0.022	0.014	0.00	0.00	9.40	7.24
32	0.008	-0.002	0.004	0.024	0.00	0.00	1.27	0.539
33	0.001	-0.007	-0.023	0.018	0.00	0.00	0.633E-01	0.225E-01
34	0.003	-0.005	-0.024	0.024	0.00	0.00	0.464E-01	0.163E-01
35	-0.001	-0.007	0.017	0.012	0.00	0.00	5.05	1.29
36	-0.010	-0.006	0.015	0.014	0.00	0.00	7.37	2.39
37	-0.009	-0.006	-0.004	0.002	0.00	0.00	0.429	0.167
38	0.001	0.008	-0.005	0.020	0.00	0.00	0.201	0.740E-01
39	0.000	-0.001	-0.004	0.025	0.00	0.00	0.155	0.613E-01
40	0.003	-0.009	-0.011	0.009	0.00	0.00	0.174	0.647E-01
41	-0.000	0.004	0.011	-0.006	0.00	0.00	4.88	6.83
42	0.008	-0.005	-0.010	0.010	0.00	0.00	0.185	0.678E-01
43	-0.009	-0.008	0.012	-0.021	0.00	0.00	0.824	0.566
44	0.003	-0.007	-0.021	0.015	0.00	0.00	0.780E-01	0.278E-01
45	0.001	0.007	-0.016	0.012	0.00	0.00	0.117	0.401E-01
46	-0.008	0.004	0.005	0.015	0.00	0.00	7.61	3.71
47	-0.001	-0.002	0.015	0.010	0.00	0.00	3.31	1.07
48	-0.007	0.009	0.018	-0.002	0.00	0.00	1.11	0.993
49	0.006	-0.005	0.004	0.008	0.00	0.00	39.4	27.1
50	-0.006	0.003	-0.011	-0.021	0.00	0.00	2.60	1.68

Table 5.1.27 Step Response Table for Third Degree Full Model versus Third Degree Reduced Model



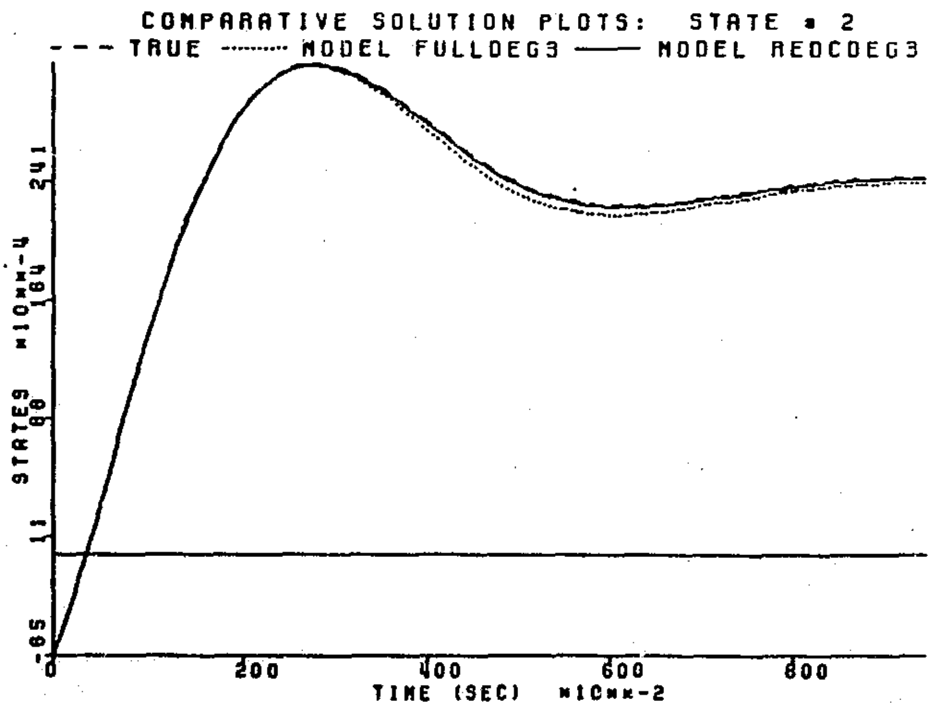
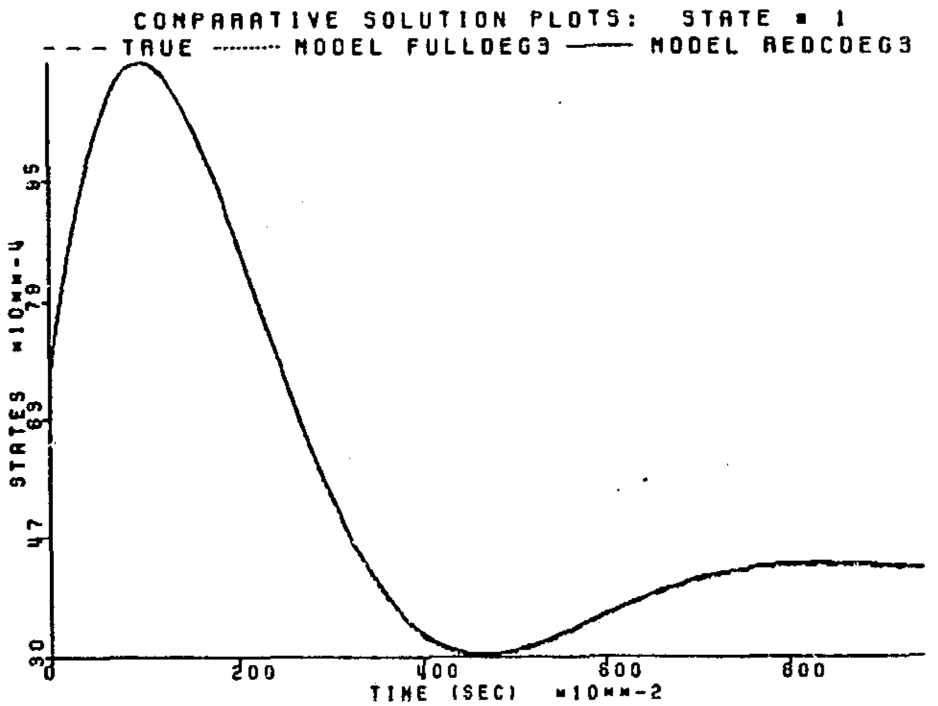


Figure 5.1.42 Simulation Number 13 of Table 5.1.27



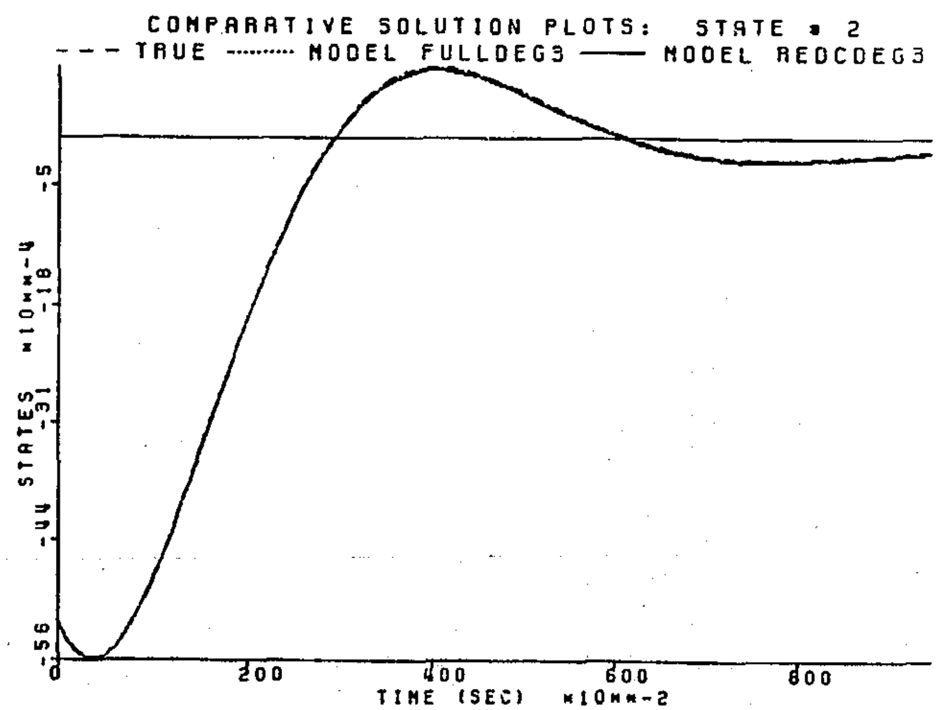


Figure 5.1.43 Simulation Number 49 of Table 5.1.27

duced model behavior is actually better than the full model behavior. So a compromise between the full and reduced models is met, the full model is stable for bigger amplitudes (that is, a larger region of stability), but the reduced model has better low frequency behavior. It is not the purpose here to resolve this decision, but rather to demonstrate that a significantly reduced model can approximate the full model without much loss of higher degree dynamics. Indeed, this was shown.

5.2 THREE STATE, THREE INPUT SYSTEM

In this example, the second degree approximation of a third degree, strictly polynomial, three state system is identified and reduced. The system is

$$\dot{x}_1 = -\frac{1}{2}x_1 + u_3 - 4x_2^2 - 2x_1x_3 + x_2u_1,$$

$$+ 3x_1x_2x_3 + 8x_3u_1u_2 - 5u_3,$$

$$\dot{x}_2 = -x_2 + u_1 + 3x_1x_2 - 2x_2x_3 + u_1u_2$$

$$+ 3x_1^3 - 2x_2x_3 + 4x_2u_2^2,$$

$$\dot{x}_3 = -\frac{3}{2}x_3 + u_2 - 5x_2x_3 + u_1u_3 - 2x_3u_2$$

$$+ 8x_1^3 - 2x_1x_2^2 + 3x_1u_1^2.$$

Again the origin is the point of expansion since it is a stable, equilibrium point. The full second degree approximation will contain terms associated with the following 27 products

$$x_1,\ x_2,\ x_3,\ u_1,\ u_2,\ u_3,\ x_1x_1,\ x_1x_2,\ x_1x_3,$$

$$x_2x_2,\ x_2x_3,\ x_3x_3,\ x_1u_1,\ x_1u_2,\ x_1u_3,\ x_2u_1,$$

$$x_2u_2,\ x_2u_3,\ x_3u_1,\ x_3u_2,\ x_3u_3,\ u_1u_1,\ u_1u_2,$$

$$u_1u_3,\ u_2u_2,\ u_2u_3,\ u_3u_3.$$

A good full model was identified at the initial state conditions (0.03, 0.04, 0.04), and excited with inputs

$$u_1(t) = 0.025 \cos(2\pi t \cdot \phi_1),$$

$$u_2(t) = 0.015 \cos(2\pi t \cdot \phi_2),$$

$$u_3(t) = 0.02 \cos(2\pi t \cdot \phi_3),$$

where (ϕ_1, ϕ_2, ϕ_3) were determined via optimization to be (0.7099, 1.9618, 2.9415) given the starting frequencies (0.75, 1, 3). This starting model and its corresponding reduction test appear in Figure 5.2.1. Directing our attention to this reduction test, we note that there are 11 second degree terms which appear to be good candidates for removal. Note also that the 3 linear input terms appear relatively insignificant, but it was already decided that the linear terms (both state and input terms) should be kept to preserve good local behavior. The 11 terms removed from the model structure are

$$x_1u_1,\ x_1u_2,\ x_1u_3,\ x_2u_2,\ x_2u_3,$$

$$x_3u_1,\ x_3u_3,\ u_1u_1,\ u_2u_2,\ u_2u_3,$$

$$u_3u_3.$$

Note that all terms corresponding to the squares of the states were kept. These terms seem to reflect the appearance of the third degree system nonlinearities. This behavior was also noted in [2] where the example was first studied. The significance of these terms is obvious.

After removal of the set of 11 terms, re-identification was performed. The same initial state conditions and input amplitudes were used, and an optimization over frequencies was performed. The starting frequencies were those used to identify the full model and the local optimization routine yielded a frequency set very near the original set. Again, this bodes well,

```
ENTER INITIAL CONDITIONS FOR THE 3 STATES:
     0.0300       0.0400       0.0400

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100    0.0600    0.0075

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH COSINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          1           1           1

ENTER THE   3 INITIAL INPUT AMPLITUDES:
 0.249999999999999996E-01  0.150000000000000003E-01  0.200000000000000004E-01

ENTER THE 3 FREQUENCY WEIGHTS:
     0.7099       1.9618       2.9415

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N


ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  27 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:          0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



TSO SPEAKEASY III FI+ 11:13 PM FEBRUARY 29, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
```

Figure 5.2.1a Full Second Degree Model and Reduction Test (First Pass)

```
PARTITION NUMBER 1.
  PRTITION (A 3 BY 3 ARRAY)
 -.50043     -2.9762E-4 -.0028604
 -6.0605E-4 -.99939      4.3545E-4
 -6.5976E-5 -5.3277E-4 -1.4982

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 3 COMPONENTS)
 -1.4982 -.99939 -.50043


PARTITION NUMBER 2.
  PRTITION (A 3 BY 3 ARRAY)
  9.1863E-5 -6.8572E-4  .9969
  .99983     -3.1204E-5 -1.928E-5
  4.0466E-6  .99868     -1.1076E-6


PARTITION NUMBER 3.
  PRTITION (A 3 BY 6 ARRAY)
  .053613   .037125 -1.7381   -3.9029   -.49871    .37027
  .084004   3.0326  -.15235   -.042012 -1.9158    .054736
  .097414   .23472  -.21056   -.10562  -5.2205    .37405


PARTITION NUMBER 4.
  PRTITION (A 3 BY 9 ARRAY)
ROW 1  -.028705  -.042196  -.040007   1.0099     .022295   .011026   .019238
        .0030023  .071073
ROW 2  -.0035305  .012927   .022232   .0044049 -.0032042 -.018316  -.010389
       -.014618  -.058518
ROW 3   .0079307 -.033548  -.0011774  .0056223  .034948   .0045049 -.049517
       -2.0196   -.0037745


PARTITION NUMBER 5.
  PRTITION (A 3 BY 6 ARRAY)
  .0071867   .01124    -4.6141E-5  .0028277  -.0037576  -.01968
 -.0026629   .99769     .001181    .020343   -2.9842E-4 -.0010876
  .01167    -.0027814   .9962     -.0015025  -.0041424  -.0015666


  S (A 27 COMPONENT ARRAY)
  10.112    7.5281   7.4573   7.3082   5.5485   5.0816   4.972    4.7653
  3.5474    3.5094   2.6157   2.5262   2.4238   1.1098   .82666   .55558
  .51718    .48339   .43195   .39965   .31808   .30539   .17233   .15784
  .044619   .025358  .029926
 THE MAXIMUM SINGULAR VALUE:
MAX =  10.112
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .025358

 ...AND THEIR RATIO:
RTIO =  398.76


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:    328.5908
FULL STATE ERRORS:    139.9642      40.8687     147.7579

TERM:          X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:    -29.4887
CHANGE IN STATE ERRORS:    -29.4959       0.0107       -0.0035

TERM:          X2.  COLUMN #:   2.
```

Figure 5.2.1b Full Second Degree Model and Reduction Test (First Pass)

```
CHANGE IN SYSTEM ERROR:        26.2132
CHANGE IN STATE ERRORS:        -0.0187       26.2604       -0.0286

TERM:          X3.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:       -55.0800
CHANGE IN STATE ERRORS:        -0.1408       -0.0054      -54.9338

TERM:          U1.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:         6.0935
CHANGE IN STATE ERRORS:         0.0001        6.0933        0.0001

TERM:          U2.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:        -3.9866
CHANGE IN STATE ERRORS:         0.0002       -0.0001       -3.9866

TERM:          U3.  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:         0.9497
CHANGE IN STATE ERRORS:         0.9499       -0.0001       -0.0000

TERM:       X1.X1.  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:         6.0076
CHANGE IN STATE ERRORS:         2.7131       -1.0940        4.3885

TERM:       X1.X2.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:        18.7248
CHANGE IN STATE ERRORS:         1.8695        6.0373       10.8180

TERM:       X1.X3.  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:       -66.5199
CHANGE IN STATE ERRORS:       -60.3416        1.6895       -7.8678

TERM:       X2.X2.  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:      -125.1554
CHANGE IN STATE ERRORS:      -120.9146        0.5453       -4.7861

TERM:       X2.X3.  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:       -93.8660
CHANGE IN STATE ERRORS:       -19.7353       33.1097     -107.2404

TERM:       X3.X3.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:        25.6494
CHANGE IN STATE ERRORS:        13.1292       -0.4948       13.0150

TERM:       X1.U1.  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:        -0.0621
CHANGE IN STATE ERRORS:        -0.2455        0.0428        0.1405

TERM:       X1.U2.  COLUMN #:  14.
CHANGE IN SYSTEM ERROR:         0.2876
CHANGE IN STATE ERRORS:         0.0456        0.0120        0.2300

TERM:       X1.U3.  COLUMN #:  15.
CHANGE IN SYSTEM ERROR:         0.5834
CHANGE IN STATE ERRORS:         0.5657        0.0146        0.0031

TERM:       X2.U1.  COLUMN #:  16.
CHANGE IN SYSTEM ERROR:        16.1320
CHANGE IN STATE ERRORS:        16.0601       -0.0387        0.1105

TERM:       X2.U2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:        -0.1156
CHANGE IN STATE ERRORS:        -0.0386        0.0019       -0.0788

TERM:       X2.U3.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:        -0.1334
CHANGE IN STATE ERRORS:        -0.1107       -0.0002       -0.0225

TERM:       X3.U1.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:        -0.6325
```

Figure 5.2.1c Full Second Degree Model and Reduction Test (First Pass)

```
CHANGE IN STATE ERRORS:        0.2598         0.0735        -0.9658

TERM:      X3.U2.  COLUMN #:  20.
CHANGE IN SYSTEM ERROR:        8.5980
CHANGE IN STATE ERRORS:       -0.0023         0.0002         8.6001

TERM:      X3.U3.  COLUMN #:  21.
CHANGE IN SYSTEM ERROR:       -0.4704
CHANGE IN STATE ERRORS:       -0.5187         0.0362         0.0121

TERM:      U1.U1.  COLUMN #:  22.
CHANGE IN SYSTEM ERROR:        0.7705
CHANGE IN STATE ERRORS:        0.3057         0.0370         0.4278

TERM:      U1.U2.  COLUMN #:  23.
CHANGE IN SYSTEM ERROR:      -22.4779
CHANGE IN STATE ERRORS:       -0.0165       -22.4503        -0.0111

TERM:      U1.U3.  COLUMN #:  24.
CHANGE IN SYSTEM ERROR:      -21.4666
CHANGE IN STATE ERRORS:        0.0001         0.0028       -21.4695

TERM:      U2.U2.  COLUMN #:  25.
CHANGE IN SYSTEM ERROR:       -0.1601
CHANGE IN STATE ERRORS:        0.1269        -0.2379        -0.0492

TERM:      U2.U3.  COLUMN #:  26.
CHANGE IN SYSTEM ERROR:        0.0087
CHANGE IN STATE ERRORS:        0.0146        -0.0017        -0.0042

TERM:      U3.U3.  COLUMN #:  27.
CHANGE IN SYSTEM ERROR:       -0.9049
CHANGE IN STATE ERRORS:       -0.8697         0.0129        -0.0481

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 16

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(10:1,2,3,4,5,6,7,8,9,10)

A = ARRAY(6:11,12,16,20,23,24)

MANUAL MODE
 SPACE USED  99 K NOW,  118 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.2.1d Full Second Degree Model and Reduction Test (First Pass) since we do not wish much change. The resulting sixteen term model and its reduction test appear in Figure 5.2.2. With the exception of the linear control inputs, there are only two terms which appear to be the next candidates for removal. They are the x_1x_1 term and the x_3u_2 term. These terms were removed and the model was re-identified, but simulations showed too much decline in model performance, especially in the third state. (This loss of performance in the third state might have been anticipated since the error test for the third state for each of the two terms contained a more significant entry.) So, we take the 16 term model as the final one. The reduction then is 11 terms removed out of a possible 27; a 41% reduction.

Simulation will be presented in the same fashion as the presentation made for the previous example of second degree model reduction. That is, several tables of simulations of the exact linear model versus the second degree reduced model are given. Following that, simulation tables for the second degree full model versus the second degree reduced model are presented. In all cases, each table is followed by at least one set of plots picked as a representative of those simulations in the table. To facilitate the output, only two of the three states are shown. Usually this includes a worst case and a typical comparison plot.

Tables 5.2.1 through Table 5.2.5 are simulation tables of the exact linear model versus the reduced second degree model. In each of these tables MODEL1 refers to the reduced model, and the error ratio is the mean square error of the reduced model over the mean square error of the linear model. The linear model for this system is

$$L_{10} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}, \quad L_{01} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Table 5.2.1 and Table 5.2.2 contain simulations which hold the frequencies constant while the cosinusoidal input amplitudes are varied along with the initial conditions. Table 5.2.3 studies the models at higher frequencies. Table 5.2.4 varies the control amplitudes and frequencies while holding the initial conditions constant. Note that the third state error ratio of simulation 3 indicates that the linear model outperformed the reduced model. This does not cause much concern as the plot shows the two curves are actually very close. Table 5.2.5 is a low frequency table. The corresponding plots show a typical step response and the worst case response to a very low frequency input. Despite an occasional simulation instance, in each of the five tables, the reduced second degree model outperforms the linear model satisfactorily.

As the final test, the reduced model is compared against the full second degree model from which it was obtained. Now in these tables the labels MODEL1 and MODEL2 refer to the full and reduced models respectively. The error ratios are the mean square errors of the reduced model over the mean square error of the full model. The same five tables of simulations as were calculated previously will be run here. To review, Table 5.2.6 and Table 5.2.7 vary the initial conditions and input amplitudes while the frequencies are held constant. Table 5.2.8 contains simulations of higher frequencies. Table 5.2.9 varies the frequencies while the initial state conditions are held constant. Finally Table 5.2.10 demonstrates the low frequency behavior of the models. Note that in all cases the 16 term model is a good approximation of the full 27 term model. Again it appears that the reduction method works quite well as both of the criteria for judging reduced models were met.

```
ENTER INITIAL CONDITIONS FOR THE 3 STATES:
     0.0300       0.0400       0.0400

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100    0.0600    0.0075

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH COSINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          1            1            1

ENTER THE   3 INITIAL INPUT AMPLITUDES:
 0.249999999999999996E-01  0.150000000000000003E-01  0.200000000000000004E-01

ENTER THE 3 FREQUENCY WEIGHTS:
     0.7099       1.9618       2.9415

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y

ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.999999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2     200    2

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.7076123D+00  0.1970220D+01  0.2940915D+01
THE CONDITION NUMBER:  0.313D+08
OBJECTIVE FUNCTION VALUE: 0.1669D+04

 40 CALLS OF FCN.....
THE PARAMETERS:
 0.7087339D+00  0.1969666D+01  0.2941748D+01
THE CONDITION NUMBER:  0.313D+08
OBJECTIVE FUNCTION VALUE: 0.1668D+04

 60 CALLS OF FCN.....
THE PARAMETERS:
 0.7086491D+00  0.1969523D+01  0.2941611D+01
THE CONDITION NUMBER:  0.313D+08
OBJECTIVE FUNCTION VALUE: 0.1668D+04
```

Figure 5.2.2a Reduced Second Degree Model and Reduction Test (Second Pass)

```
CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  16 ROWS AND 100 COLUMNS.


NUMBER OF TIMES COST FUNCTION WAS EVALUATED:       68

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  0.709     FREQ(2)=  1.970     FREQ(3)=  2.942

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



TSO SPEAKEASY III PI+  3:02 PM MARCH 2, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT
EXECUTION STARTED
  COLSKEPT (A 16 COMPONENT ARRAY)
  1  2  3  4  5  6  7  8  9  10  11  12  16  20  23  24


PARTITION NUMBER 1.
  PRTITION (A 3 BY 3 ARRAY)
 -.5004       -4.9738E-4 -.0035303
 -3.633E-4   -.99922      .0011023
  .0022912  -9.4237E-4 -1.499

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 3 COMPONENTS)
 -1.499  -.99922 -.50041


PARTITION NUMBER 2.
  PRTITION (A 3 BY 3 ARRAY)
  1.0112E-5 -8.3356E-4  .99691
  .99981      2.3861E-5 -1.6467E-4
  1.1036E-5  .99857      8.323E-7


PARTITION NUMBER 3.
  PRTITION (A 3 BY 6 ARRAY)
 -9.8639E-5  .14795    -1.6646    -3.9437    -.55746    .39095
  .11372     2.9703    -.27546    -.05058    -1.7861    .028917
 -.18827     .42369     .074939   -.20082    -5.174     .19199


PARTITION NUMBER 4.
  PRTITION (A 3 BY 9 ARRAY)
ROW 1   0           0          0         .99666     0        0        0
       -.0075251    0
ROW 2   0           0          0        -.0021199   0        0        0
       -.0056813    0
ROW 3   0           0          0        -.0082638   0        0        0
       -2.0038      0


PARTITION NUMBER 5.
  PRTITION (A 3 BY 6 ARRAY)
  0          .016329    9.8794E-4  0          0          0
```

Figure 5.2.2b Reduced Second Degree Model and Reduction Test (Second Pass)

```
 0          .99372    -.0011691   0          0          0
 0          3.1774E-4  .99658     0          0          0


 S (A 16 COMPONENT ARRAY)
 7.3001    6.6372    7.0392    7.1091    4.9926    4.8108    1.9791    1.3043
 1.3379    .80738    .3557     .24935    .18817    .052128   .028725   .036267
 THE MAXIMUM SINGULAR VALUE:
MAX =  7.3001
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .028725

 ...AND THEIR RATIO:
RTIO =  254.14


WANT TO TRY REDUCTION TEST?[Y/N]: Y

FULL SYSTEM ERROR:      326.9198
FULL STATE ERRORS:      137.6369       39.8868       149.3960

TERM:            X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     -29.0252
CHANGE IN STATE ERRORS:     -29.1570       0.0061        0.1258

TERM:            X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:      25.4363
CHANGE IN STATE ERRORS:      -0.0310      25.5185       -0.0513

TERM:            X3.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:     -55.9796
CHANGE IN STATE ERRORS:      -0.1721      -0.0130      -55.7944

TERM:            U1.  COLUMN #:   4.
CHANGE IN SYSTEM ERROR:       6.4543
CHANGE IN STATE ERRORS:       0.0000       6.4541        0.0002

TERM:            U2.  COLUMN #:   5.
CHANGE IN SYSTEM ERROR:      -4.0565
CHANGE IN STATE ERRORS:       0.0010       0.0001       -4.0576

TERM:            U3.  COLUMN #:   6.
CHANGE IN SYSTEM ERROR:       0.8909
CHANGE IN STATE ERRORS:       0.8917      -0.0009        0.0000

TERM:         X1.X1.  COLUMN #:   7.
CHANGE IN SYSTEM ERROR:      -9.7038
CHANGE IN STATE ERRORS:      -0.0049      -1.3848       -8.3140

TERM:         X1.X2.  COLUMN #:   8.
CHANGE IN SYSTEM ERROR:      34.5059
CHANGE IN STATE ERRORS:       7.4622       6.9057       20.1380

TERM:         X1.X3.  COLUMN #:   9.
CHANGE IN SYSTEM ERROR:     -51.6266
CHANGE IN STATE ERRORS:     -57.5502       3.0254        2.8983

TERM:         X2.X2.  COLUMN #:  10.
CHANGE IN SYSTEM ERROR:    -127.9159
CHANGE IN STATE ERRORS:    -119.4493       0.6269       -9.0936

TERM:         X2.X3.  COLUMN #:  11.
CHANGE IN SYSTEM ERROR:    -101.3486
CHANGE IN STATE ERRORS:     -21.7269      29.1325     -108.7542

TERM:         X3.X3.  COLUMN #:  12.
CHANGE IN SYSTEM ERROR:      20.1306
CHANGE IN STATE ERRORS:      13.7498      -0.2507        6.6315
```

Figure 5.2.2c Reduced Second Degree Model and Reduction Test (Second Pass)

```
TERM:       X2.U1.   COLUMN #:  16.
CHANGE IN SYSTEM ERROR:      15.4849
CHANGE IN STATE ERRORS:      15.6285       0.0179       -0.1614

TERM:       X3.U2.   COLUMN #:  20.
CHANGE IN SYSTEM ERROR:       8.7532
CHANGE IN STATE ERRORS:       0.0066      -0.0001        8.7487

TERM:       U1.U2.   COLUMN #:  23.
CHANGE IN SYSTEM ERROR:     -22.3259
CHANGE IN STATE ERRORS:      -0.0164     -22.3106        0.0011

TERM:       U1.U3.   COLUMN #:  24.
CHANGE IN SYSTEM ERROR:     -21.2391
CHANGE IN STATE ERRORS:      -0.0014      -0.0020      -21.2356

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N
MANUAL MODE
 SPACE USED  71 K NOW,  83 K PEAK,  SIZE  500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N
DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.2.2d Reduced Second Degree Model and Reduction Test (Second Pass)

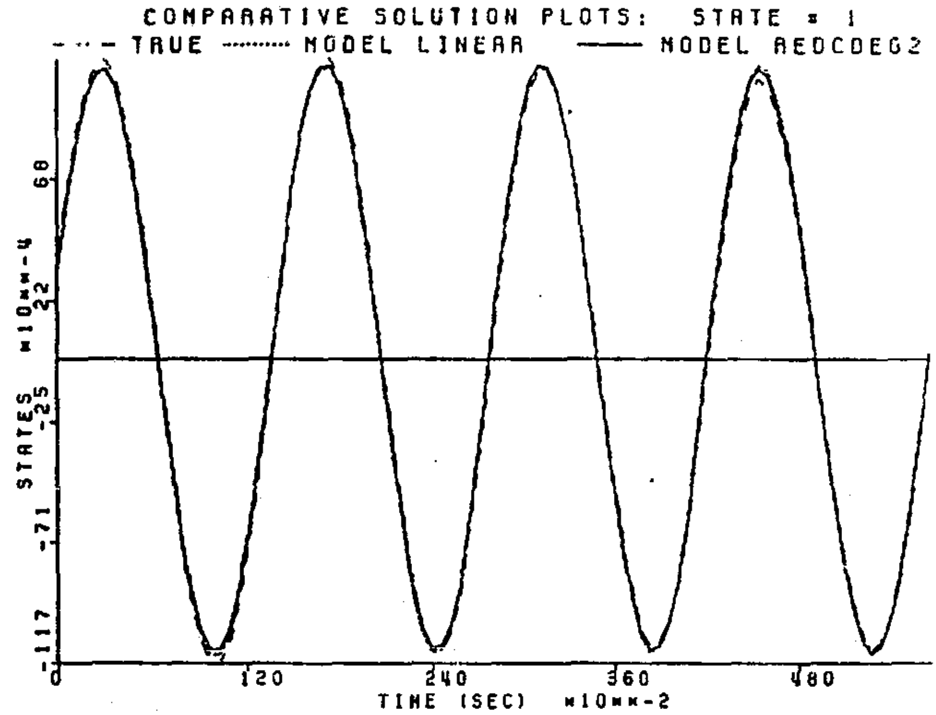
****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 3
OF INPUTS: 3
OF TERMS IN MODEL 1: 16
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.1000E-02	0.1000E-02	0.1000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	0.4513E-01	0.2112E-02	0.1085E-01
2	0.1000E-02	0.1000E-02	0.1000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.1161	0.1327E-02	0.2266E-01
3	0.1000E-02	0.1000E-02	0.1000E-02	0.050	0.050	0.050	0.50	0.60	0.70	0.4214E-01	0.9359E-03	0.1961E-02
4	0.1000E-02	0.1000E-02	0.1000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.3498E-01	0.6510E-03	0.2247E-02
5	0.1000E-02	0.1000E-02	0.1000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.6242E-01	0.2225E-02	0.1788E-02
6	0.1000E-02	0.1000E-02	0.1000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.7869	0.2729E-02	0.2626E-02
7	0.1000E-02	0.1000E-02	0.1000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.8895E-02	0.2838E-02	0.2126E-02
8	0.1000E-02	0.1000E-02	0.1000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.1195E-01	0.1012E-02	0.6083E-02
9	0.1000E-02	0.1000E-02	-.1000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	0.6107E-01	0.2575E-02	0.1108E-01
10	0.1000E-02	0.1000E-02	-.1000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.1263	0.9690E-03	0.2176E-01
11	0.1000E-02	0.1000E-02	-.1000E-02	0.050	0.050	0.050	0.50	0.60	0.70	0.4358E-01	0.9399E-03	0.1963E-02
12	0.1000E-02	0.1000E-02	-.1000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.3449E-01	0.6442E-03	0.2252E-02
13	0.1000E-02	0.1000E-02	-.1000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.6268E-01	0.2224E-02	0.1803E-02
14	0.1000E-02	0.1000E-02	-.1000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.7818	0.2714E-02	0.2628E-02
15	0.1000E-02	0.1000E-02	-.1000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.8686E-02	0.2818E-02	0.2150E-02
16	0.1000E-02	0.1000E-02	-.1000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.1228E-01	0.1007E-02	0.6384E-02
17	-.5000E-02	0.5000E-02	0.5000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	0.5158E-01	0.1110E-01	0.6563E-01
18	-.5000E-02	0.5000E-02	0.5000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.1554	0.3678E-01	0.7552E-01
19	-.5000E-02	0.5000E-02	0.5000E-02	0.050	0.050	0.050	0.50	0.60	0.70	0.3714E-01	0.9543E-03	0.1747E-02
20	-.5000E-02	0.5000E-02	0.5000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.3540E-01	0.8008E-03	0.2380E-02
21	-.5000E-02	0.5000E-02	0.5000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.6275E-01	0.2274E-02	0.1420E-02
22	-.5000E-02	0.5000E-02	0.5000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.7872	0.2800E-02	0.2793E-02
23	-.5000E-02	0.5000E-02	0.5000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.9202E-02	0.3392E-02	0.1784E-02
24	-.5000E-02	0.5000E-02	0.5000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.1148E-01	0.1049E-02	0.4338E-02

Table 5.2.1 Simulation Table for Linear Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	-.5000E-02	-.5000E-02	0.5000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	0.4602E-01	0.1368E-01	0.1667E-01
2	-.5000E-02	-.5000E-02	0.5000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.5347E-01	0.3101E-02	0.4939E-01
3	-.5000E-02	-.5000E-02	0.5000E-02	0.050	0.050	0.050	0.50	0.60	0.70	0.4424E-01	0.9840E-03	0.1763E-02
4	-.5000E-02	-.5000E-02	0.5000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.3679E-01	0.1070E-02	0.2125E-02
5	-.5000E-02	-.5000E-02	0.5000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.6142E-01	0.2018E-02	0.1335E-02
6	-.5000E-02	-.5000E-02	0.5000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.8405	0.5664E-02	0.2898E-02
7	-.5000E-02	-.5000E-02	0.5000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.9917E-02	0.3623E-02	0.1757E-02
8	-.5000E-02	-.5000E-02	0.5000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.1243E-01	0.1111E-02	0.3028E-02
9	-.7500E-01	-.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	0.2301E-01	0.4194E-01	0.4320E-01
10	-.7500E-01	-.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	0.2175E-01	0.4343E-01	0.3713E-01
11	-.7500E-01	-.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	0.3249E-01	0.3987E-01	0.6480E-01
12	-.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	0.1458E-01	0.9203E-01	0.6595E-01
13	-.7500E-01	-.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.4143E-01	0.4989E-01	0.4772E-01
14	-.7500E-01	-.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	0.2474E-01	0.2454	0.3273E-01
15	-.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	0.1396E-01	0.1010	0.3836E-01
16	-.7500E-01	-.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	0.3920E-01	0.2655E-01	0.8108E-01
17	0.7500E-01	-.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	0.6127E-01	0.6150E-02	0.1305
18	0.7500E-01	-.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	0.5673E-01	0.6097E-02	0.1298
19	0.7500E-01	-.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	0.2031E-01	0.9761E-02	0.3337
20	0.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	0.4871E-01	0.7634E-02	0.1597
21	0.7500E-01	-.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.5543E-02	0.1905E-01	0.6717E-01
22	0.7500E-01	-.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	0.1074	0.1629E-01	0.1213
23	0.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	0.5075E-01	0.4430E-02	0.1886
24	0.7500E-01	-.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	0.7857E-02	0.1121E-01	0.2775
25	-.7500E-01	0.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	0.1862E-01	0.1296E-01	0.8064E-03
26	-.7500E-01	0.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	0.2036E-01	0.1374E-01	0.8833E-03
27	-.7500E-01	0.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	0.2132E-01	0.1471E-01	0.8417E-02
28	-.7500E-01	0.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	0.1167E-01	0.6982E-02	0.1494E-01
29	-.7500E-01	0.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.4237E-01	0.1182E-01	0.5298E-01
30	-.7500E-01	0.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	0.2552E-01	0.1267E-01	0.2356E-02
31	-.7500E-01	0.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	0.1671E-01	0.1289E-01	0.1007E-01
32	-.7500E-01	0.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	0.2057E-01	0.1128E-01	0.1657E-01

Table 5.2.2 Simulation Table for Linear Model versus Second Degree Reduced Model



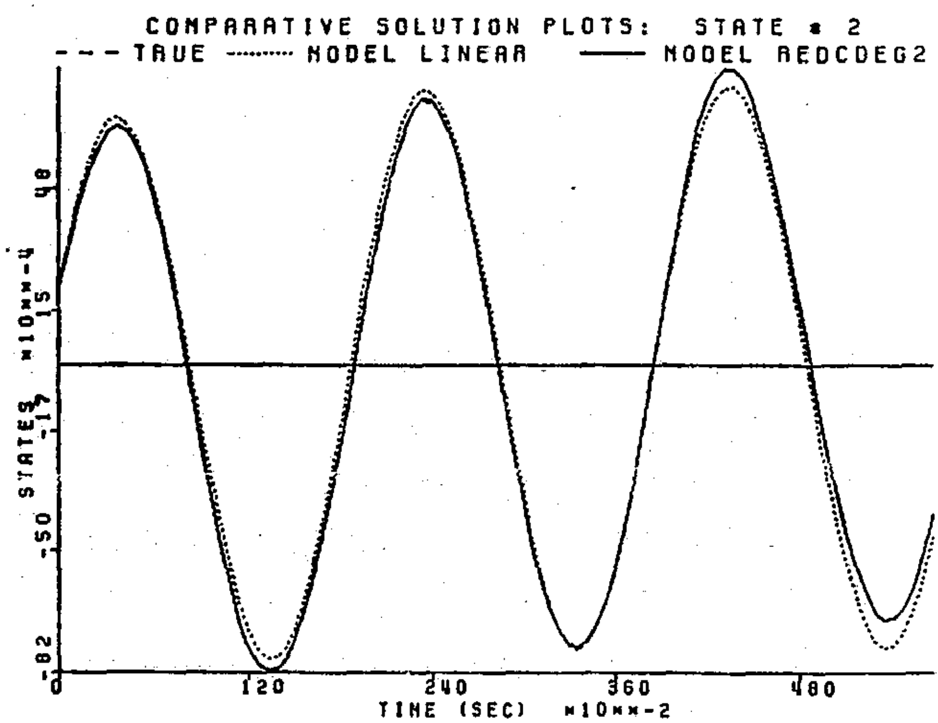


Figure 5.2.3 Simulation Number 6 of Table 5.2.1

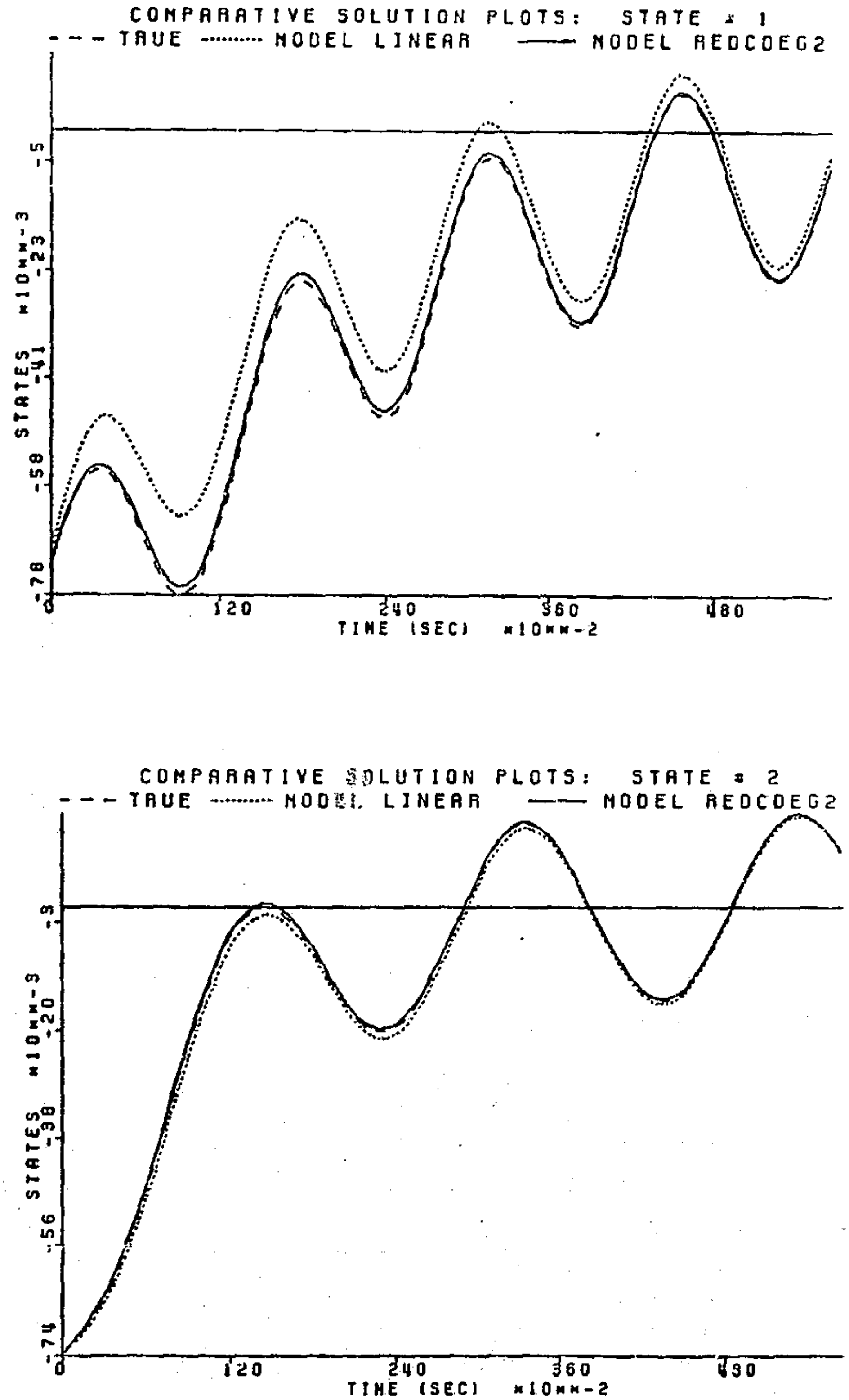


Figure 5.2.4 Simulation Number 15 of Table 5.2.2

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 3
OF INPUTS: 3
OF TERMS IN MODEL 1: 16
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	1.00	2.00	1.00	0.7639E-03	0.1131E-01	0.6534E-02
2	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	2.00	2.00	2.00	0.9999E-04	0.2969E-02	0.5413E-02
3	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	0.50	1.00	1.50	0.5218E-03	0.1303E-01	0.4332E-02
4	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	1.50	1.00	0.50	0.1279E-02	0.1074E-01	0.1789E-01
5	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	1.00	2.00	1.00	0.1105E-02	0.2857E-02	0.1011E-01
6	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	2.00	2.00	2.00	0.2859E-02	0.2811E-02	0.9392E-02
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	0.50	1.00	1.50	0.2052E-02	0.4268E-02	0.3848E-01
8	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	1.50	1.00	0.50	0.1139E-02	0.2861E-02	0.2673E-01
9	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	1.00	2.00	1.00	0.1743E-01	[illegible]	0.2311E-01
10	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	2.00	2.00	2.00	0.3096E-01	0.4764E-02	0.3173E-01
11	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	0.50	1.00	1.50	0.2030E-01	[illegible]	0.3068E-01
12	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	1.50	1.00	0.50	0.1710E-01	0.6896E-01	0.2031E-01
13	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	1.00	2.00	1.00	0.1457E-01	0.1444E-01	0.4192E-01
14	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	2.00	2.00	2.00	0.8380E-02	0.2242E-01	[illegible]
15	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	0.50	1.00	1.50	0.8327E-02	0.1581E-01	0.2163E-01
13	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	1.50	1.00	0.50	0.1750E-01	0.2464E-01	0.5762E-01

Table 5.2.3 Simulation Table for Linear Model versus Second Degree Reduced Model

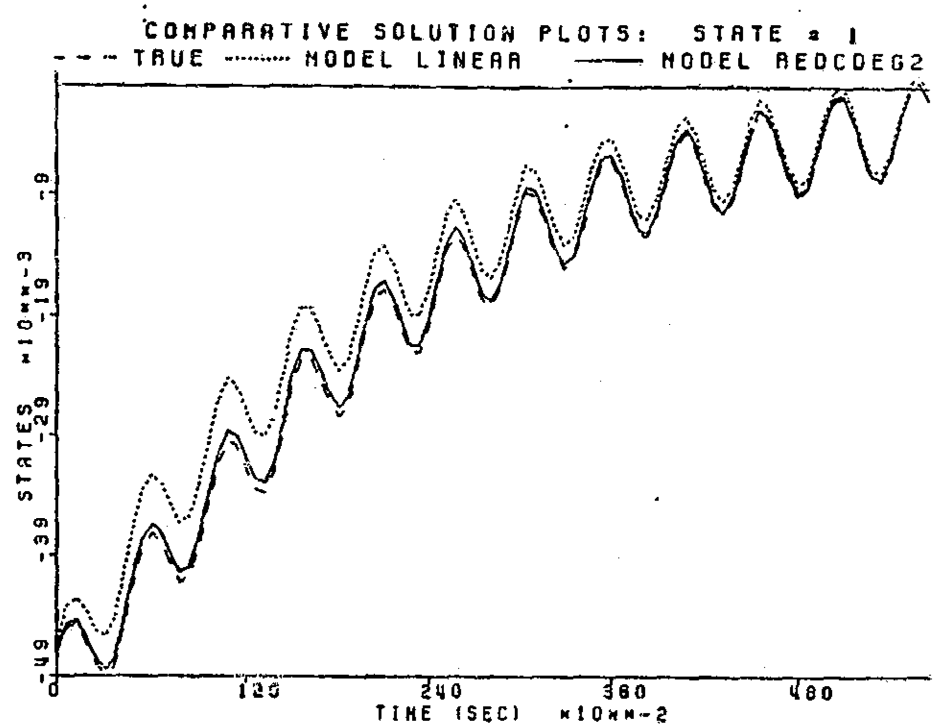
S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.00	0.25	0.50	0.4668E-03	0.2337E-01	0.1146E-01
2	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.50	0.00	0.25	0.2041E-01	0.3175E-01	0.5587E-02
3	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.25	0.50	0.00	0.7300E-02	0.4695E-01	1.038
4	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.25	1.00	0.50	0.4319E-03	0.1147E-01	0.1388E-01
5	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	1.00	0.50	0.25	0.1455E-02	0.1215E-01	0.3617E-01
6	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.50	0.25	1.00	0.4849E-03	0.2027E-01	0.2464E-02
7	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.00	0.25	0.50	0.1058E-02	0.7085E-01	0.4997E-02
8	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.50	0.00	0.02	0.8773E-01	0.1174	0.1341E-02
9	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.25	0.50	0.00	0.2047E-02	0.1754E-01	0.7117E-01
10	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.25	1.00	0.50	0.9068E-03	0.2819E-01	0.6591E-02
11	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	1.00	0.50	0.25	0.2984E-02	0.2663E-01	0.7651E-02
12	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.50	0.25	1.00	0.6246E-02	0.4477E-01	0.9181E-03

Table 5.2.4 Simulation Table for Linear Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.00	0.00	0.00	0.1508E-03	0.2628E-01	0.3399
2	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.02	0.02	0.02	0.1760E-03	0.1754E-01	0.1457
3	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.05	0.05	0.05	0.7053E-03	0.8649E-02	0.4244
4	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.10	0.10	0.10	0.1053E-02	0.1123E-01	0.6259
5	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.02	0.02	0.02	0.2579	0.5215	0.7814
6	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.05	0.05	0.05	0.5982E-01	0.5597E-01	0.7137
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.10	0.10	0.10	0.8089E-02	0.6664E-02	1.416
8	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.00	0.00	0.00	0.4316E-01	0.3526E-01	0.5143E-01
9	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.02	0.02	0.02	0.2966E-01	0.3025E-01	0.3606E-01
10	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.05	0.05	0.05	0.2290E-01	0.2908E-01	0.2374E-01
11	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.10	0.10	0.10	0.3585E-01	0.3734E-01	0.3147E-01
12	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.00	0.00	0.00	0.1200E-02	0.3633E-01	0.4469
13	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.02	0.02	0.02	0.1910E-02	0.3540E-01	0.4788
14	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.05	0.05	0.05	0.2962E-02	0.3492E-01	0.6230
15	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.10	0.10	0.10	0.3721E-02	0.3220E-01	0.7801
16	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.02	0.02	0.02	0.3760	3.549	0.8639
17	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.05	0.05	0.05	0.2039	0.9060	0.8624
18	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.10	0.10	0.10	0.6600E-01	0.1830	0.9858
19	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.00	0.00	0.00	0.4025E-01	0.3138E-01	0.3277E-01
20	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.02	0.02	0.02	0.2503E-01	0.1465E-01	0.1175E-01
21	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.05	0.05	0.05	0.2038E-01	0.9864E-02	0.1934E-01
22	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.10	0.10	0.10	0.3452E-01	0.3809E-01	0.5811E-01

Table 5.2.5 Low Frequency Table for Linear Model versus Second Degree Reduced Model

ORIGINAL PAGE IS
OF POOR QUALITY



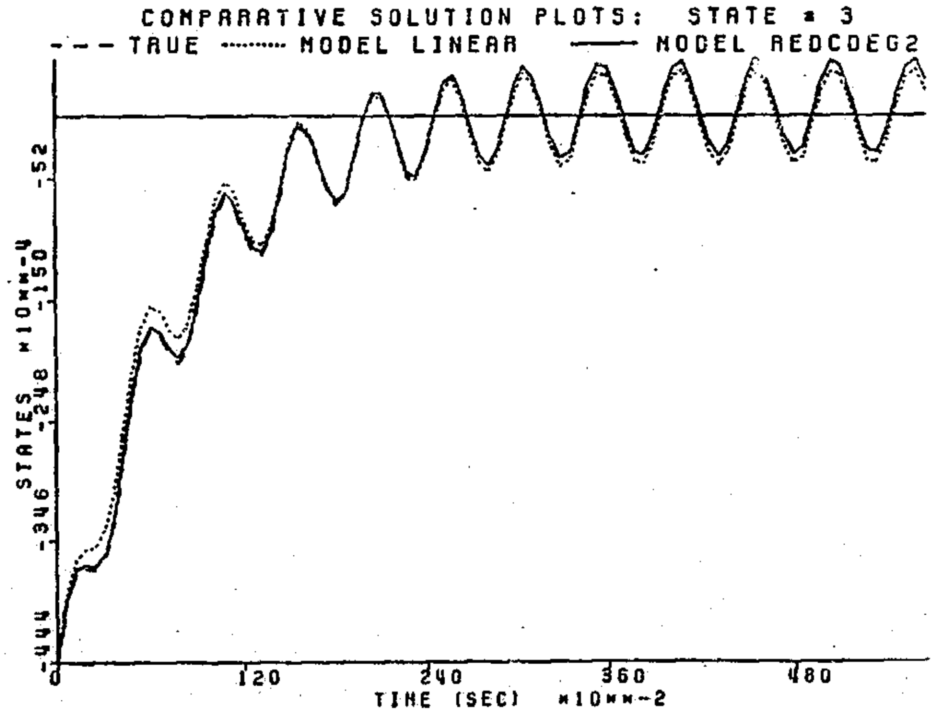
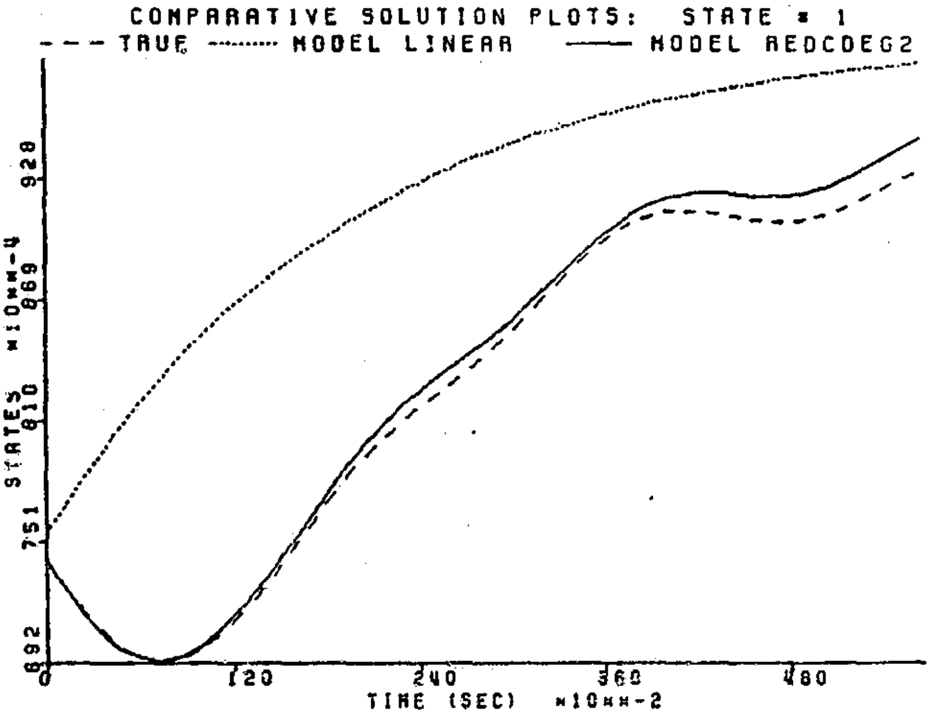


Figure 5.2.5 Simulation Number 10 of Table 5.2.3



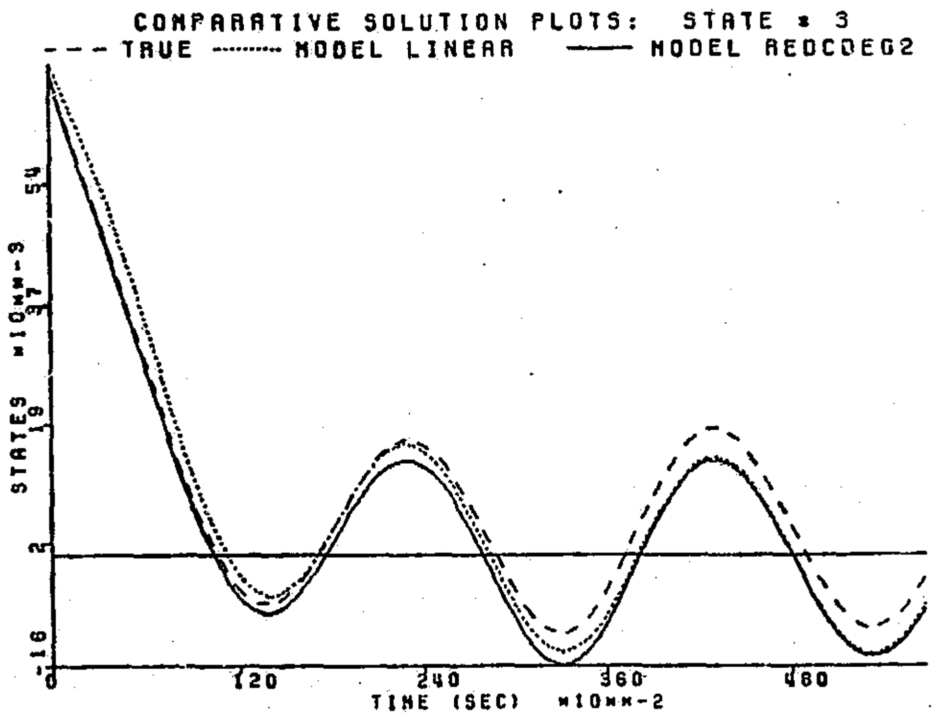
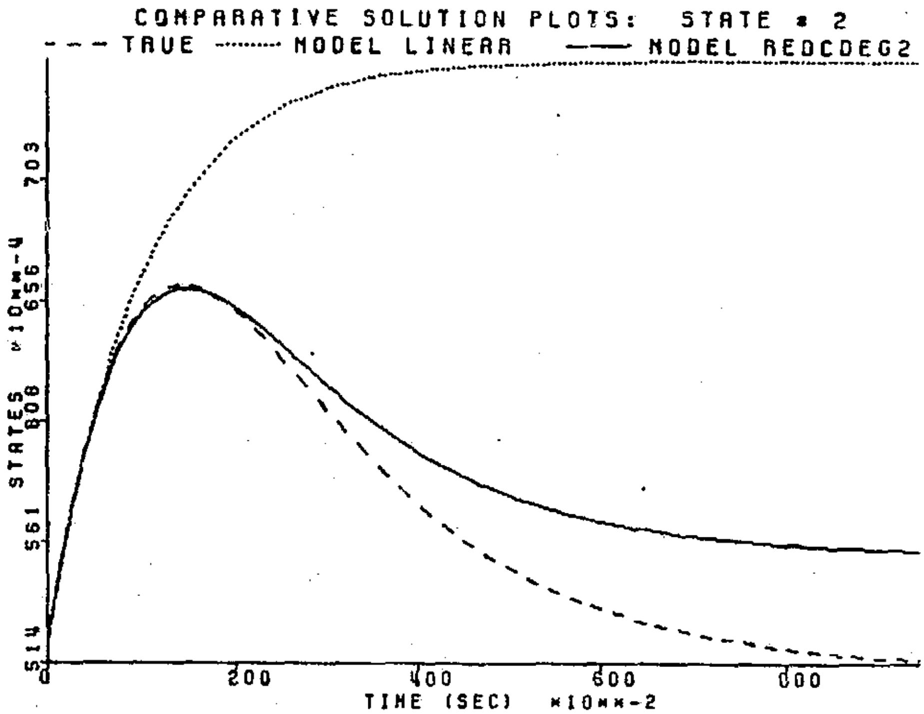


Figure 5.2.6 Simulation Number 3 of Table 5.2.4



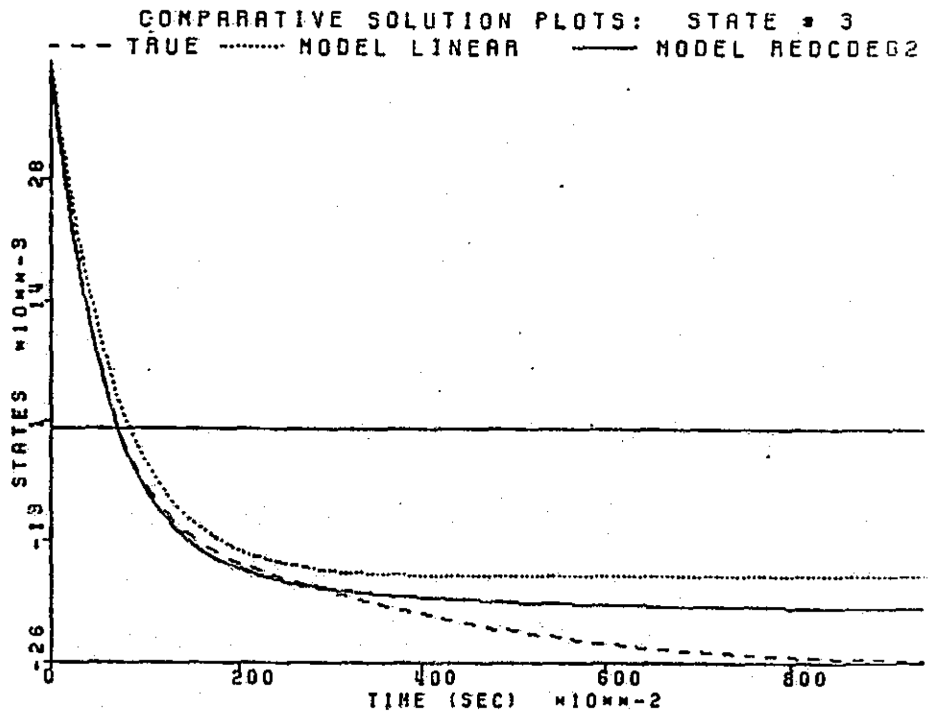
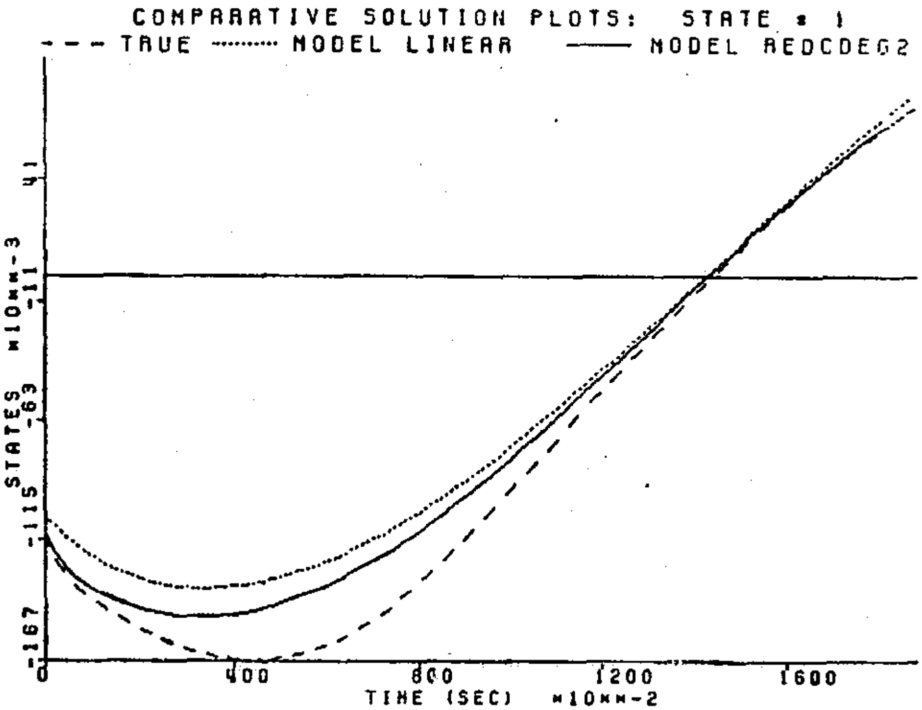


Figure 5.2.7 Simulation Number 1 of Table 5.2.5



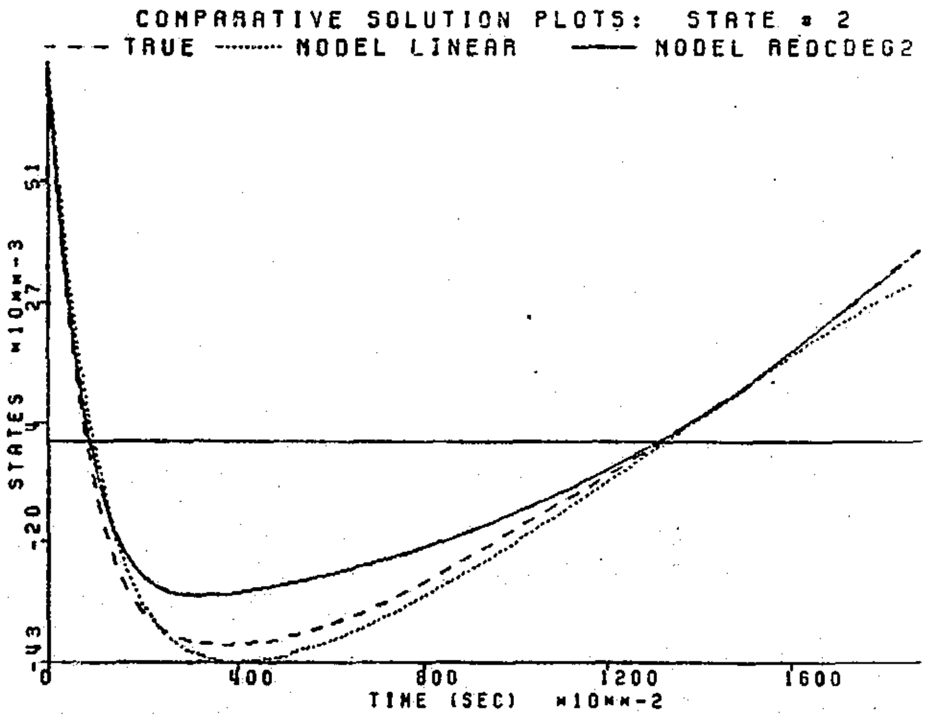


Figure 5.2.8 Simulation Number 16 of Table 5.2.5

ORIGINAL PAGE IS
OF POOR QUALITY

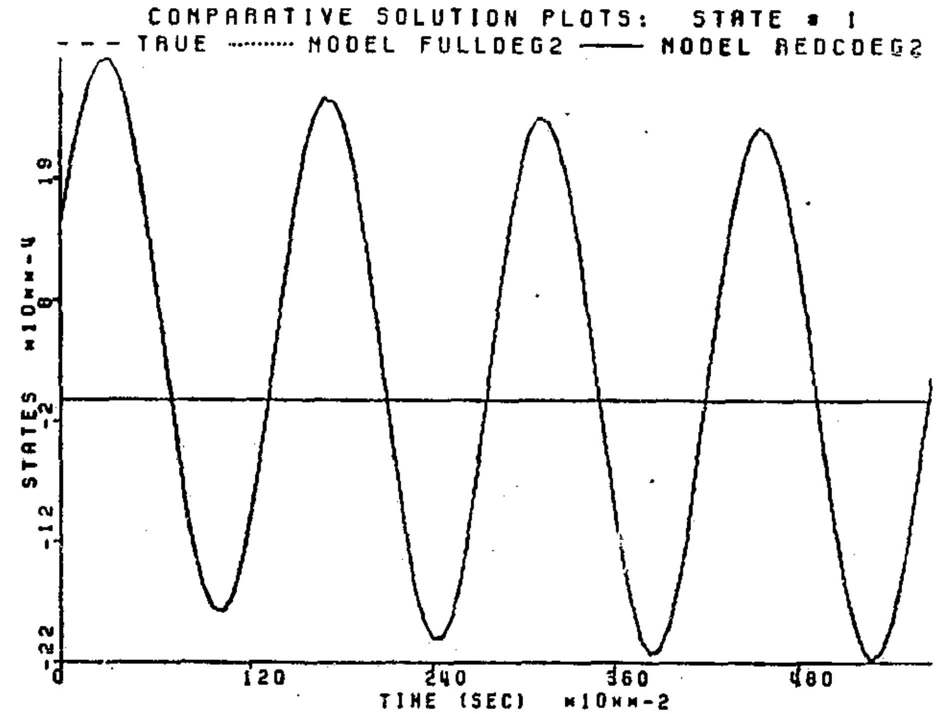
****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 3
OF INPUTS: 3
OF TERMS IN MODEL 1: 27
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 16
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.1000E-02	0.1000E-02	0.1000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	1.006	0.5139	0.9057
2	0.1000E-02	0.1000E-02	0.1000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	1.039	2.517	1.742
3	0.1000E-02	0.1000E-02	0.1000E-02	0.050	0.050	0.050	0.50	0.60	0.70	1.355	0.2341	0.4375
4	0.1000E-02	0.1000E-02	0.1000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.7310	0.6318	0.3957
5	0.1000E-02	0.1000E-02	0.1000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.105	0.5635	0.5476
6	0.1000E-02	0.1000E-02	0.1000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.9179	0.2275	0.1659
7	0.1000E-02	0.1000E-02	0.1000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.4727	0.5705	1.445
8	0.1000E-02	0.1000E-02	0.1000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.5525	1.588	1.868
9	0.1000E-02	0.1000E-02	-.1000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	1.143	0.6354	0.9311
10	0.1000E-02	0.1000E-02	-.1000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.9802	2.150	1.787
11	0.1000E-02	0.1000E-02	-.1000E-02	0.050	0.050	0.050	0.50	0.60	0.70	1.398	0.2373	0.4400
12	0.1000E-02	0.1000E-02	-.1000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.7305	0.6385	0.4130
13	0.1000E-02	0.1000E-02	-.1000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.128	0.5625	0.5489
14	0.1000E-02	0.1000E-02	-.1000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.9181	0.2252	0.1656
15	0.1000E-02	0.1000E-02	-.1000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.4453	0.5741	1.469
16	0.1000E-02	0.1000E-02	-.1000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.5461	1.603	1.865
17	-.5000E-02	0.5000E-02	0.5000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	1.862	1.155	5.172
18	-.5000E-02	0.5000E-02	0.5000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	1.621	1.799	7.107
19	-.5000E-02	0.5000E-02	0.5000E-02	0.050	0.050	0.050	0.50	0.60	0.70	1.255	0.2226	0.3715
20	-.5000E-02	0.5000E-02	0.5000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.7064	0.6887	0.2552
21	-.5000E-02	0.5000E-02	0.5000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.054	0.5909	0.4456
22	-.5000E-02	0.5000E-02	0.5000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.9054	0.2706	0.1587
23	-.5000E-02	0.5000E-02	0.5000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.5162	0.5444	0.7739
24	-.5000E-02	0.5000E-02	0.5000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.5780	1.544	1.177

Table 5.2.6 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	-.5000E-02	-.5000E-02	0.5000E-02	0.010	-0.010	-0.010	0.50	0.60	0.70	1.185	0.4955	0.3688
2	-.5000E-02	-.5000E-02	0.5000E-02	-0.010	-0.010	0.010	0.50	0.60	0.70	0.9711	0.9145	0.6242
3	-.5000E-02	-.5000E-02	0.5000E-02	0.050	0.050	0.050	0.50	0.60	0.70	1.298	0.1848	0.3612
4	-.5000E-02	-.5000E-02	0.5000E-02	-0.050	0.050	-0.050	0.50	0.60	0.70	0.7292	0.5173	0.2126
5	-.5000E-02	-.5000E-02	0.5000E-02	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.066	0.4213	0.4112
6	-.5000E-02	-.5000E-02	0.5000E-02	0.025	-0.075	0.050	0.50	0.60	0.70	0.9098	0.3061	0.1510
7	-.5000E-02	-.5000E-02	0.5000E-02	-0.050	0.025	0.075	0.50	0.60	0.70	0.6557	0.5255	0.6750
8	-.5000E-02	-.5000E-02	0.5000E-02	0.075	0.050	-0.025	0.50	0.60	0.70	0.5734	1.314	0.6448
9	-.7500E-01	-.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	1.065	0.5965	0.6172
10	-.7500E-01	-.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	1.054	0.6114	0.6003
11	-.7500E-01	-.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	1.127	0.5948	0.6201
12	-.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	0.9956	0.6737	0.6169
13	-.7500E-01	-.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.087	0.6393	0.6971
14	-.7500E-01	-.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	0.9884	0.6693	0.5352
15	-.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	1.168	0.6626	0.5683
16	-.7500E-01	-.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	0.9729	0.6243	0.6442
17	0.7500E-01	-.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	2.570	33.18	10.02
18	0.7500E-01	-.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	2.593	33.67	9.474
19	0.7500E-01	-.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	3.901	6.385	5.753
20	0.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	2.203	6.294	7.693
21	0.7500E-01	-.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	1.641	2.863	6.225
22	0.7500E-01	-.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	2.113	2.930	12.22
23	0.7500E-01	-.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	1.950	19.35	5.564
24	0.7500E-01	-.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	3.383	10.48	5.724
25	-.7500E-01	0.7500E-01	-.7500E-01	0.010	-0.010	-0.010	0.50	0.60	0.70	0.6515	0.7081	0.8672E-02
26	-.7500E-01	0.7500E-01	-.7500E-01	-0.010	-0.010	0.010	0.50	0.60	0.70	0.6680	0.7347	0.9461E-02
27	-.7500E-01	0.7500E-01	-.7500E-01	0.050	0.050	0.050	0.50	0.60	0.70	0.7228	0.6521	0.1069
28	-.7500E-01	0.7500E-01	-.7500E-01	-0.050	0.050	-0.050	0.50	0.60	0.70	0.5186	0.6658	0.8523E-01
29	-.7500E-01	0.7500E-01	-.7500E-01	-0.075	-0.075	-0.075	0.50	0.60	0.70	0.8610	0.4370	0.2356
30	-.7500E-01	0.7500E-01	-.7500E-01	0.025	-0.075	0.050	0.50	0.60	0.70	0.7107	1.171	0.2476E-01
31	-.7500E-01	0.7500E-01	-.7500E-01	-0.050	0.025	0.075	0.50	0.60	0.70	0.7328	0.7840	0.6211E-01
32	-.7500E-01	0.7500E-01	-.7500E-01	0.075	0.050	-0.025	0.50	0.60	0.70	0.6189	0.6525	0.9127E-01

Table 5.2.7 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



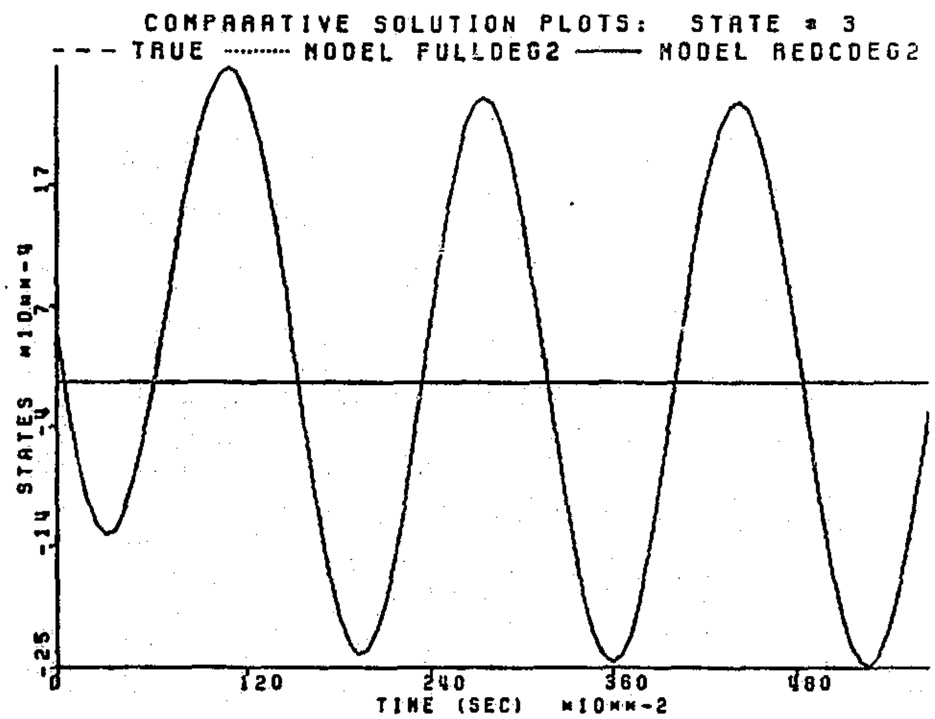


Figure 5.2.9 Simulation Number 2 of Table 5.2.6

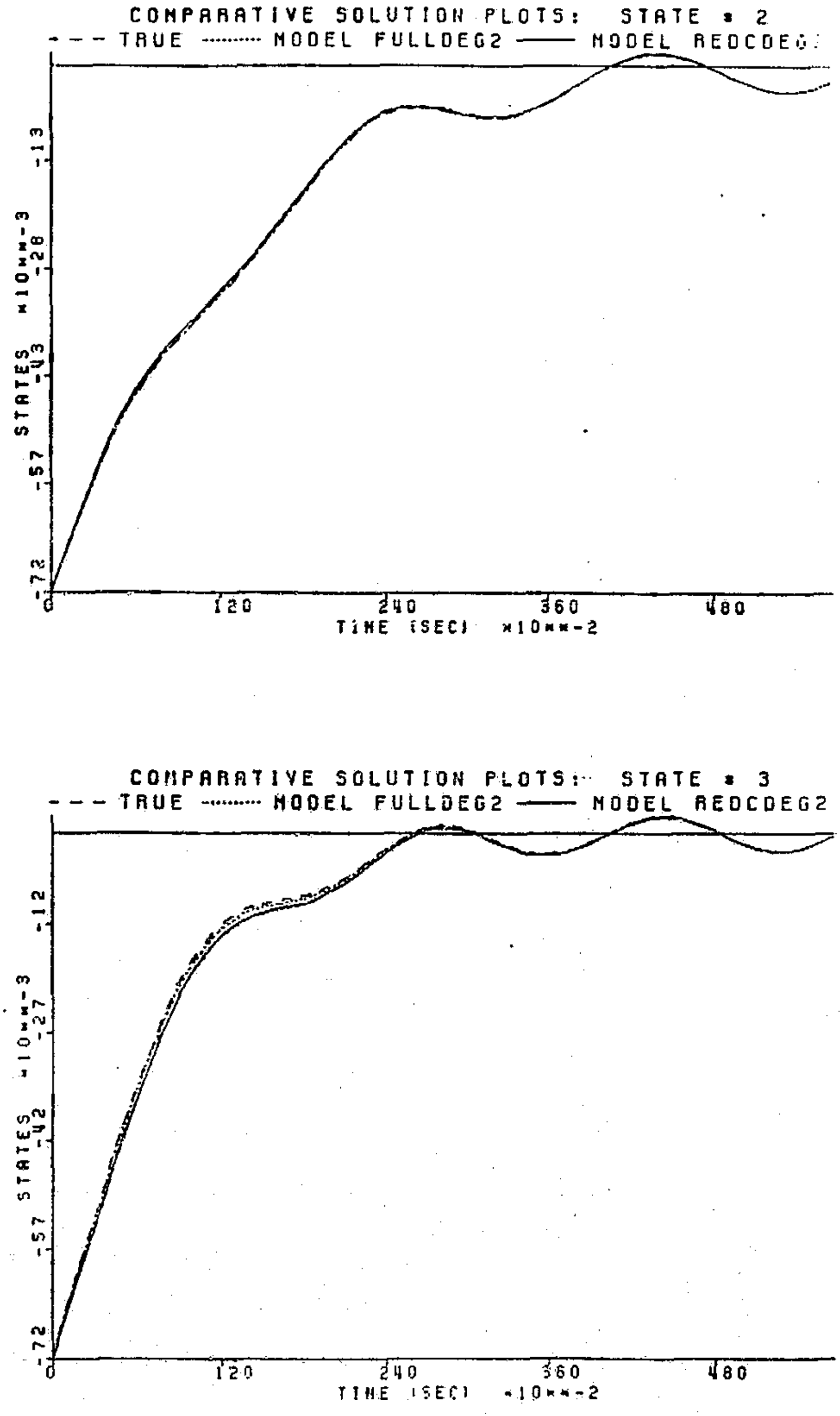


Figure 5.2.10 Simulation Number 17 of Table 5.2.7

##
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 3
OF INPUTS: 3
OF TERMS IN MODEL 1: 27
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 16
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
##

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	1.00	2.00	1.00	1.934	1.705	1.375
2	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	2.00	2.00	2.00	0.2240	1.844	1.174
3	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	0.50	1.00	1.50	1.296	1.616	1.480
4	0.5000E-01	0.5000E-01	0.5000E-01	0.050	0.050	0.050	1.50	1.00	0.50	1.329	1.433	1.737
5	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	1.00	2.00	1.00	1.727	2.044	1.579
6	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	2.00	2.00	2.00	1.193	1.903	1.579
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.075	0.025	-0.050	0.50	1.00	1.50	1.134	1.792	2.118
8	0.5000E-01	0.5000E-01	0.5000E-01	-0.375	0.025	-0.050	1.50	1.00	0.50	1.394	1.702	2.093
9	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	1.00	2.00	1.00	1.215	0.4919	0.4672
10	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	2.00	2.00	2.00	1.194	0.4525	0.5042
11	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	0.50	1.00	1.50	1.182	0.4456	0.4647
12	-.5000E-01	-.5000E-01	-.5000E-01	0.050	0.050	0.050	1.50	1.00	0.50	1.165	0.5661	0.3663
13	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	1.00	2.00	1.00	1.129	0.4958	0.5311
14	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	2.00	2.00	2.00	1.157	0.5194	0.5184
15	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	0.50	1.00	1.50	0.9804	0.5222	0.4765
16	-.5000E-01	-.5000E-01	-.5000E-01	-0.075	0.025	-0.050	1.50	1.00	0.50	1.085	0.5906	0.5697

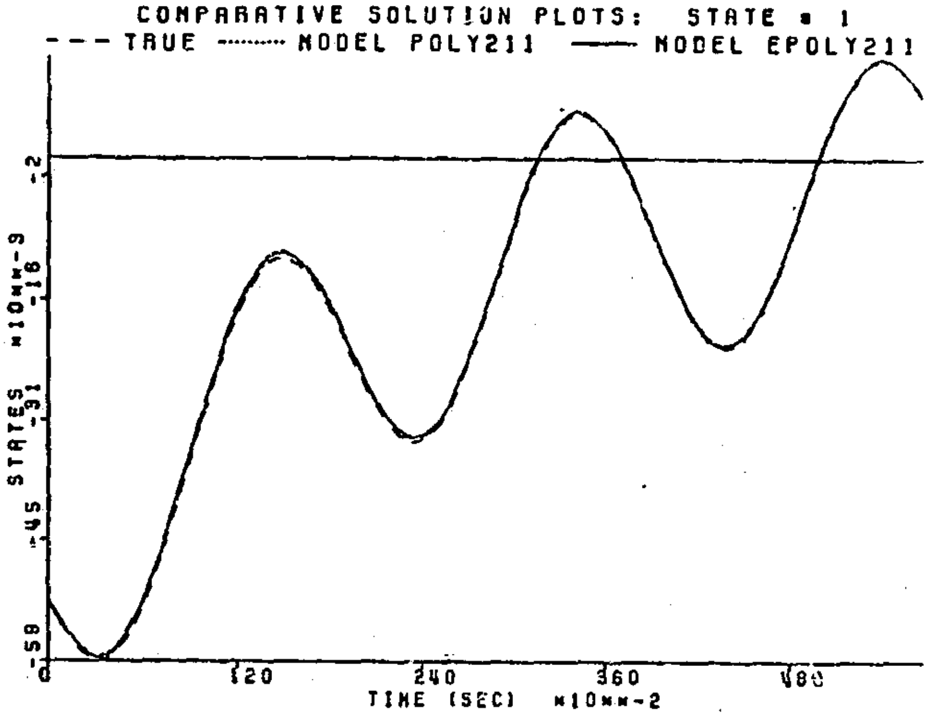
Table 5.2.8 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.00	0.25	0.50	0.5774	1.440	2.418
2	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.50	0.00	0.25	1.043	1.476	0.9911
3	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.25	0.50	0.00	1.143	0.8409	2.006
4	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.25	1.00	0.50	1.229	1.374	1.437
5	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	1.00	0.50	0.25	1.167	1.188	1.689
6	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.050	0.050	0.50	0.25	1.00	1.169	1.408	1.111
7	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.00	0.25	0.50	0.6098	1.368	2.778
8	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.50	0.00	0.02	1.080	1.935	1.140
9	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.25	0.50	0.00	0.9867	1.278	2.955
10	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.25	1.00	0.50	0.9601	1.376	1.404
11	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	1.00	0.50	0.25	1.074	1.363	1.650
12	0.7500E-01	0.7500E-01	0.7500E-01	0.050	0.075	0.025	0.50	0.25	1.00	0.9951	1.414	0.6005

Table 5.2.9 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES			FREQUENCIES			ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.00	0.00	0.00	0.1327E-01	0.9726	0.3866
2	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.02	0.02	0.02	0.2260E-01	1.009	0.1816
3	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.05	0.05	0.05	0.1406	1.091	1.402
4	0.5000E-01	0.5000E-01	0.5000E-01	0.075	-0.025	-0.050	0.10	0.10	0.10	0.2674	1.143	3.751
5	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.02	0.02	0.02	0.8874	0.9328	0.6507
6	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.05	0.05	0.05	0.9950	0.9016	0.5904
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.050	-0.025	-0.075	0.10	0.10	0.10	1.997	0.9788	1.537
8	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.00	0.00	0.00	1.328	1.440	1.241
9	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.02	0.02	0.02	1.347	1.459	1.303
10	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.05	0.05	0.05	1.351	1.156	0.9469
11	0.5000E-01	0.5000E-01	0.5000E-01	0.025	0.075	0.050	0.10	0.10	0.10	1.175	0.7408	0.6745
12	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.00	0.00	0.00	0.5226E-01	0.9651	0.4389
13	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.02	0.02	0.02	0.8691E-01	0.9814	0.3905
14	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.05	0.05	0.05	0.1517	0.9906	0.3894
15	-.1000	0.1000	-.1000	0.075	-0.025	-0.050	0.10	0.10	0.10	0.2607	0.9788	0.3822
16	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.02	0.02	0.02	0.8700	0.9653	0.7350
17	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.05	0.05	0.05	0.8243	0.9592	0.6784
18	-.1000	0.1000	-.1000	-0.050	-0.025	-0.075	0.10	0.10	0.10	0.7764	0.9535	0.6230
19	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.00	0.00	0.00	1.038	1.527	1.153
20	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.02	0.02	0.02	0.7521	1.733	0.9365
21	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.05	0.05	0.05	0.6093	0.5344	0.6779
22	-.1000	0.1000	-.1000	0.025	0.075	0.050	0.10	0.10	0.10	0.8177	0.6084	0.6423

Table 5.2.10 Low Frequency Table for Second Degree Full Model versus Second Degree Reduced Model



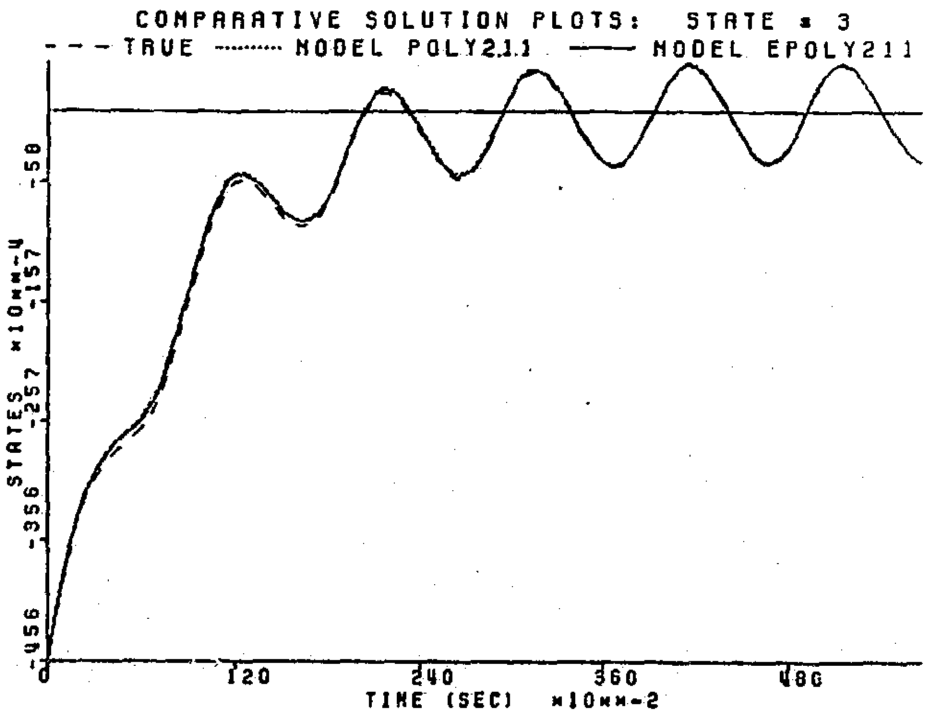
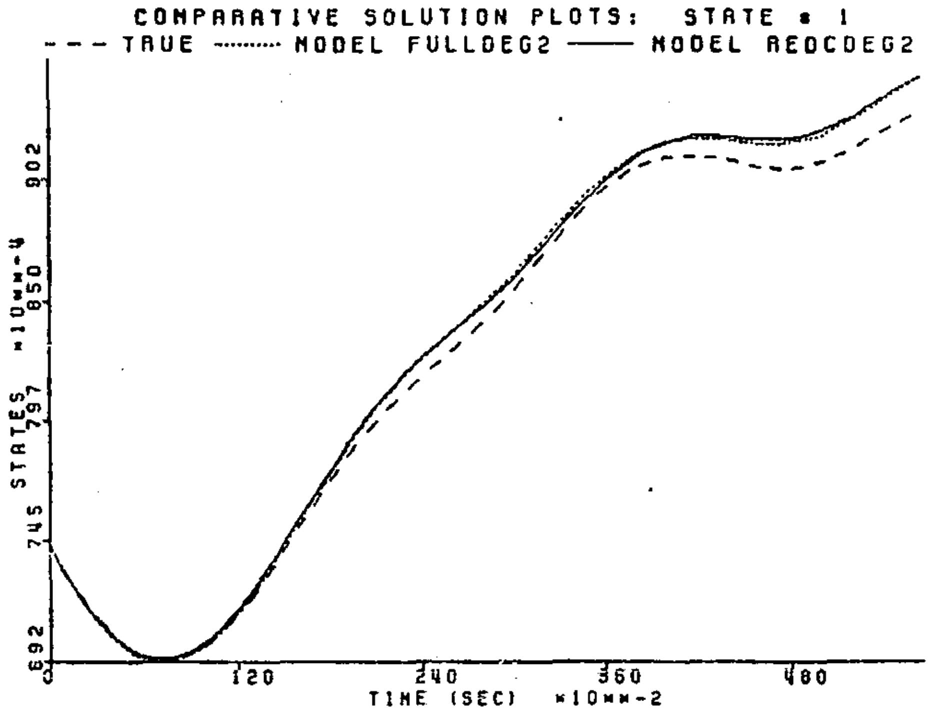


Figure 5.2.11 Simulation Number 16 of Table 5.2.8



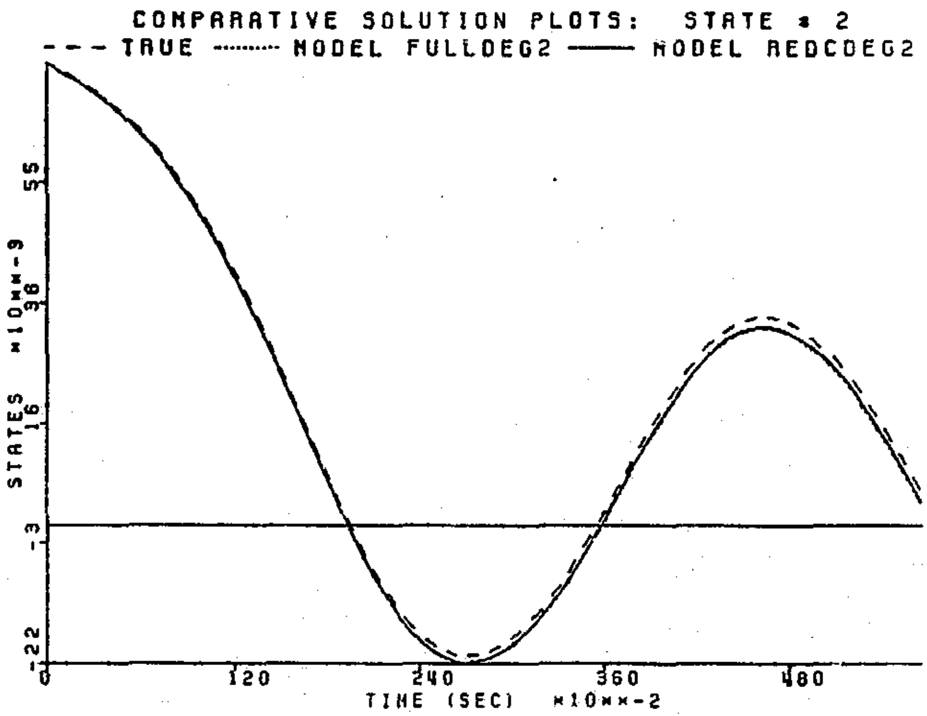
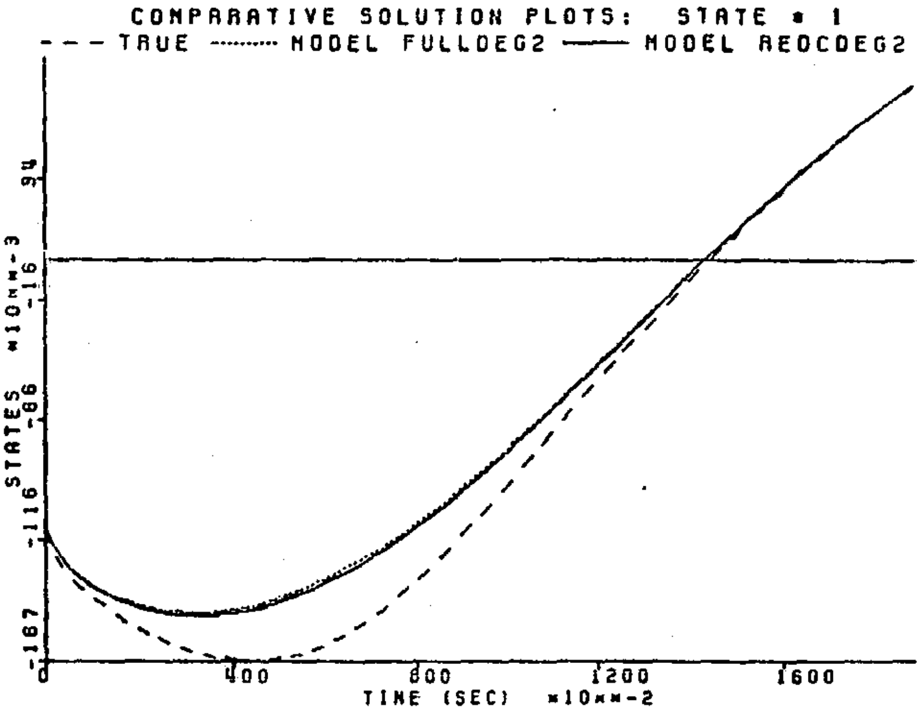


Figure 5.2.12 Simulation Number 9 of Table 5.2.9



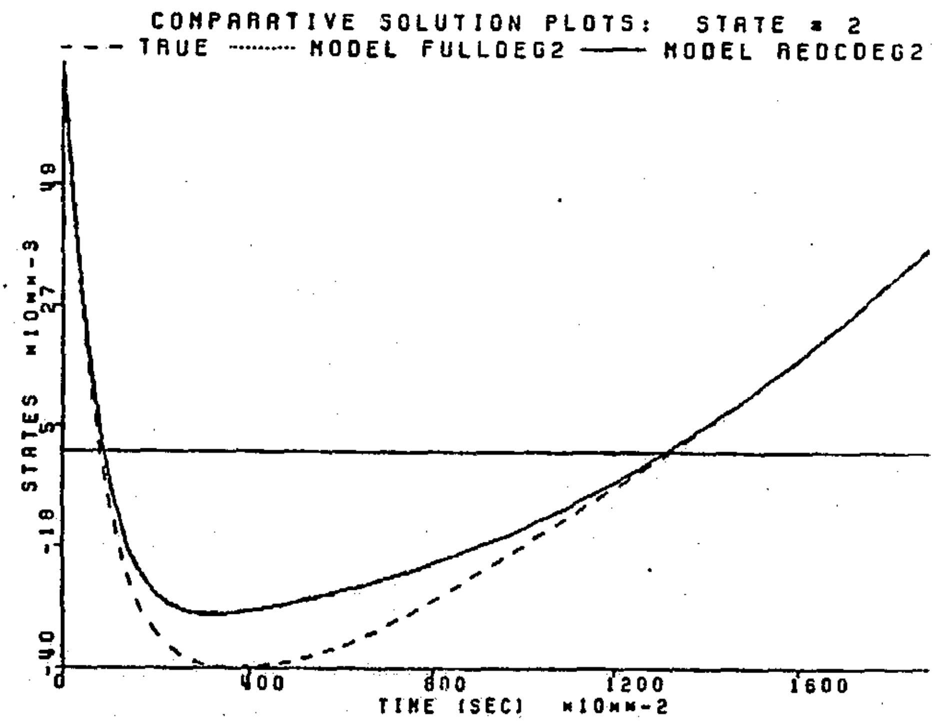
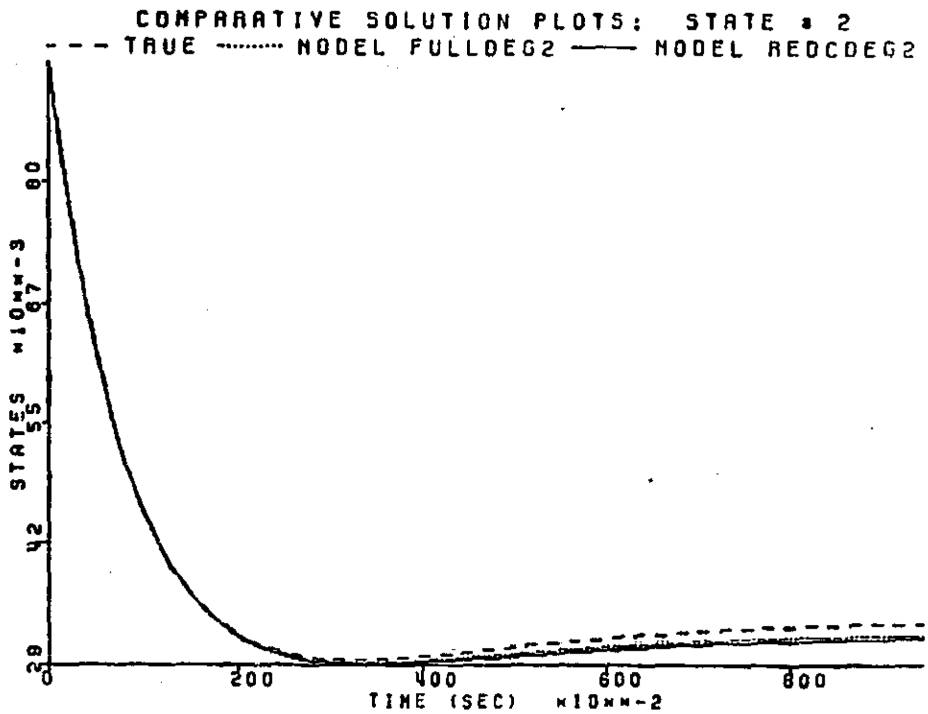


Figure 5.2.13 Simulation Number 16 of Table 5.2.10



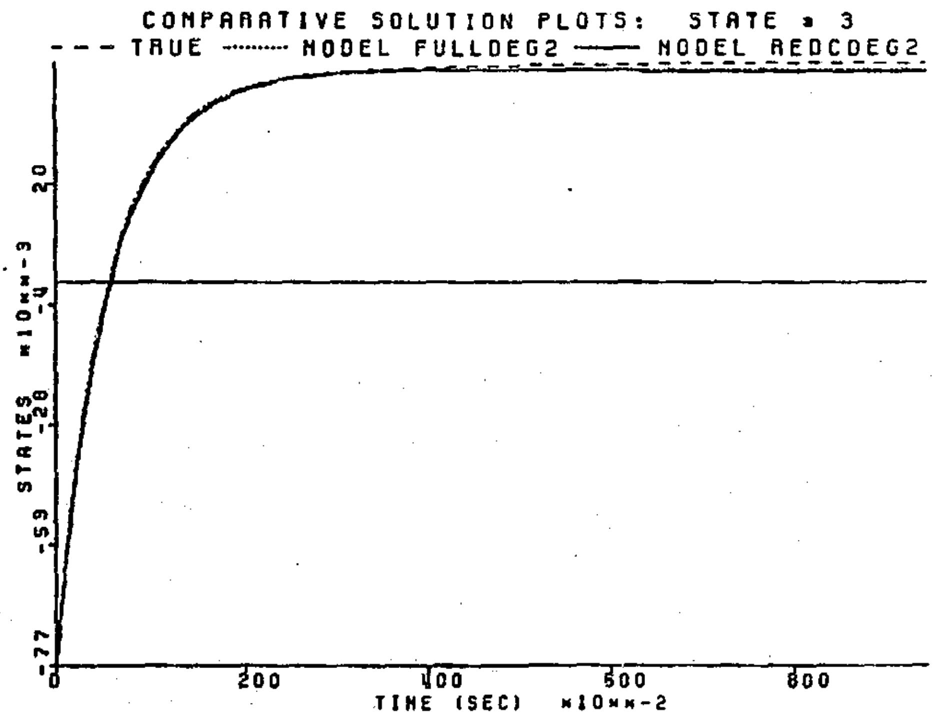


Figure 5.2.14 Simulation Number 19 of Table 5.2.10

5.3 THREE STATE, TWO INPUT SYSTEM

In this section the degree two approximation of a three state, two input system is studied. The system of three differential equations is again chosen to have the origin as a point of stable equilibrium from which it will be perturbed. The set of three equations is

$$\dot{x}1 = -x_1 + \cosh(x_1) + \sinh(x_2) - u_1e^{-u_2} + u_2$$

$$+ x_1^2x_2x_3 - e^{-x_3}\cosh(x_2) + 5x_1u_1u_2$$

$$- 2u_1^2 + x_2x_3,$$

$$\dot{x}_2 = -\frac{1}{2}x_2 + 6x_1^2 - 2e^{-x_3} - x_2^2x_3 + x_1u_2e^{-x_3}$$

$$- \sinh(x_2) - \sinh(4x_3) + 2u_1 + 3u_1^2u_2 + 2,$$

$$\dot{x}_3 = -2x_3 + 5x_3u_2 - 2\cosh(3u_1) + \cosh(x_1)$$

$$+ \cosh(x_3) + x_1u_2^{-3x_3} - \frac{1}{2}u_2 + u_2^2.$$

The second degree expansion contains 20 terms corresponding to the products

$$x_1,\ x_2,\ x_3,\ u_1,\ u_2,\ x_1x_1,\ x_1x_2,$$

$$x_1x_3,\ x_2x_2,\ x_2x_3,\ x_3x_3,\ x_1u_1,\ x_1u_2,$$

$$x_2u_1,\ x_2u_2,\ x_3u_1,\ x_3u_2,\ u_1u_1,\ u_1u_2,$$

$$u_2u_2.$$

An acceptable model was identified at the initial conditions (0.01, 0.04, -0.02). And the input excitations are

$$u_1(t) = 0.025\sin(2\pi t\cdot\phi_1) + 0.025\sin(2\pi t\cdot 2\phi_1),$$

$$u_2(t) = -0.03\sin(2\pi t\cdot\phi_2) - 0.03\sin(2\pi t\cdot 2\phi_2).$$

Where (ϕ_1,ϕ_2) was determined by local optimization to be (1.9941, 1.5518) from the starting frequencies (2.0, 1.6). This model and the corresponding reduction test appear in Figure 5.3.1.

```
ENTER INITIAL CONDITIONS FOR THE 3 STATES:
     0.0100       0.0400      -0.0200

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100     0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2            2

ENTER THE    4 INITIAL INPUT AMPLITUDES:
 0.249999999999999996E-01 -0.299999999999999997E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     1.9941       1.5518

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
N

ENTER OPTIMIZATION OPTION:
         1 - NO OPTIMIZATIONS
         2 - AMPLITUDES ONLY
         3 - FREQUENCIES ONLY
         4 - INITIAL CONDITIONS ONLY
         5 - AMPLITUDES AND FREQUENCIES
         6 - AMPLITUDES AND INITIAL CONDITIONS
         7 - FREQUENCIES AND INITIAL CONDITIONS
         8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
          1

DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y

THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  20 ROWS AND 100 COLUMNS.

NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        0

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N

TSO SPEAKEASY III PI+  8:14 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED

PARTITION NUMBER 1.
  PRTITION (A 3 BY 3 ARRAY)
```

Figure 5.3.1a Full Second Degree Model and Reduction Test (First Pass)

```
 -.99992     .99926    1.0047
 -.0072833 -1.4999    -2.0127
 -.0022769  .0019061 -1.9879

 ...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 3 COMPONENTS)
 -1.9676 -1.5126 -1.0075


 PARTITION NUMBER 2.
  PRTITION (A 3 BY 2 ARRAY)
 -.99886      .99926
  1.9965      .0011641
  4.1191E-5 -.49957


 PARTITION NUMBER 3.
  PRTITION (A 3 BY 6 ARRAY)
  .47926     .060162  -.28906    -.44635     1.2155     -.14853
  6.0639    -.043156  -1.0172    -.24907    -1.416      -3.8262
  .55542    -.087974  -.44303     .0076769   .0016053   .83691

 PARTITION NUMBER 4.
  PRTITION (A 3 BY 6 ARRAY)
  .021255   -.046279     .025311   -6.1793E-4  .069515   -.022372
  .060328    1.04        .04484    -.13033     .19197    -.37207
  .0082058   1.009       .010457   -.012593    .05819     4.9249

 PARTITION NUMBER 5.
  PRTITION (A 3 BY 3 ARRAY)
 -1.9896     1.0341     .0023039
  .0063837 -.0042395 -.0059681
 -8.9742     3.753E-4  1.0006


  S (A 20 COMPONENT ARRAY)
  9.7773    6.9733    6.7771    4.317     3.7318    3.5834    3.0317    1.8302
  1.5996    1.3777    1.023     .91586    .81555    .6472     .50885    .44339
  .36988    .22983    .072894   .13248
 THE MAXIMUM SINGULAR VALUE:
MAX =  9.7773
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .072894

 ...AND THEIR RATIO:
RTIO =  134.13


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:    2375.2090
FULL STATE ERRORS:     229.7720      750.5278     1394.9093

TERM:           X1.  COLUMN #:   1.
CHANGE IN SYSTEM ERROR:     -76.4391
CHANGE IN STATE ERRORS:     -78.0076        1.9494        -0.3809

TERM:           X2.  COLUMN #:   2.
CHANGE IN SYSTEM ERROR:     281.7762
CHANGE IN STATE ERRORS:      90.6889      191.0099         0.0774

TERM:           X3.  COLUMN #:   3.
CHANGE IN SYSTEM ERROR:     -81.4371
CHANGE IN STATE ERRORS:     -66.0698     -166.4523       151.0850
```

Figure 5.3.1b   Full Second Degree Model and Reduction Test (First Pass)

```
TERM:          U1.  COLUMN #:    4.
CHANGE IN SYSTEM ERROR:      -5.8238
CHANGE IN STATE ERRORS:     -10.2078        4.3847       -0.0008

TERM:          U2.  COLUMN #:    5.
CHANGE IN SYSTEM ERROR:      -25.[illegible]
CHANGE IN STATE ERRORS:      -3.172[illegible]       -0.0723      -22.1175

TERM:       X1.X1.  COLUMN #:    6.
CHANGE IN SYSTEM ERROR:    -576.5861
CHANGE IN STATE ERRORS:      37.3129     -679.8017       65.9026

TERM:       X1.X2.  COLUMN #:    7.
CHANGE IN SYSTEM ERROR:       3.2733
CHANGE IN STATE ERRORS:       2.3766        3.7719       -2.8752

TERM:       X1.X3.  COLUMN #:    8.
CHANGE IN SYSTEM ERROR:     -52.8012
CHANGE IN STATE ERRORS:      11.4518      -81.3662       17.1132

TERM:       X2.X2.  COLUMN #:    9.
CHANGE IN SYSTEM ERROR:     -10.0494
CHANGE IN STATE ERRORS:     -21.7582       11.6903        0.0185

TERM:       X2.X3.  COLUMN #:   10.
CHANGE IN SYSTEM ERROR:     -90.7143
CHANGE IN STATE ERRORS:     -49.0060      -41.7053       -0.0030

TERM:       X3.X3.  COLUMN #:   11.
CHANGE IN SYSTEM ERROR:     207.7640
CHANGE IN STATE ERRORS:      -6.7590      205.9084        8.6146

TERM:       X1.U1.  COLUMN #:   12.
CHANGE IN SYSTEM ERROR:      -1.5119
CHANGE IN STATE ERRORS:       0.3419       -1.7045       -0.1493

TERM:       X1.U2.  COLUMN #:   13.
CHANGE IN SYSTEM ERROR:      10.4381
CHANGE IN STATE ERRORS:       0.5890      -33.6165       43.4656

TERM:       X2.U1.  COLUMN #:   14.
CHANGE IN SYSTEM ERROR:      -0.2625
CHANGE IN STATE ERRORS:       0.6154       -0.8328       -0.0450

TERM:       X2.U2.  COLUMN #:   15.
CHANGE IN SYSTEM ERROR:       2.6908
CHANGE IN STATE ERRORS:       0.0156        3.0345       -0.3592

TERM:       X3.U1.  COLUMN #:   16.
CHANGE IN SYSTEM ERROR:       2.2068
CHANGE IN STATE ERRORS:      -1.7913        3.7189        0.2792

TERM:       X3.U2.  COLUMN #:   17.
CHANGE IN SYSTEM ERROR:     -55.0940
CHANGE IN STATE ERRORS:      -0.5280       -9.1829      -45.3831

TERM:       U1.U1.  COLUMN #:   18.
CHANGE IN SYSTEM ERROR:   -1282.1752
CHANGE IN STATE ERRORS:    -153.6015       -0.8254    -1127.7482

TERM:       U1.U2.  COLUMN #:   19.
CHANGE IN SYSTEM ERROR:     -15.4678
CHANGE IN STATE ERRORS:     -15.5732        0.0976        0.0077

TERM:       U2.U2.  COLUMN #:   20.
CHANGE IN SYSTEM ERROR:     105.8121
CHANGE IN STATE ERRORS:       0.1749        0.8234      104.8139
```

Figure 5.3.1c Full Second Degree Model and Reduction Test (First Pass)

```
DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: Y

HOW MANY TERMS WILL BE KEPT? 15

ENTER THE COLUMN NUMBERS OF COLUMNS WHICH ARE TO BE KEPT.
ENTRY OF COLUMN NUMBERS WILL TAKE PLACE IN BLOCKS OF 10.
WHEN ASKED, ENTER A AS AN ARRAY WITH A MAXIMUM OF TEN ELEMENTS.
A = ARRAY(10:1,2,3,4,5,6,8,9,10,11)

A = ARRAY(5:13,17,18,19,20)

MANUAL MODE
 SPACE USED  82 K NOW,  95 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:Y
```

Figure 5.3.1d Full Second Degree Model and Reduction Test (First Pass)

From the reduction test, there are 5 columns of the model which appear to be relatively insignificant. They are

$$x_1x_2,\ x_1u_1,\ x_2u_1,\ x_2u_2,\ x_3u_1.$$

These terms were removed and another optimization over frequencies was performed. This time, given the starting frequency set (1.994, 1.5518), the routine converged to frequencies (2.005, 1.553). This model and the reduction test appear in Figure 5.3.2. The test indicates that there remain no terms whose effect is negligible. From this point, reduction can proceed with careful choice of any term to be removed. For the purpose of demonstration, though, we accept the 5 term reduction as the final one. This is a 25% reduction and again all linear terms were kept.

With the model identified, simulation remains. As in the previous section, for the purpose of presentation only two of the three states will be plotted for any given simulation. Those chosen are generally a worst case and a typical plot. The first five tables and their corresponding plots are simulations of the exact linear model versus the reduced second degree model. Again the ratio is the mean square error of the reduced model over the mean square error of the linear approximation. In this case, the exact linear model appears

$$L_{10} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1.5 & -2 \\ 0 & 0 & -2 \end{bmatrix}, \quad L_{01} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

Table 5.3.1 and Table 5.3.2 show the simulation data for moderately sized excitations. Table 5.3.3 stretches the amplitudes out further than the previous tables. In Table 5.3.4 random excitations were chosen. The initial conditions were chosen in the range (-0.075, 0.075). And the cosinusoidal input

```
ENTER INITIAL CONDITIONS FOR THE 3 STATES:
     0.0100      0.0400     -0.0200

ENTER SAMPLING PARAMETERS:
          APPROXIMATION DEGREE
          # SAMPLE POINTS
          INTERVAL BETWEEN SAMPLES
          INTEGRATION STEPSIZE
   2  100    0.0400    0.0050

SHOULD THE EXCITATION BE SINUSOIDAL OR COSINUSOIDAL?[S/C]:

IDENTIFICATION WILL BE DONE WITH SINES.

ENTER THE NUMBER OF (CO)SINUSOIDS PER INPUT:
          2          2

ENTER THE   4 INITIAL INPUT AMPLITUDES:
 0.24999999999999996E-01 -0.29999999999999997E-01

ENTER THE 2 FREQUENCY WEIGHTS:
     1.9941      1.5518

SHOULD THE IDENTIFICATION BE PERFORMED USING DATA THAT IS REDUCED?[Y/N]:
Y

ENTER OPTIMIZATION OPTION:
          1 - NO OPTIMIZATIONS
          2 - AMPLITUDES ONLY
          3 - FREQUENCIES ONLY
          4 - INITIAL CONDITIONS ONLY
          5 - AMPLITUDES AND FREQUENCIES
          6 - AMPLITUDES AND INITIAL CONDITIONS
          7 - FREQUENCIES AND INITIAL CONDITIONS
          8 - AMPLITUDES, FREQUENCIES, AND INITIAL CONDITIONS
           3

ENTER THE TRACE OF THE COVARIANCE MATRIX:
 0.99999999999999954E-06

ENTER THE OPTIMIZATION METHOD: 1,2,3 OR 4
          4

ENTER # OF SIGNIFICANT DIGITS FOR CONVERGENCE,
MAXIMUM # OF FUNCTION CALLS,
AND THE ZXMIN OPTION # (0,1,2,3)
2    200    2

 10 CALLS OF FCN.....
THE PARAMETERS:
 0.2002840D+01  0.1550755D+01
THE CONDITION NUMBER:  0.722D+08
OBJECTIVE FUNCTION VALUE: 0.9556D+03

 20 CALLS OF FCN.....
THE PARAMETERS:
 0.2004800D+01  0.1552782D+01
THE CONDITION NUMBER:  0.721D+08
OBJECTIVE FUNCTION VALUE: 0.9551D+03

 30 CALLS OF FCN.....
THE PARAMETERS:
 0.2004831D+01  0.1552813D+01
THE CONDITION NUMBER:  0.720D+08
OBJECTIVE FUNCTION VALUE: 0.9551D+03
```

Figure 5.3.2a Reduced Second Degree Model and Reduction Test (Second Pass)

```
 40 CALLS OF FCN.....
THE PARAMETERS:
 0.2004810D+01  0.1552793D+01
THE CONDITION NUMBER:  0.721D+08
OBJECTIVE FUNCTION VALUE: 0.9551D+03

CONVERGENCE WAS ACHIEVED AND NO ERRORS OCCURRED.


DO YOU WISH TO NORMALIZE THE DATA?[Y/N]:
Y



THE MATRIX OF SAMPLED MONOMIAL TERMS
HAS  15 ROWS AND 100 COLUMNS.

NUMBER OF TIMES COST FUNCTION WAS EVALUATED:        47

THE OPTIMUM FREQUENCIES ARE:
FREQ(1)=  2.005     FREQ(2)=  1.553     FREQ(3)=  0.000

WOULD YOU LIKE TO SEE PLOTS OF THE INPUTS?
IF SO, LINE UP THE CARRIAGE.
N



 MESSAGE SUMMARY: MESSAGE NUMBER - COUNT
                              208      33

TSO SPEAKEASY III PI+  9:26 PM FEBRUARY 22, 1984
:_SIZE=500;GET IDENT;IDENT;QUIT

EXECUTION STARTED
  COLSKEPT (A 15 COMPONENT ARRAY)
  1   2   3   4   5   6   8   9   10  11  13  17  18  19  20


PARTITION NUMBER 1.
  PRTITION (A 3 BY 3 ARRAY)
 -.99958      .99946     1.006
 -.0081558  -1.5014     -2.0171
 -.0025067   9.6701E-4 -1.9909

...WITH EIGENVALUES:
  VALUES (A VECTOR WITH 3 COMPONENTS)
 -1.9736 -1.5101 -1.0082


PARTITION NUMBER 2.
  PRTITION (A 3 BY 2 ARRAY)
 -.99869      .99925
  1.9967      .0015374
 -7.3392E-6 -.49957


PARTITION NUMBER 3.
  PRTITION (A 3 BY 6 ARRAY)
  .4238    0       -.56573  -.43827   1.2253  -.20865
  6.0525   0       -1.1585  -.10333  -.88992  -3.875
  .58506   0       -.13797   .024684  .040922  .78135


PARTITION NUMBER 4.
```

Figure 5.3.2b   Reduced Second Degree Model and Reduction Test (Second Pass)

```
  PRTITION (A 3 BY 6 ARRAY)
  0        -.040326  0        0        0        -.015627
  0        1.0415    0        0        0        .061985
  0        1.0091    0        0        0        4.9553


PARTITION NUMBER 5.
  PRTITION (A 3 BY 3 ARRAY)
 -1.9891     1.0348      .0031636
  .0041551  -.0061485  -.0015756
 -8.9747    -2.4192E-4  1.0005


  S (A 15 COMPONENT ARRAY)
  9.5482  6.7397  5.7757  4.2381  3.6701  3.5553  2.9761  1.2907  1.4226
  .80701  .72912  .4787   .41589  .0802   .28158
 THE MAXIMUM SINGULAR VALUE:
MAX =  9.5482
 THE MINIMUM NONZERO SINGULAR VALUE:
MIN =  .0802

 ...AND THEIR RATIO:
RTIO =  119.06


WANT TO TRY REDUCTION TEST?[Y/N]: Y


FULL SYSTEM ERROR:   2360.2953
FULL STATE ERRORS:    227.9609     737.6836    1394.6509

TERM:          X1.  COLUMN #:  1.
CHANGE IN SYSTEM ERROR:    -75.2443
CHANGE IN STATE ERRORS:    -76.9824        2.1605       -0.4224

TERM:          X2.  COLUMN #:  2.
CHANGE IN SYSTEM ERROR:    276.6712
CHANGE IN STATE ERRORS:     89.9027      186.7268        0.0416

TERM:          X3.  COLUMN #:  3.
CHANGE IN SYSTEM ERROR:    -73.4891
CHANGE IN STATE ERRORS:    -65.3272     -162.6399      154.4780

TERM:          U1.  COLUMN #:  4.
CHANGE IN SYSTEM ERROR:     -7.9837
CHANGE IN STATE ERRORS:    -10.6276        2.6437        0.0001

TERM:          U2.  COLUMN #:  5.
CHANGE IN SYSTEM ERROR:    -26.4766
CHANGE IN STATE ERRORS:     -4.3680       -0.0868      -22.0218

TERM:       X1.X1.  COLUMN #:  6.
CHANGE IN SYSTEM ERROR:   -566.2388
CHANGE IN STATE ERRORS:     32.1767     -668.7979       70.3824

TERM:       X1.X3.  COLUMN #:  8.
CHANGE IN SYSTEM ERROR:    -62.6014
CHANGE IN STATE ERRORS:     22.6173      -90.5377        5.3191

TERM:       X2.X2.  COLUMN #:  9.
CHANGE IN SYSTEM ERROR:    -16.5149
CHANGE IN STATE ERRORS:    -21.1498        4.5312        0.1037

TERM:       X2.X3.  COLUMN #: 10.
CHANGE IN SYSTEM ERROR:    -76.0134
CHANGE IN STATE ERRORS:    -48.6315      -27.2563       -0.1257

TERM:       X3.X3.  COLUMN #: 11.
```

Figure 5.3.2c Reduced Second Degree Model and Reduction Test (Second Pass)

```
CHANGE IN SYSTEM ERROR:     204.0818
CHANGE IN STATE ERRORS:      -9.3087      204.6296        8.7610

TERM:       X1.U2.  COLUMN #:  13.
CHANGE IN SYSTEM ERROR:      14.3135
CHANGE IN STATE ERRORS:       0.5307     -30.3188       44.1017

TERM:       X3.U2.  COLUMN #:  17.
CHANGE IN SYSTEM ERROR:     -43.2243
CHANGE IN STATE ERRORS:      -0.3695       1.5486      -44.4034

TERM:       U1.U1.  COLUMN #:  18.
CHANGE IN SYSTEM ERROR:   -1290.6057
CHANGE IN STATE ERRORS:    -152.7579      -0.5303    -1137.3176

TERM:       U1.U2.  COLUMN #:  19.
CHANGE IN SYSTEM ERROR:     -16.7355
CHANGE IN STATE ERRORS:     -16.8697       0.1380       -0.0037

TERM:       U2.U2.  COLUMN #:  20.
CHANGE IN SYSTEM ERROR:     104.6273
CHANGE IN STATE ERRORS:       0.2396       0.2167      104.1710

DO YOU WANT TO DISCARD ANY TERMS AND RE-OPTIMIZE?[Y/N]: N

MANUAL MODE
 SPACE USED  68 K NOW,  80 K PEAK,  SIZE   500 K

DO YOU WISH TO SAVE THIS MODEL?[Y/N]:N

DO YOU WISH TO IDENTIFY ANOTHER MODEL?[Y/N]:N
```

Figure 5.3.2d   Reduced Second Degree Model and Reduction Test (Second Pass)

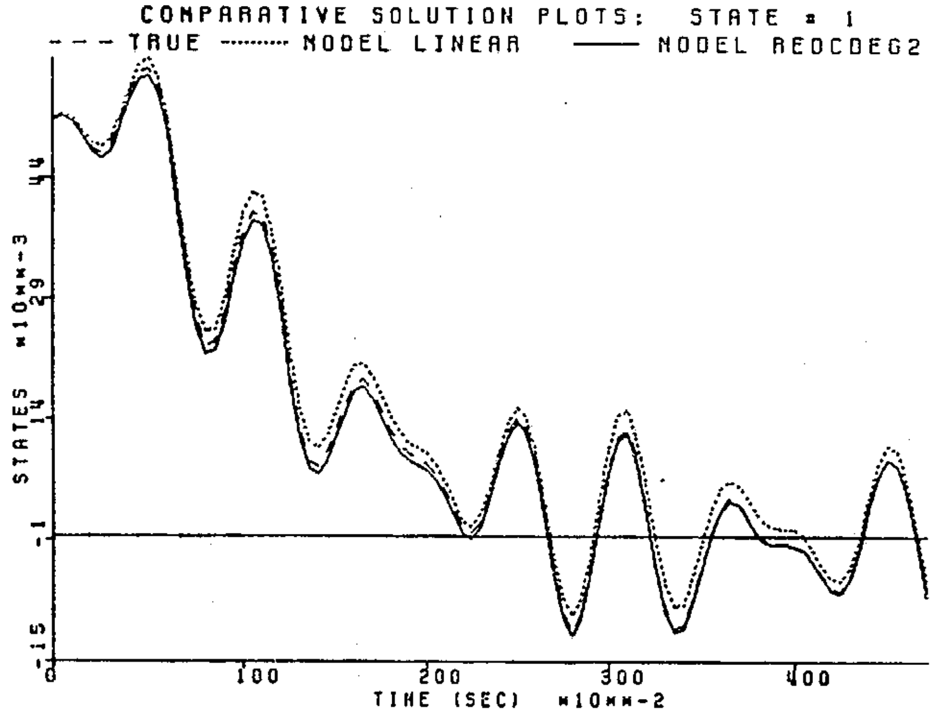
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 3
OF INPUTS: 2
OF TERMS IN MODEL 1: 15
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	0.1299	0.3807E-02	0.6926E-04
2	0.5000E-01	0.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	0.9749E-01	0.3371E-02	0.7737E-04
3	0.5000E-01	0.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	0.1073	0.3553E-02	0.7374E-04
4	0.5000E-01	0.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	0.1116	0.3484E-02	0.6792E-04
5	0.5000E-01	0.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	0.1118E-01	0.4354E-02	0.3663E-03
6	0.5000E-01	0.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	0.1117E-01	0.4164E-02	0.3715E-03
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	0.1115E-01	0.4282E-02	0.3714E-03
8	0.5000E-01	0.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	0.1046E-01	0.4206E-02	0.3643E-03
9	0.5000E-01	0.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.9706E-03	0.1936E-03	0.7065E-04
10	0.5000E-01	0.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.1537E-02	0.1289E-03	0.6609E-04
11	0.5000E-01	0.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.1759E-02	0.1807E-03	0.6671E-04
12	0.5000E-01	0.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.1145E-02	0.1517E-03	0.7117E-04
13	0.5000E-01	0.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	0.8650E-02	0.5129E-02	0.3970E-03
14	0.5000E-01	0.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	0.9064E-02	0.5048E-02	0.3902E-03
15	0.5000E-01	0.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	0.8951E-02	0.5053E-02	0.3595E-03
16	0.5000E-01	0.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	0.8493E-02	0.5117E-02	0.3974E-03
17	0.5000E-01	-.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	0.1191E-01	0.1369E-02	0.6309E-04
18	0.5000E-01	-.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	0.6629E-02	0.1210E-02	0.6231E-04
19	0.5000E-01	-.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	0.7677E-02	0.1274E-02	0.6772E-04
20	0.5000E-01	-.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	0.9801E-02	0.1229E-02	0.5742E-04
21	0.5000E-01	-.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	0.1574E-01	0.4525E-02	0.3730E-03
22	0.5000E-01	-.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	0.1543E-01	0.4472E-02	0.3710E-03
23	0.5000E-01	-.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	0.1550E-01	0.4554E-02	0.3756E-03
24	0.5000E-01	-.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	0.1499E-01	0.4414E-02	0.3677E-03
25	0.5000E-01	-.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.6210E-02	0.1874E-02	0.3797E-04
26	0.5000E-01	-.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.4325E-02	0.1718E-02	0.3717E-04
27	0.5000E-01	-.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.5055E-02	0.1914E-02	0.3647E-04
28	0.5000E-01	-.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.5585E-02	0.1819E-02	0.4203E-04
29	0.5000E-01	-.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	0.1799E-01	0.6099E-02	0.3939E-03
30	0.5000E-01	-.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	0.1679E-01	0.6224E-02	0.3968E-03
31	0.5000E-01	-.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	0.1677E-01	0.6147E-02	0.3944E-03
32	0.5000E-01	-.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	0.1773E-01	0.6186E-02	0.3973E-03

Table 5.3.1 Simulation Table for Linear Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	-.5000E-01	0.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	0.2436E-02	0.9744E-03	0.1277E-03
2	-.5000E-01	0.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	0.2329E-02	0.9892E-03	0.1805E-03
3	-.5000E-01	0.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	0.1817E-02	0.8766E-03	0.1355E-03
4	-.5000E-01	0.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	0.2815E-02	0.9712E-03	0.1631E-03
5	-.5000E-01	0.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	0.1319E-01	0.4397E-02	0.3802E-03
6	-.5000E-01	0.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	0.1485E-01	0.4320E-02	0.3984E-03
7	-.5000E-01	0.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	0.1404E-01	0.4378E-02	0.3860E-03
8	-.5000E-01	0.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	0.1343E-01	0.4317E-02	0.3913E-03
9	-.5000E-01	0.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.1346E-01	0.1875E-02	0.4673E-04
10	-.5000E-01	0.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.1469E-01	0.1386E-02	0.3976E-04
11	-.5000E-01	0.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.1576E-01	0.1796E-02	0.4507E-04
12	-.5000E-01	0.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.1188E-01	0.1523E-02	0.4269E-04
13	-.5000E-01	0.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	0.1748E-01	0.5829E-02	0.3977E-03
14	-.5000E-01	0.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	0.1788E-01	0.5692E-02	0.3917E-03
15	-.5000E-01	0.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	0.1744E-01	0.5719E-02	0.3927E-03
16	-.5000E-01	0.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	0.1734E-01	0.5813E-02	0.3994E-03
17	-.5000E-01	-.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	0.1022E-01	0.1834E-03	0.2102E-03
18	-.5000E-01	-.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	0.1177E-01	0.1957E-03	0.2513E-03
19	-.5000E-01	-.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	0.1036E-01	0.1960E-03	0.2251E-03
20	-.5000E-01	-.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	0.1286E-01	0.1854E-03	0.2368E-03
21	-.5000E-01	-.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	0.2497E-01	0.4129E-02	0.4135E-03
22	-.5000E-01	-.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	0.2631E-01	0.4189E-02	0.4247E-03
23	-.5000E-01	-.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	0.2514E-01	0.4232E-02	0.4182E-03
24	-.5000E-01	-.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	0.2566E-01	0.4069E-02	0.4213E-03
25	-.5000E-01	-.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.2126	0.3367E-02	0.8401E-04
26	-.5000E-01	-.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.1604	0.2681E-02	0.7603E-04
27	-.5000E-01	-.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.1757	0.3283E-02	0.8321E-04
28	-.5000E-01	-.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.1937	0.2981E-02	0.8209E-04
29	-.5000E-01	-.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	0.4082E-01	0.5966E-02	0.4294E-03
30	-.5000E-01	-.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	0.3671E-01	0.5954E-02	0.4218E-03
31	-.5000E-01	-.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	0.3639E-01	0.5942E-02	0.4221E-03
32	-.5000E-01	-.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	0.4057E-01	0.6023E-02	0.4341E-03

Table 5.3.2 Simulation Table for Linear Model versus Second Degree Reduced Model



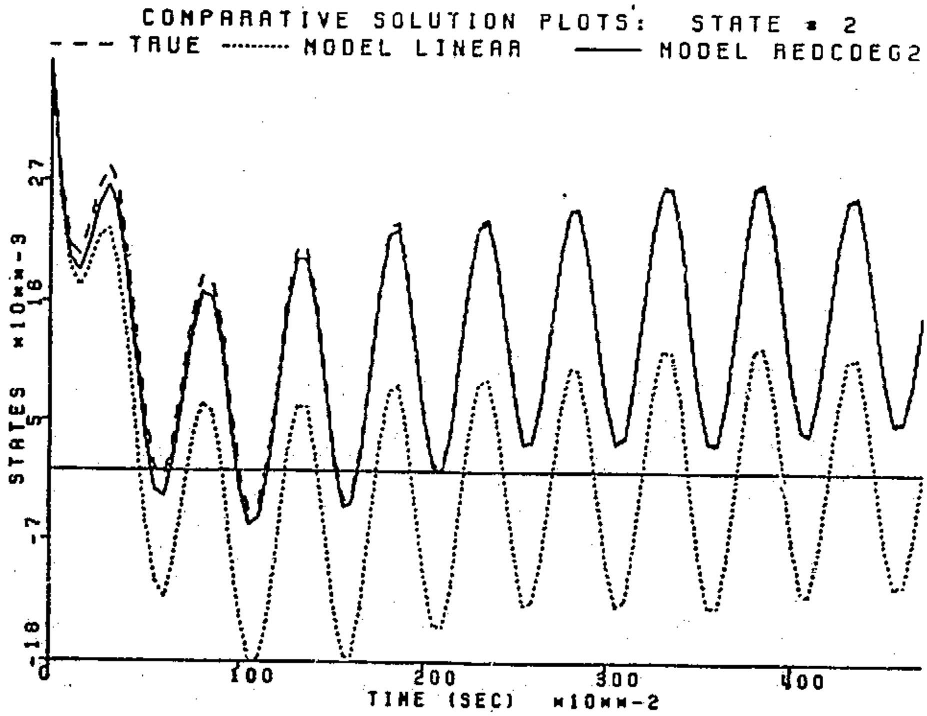
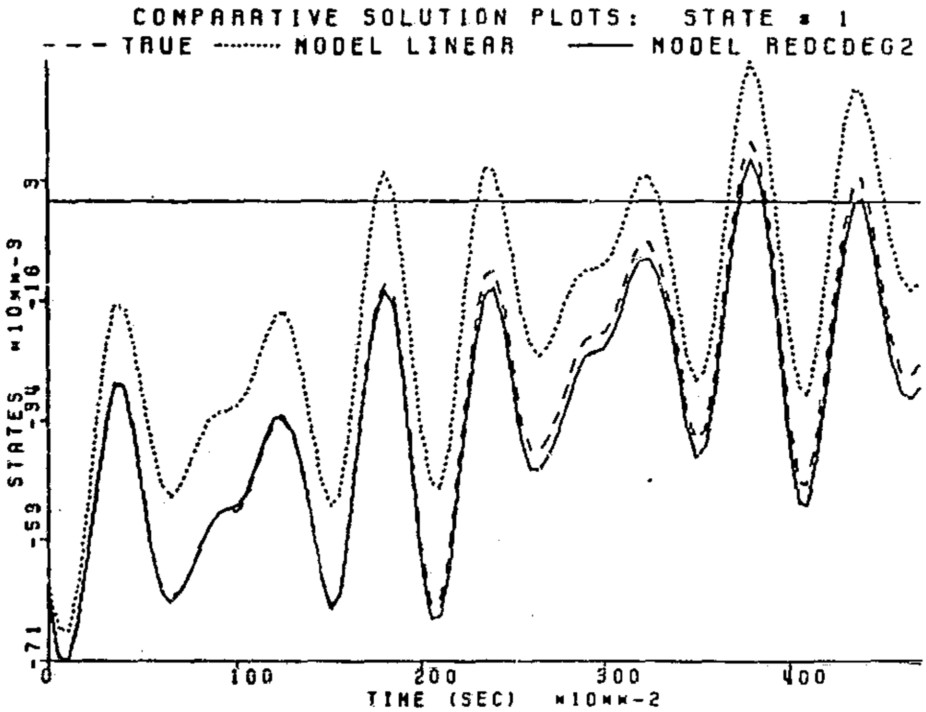


Figure 5.3.3 Simulation Number 4 of Table 5.3.1



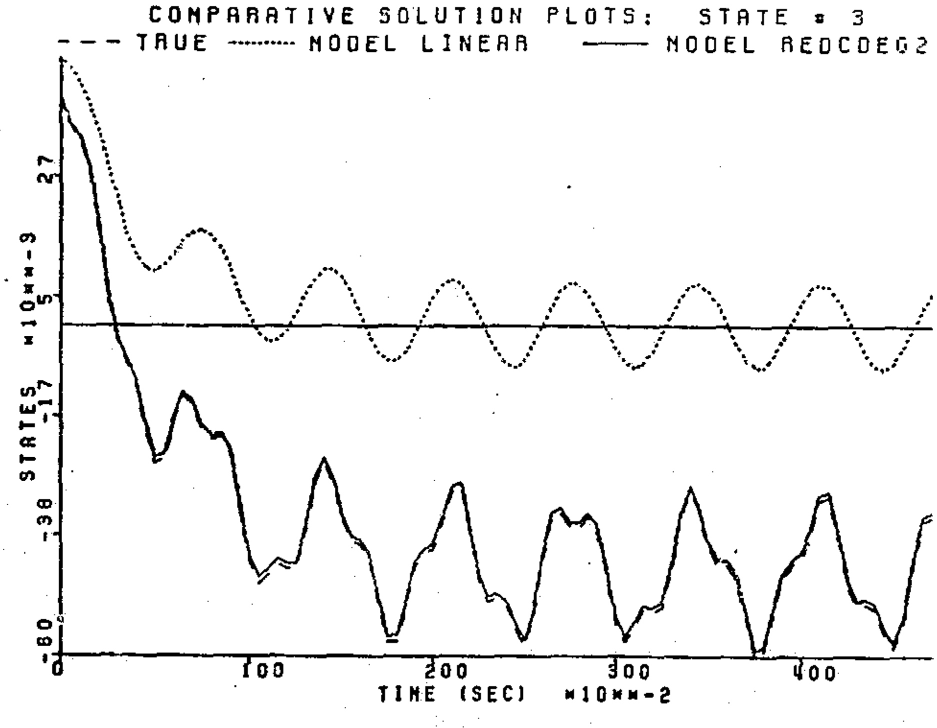


Figure 5.3.4 Simulation Number 22 of Table 5.3.2

ORIGINAL PAGE IS OF POOR QUALITY

****

PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 3
OF INPUTS: 2
OF TERMS IN MODEL 1: 15
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE

****

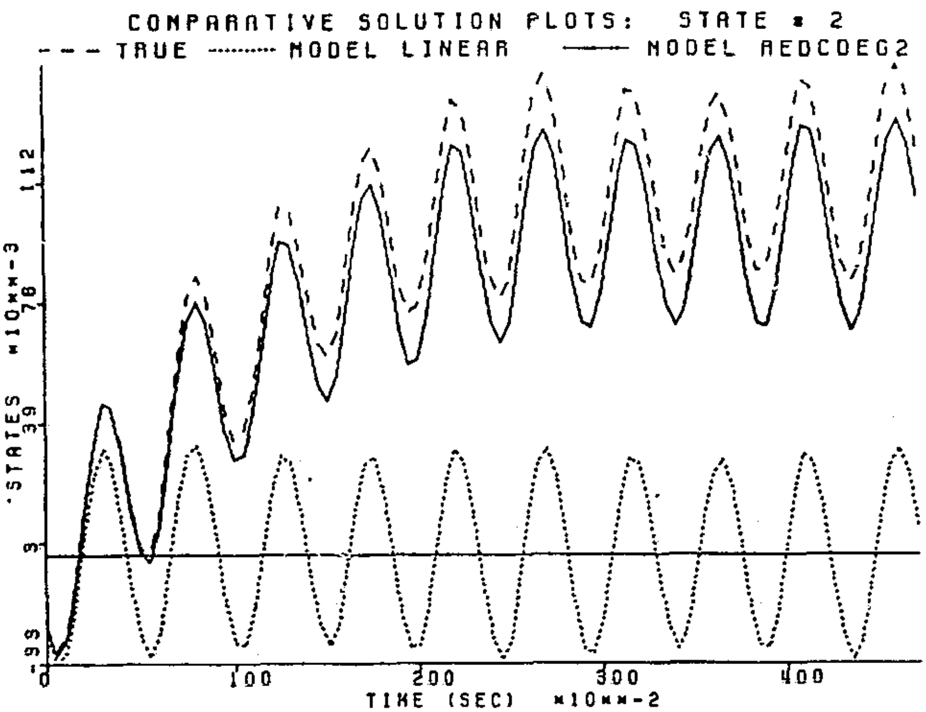
S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.1000E-01	0.1000E-01	0.1000E-01	0.2000	0.2000	2.10	1.60	0.3361E-01	0.1687E-01	0.1118E-02
2	0.1000E-01	0.1000E-01	0.1000E-01	0.2000	-0.2000	2.10	1.60	0.3526E-01	0.1739E-01	0.1117E-02
3	0.1000E-01	0.1000E-01	0.1000E-01	-0.2000	0.2000	2.10	1.60	0.3516E-01	0.1697E-01	0.1113E-02
4	0.1000E-01	0.1000E-01	0.1000E-01	-0.2000	-0.2000	2.10	1.60	0.3365E-01	0.1729E-01	0.1122E-02
5	0.1000E-01	0.1000E-01	0.1000E-01	0.3000	0.3000	2.10	1.60	0.7370E-01	0.9237E-01	0.5608E-02
6	0.1000E-01	0.1000E-01	0.1000E-01	0.3000	-0.3000	2.10	1.60	0.7928E-01	0.9442E-01	0.5616E-02
7	0.1000E-01	0.1000E-01	0.1000E-01	-0.3000	0.3000	2.10	1.60	0.7925E-01	0.9302E-01	0.5579E-02
8	0.1000E-01	0.1000E-01	0.1000E-01	-0.3000	-0.3000	2.10	1.60	0.7362E-01	0.9395E-01	0.5647E-02
9	0.1000E-01	-.1000E-01	-.1000E-01	0.2000	0.2000	2.10	1.60	0.3652E-01	0.1726E-01	0.1134E-02
10	0.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	2.10	1.60	0.3762E-01	0.1791E-01	0.1139E-02
11	0.1000E-01	-.1000E-01	-.1000E-01	-0.2000	0.2000	2.10	1.60	0.3747E-01	0.1743E-01	0.1126E-02
12	0.1000E-01	-.1000E-01	-.1000E-01	-0.2000	-0.2000	2.10	1.60	0.3667E-01	0.1775E-01	0.1140E-02
13	0.1000E-01	-.1000E-01	-.1000E-01	0.3000	0.3000	2.10	1.60	0.7631E-01	0.9267E-01	0.5664E-02
14	0.1000E-01	-.1000E-01	-.1000E-01	0.3000	-0.3000	2.10	1.60	0.8128E-01	0.9503E-01	0.5661E-02
15	0.1000E-01	-.1000E-01	-.1000E-01	-0.3000	0.3000	2.10	1.60	0.8114E-01	0.9354E-01	0.5620E-02
16	0.1000E-01	-.1000E-01	-.1000E-01	-0.3000	-0.3000	2.10	1.60	0.7640E-01	0.9436E-01	0.5706E-02
17	-.1000E-01	-.1000E-01	0.1000E-01	0.2000	0.2000	2.10	1.60	0.3823E-01	0.1652E-01	0.1133E-02
18	-.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	2.10	1.60	0.3983E-01	0.1714E-01	0.1130E-02
19	-.1000E-01	-.1000E-01	0.1000E-01	-0.2000	0.2000	2.10	1.60	0.3954E-01	0.1671E-01	0.1126E-02
20	-.1000E-01	-.1000E-01	0.1000E-01	-0.2000	-0.2000	2.10	1.60	0.3847E-01	0.1595E-01	0.1138E-02
21	-.1000E-01	-.1000E-01	0.1000E-01	0.3000	0.3000	2.10	1.60	0.7810E-01	0.9069E-01	0.5645E-02
22	-.1000E-01	-.1000E-01	0.1000E-01	0.3000	-0.3000	2.10	1.60	0.8367E-01	0.9299E-01	0.5651E-02
23	-.1000E-01	-.1000E-01	0.1000E-01	-0.3000	0.3000	2.10	1.60	0.8347E-01	0.9157E-01	0.5611E-02
24	-.1000E-01	-.1000E-01	0.1000E-01	-0.3000	-0.3000	2.10	1.60	0.7819E-01	0.9227E-01	0.5699E-02
25	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	0.2000	2.10	1.60	0.3893E-01	0.1704E-01	0.1144E-02
26	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	2.10	1.60	0.4010E-01	0.1772E-01	0.1135E-02
27	-.1000E-01	-.1000E-01	-.1000E-01	-0.2000	0.2000	2.10	1.60	0.3978E-01	0.1722E-01	0.1133E-02
28	-.1000E-01	-.1000E-01	-.1000E-01	-0.2000	-0.2000	2.10	1.60	0.3924E-01	0.1753E-01	0.1150E-02
29	-.1000E-01	-.1000E-01	-.1000E-01	0.3000	0.3000	2.10	1.60	0.7828E-01	0.9174E-01	0.5680E-02
30	-.1000E-01	-.1000E-01	-.1000E-01	0.3000	-0.3000	2.10	1.60	0.8338E-01	0.9424E-01	0.5690E-02
31	-.1000E-01	-.1000E-01	-.1000E-01	-0.3000	0.3000	2.10	1.60	0.8308E-01	0.9269E-01	0.5636E-02
32	-.1000E-01	-.1000E-01	-.1000E-01	-0.3000	-0.3000	2.10	1.60	0.7651E-01	0.9345E-01	0.5729E-02

Table 5.3.3 Simulation Table for Linear Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	-.3562E-01	-.6893E-01	0.2058E-02	0.1068	-0.2188	3.43	2.72	0.3651E-02	0.8128E-03	0.4056E-03
2	-.6859E-01	-.6320E-01	-.2565E-01	-0.0475	0.1321	4.66	2.21	0.7719E-02	0.1853E-02	0.1349E-01
3	0.7058E-01	-.2950E-02	0.6901E-01	0.0366	-0.1964	1.97	1.07	0.2206	0.9631E-01	0.5867E-03
4	-.4633E-02	-.1100E-01	-.3541E-01	-0.2272	-0.0738	3.26	4.81	0.6521E-01	0.3452E-01	0.1664E-02
5	-.3905E-01	0.1766E-01	-.4348E-01	0.1093	0.0399	3.09	1.79	0.1206E-01	0.1752E-02	0.9641E-04
6	-.4442E-01	0.2990E-01	-.4312E-01	0.2190	-0.2008	1.76	4.89	0.4911E-01	0.2625E-01	0.1593E-02
7	0.4364E-02	-.7165E-01	-.2782E-01	0.0304	0.0827	3.05	3.42	0.8346E-01	0.5653E-01	0.2195E-02
8	0.3455E-01	-.6117E-02	0.3014E-01	-0.1400	0.1952	3.42	3.90	0.6300E-02	0.2584E-02	0.3268E-03
9	-.5552E-01	-.3124E-01	0.2795E-01	0.1417	-0.2474	4.78	3.69	0.7539E-02	0.2257E-02	0.5434E-03
10	0.3747E-01	-.7435E-02	-.2476E-01	-0.2217	-0.0448	2.87	1.75	0.5711E-01	0.3202E-01	0.1481E-02
11	0.7219E-01	0.2200E-01	-.4525E-01	0.2317	-0.0048	1.20	4.76	0.5904E-01	0.4115E-01	0.1809E-02
12	-.6679E-01	0.2379E-01	0.1986E-01	-0.0655	-0.0328	3.14	1.74	0.3002E-02	0.4315E-04	0.9340E-04
13	0.6943E-01	0.3078E-02	-.1860E-01	0.1525	-0.1689	3.83	1.79	0.1019E-01	0.4746E-02	0.4086E-03
14	-.4655E-01	-.4825E-01	0.3025E-01	-0.0648	0.0120	4.70	4.88	0.8421E-02	0.5139E-04	0.1243E-03
15	-.3726E-02	-.6579E-01	-.3959E-01	-0.0513	-0.1143	2.75	1.76	0.4452E-01	0.1112E-01	0.2216E-03
16	-.7052E-01	-.3124E-01	0.5198E-01	-0.2166	0.0507	4.32	3.47	0.7632E-01	0.2412E-01	0.1344E-02
17	-.3934E-01	-.5829E-01	0.4018E-01	-0.0745	-0.0818	2.78	1.88	0.5859E-02	0.1392E-03	0.1723E-03
18	-.6317E-01	-.1049E-01	0.6379E-01	-0.2284	-0.2463	3.46	1.48	0.6203E-01	0.2723E-01	0.1992E-02
19	-.5110E-02	0.7268E-01	0.3991E-01	0.2218	0.1643	1.14	3.12	0.4533E-01	0.2785E-01	0.1568E-02
20	-.4693E-01	0.2372E-01	-.7699E-02	0.0951	0.0494	4.98	3.63	0.3866E-02	0.4675E-03	0.5720E-04
21	0.5610E-01	0.3104E-01	-.5243E-02	0.1115	0.1216	3.88	4.88	0.1808E-02	0.7144E-03	0.1242E-03
22	-.7456E-01	0.6746E-01	0.6990E-01	-0.1877	-0.1140	4.42	4.91	0.3529E-01	0.1248E-01	0.7834E-03
23	-.7214E-01	-.2059E-01	0.6127E-01	0.0652	-0.1008	2.19	3.54	0.6866E-01	0.5550E-03	0.8759E-03
24	0.2872E-01	0.3261E-01	0.5939E-01	-0.0492	-0.0861	4.88	4.62	0.3605	0.4228E-01	0.4765E-03
25	0.5666E-01	-.3530E-01	0.4306E-01	0.0973	0.1462	1.43	4.37	0.3756E-02	0.9729E-03	0.1152E-03
26	-.8721E-02	-.2256E-01	0.6255E-01	-0.0374	0.1687	4.06	1.98	0.2740E-01	0.2207E-01	0.1322E-02
27	0.3171E-01	-.4486E-01	0.2218E-01	0.1942	-0.2352	1.97	4.82	0.2704E-01	0.1290E-01	0.1054E-02
28	-.7419E-01	-.3983E-01	-.4984E-01	0.1164	0.2291	2.28	4.73	0.3349E-01	0.3965E-02	0.4357E-03
29	0.3679E-01	-.5320E-01	-.8601E-02	-0.2467	-0.1628	2.94	4.74	0.6962E-01	0.4398E-01	0.2366E-02
30	0.4039E-01	0.4743E-01	-.1621E-02	0.0480	0.2108	2.18	4.87	0.9457E-02	0.3593E-02	0.8989E-04
31	0.4839E-01	0.2203E-01	0.3802E-01	-0.1399	0.2236	1.25	3.95	0.4001E-02	0.2530E-02	0.3991E-03
32	-.4102E-01	-.5523E-01	0.6202E-01	-0.1598	0.2165	4.97	3.22	0.1781E-01	0.4495E-02	0.5932E-03
33	-.7302E-01	0.6748E-01	-.9008E-02	0.1904	-0.1361	1.74	3.32	0.4070E-01	0.1617E-01	0.8504E-03
34	-.5007E-01	-.3751E-01	0.5766E-01	0.2419	-0.2383	3.10	1.46	0.7909E-01	0.3627E-01	0.2417E-02
35	0.2560E-01	-.4823E-02	-.6368E-01	0.0349	-0.1240	3.92	2.27	0.6021E-01	0.2646	0.4562E-03

Table 5.3.4 Simulation Table for Linear Model versus Second Degree Reduced Model

ORIGINAL PAGE IS
OF POOR QUALITY



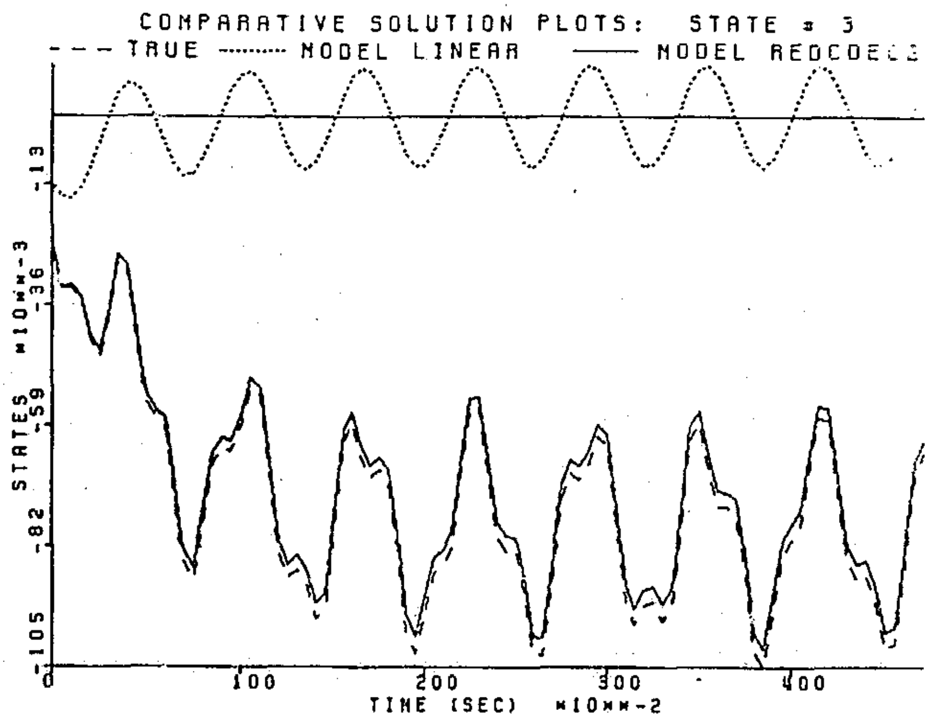
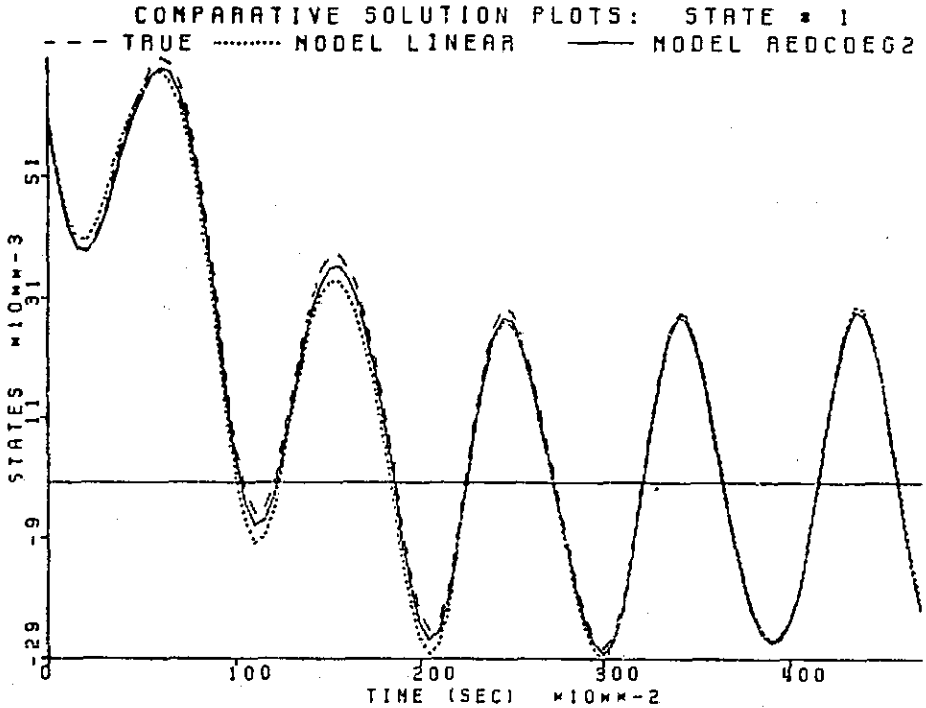


Figure 5.3.5 Simulation Number 11 of Table 5.3.3



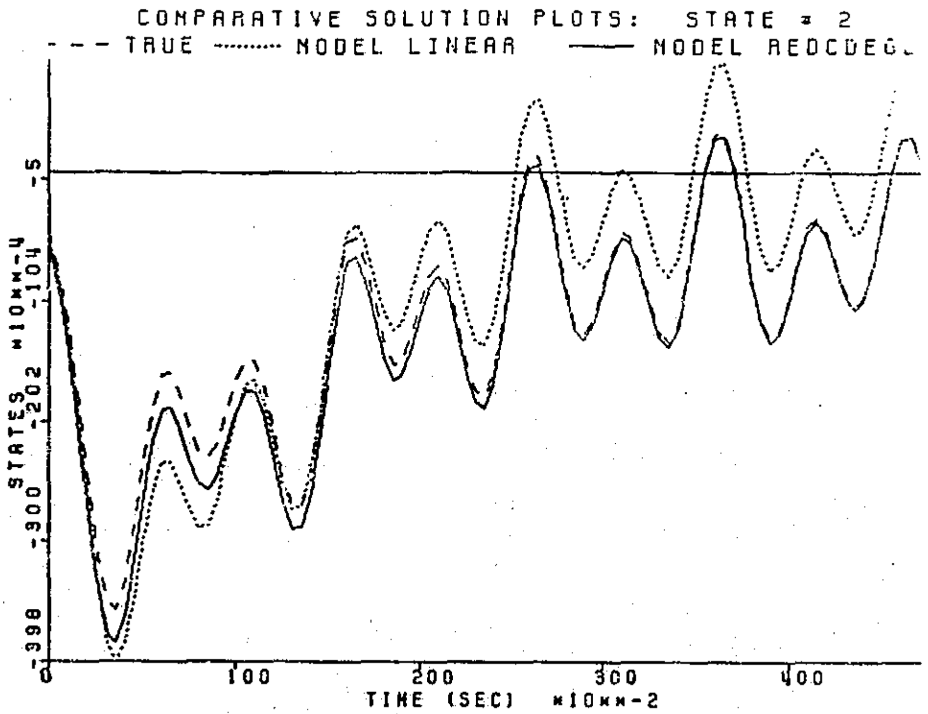
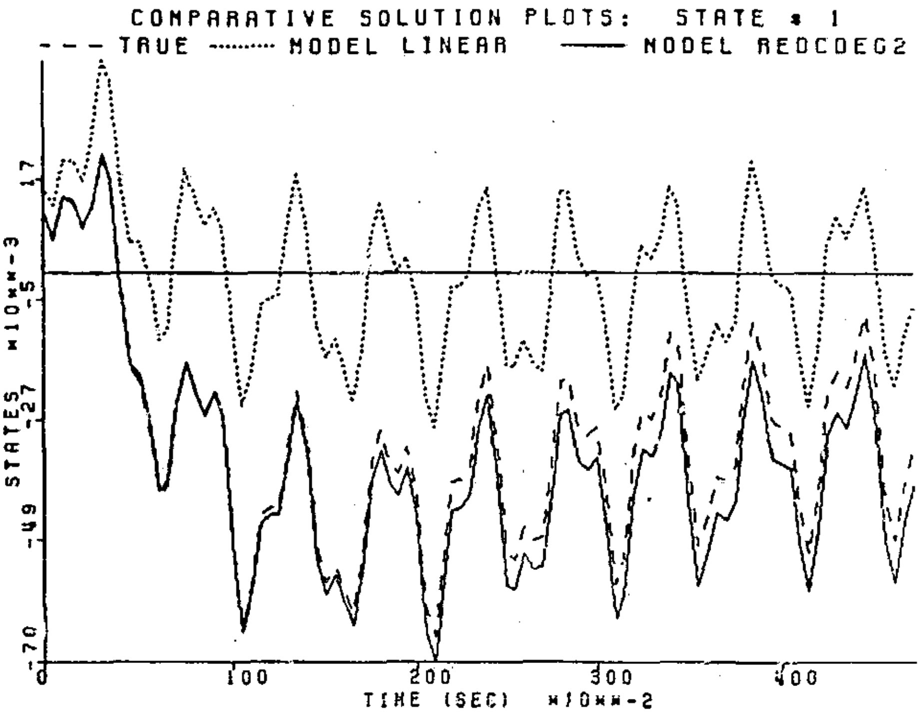


Figure 5.3.6 Simulation Number 3 of Table 5.3.4



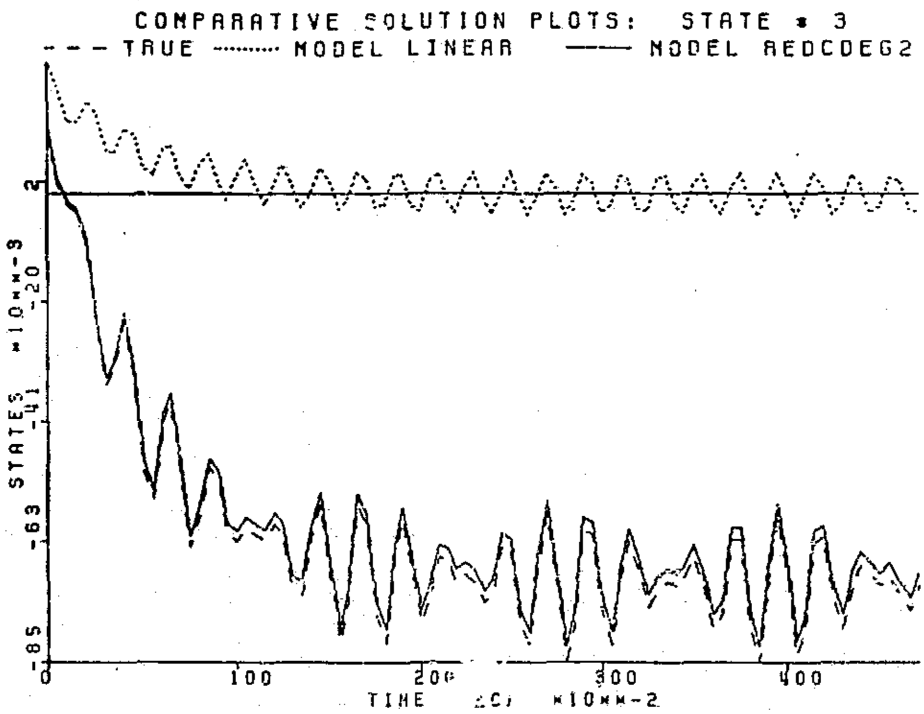


Figure 5.3.7 Simulation Number 27 of Table 5.3.4

amplitudes and frequencies were chosen in the ranges (-0.25, 0.25) and (1,5) respectively. Finally, Table 5.3.5 tests the d.c. and low frequency behavior of the models. In all cases, the reduced model outperforms the linear model.

As the final test of model degradation, the same simulations are performed for the full second degree model versus the reduced second degree model. Again in each of the five tables, the error ratio is the mean square error of the reduced model over the mean square error of the full model. In brief, Table 5.3.6 and Table 5.3.7 contain excitations of moderate size, while Table 5.3.8 moves the amplitudes out farther away from the origin. Table 5.3.9 is a table of randomly chosen excitations, and Table 5.3.10 is a low frequency test. Note in all cases, with the exception of the Table 5.3.10, the error ratios are near unity. This implies that almost no information was lost with the removal of the five terms. In Table 5.3.10, it appears that the third state has been hurt the most by the reduction, but Figure 5.3.14 shows that even a worst case example is not bad. All things considered, the reduction, although only five terms, has tested out well.

5.4 REMARKS

In this chapter three examples are studied to demonstrate the reduction method. For each example system, a reduction of the second degree model is carried out, and for the first example, the degree three approximation is also reduced. In each exercise the reduction method yields a smaller model while retaining the nonlinearities associated with the additional degree of approximation.

As a note on the computation involved in the reduction test, it is worth mentioning that the current version of the reduction test software is written

```
************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
# OF STATES: 3
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 15
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
************************************
```

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.00	0.00	0.5443E-02	0.1435E-02	0.9056E-04
2	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.01	0.01	0.6009E-02	0.1652E-02	0.1597E-03
3	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.03	0.03	0.1994E-01	0.3448E-02	0.7629E-03
4	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.00	0.00	0.6659E-02	0.3945E-04	0.3514E-03
5	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.01	0.01	0.5931E-02	0.3183E-04	0.3904E-03
6	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.03	0.03	0.4311E-02	0.3701E-04	0.4373E-03
7	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.00	0.00	0.2106E-01	0.1055E-01	0.9423E-03
8	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.01	0.01	0.1557E-01	0.1333E-01	0.1334E-02
9	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.03	0.03	0.6689E-02	0.7130E-02	0.1307E-02
10	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.00	0.00	0.2501	0.9865E-02	0.3954E-02
11	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.01	0.01	0.2257	0.7934E-02	0.3388E-02
12	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.03	0.03	0.1196	0.6541E-02	0.2533E-02
13	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.00	0.00	0.4264E-02	0.1054E-02	0.6321E-04
14	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.01	0.01	0.4693E-02	0.1158E-02	0.1452E-03
15	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.03	0.03	0.8015E-02	0.2125E-02	0.5409E-03
16	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.00	0.00	0.4711E-02	0.3970E-04	0.3524E-03
17	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.01	0.01	0.5984E-02	0.3212E-04	0.3922E-03
18	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.03	0.03	0.4395E-02	0.4013E-04	0.4433E-03
19	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.00	0.00	0.2164E-01	0.1094E-01	0.8553E-03
20	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.01	0.01	0.1914E-01	0.1364E-01	0.1213E-02
21	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.03	0.03	0.7183E-02	0.7647E-02	0.1217E-02
22	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.00	0.00	0.2585	0.9915E-02	0.3979E-02
23	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.01	0.01	0.2362	0.7949E-02	0.3417E-02
24	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.03	0.03	0.1299	0.6797E-02	0.2997E-02
25	0.1000E-01	-.1000E-01	0.1000E-01	0.0100	0.0100	0.00	0.00	0.6534E-02	0.1866E-02	0.1355E-03
26	0.1000E-01	-.1000E-01	0.1000E-01	0.0100	0.0100	0.01	0.01	0.7312E-02	0.2208E-02	0.2429E-03
27	0.1000E-01	-.1000E-01	0.1000E-01	0.0100	0.0100	0.03	0.03	0.1529E-01	0.5603E-02	0.8865E-03
28	0.1000E-01	-.1000E-01	0.1000E-01	-0.0750	-0.0750	0.00	0.00	0.6604E-02	0.3755E-04	0.3579E-03
29	0.1000E-01	-.1000E-01	0.1000E-01	-0.0750	-0.0750	0.01	0.01	0.5879E-02	0.2953E-04	0.3973E-03
30	0.1000E-01	-.1000E-01	0.1000E-01	-0.0750	-0.0750	0.03	0.03	0.4256E-02	0.3277E-04	0.4530E-03
31	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.00	0.00	0.2074E-01	0.1014E-01	0.9129E-03
32	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.01	0.01	0.1864E-01	0.1272E-01	0.1302E-02
33	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.03	0.03	0.6473E-02	0.6860E-02	0.1254E-02
34	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.00	0.00	0.2580	0.9659E-02	0.3952E-02
35	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.01	0.01	0.2349	0.7883E-02	0.3387E-02
36	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.03	0.03	0.1247	0.6637E-02	0.2930E-02
37	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.00	0.00	0.4295E-02	0.1041E-02	0.5794E-04
38	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.01	0.01	0.5444E-02	0.1181E-02	0.1326E-03
39	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.03	0.03	0.1033E-01	0.2129E-02	0.4444E-03
40	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.00	0.00	0.6653E-02	0.3653E-04	0.3591E-03
41	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.01	0.01	0.5961E-02	0.2839E-04	0.4010E-03
42	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.03	0.03	0.4373E-02	0.3038E-04	0.4659E-03
43	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.00	0.00	0.2246E-01	0.1081E-01	0.7579E-03
44	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.01	0.01	0.1933E-01	0.1338E-01	0.1080E-02
45	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.03	0.03	0.7549E-02	0.7991E-02	0.1107E-02
46	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.00	0.00	0.2759	0.9961E-02	0.4003E-02
47	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.01	0.01	0.2580	0.8014E-02	0.3444E-02
48	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.03	0.03	0.1479	0.6953E-02	0.3059E-02

Table 5.3.5 Low Frequency Table for Linear Model versus Second Degree Reduced Model

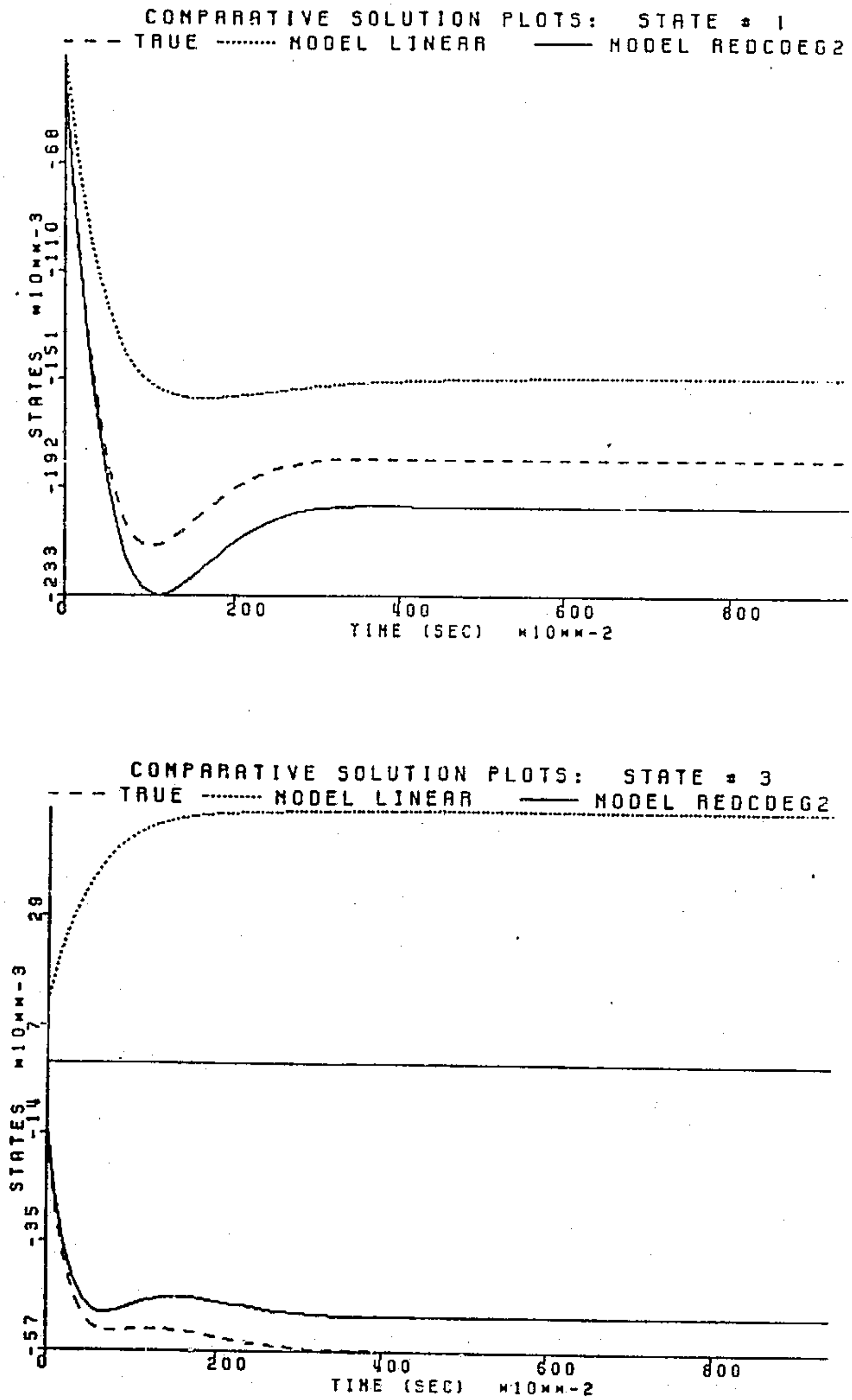
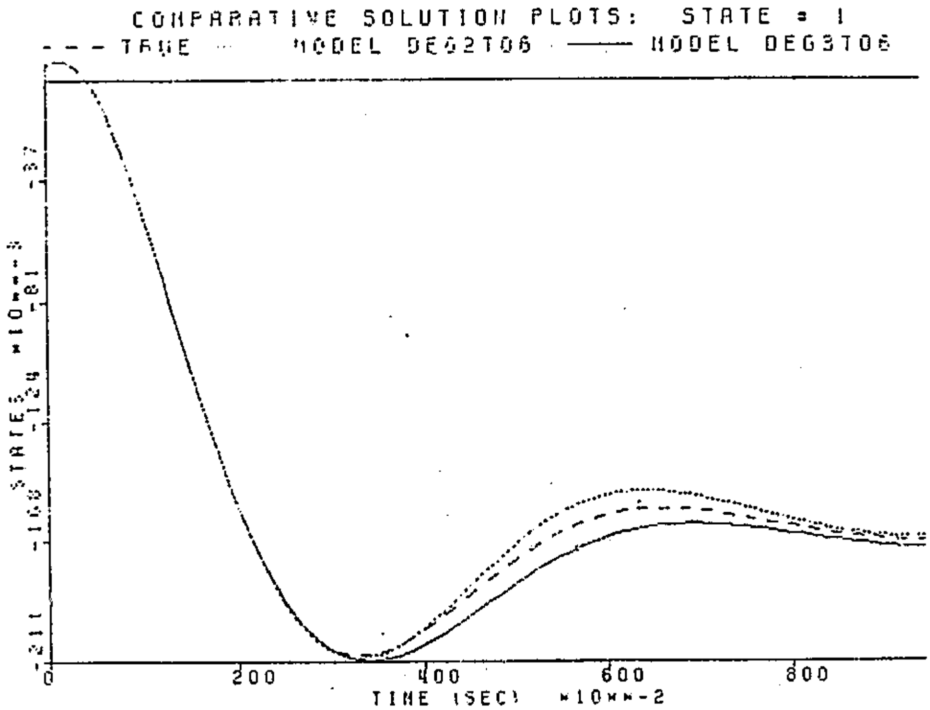


Figure 5.3.8 Simulation Number 10 of Table 5.3.5



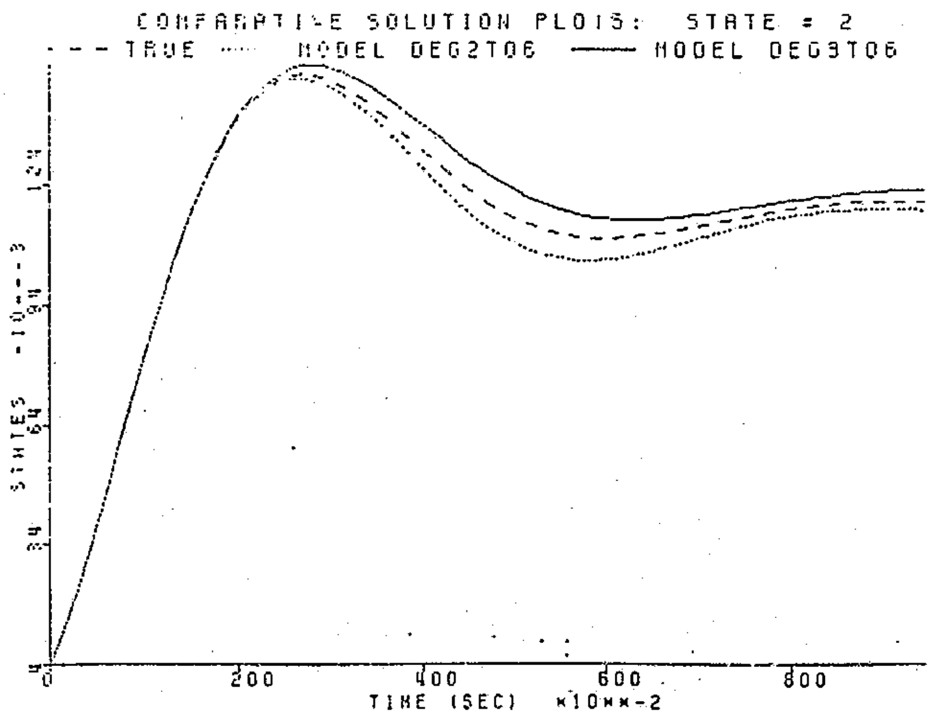
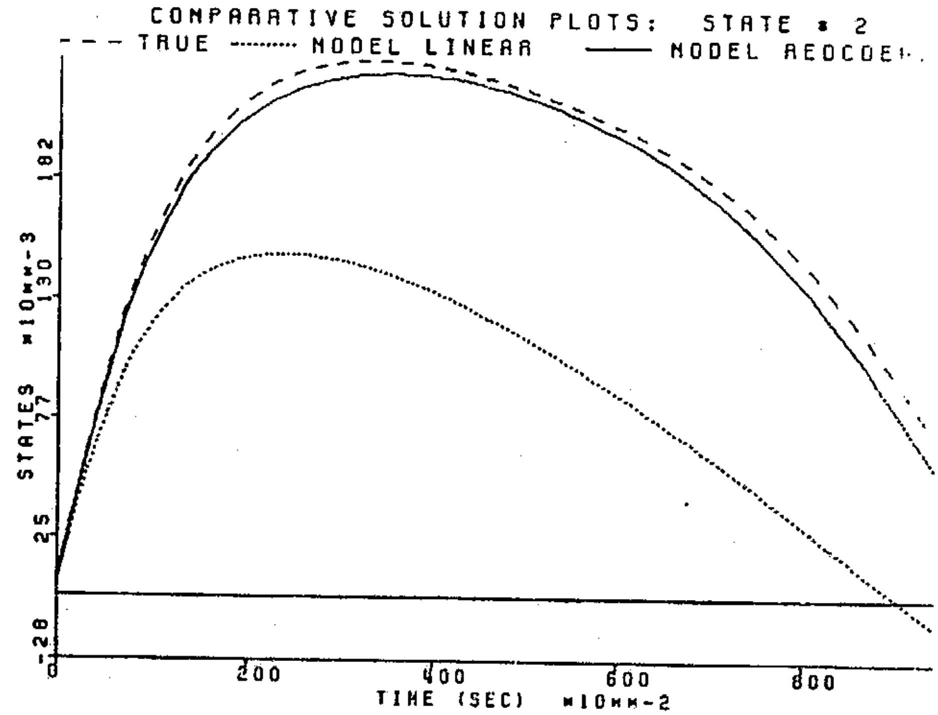


Figure A.33 S#5, Table A.18: Compare to Figure A.30



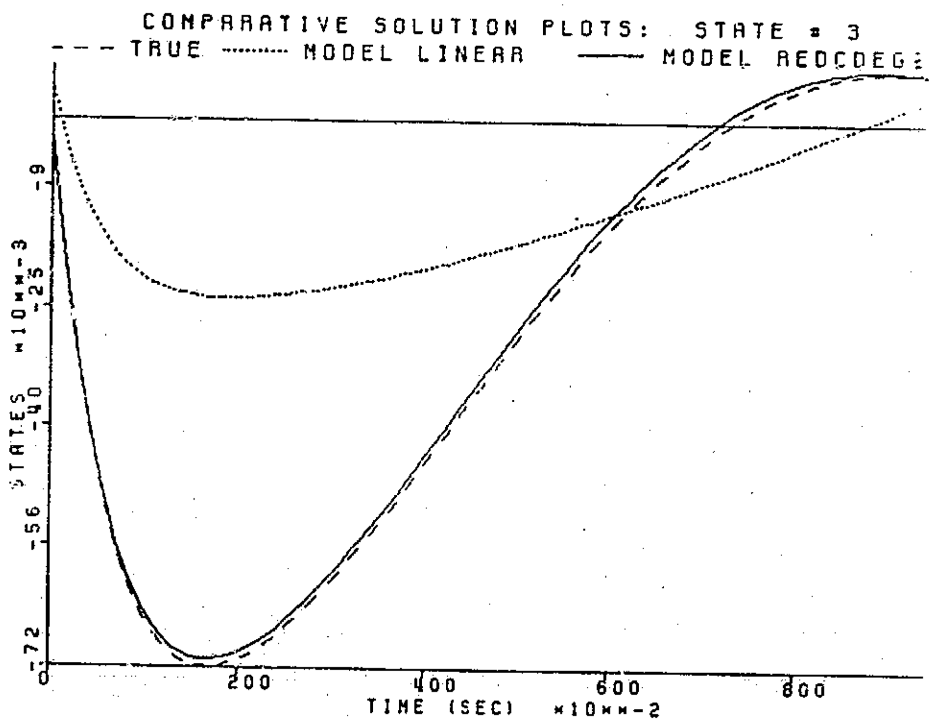


Figure 5.3.9 Simulation Number 33 of Table 5.3.5

****

PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1:MODEL2
OF STATES: 3
OF INPUTS: 2
OF TERMS IN MODEL 1: 20
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 15
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE

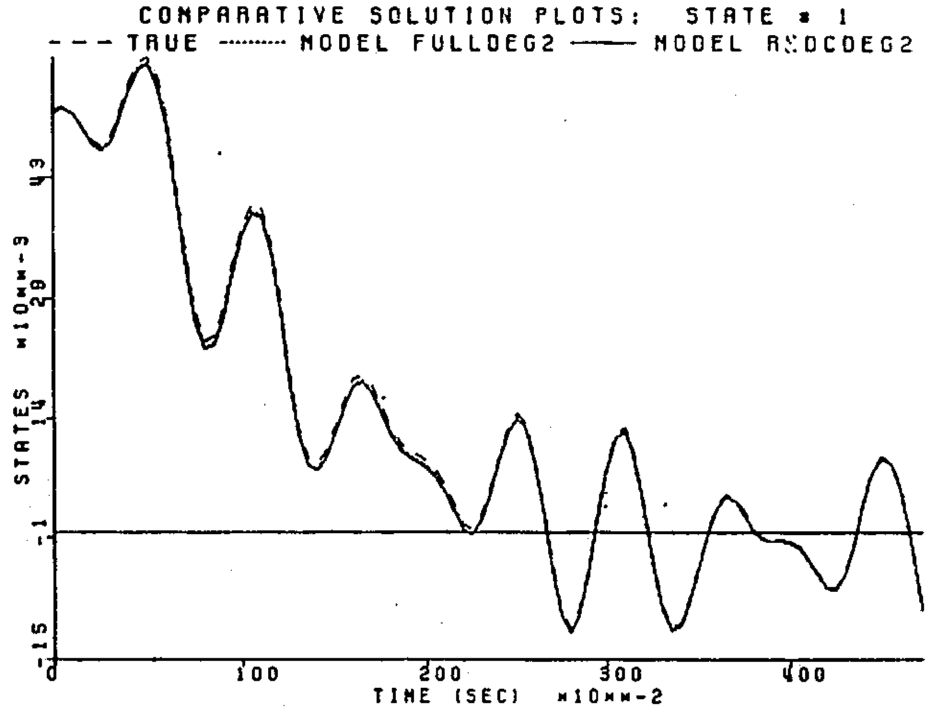
****

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.5000E-01	0.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	1.434	1.132	2.821
2	0.5000E-01	0.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	1.530	1.220	2.585
3	0.5000E-01	0.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	1.462	1.198	2.903
4	0.5000E-01	0.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	1.567	1.219	2.626
5	0.5000E-01	0.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	1.815	1.318	1.028
6	0.5000E-01	0.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	1.827	1.333	1.008
7	0.5000E-01	0.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	1.836	1.332	1.027
8	0.5000E-01	0.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	1.903	1.330	1.017
9	0.5000E-01	0.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.6764	1.402	0.5404
10	0.5000E-01	0.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.8494	1.386	[illegible]
11	0.5000E-01	0.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.7581	1.514	0.5463
12	0.5000E-01	0.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.8137	1.176	0.5760
13	0.5000E-01	0.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	1.935	1.344	0.8976
14	0.5000E-01	0.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	1.910	1.349	0.9003
15	0.5000E-01	0.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	1.914	1.352	0.9013
16	0.5000E-01	0.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	1.981	1.336	0.8999
17	0.5000E-01	-.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	1.741	1.463	1.625
18	0.5000E-01	-.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	2.134	1.648	1.439
19	0.5000E-01	-.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	2.028	1.579	1.601
20	0.5000E-01	-.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	1.895	1.595	1.519
21	0.5000E-01	-.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	1.776	1.346	1.156
22	0.5000E-01	-.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	1.760	1.341	1.145
23	0.5000E-01	-.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	1.774	1.347	1.159
24	0.5000E-01	-.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	1.819	1.346	1.153
25	0.5000E-01	-.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	0.7948	0.7526	0.6126
26	0.5000E-01	-.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	0.8171	0.7437	0.7092
27	0.5000E-01	-.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	0.8334	0.7539	0.5668
28	0.5000E-01	-.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	0.7459	0.7052	0.6810
29	0.5000E-01	-.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	1.648	1.279	1.048
30	0.5000E-01	-.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	1.702	1.280	1.064
31	0.5000E-01	-.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	1.714	1.282	1.055
32	0.5000E-01	-.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	1.634	1.266	1.059

Table 5.3.6 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	-.5000E-01	0.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	0.9373	0.6711	0.6049
2	-.5000E-01	0.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	0.8587	0.7159	0.5735
3	-.5000E-01	0.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	0.8475	0.7225	0.6173
4	-.5000E-01	0.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	1.017	0.7282	0.5870
5	-.5000E-01	0.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	1.864	1.357	1.087
6	-.5000E-01	0.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	1.822	1.350	1.049
7	-.5000E-01	0.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	1.833	1.363	1.087
8	-.5000E-01	0.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	1.867	1.353	1.068
9	-.5000E-01	0.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	2.228	1.817	1.595
10	-.5000E-01	0.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	2.121	1.997	1.411
11	-.5000E-01	0.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	2.160	1.861	1.551
12	-.5000E-01	0.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	2.226	1.814	1.481
13	-.5000E-01	0.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	1.980	1.425	1.104
14	-.5000E-01	0.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	1.989	1.436	1.103
15	-.5000E-01	0.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	2.004	1.433	1.105
16	-.5000E-01	0.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	1.975	1.421	1.110
17	-.5000E-01	-.5000E-01	0.5000E-01	0.0750	0.0750	2.00	1.50	1.213	1.178	1.005
18	-.5000E-01	-.5000E-01	0.5000E-01	0.0750	-0.0750	2.00	1.50	1.032	1.146	0.9980
19	-.5000E-01	-.5000E-01	0.5000E-01	-0.0750	0.0750	2.00	1.50	1.076	1.240	1.005
20	-.5000E-01	-.5000E-01	0.5000E-01	-0.0750	-0.0750	2.00	1.50	1.076	1.160	0.9449
21	-.5000E-01	-.5000E-01	0.5000E-01	0.1500	0.1500	2.00	1.50	1.857	1.399	1.275
22	-.5000E-01	-.5000E-01	0.5000E-01	0.1500	-0.1500	2.00	1.50	1.817	1.378	1.243
23	-.5000E-01	-.5000E-01	0.5000E-01	-0.1500	0.1500	2.00	1.50	1.825	1.392	1.274
24	-.5000E-01	-.5000E-01	0.5000E-01	-0.1500	-0.1500	2.00	1.50	1.844	1.388	1.263
25	-.5000E-01	-.5000E-01	-.5000E-01	0.0750	0.0750	2.00	1.50	1.535	1.099	2.307
26	-.5000E-01	-.5000E-01	-.5000E-01	0.0750	-0.0750	2.00	1.50	1.626	1.140	1.637
27	-.5000E-01	-.5000E-01	-.5000E-01	-0.0750	0.0750	2.00	1.50	1.595	1.085	2.230
28	-.5000E-01	-.5000E-01	-.5000E-01	-0.0750	-0.0750	2.00	1.50	1.506	1.068	1.901
29	-.5000E-01	-.5000E-01	-.5000E-01	0.1500	0.1500	2.00	1.50	1.745	1.361	1.337
30	-.5000E-01	-.5000E-01	-.5000E-01	0.1500	-0.1500	2.00	1.50	1.845	1.385	1.348
31	-.5000E-01	-.5000E-01	-.5000E-01	-0.1500	0.1500	2.00	1.50	1.839	1.369	1.337
32	-.5000E-01	-.5000E-01	-.5000E-01	-0.1500	-0.1500	2.00	1.50	1.738	1.358	1.358

Table 5.3.7 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



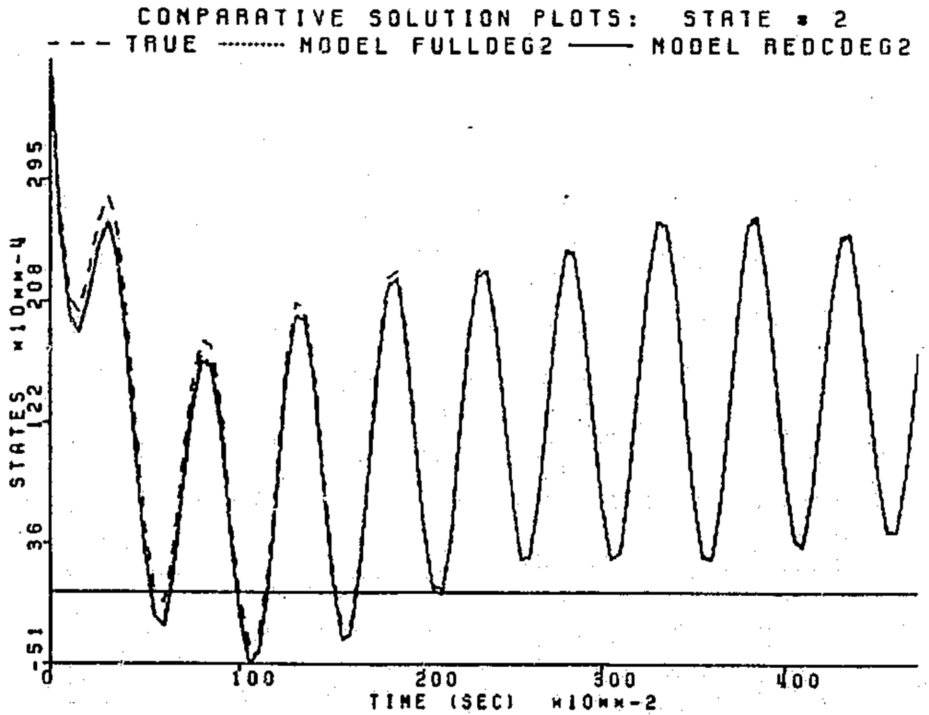
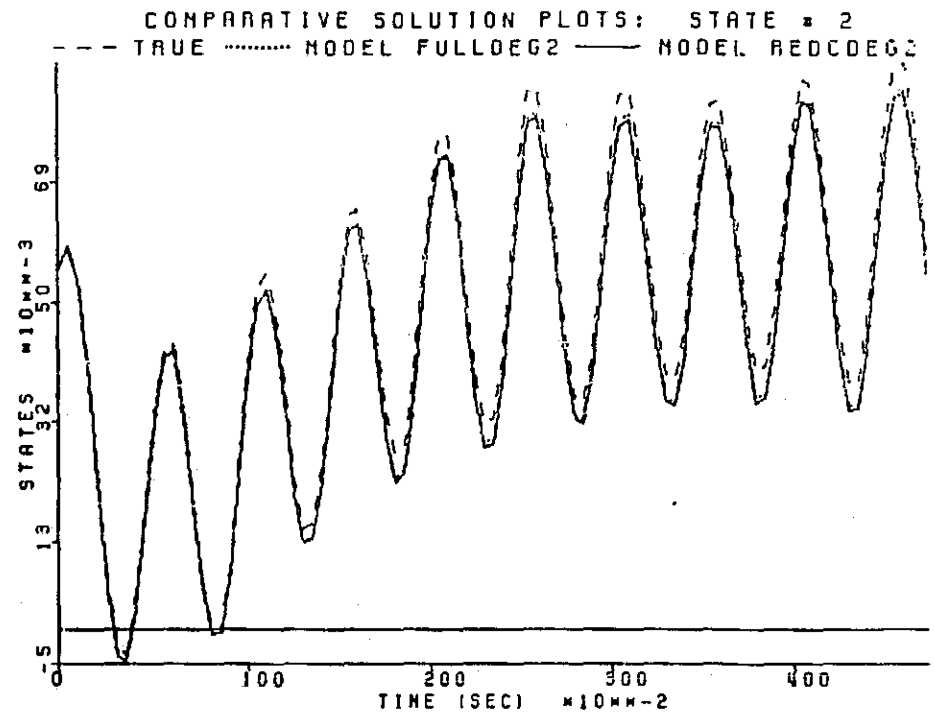


Figure 5.3.10 Simulation Number 4 of Table 5.3.6



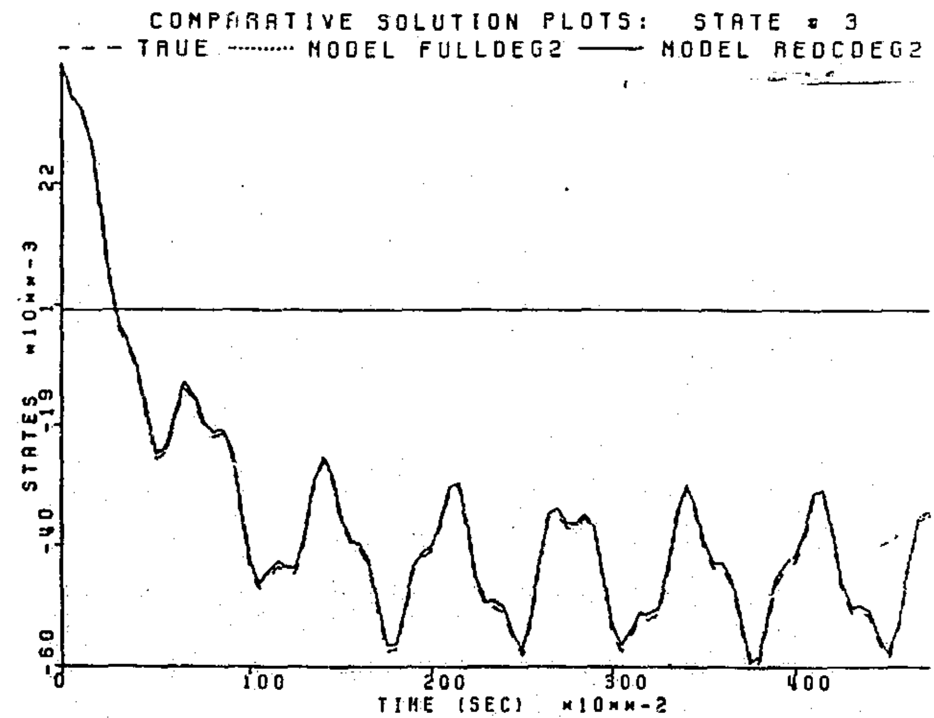


Figure 5.3.11 Simulation Number 5 of Table 5.3.7

ORIGINAL PAGE IS
OF POOR QUALITY

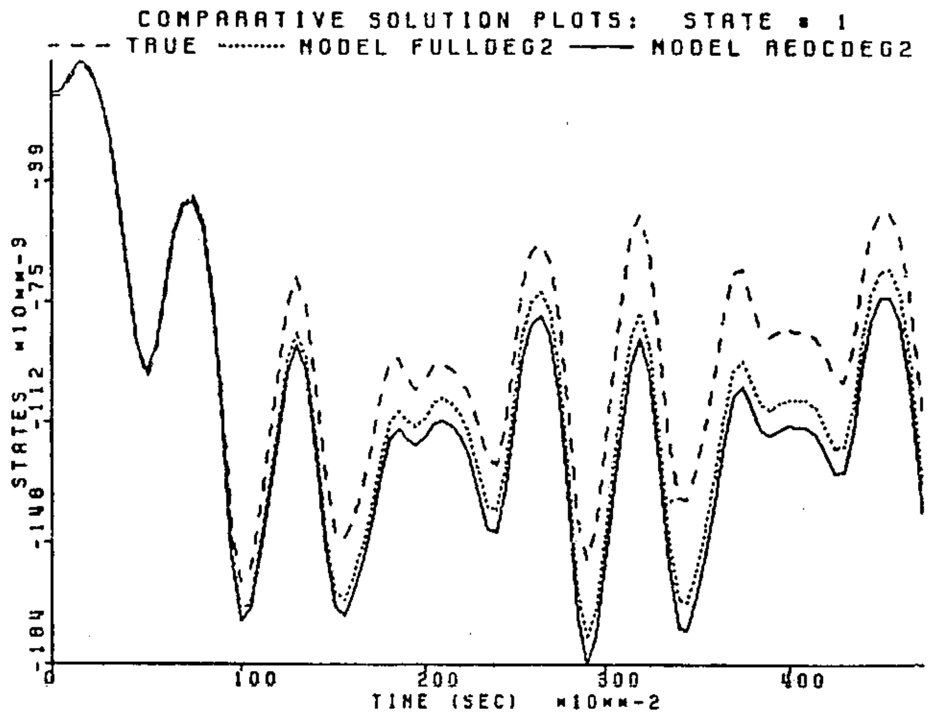
****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 3
OF INPUTS: 2
OF TERMS IN MODEL 1: 20
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 15
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.1000E-01	0.1000E-01	0.1000E-01	0.2000	0.2000	2.10	1.60	1.820	1.298	1.119
2	0.1000E-01	0.1000E-01	0.1000E-01	0.2000	-0.2000	2.10	1.60	1.780	1.297	1.111
3	0.1000E-01	0.1000E-01	0.1000E-01	-0.2000	0.2000	2.10	1.60	1.780	1.299	1.118
4	0.1000E-01	0.1000E-01	0.1000E-01	-0.2000	-0.2000	2.10	1.60	1.829	1.303	1.109
5	0.1000E-01	0.1000E-01	0.1000E-01	0.3000	0.3000	2.10	1.60	1.761	1.156	1.189
6	0.1000E-01	0.1000E-01	0.1000E-01	0.3000	-0.3000	2.10	1.60	1.701	1.153	1.187
7	0.1000E-01	0.1000E-01	0.1000E-01	-0.3000	0.3000	2.10	1.60	1.700	1.154	1.192
8	0.1000E-01	0.1000E-01	0.1000E-01	-0.3000	-0.3000	2.10	1.60	1.768	1.158	1.182
9	0.1000E-01	-.1000E-01	-.1000E-01	0.2000	0.2000	2.10	1.60	1.805	1.298	1.133
10	0.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	2.10	1.60	1.777	1.297	1.128
11	0.1000E-01	-.1000E-01	-.1000E-01	-0.2000	0.2000	2.10	1.60	1.779	1.299	1.133
12	0.1000E-01	-.1000E-01	-.1000E-01	-0.2000	-0.2000	2.10	1.60	1.810	1.303	1.124
13	0.1000E-01	-.1000E-01	-.1000E-01	0.3000	0.3000	2.10	1.60	1.754	1.155	1.194
14	0.1000E-01	-.1000E-01	-.1000E-01	0.3000	-0.3000	2.10	1.60	1.700	1.153	1.194
15	0.1000E-01	-.1000E-01	-.1000E-01	-0.3000	0.3000	2.10	1.60	1.700	1.154	1.197
16	0.1000E-01	-.1000E-01	-.1000E-01	-0.3000	-0.3000	2.10	1.60	1.759	1.158	1.187
17	-.1000E-01	-.1000E-01	0.1000E-01	0.2000	0.2000	2.10	1.60	1.814	1.304	1.155
18	-.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	2.10	1.60	1.781	1.301	1.147
19	-.1000E-01	-.1000E-01	0.1000E-01	-0.2000	0.2000	2.10	1.60	1.781	1.303	1.154
20	-.1000E-01	-.1000E-01	0.1000E-01	-0.2000	-0.2000	2.10	1.60	1.820	1.308	1.145
21	-.1000E-01	-.1000E-01	0.1000E-01	0.3000	0.3000	2.10	1.60	1.758	1.156	1.201
22	-.1000E-01	-.1000E-01	0.1000E-01	0.3000	-0.3000	2.10	1.60	1.702	1.154	1.199
23	-.1000E-01	-.1000E-01	0.1000E-01	-0.3000	0.3000	2.10	1.60	1.701	1.154	1.203
24	-.1000E-01	-.1000E-01	0.1000E-01	-0.3000	-0.3000	2.10	1.60	1.763	1.159	1.193
25	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	0.2000	2.10	1.60	1.811	1.304	1.153
26	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	2.10	1.60	1.787	1.302	1.145
27	-.1000E-01	-.1000E-01	-.1000E-01	-0.2000	0.2000	2.10	1.60	1.788	1.304	1.152
28	-.1000E-01	-.1000E-01	-.1000E-01	-0.2000	-0.2000	2.10	1.60	1.814	1.308	1.143
29	-.1000E-01	-.1000E-01	-.1000E-01	0.3000	0.3000	2.10	1.60	1.756	1.156	1.201
30	-.1000E-01	-.1000E-01	-.1000E-01	0.3000	-0.3000	2.10	1.60	1.704	1.154	1.200
31	-.1000E-01	-.1000E-01	-.1000E-01	-0.3000	0.3000	2.10	1.60	1.704	1.154	1.203
32	-.1000E-01	-.1000E-01	-.1000E-01	-0.3000	-0.3000	2.10	1.60	1.760	1.159	1.193

Table 5.3.8 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	-.3562E-01	-.6893E-01	0.2058E-02	0.1068	-0.2188	3.43	2.72	1.202	1.184	1.518
2	-.6859E-01	-.6320E-01	-.2565E-01	-0.0475	0.1321	4.66	2.21	1.766	0.7579	2.755
3	0.7058E-01	-.2950E-02	0.6901E-01	0.0386	-0.1964	1.97	1.07	2.116	1.783	0.9604
4	-.4633E-02	-.1100E-01	-.3541E-01	-0.2272	-0.0738	3.26	4.81	1.791	1.259	1.215
5	-.3905E-01	0.1766E-01	-.4348E-01	0.1093	0.0399	3.09	1.79	2.109	1.574	1.372
6	-.4442E-01	0.2890E-01	-.4312E-01	0.2190	-0.2008	1.76	4.89	1.824	1.274	1.192
7	0.4164E-02	-.7165E-01	-.2782E-01	0.0304	0.0827	3.05	3.42	0.5718	0.5023	0.3516
8	0.3455E-01	-.6817E-02	0.3014E-01	-0.1400	0.1952	3.42	3.90	1.674	1.302	1.127
9	-.5552E-01	-.3824E-01	0.2795E-01	0.1417	-0.2474	4.78	3.69	1.715	1.305	1.265
10	0.3747E-01	-.7436E-02	-.2476E-01	-0.2217	-0.0446	2.87	1.75	1.785	1.261	1.167
11	0.7219E-01	0.2200E-01	-.4525E-01	0.2317	-0.0048	1.20	4.76	1.744	1.229	1.105
12	-.6679E-01	0.2379E-01	0.1986E-01	-0.0655	-0.0328	3.14	1.74	1.484	1.136	0.7564
13	0.6943E-01	0.3078E-02	-.1860E-01	0.1525	-0.1689	3.83	1.79	1.845	1.340	0.9662
14	-.4655E-01	-.4825E-01	0.3025E-01	-0.0648	0.0120	4.70	4.88	1.071	2.154	1.453
15	-.3726E-02	-.6579E-01	-.3959E-01	-0.0513	-0.1143	2.75	1.76	0.8501	0.6855	0.9479
16	-.7052E-01	-.3124E-01	0.5198E-01	-0.2166	0.0507	4.32	3.47	1.832	1.287	1.268
17	-.3934E-01	-.5829E-01	0.4018E-01	-0.0745	-0.0618	2.78	1.88	1.156	1.365	1.261
18	-.6317E-01	-.1049E-01	0.6379E-01	-0.2284	-0.2463	3.46	1.48	1.819	1.267	1.168
19	-.5110E-02	0.7266E-01	0.3991E-01	0.2218	0.1643	1.14	3.12	1.757	1.246	1.124
20	-.4693E-01	0.2372E-01	-.7699E-02	0.0951	0.0494	4.98	3.63	2.597	1.738	1.450
21	0.5610E-01	0.3104E-01	-.5243E-02	0.1115	0.1216	3.88	4.88	1.908	1.330	0.9492
22	-.7456E-01	0.6746E-01	0.6990E-01	-0.1877	-0.1140	4.42	4.91	1.869	1.326	1.155
23	-.7214E-01	-.2059E-01	0.6127E-01	0.0652	-0.1038	2.19	3.54	1.124	0.6770	0.5957
24	0.2872E-01	0.3261E-01	0.5939E-01	-0.0492	-0.0861	4.88	4.62	1.385	1.187	2.382
25	0.5666E-01	-.3530E-01	0.4306E-01	0.0973	0.1462	1.43	4.37	1.559	1.208	1.265
26	-.8721E-02	-.2256E-01	0.6255E-01	-0.0374	0.1687	4.06	1.98	1.460	1.195	0.6496
27	0.3171E-01	-.4486E-01	0.2218E-01	0.1942	-0.2352	1.97	4.82	1.781	1.278	1.169
28	-.7419E-01	-.3983E-01	-.4984E-01	0.1164	0.2291	2.28	4.73	1.387	1.121	1.666
29	0.3679E-01	-.5320E-01	-.8601E-02	-0.2467	-0.1628	2.94	4.74	1.760	1.224	1.216
30	0.4039E-01	0.4743E-01	-.1621E-02	0.0480	0.2108	2.18	4.87	0.9722	0.9128	1.201
31	0.4539E-01	0.2203E-01	0.3802E-01	-0.1399	0.2236	1.25	3.95	1.533	1.168	1.055
32	-.4102E-01	-.5523E-01	0.6202E-01	-0.1598	0.2165	4.97	3.22	1.877	1.345	1.271
33	-.7302E-01	0.6745E-01	-.9008E-02	0.1904	-0.1361	1.74	3.32	1.765	1.291	1.152
34	-.5007E-01	-.3751E-01	0.5766E-01	0.2419	-0.2383	3.10	1.46	1.748	1.241	1.195
35	0.2660E-01	-.4823E-02	-.6368E-01	0.0349	-0.1240	3.92	2.27	0.9930	0.9585	0.3783

Table 5.3.9 Simulation Table for Second Degree Full Model versus Second Degree Reduced Model



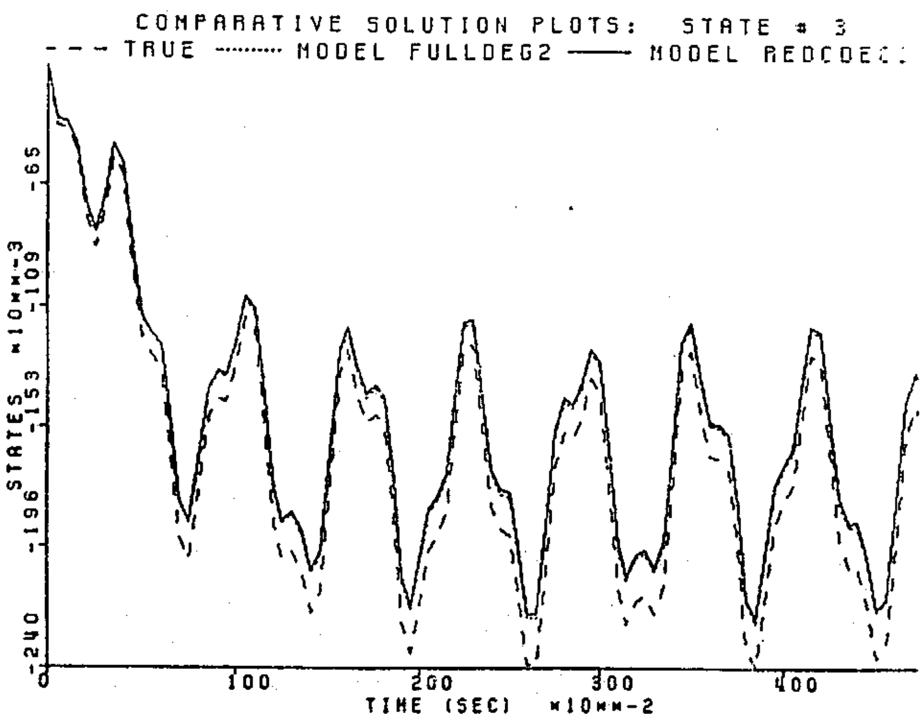
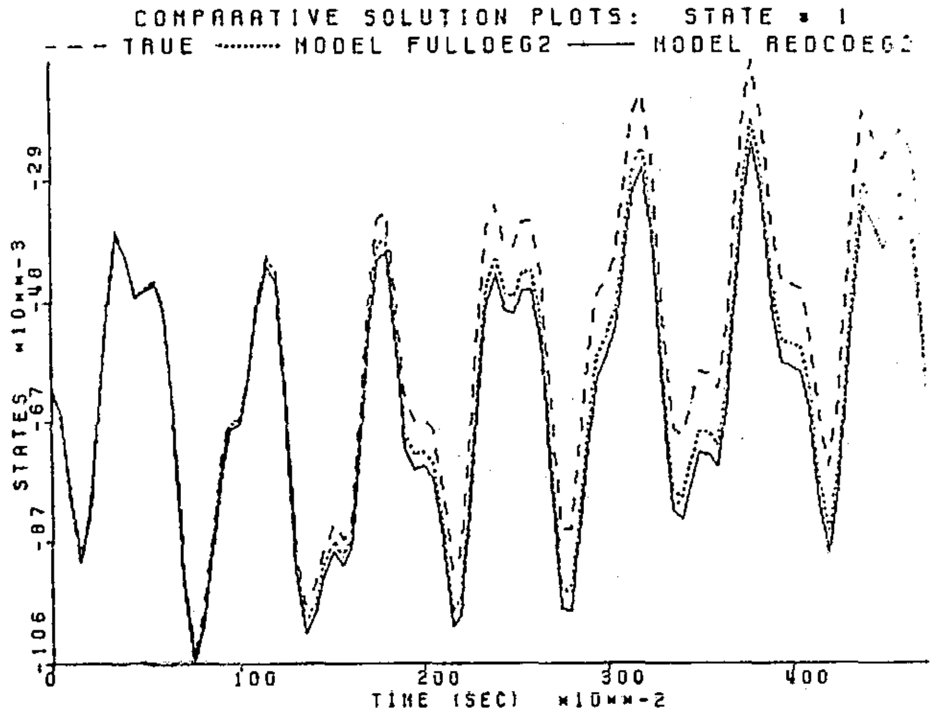


Figure 5.3.12 Simulation Number 21 of Table 5.3.8

ORIGINAL PAGE IS
OF POOR QUALITY



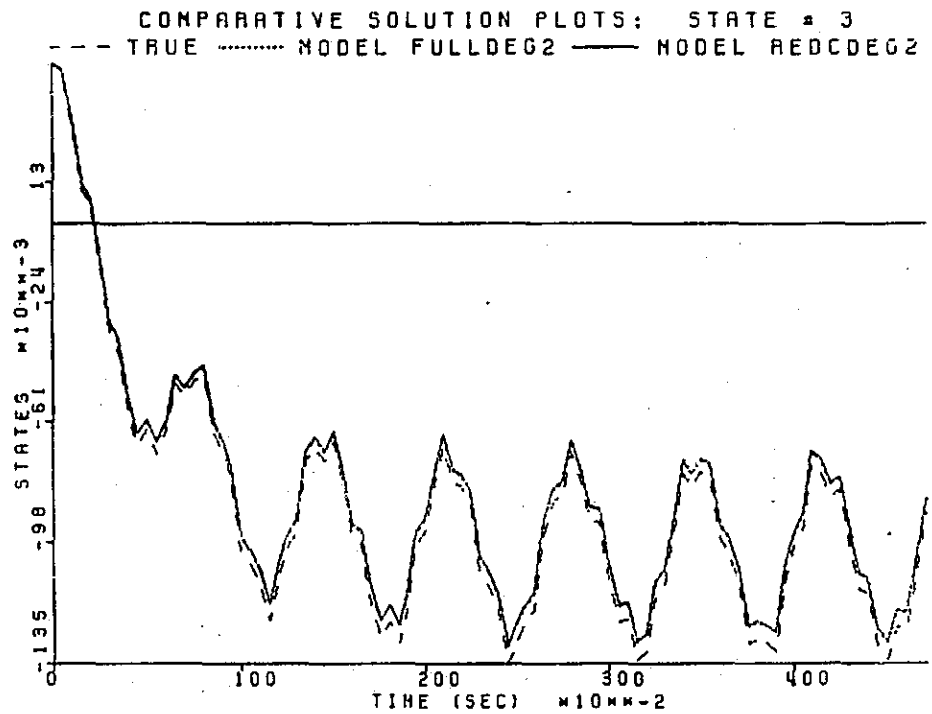


Figure 5.3.13 Simulation Number 18 of Table 5.3.9

\#

PROBLEM SUMMARY

CONFIGURATION: TRUE,MODEL1,MODEL2

\# OF STATES: 3

\# OF INPUTS: 3

\# OF TERMS IN MODEL 1: 20

DEGREE OF APPROXIMATION: 2

\# OF TERMS IN MODEL 2: 15

DEGREE OF APPROXIMATION: 2

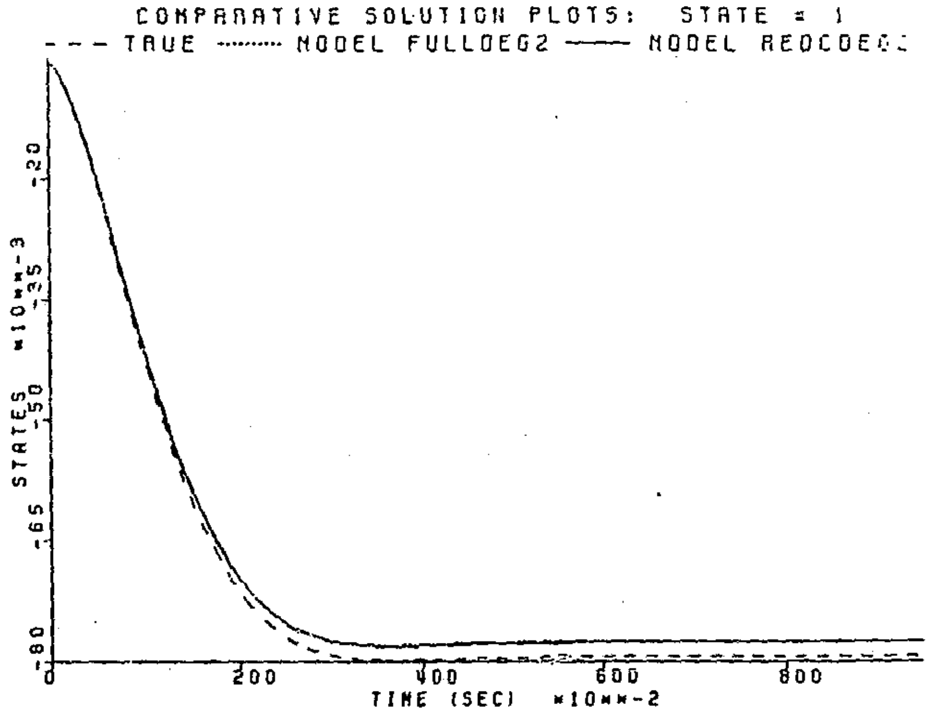
SIMULATION WITH COSINE

\#

S#	INITIAL CONDITIONS			AMPLITUDES		FREQUENCIES		ERROR RATIOS		
1	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.00	0.00	1.221	0.9927	0.2835
2	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.01	0.01	1.224	0.9092	0.4012
3	0.5000E-02	0.5000E-02	0.5000E-02	0.0100	0.0100	0.03	0.03	1.213	0.9959	0.7934
4	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.00	0.00	0.9586	0.1992	108.8
5	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.01	0.01	0.9698	0.1923	56.58
6	0.5000E-02	0.5000E-02	0.5000E-02	-0.0750	-0.0750	0.03	0.03	1.051	0.3121	14.62
7	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.00	0.00	0.5345	0.4550	0.8715E-01
8	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.01	0.01	0.5800	0.5110	0.1037
9	0.5000E-02	0.5000E-02	0.5000E-02	0.1000	0.1000	0.03	0.03	1.064	0.9850	1.584
10	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.00	0.00	0.8609	1.341	0.7389
11	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.01	0.01	0.8497	1.365	0.7287
12	0.5000E-02	0.5000E-02	0.5000E-02	0.2000	-0.2000	0.03	0.03	0.8669	1.353	0.7327
13	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.00	0.00	1.254	0.8884	0.2395
14	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.01	0.01	1.258	0.9035	0.3764
15	-.5000E-02	0.5000E-02	-.5000E-02	0.0100	0.0100	0.03	0.03	1.251	1.043	0.7934
16	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.00	0.00	0.9544	0.2013	143.9
17	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.01	0.01	0.9644	0.1969	65.77
18	-.5000E-02	0.5000E-02	-.5000E-02	-0.0750	-0.0750	0.03	0.03	1.035	0.3364	15.92
19	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.00	0.00	0.5510	0.4696	0.5276E-01
20	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.01	0.01	0.5959	0.5245	0.9903E-01
21	-.5000E-02	0.5000E-02	-.5000E-02	0.1000	0.1000	0.03	0.03	1.098	1.014	1.473
22	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.00	0.00	0.8610	1.341	0.7389
23	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.01	0.01	0.8499	1.364	0.7287
24	-.5000E-02	0.5000E-02	-.5000E-02	0.2000	-0.2000	0.03	0.03	0.8674	1.350	0.7329
25	0.1000E-01	-.1000E-01	0.1000E-01	0.0100	0.0100	0.00	0.00	1.249	0.9681	0.3835
26	0.1000E-01	-.1000E-01	0.1000E-01	0.0100	0.0100	0.01	0.01	1.254	0.9997	0.4782
27	0.1000E-01	-.1000E-01	[illegible]	0.0100	0.0100	0.03	0.03	1.253	1.112	0.8417
28	0.1000E-01	-.1000E-01	[illegible]	-0.0750	-0.0750	0.00	0.00	0.9595	0.1877	104.9
29	0.1000E-01	-.1000E-01	[illegible]	-0.0750	-0.0750	0.01	0.01	0.9709	0.1773	55.49
30	0.1000E-01	-.1000E-01	0.1000E-01	-0.0750	-0.0750	0.03	0.03	1.055	0.2537	14.95
31	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.00	0.00	0.5377	0.4567	0.9229E-01
32	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.01	0.01	0.5813	0.5105	0.1111
33	0.1000E-01	-.1000E-01	0.1000E-01	0.1000	0.1000	0.03	0.03	1.074	0.9902	1.630
34	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.00	0.00	0.8609	1.341	0.7370
35	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.01	0.01	0.8495	1.364	0.7259
36	0.1000E-01	-.1000E-01	0.1000E-01	0.2000	-0.2000	0.03	0.03	0.8668	1.351	0.7333
37	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.00	0.00	1.281	0.8669	0.1450
38	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.01	0.01	1.286	0.9034	0.2402
39	-.1000E-01	-.1000E-01	-.1000E-01	0.0100	0.0100	0.03	0.03	1.292	1.049	0.4640
40	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.00	0.00	0.9509	0.1872	155.5
41	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.01	0.01	0.9602	0.1762	74.96
42	-.1000E-01	-.1000E-01	-.1000E-01	-0.0750	-0.0750	0.03	0.03	1.023	0.2543	17.22
43	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.00	0.00	0.5708	0.4252	0.5226E-01
44	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.01	0.01	0.6134	0.5383	0.9966E-01
45	-.1000E-01	-.1000E-01	-.1000E-01	0.1000	0.1000	0.03	0.03	1.139	1.046	1.405
46	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.00	0.00	0.8610	1.337	0.7390
47	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.01	0.01	0.8550	1.362	0.7290
48	-.1000E-01	-.1000E-01	-.1000E-01	0.2000	-0.2000	0.03	0.03	0.8577	1.346	0.7337

Table 5.3.10 Low Frequency Table for Second Degree Full Model versus Second Degree Reduced Model

ORIGINAL PAGE IS
OF POOR QUALITY



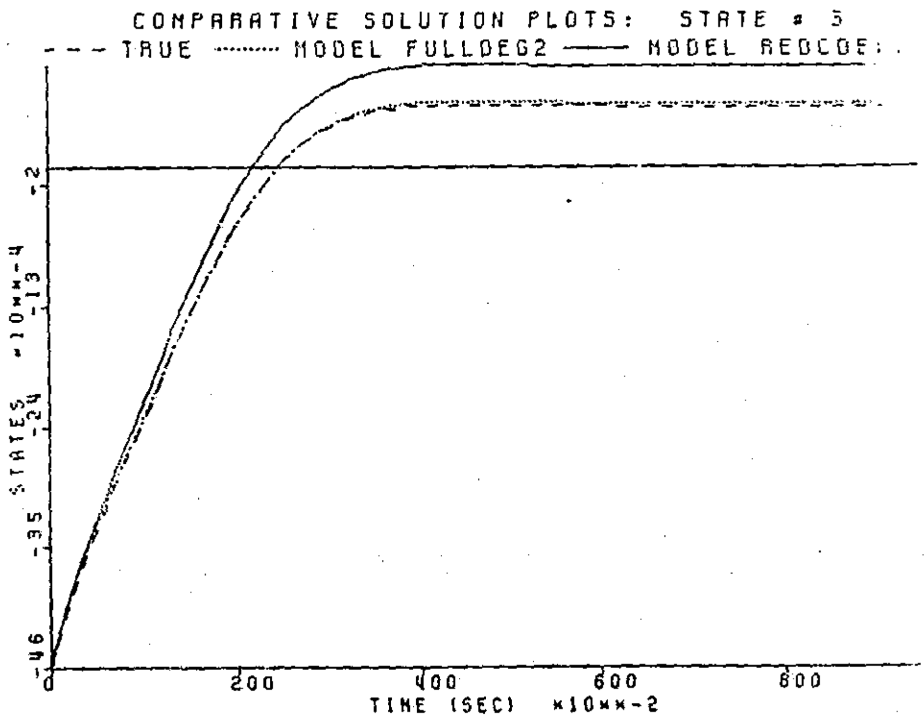
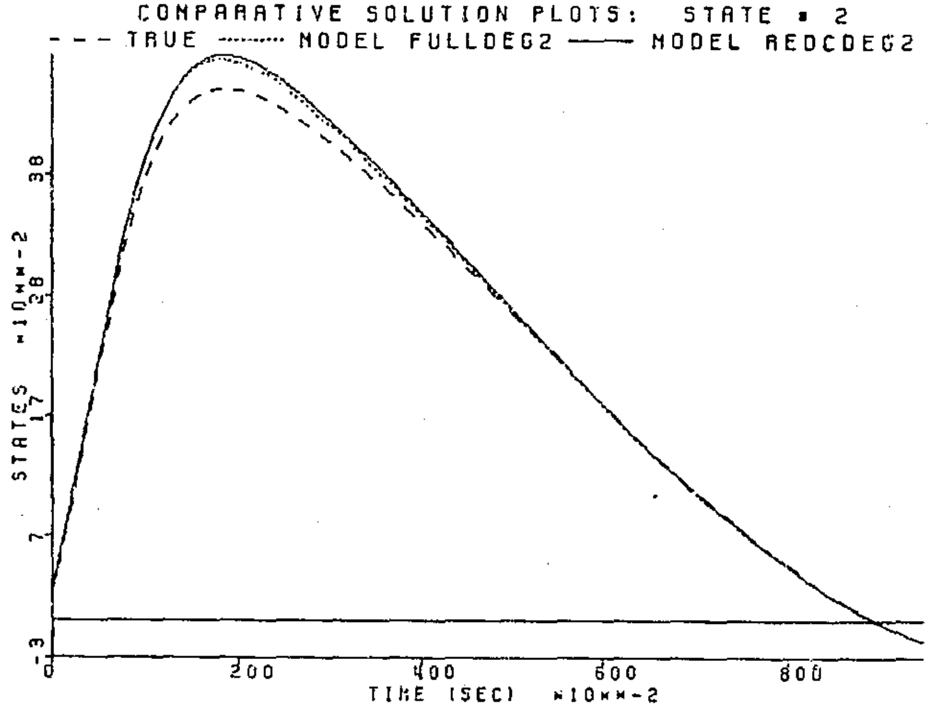


Figure 5.3.14 Simulation Number 16 of Table 5.3.10

C-3



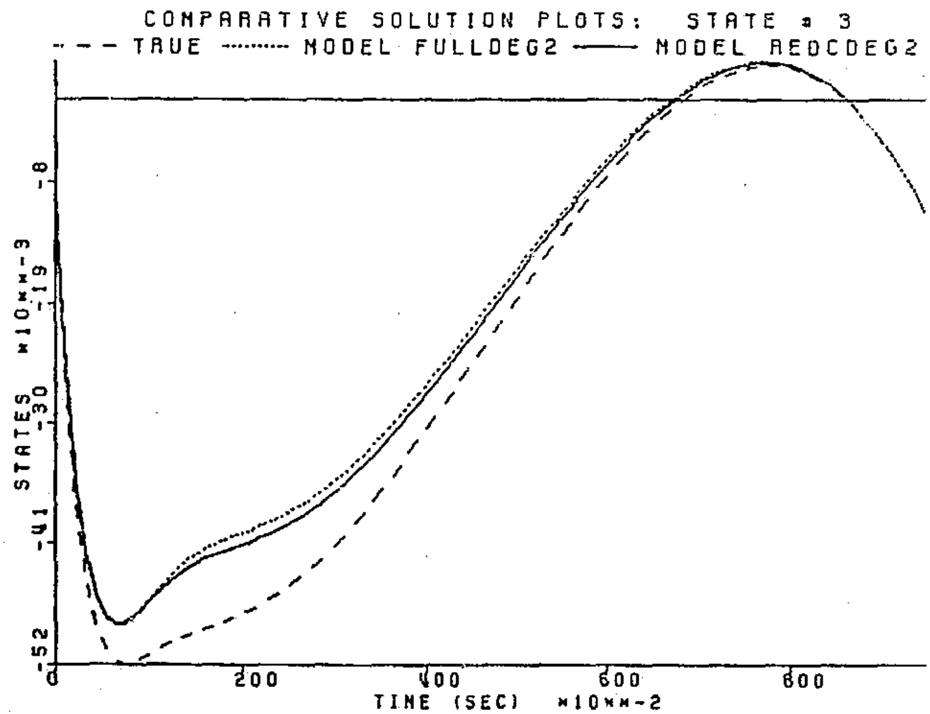


Figure 5.3.15 Simulation Number 36 of Table 5.3.10

in the FORTRAN based language SPEAKEASY. While SPEAKEASY makes the iterative elimination of rows and columns a trivial task, it is (being a very high level computer language) inherently slow. A reduction test about the size of those presented here, would take, on the average, about 5 minutes of cpu time. This time of execution might be shortened singificantly if future versions of reduction software are written directly in FORTRAN. This may even become necessary as model size increases, but for the initial efforts contained herein, the slow execution is accepted.

CHAPTER VI

CONCLUSIONS

The primary goal of this thesis was to initiate studies of and develop a methodology for the size reduction of the nonlinear models generated in the tensor framework. This goal was accomplished using an idea based on the error inherent in the model due to the least squares fit of the data. The iterative reduction scheme was developed primarily by logical argument and intuition. Based on the reasoning, a loose step by step reduction methodology was formulated. Using the reduction technique, system identification and reduction were carried out for several examples. In all cases the results were good and demonstrated that a reduced model can be identified which retains influential higher order nonlinearities while discarding the insignificant ones in an effort to decrease the size of the model.

Based on the encouraging nature of the results documented here, further research into the following ideas is warranted. First, with respect to this particular reduction method, Desrochers has suggested that cost functions can be implemented to select the appropriate terms in the appropriate combinations so as to aid in the omitting of terms. Perhaps his optimization ideas may be modified to fit this particular problem formulation as was his idea for model reduction.

A second suggestion has to do with extending the model reduction technique to the layered model identification approach. This identification scheme has been studied in [14] and concerns itself with modelling the error in some model approximation with the addition of another degree of nonlinearity. The outlook for this identification technique is promising.

The current reduction method could be modified to enable a layer by layer reduction. Or if an entire layered model is to be reduced, provisions would have to be made to save the data from each previously modelled layer, as the current reduction method requires it for the error test. In either case, the application of the reduction test to the layered modelling approach is a problem that deserves study.

In sum, based on the findings of this thesis, further research is justified not only in the extension of the reduction method to other identification approaches, but also into the improvement and advancement of the current reduction scheme.

APPENDIX A [See Acknowledgments, page iii]
NONLINEAR TENSOR MODELS:
A CASE STUDY

A.1 INTRODUCTION

So often schemes for various procedures in nonlinear system theory such as parameter estimation, inverse systems, and so on, are discussed, whether it be in textbooks or technical papers, but are left void of any substantial convincing proof of validity or utility. Indeed, even some of the simplest of nonlinear systems present problems which require extensive illustration of a technique to produce an acceptable level of reliability. It is this proof-in-the-pudding spirit to which we adhere in this report to study the tensor parameterization approach.

Let us begin by presenting the nonlinear system to be modeled. Two coupled nonlinear differential equations given by

$$\begin{aligned} \dot{x}_1 &= u_2\cosh(x_1x_2) - 3\sinh(x_2) - e^{2u_1}\sinh(2x_1) \\ \dot{x}_2 &= \sinh(x_2) + e^{u_1u_2}\sinh(x_1) - e^{u_1}u_1\cosh(x_1^2) \end{aligned} \tag{A.1.1}$$

serve as our example system, where as usual x_1 and x_2 represent the states, $\dot{x}_1$ and $\dot{x}_2$ their time derivatives, and u_1 and u_2 the system inputs. In order to utilize the tensor parameterization modeling approach, the allowable input set must be bounded in some reasonable sense, particularly for low frequency cases. In fact, the system (A.1.1) exhibits finite escape for a relatively small step input in the first input channel. The L_{10} matrix (that is, the matrix of parameters associated with linear terms in x alone) of the linear approximation for (A.1.1),

$$\begin{aligned} \dot{x}_1 &= u_2 - (2x_1 + 3x_2) \\ \dot{x}_2 &= x_1 + x_2 - u_1, \end{aligned} \tag{A.1.2}$$

has characteristic values

$$\lambda_i = -\frac{1}{2} \pm j\sqrt{3}/2,\ i = 1,2. \tag{A.1.3}$$

Transient responses to step inputs are of five to six seconds in duration. Note further that the origin, $(x,u) = 0$, is a stable equilibrium point.

In the section to follow specifics relative to the actual identification scheme are discussed, and we comment on the extensive software capabilities which have been developed to facilitate these modeling studies. Following this are two sections illustrating a second degree approximation model and a third degree approximation model. Verification studies in the form of simulation tables and representative plots make up the bulk of these sections. In the final section we discuss some further considerations of these particular models, and dwell on possibilities for improved performance.

A.2 ALGORITHM AND SOFTWARE CONSIDERATIONS

The overall modeling procedure can be separated into three basic stages: problem formulation, model identification, and model verification. In the first stage, the system is formulated in the differential equation description; in the general case we assume this form of system description, and assume that all states are measurable. Most of the software for the overall numerical procedure is involved in the second stage, model identification, where much attention is paid to exciting the system to be measured with the "best" input. That is, it is crucial to produce independent data in the sampling procedure. The final stage, model verification, requires numerous simulation studies and comparative error analyses. One basic course of action in these studies is to emphasize a region of model validity, or region of acceptable performance; the burden of proof, then, lies in showing that as the degree of the model gets larger, so does the region of validity.

It is important to note from the onset that the overshadowing spirit in these modeling studies is to regard the system to be modeled as a "black box" whose dynamics are known to us only by way of state trajectory response curves resulting from various input excitations. We therefore view the system (A.1.1) in this light and attempt to construct a model consisting of a system of nonlinear differential equations, each of which being a truncated series expansion with tensor coefficients.

Consider the nonlinear dynamical equation

$$\dot{x} = f(x,u), \tag{A.2.1}$$

which is the general expression for (A.1.1), for $f : X \times U \to X$, X the n-dimensional state space and U the m-dimensional input space. Using the tensor parameterization in an expansion for the right side of (A.2.1) such as

$$f(x,u) = [L_{10} \mid L_{01} \mid L_{20} \mid L_{11} \mid L_{02} \mid L_{30} \mid \ldots]x_s, \tag{A.2.2}$$

where the vector x_s contains symmetric powers of x and u,

$$x_s = (x,\ u,\ x \vee x,\ x \vee u,\ u \vee u,\ x \vee x \vee x, \ldots)^T, \tag{A.2.3}$$

the true system (A.2.1) may be integrated, and solutions sampled, to form the matrix equation, arising from (A.2.2), as

$$\dot{X} = LX_s. \tag{A.2.4}$$

Here the matrices $\dot{X}$ and X_s each have p columns, p being the number of samples taken, where the elements of $\dot{X}$ are derivative estimates of the states taken along the state trajectories for some input. A lexicographic ordering algorithm is used for forming the matrix X_s from (A.2.3). Critical parameters in the formulation of (A.2.4) are model degree, sampling interval, the total experiment time, integration stepsize (since the derivatives are to be estimated, a reasonably small stepsize must be used to ensure accuracy), and most important, the choice of input. This leads us to the identification stage of the procedure.

With regard to the matrix Equation (A.2.4), the problem of identifying the model parameters embodied in the matrix L is a straightforward exercise using linear least squares techniques. Indeed, this is one of the primary organizational advantages of the tensor parameterization. The difficulty arises, however, in the procedure used to choose the input to excite enough of the nonlinear modes of the system. A linear combination of sinusoids forms the general structure of the input,

$$u_j = \sum_{i=1}^{r} c_{ij} \sin\omega_{ij}t \quad , \quad j = 1,2,\ldots,m, \tag{A.2.5}$$

where the c_{ij} are input amplitudes and

$$\omega_{ij} = (2\pi f_j)i, \tag{A.2.6}$$

for the frequency f_j in hertz. The design of the inputs (A.2.5) is carried out through use of optimization techniques where the quadratic data matrix M (analogous to the information matrix in the information theoretic approach to optimal input design) is made small in some sense. The cost criterion chosen here for accomplishing this is given by

$$J = \mathrm{tr}\,(M^{-1}), \tag{A.2.7}$$

so that the design problem may be simply stated as

$$u = \min_{\bar{u}} \{J\}, \tag{A.2.8}$$

for $\bar{u}$ an element of some admissible set of input trajectories.

A brief discussion of the software package used in this overall scheme will aid in clarifying these procedures. The routine used for formulating Equation (A.2.4) for the least squares identification of L has the following program inputs:

- initial state conditions
- integration stepsize

- model approximation degree
- number of samples (h)
- number of sinusoids per input (r)
- input amplitudes (c_{ij})
- input frequencies (f_j)
- optimization option
- optimization method with associated parameters.

The optimization option is a selection parameter by which the user may choose to optimize over input amplitudes, input frequencies, initial state conditions, or any combination thereof. With the optimization method parameter the user has a choice of two techniques each for the constrained or unconstrained optimization problem. For reasons to be explained later, we concern ourselves here with the unconstrained problem such as that given in (A.2.8). One method choice employs the conjugate gradient approach to nonlinear optimization. The main advantage of this technique is that it provides a fast rate of convergence without need for storage of any large matrices. But since this indirect optimization approach requires either a good estimate of the cost function gradient or a closed form solution thereof, it is not employed in the example of this case study. Instead, a direct method approach to solution of (A.2.8) is used via a routine from the IMSL support software on the IBM 370/3033. In this technique, the quasi-Newton method, it is assumed that the gradient vector and the Hessian matrix exist, but the user is not required to supply formulas for their evaluation. Several free parameters may be specified for use of this routine, such as maximum number of cost function evaluations (to limit CPU time), convergence criterion, and choice of the initial Hessian matrix.

Once all program inputs are specified for the equation formulation, the input design begins with an integration of the true system; a fourth order Runge Kutta algorithm is used for this purpose. The data matrix M is formed initially with the start-up values, and then with values returned by the optimization routine for each subsequent evaluation of the cost function.

Typical optimizations require anywhere from 50 to 150 evaluations of the cost function. The IMSL routine has a built-in stopping mechanism in the event that further progress is impossible. For instance, the iterative procedure may be terminated if it is determined that roundoff errors become dominant. Otherwise, the routine either terminates with convergence (according to a prespecified threshold) or when the user-specified maximum number of cost function calculations has been reached. Along with the value of the objective function, an important quantity to observe in this iterative procedure is the condition[1] number of the data matrix. It is clear that as the matrix approaches a singular matrix the reliability of the optimization deteriorates. For purposes of accuracy, double precision is used throughout.

Once the matrix Equation (A.2.4) is loaded, using data resulting from the designed input, linear least squares is performed to produce the parameters which comprise the nonlinear model. A singular value decomposition routine for solution of simultaneous matrix equations, from the SPEAKEZY library on the IBM 370/3033, is used for this purpose and returns the singular values of X_s so that its condition number may be calculated. This serves as another measure of the independence of the sampled data. It should be noted that in this identification step, prior to entering the SPEAKEZY routine, the user is given the option to normalize the data in X_s in order to enhance the capabilities of the numerical routines which follow. When this is done, the result from the SPEAKEZY routine (that is, the matrix L in (A.2.4)) is then transformed back in a reverse process.

[1]Recall that the condition number of a matrix is simply the ratio of the largest to smallest singular value of the matrix. An EISPAK SVD routine is used in this application.

The verification stage of the overall procedure is carried out by way of two separate routines in two separate computing environments. To take advantage of the speed and efficiency of the IBM 370/3033, the large exhaustive simulation exercises are done there, while the graphics capabilities and immediate turn-around time opportunities afforded by the DEC PDP 11/44 are utilized for interactive viewing of response curves and hardcopy plotting. In either case, comparative studies are done for model performance.

Simulation parameters (that is, initial state conditions, input amplitudes, and input frequencies) are loaded into a sequential data set for the bulk simulation exercises. The user specifies the system configuration as true system versus either one or two models of the system. These systems are then integrated, again using a fourth-order Runge Kutta algorithm, only now calculations are carried out in single precision for added speed.

If two models are compared against the true system, an error analysis is performed in which a ratio of mean square errors over the integration interval is calculated. To clarify this, consider the discrete version of our problem and suppose that $x_j(k)$ represents the jth state of the true system solution, $x1_j(k)$ the jth state of the first model solution and $x2_j(k)$ the jth state of the second model solution, each at the timepoint k. Then for the jth state the error ratio ε_j is given by

$$\varepsilon_j = \frac{\sum_{k=1}^{q} (x_j(k) - x2_j(k))^2}{\sum_{k=1}^{q} (x_j(k) - x1_j(k))^2}, \qquad (A.2.9)$$

where q is the total number of time steps in the simulation interval. Therefore, in view of (A.2.9), if

$$\varepsilon_j < 1 \tag{A.2.10}$$

is true, the second model has outperformed the first model in the jth state for that particular set of simulation parameters. In this way a region of model validity may be mapped out when a sufficient amount of simulation results has been generated. Typical simulation tables, as we shall see in the sections to follow, consist of initial state conditions and input amplitudes which increase in some organized fashion about the equilibrium point values. Frequencies of the inputs are also varied, typically from 0 to 10 hertz. Another good test of model region employed throughout the validation studies uses randomly generated (within specified limits) simulation parameters.

Since no one error criterion tells the entire story for any given simulation, another measure is used for these comparative studies. The maximum absolute error over both models compared is multiplied by the total number of time steps q; this value is retained as the error normalization factor (enf). Then the normalized errors

$$\varepsilon 1_j = \frac{\sum_{k=1}^{q} (x_j(k) - x1_j(k))^2}{\text{enf}}, \tag{A.2.11}$$

$$\varepsilon 2_j = \frac{\sum_{k=1}^{q} (x_j(k) - x2_j(k))^2}{\text{enf}}, \tag{A.2.12}$$

are printed out for each state, for each simulation, in tabular form. These individual errors alert the user to a situation in which, for instance, an error ratio near unity results but when both model errors are unacceptable. The relative size of the enf, along with the individual normalized errors (A.2.11) and (A.2.12), gives added insight to accompany the error ratio analyses. In general, the larger the enf is, the more relative significance the error ratio carries.

The final and most revealing test for model performance lies in the viewing of state trajectories for simulation parameters within the regions of interest. For this purpose the DEC PDP11/44 computing system is used with the Tektronix 4025A video graphics terminal to display plots for various simulation parameter sets. While the actual time involved for simulations is longer on this machine than on the IBM 370/3033, the trade-off in turnaround time for interactive and hardcopy plotting is worthwhile.

While practical limitations prohibit a truly exhaustive search of regions of validity, the combination of results from simulation studies such as these gives a reasonably complete picture of model performance. An illustration of these ideas appears in the section to follow for the comparative study of a second degree approximation model of the system (A.1.1) versus a linear approximation model, as given in (A.1.2).

A.3 SECOND DEGREE MODELING

We return now to the original system (A.1.1) for this case study, and illustrate the application of the model formulation, identification, and verification as described in the preceding section.

With an integration timestep of 0.005 seconds the system is integrated with initial conditions of 0.005 and -0.005 for x_1 and x_2, respectively, and sampled every 0.04 seconds for 100 points. A second degree truncation approximation will be considered first, so that the nonlinear model has a total of 28 parameters, 14 for each equation. The input for this model identification consists of a sum of two sinusoids whose amplitudes are held constant at 0.025; after a few trials it was determined that these amplitudes give very good results. Optimization is done over the input frequencies, and the startup values are set at 0.75 and 1.0 hertz for u_1 and u_2, respectively. After 62 evaluations of the cost function convergence is achieved with a cost function reduction of approximately 35%. At this point a local optimization is achieved and the final values of the frequencies are 0.742 and 0.902. The parameters which result from the least squares identification using the data generated with these inputs are shown in Figure A.1, separated according to the individual L_{ij} partitions as depicted in (A.2.2). The eigenvalues of L_{10} are within one percent of those in (A.1.3), and notice that L_{10} and L_{01} embody almost exactly the parameters of the linear approximation (A.1.2). We also make note of the fact that the condition number of the matrix X_s for this application is approximately 19, well within the bounds dictated by limits of double precision calculations. In fact, condition numbers of this magnitude are about as small as can be expected, indicating to a degree the independence of the data achieved with the designed input.

For purposes of comparison a linear model as given in the exact expression (A.1.2) is integrated (forming the solution x1(k) in (A.2.9)) alongside the second degree model of Figure A.1 (forming the solution x2(k) in (A.2.9)). We note that if the identification were to be carried out for the linear approximation as with the second degree approximation, the resulting parameters would be practically identical to those of the exact expression.

In the pages to follow we offer numerous simulation results in the form of tables and plots, as discussed in the preceding section. The scheme for displaying these results, in this section and the next, is explained in the following.

$$L_{10} = \begin{bmatrix} -2.0009 & -2.9978 \\ 0.9998 & 0.9998 \end{bmatrix}$$

$$L_{01} = \begin{bmatrix} 0.0007 & 0.9993 \\ -1.0003 & -0.0000 \end{bmatrix}$$

$$L_{20} = \begin{bmatrix} 0.0019 & -0.0810 & 0.0340 \\ 0.0053 & 0.0009 & -0.0004 \end{bmatrix}$$

$$L_{11} = \begin{bmatrix} -4.0686 & -0.0063 & -0.0305 & -0.0249 \\ -0.0190 & -0.0115 & -0.0122 & -0.0117 \end{bmatrix}$$

$$L_{02} = \begin{bmatrix} -0.0339 & -0.0006 & 0.0049 \\ -0.9991 & 0.0066 & 0.0001 \end{bmatrix}$$

Figure A.1 Second Degree Model

A table is given and accompanied by plots corresponding to one or more simulations, identifiable by the associated number given leftmost in the table. Each table offers different combinations of input amplitudes and frequencies, and initial state conditions. And, as mentioned in the last section, several sets of randomly generated simulation parameters are given. An effort is made to display typical as well as unusual behavior exhibited in the error ratio result.

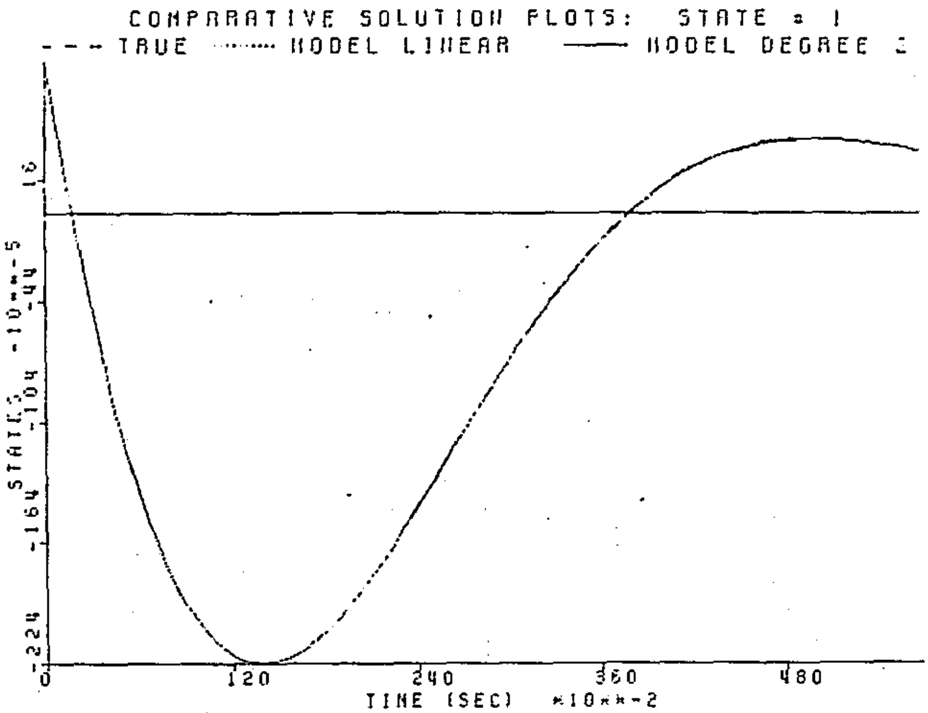
As an example, recalling from (A.2.9) that if the error ratio for a particular state in a given simulation is less than one, we conclude that the second model (in this case the second degree approximation) outperforms the first model (the linear approximation). There may be cases such as simulations 1, 10, 19, and 28 of Tables A.1, A.2, and A.3 for zero input amplitudes in which the error ratios are very large, indicating that the mean square error for the linear model is several orders of magnitude smaller than that of the second degree model. A view of the state trajectories given in Figure A.2 (for simulation 1 in Table A.1), however, illustrates the fact that all three solution curves--true, linear, and second degree--lie atop one another. To make use of another analysis tool we note that the error normalization factor (enf) for each state in this simulation is of the order of 10^{-8}. But the enf for simulation 12 from Table A.1, whose solution curves are shown in Figure A.3, is of the order of 10^{-1} in each state. This serves to reinforce the statement we made earlier concerning the enf: In general, the larger the enf is, the more relative significance the error ratio carries.

As a final note, we point out that the simulations for the first five tables represent data from integrations out to six seconds. For lower frequency inputs though, as in the final three Tables A.6, A.7, and A.8, the simulations are for integrations out to ten seconds.

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.105E+07	0.839E+06
2	0.001	0.001	0.050	0.050	0.75	1.00	0.639E-05	0.658E-04
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.967E-05	0.951E-04
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.536E-05	0.564E-04
5	0.001	0.001	-0.050	0.050	0.75	1.00	0.266E-05	0.247E-04
6	0.001	0.001	0.150	0.150	0.75	1.00	0.492E-03	0.236E-02
7	0.001	0.001	0.150	-0.150	0.75	1.00	0.390E-03	0.311E-02
8	0.001	0.001	-0.150	-0.150	0.75	1.00	0.833E-03	0.450E-02
9	0.001	0.001	-0.150	0.150	0.75	1.00	0.610E-03	0.389E-02
10	0.001	-0.001	0.000	0.000	0.75	1.00	0.499E+07	0.604E+07
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.499E-05	0.454E-04
12	0.001	-0.001	0.050	-0.050	0.75	1.00	0.993E-05	0.101E-03
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.492E-05	0.455E-04
14	0.001	-0.001	-0.050	0.050	0.75	1.00	0.216E-05	0.185E-04
15	0.001	-0.001	0.150	0.150	0.75	1.00	0.570E-03	0.266E-02
16	0.001	-0.001	0.150	-0.150	0.75	1.00	0.435E-03	0.333E-02
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	0.955E-03	0.504E-02
18	0.001	-0.001	-0.150	0.150	0.75	1.00	0.647E-03	0.400E-02
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.833E+06	0.666E+06
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.518E-05	0.430E-04
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	0.866E-05	0.894E-04
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.546E-05	0.482E-04
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	0.176E-05	0.153E-04
24	-0.001	-0.001	0.150	0.150	0.75	1.00	0.617E-03	0.284E-02
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	0.443E-03	0.341E-02
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	0.103E-02	0.534E-02
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	0.689E-03	0.415E-02
28	-0.001	0.001	0.000	0.000	0.75	1.00	0.569E+07	0.688E+07
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.630E-05	0.604E-04
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.840E-05	0.836E-04
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.542E-05	0.525E-04
32	-0.001	0.001	-0.050	0.050	0.75	1.00	0.211E-05	0.210E-04
33	-0.001	0.001	0.150	0.150	0.75	1.00	0.533E-03	0.252E-02
34	-0.001	0.001	0.150	-0.150	0.75	1.00	0.396E-03	0.319E-02
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	0.897E-03	0.478E-02
36	-0.001	0.001	-0.150	0.150	0.75	1.00	0.650E-03	0.403E-02

Table A.1 Linear vs. Degree 2: Constant input frequencies



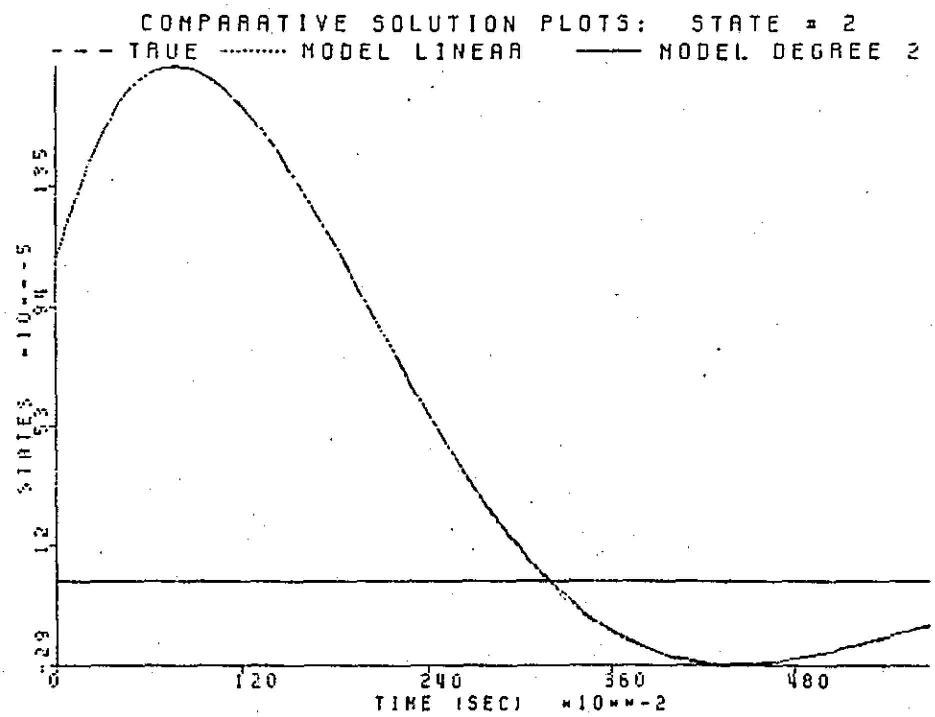
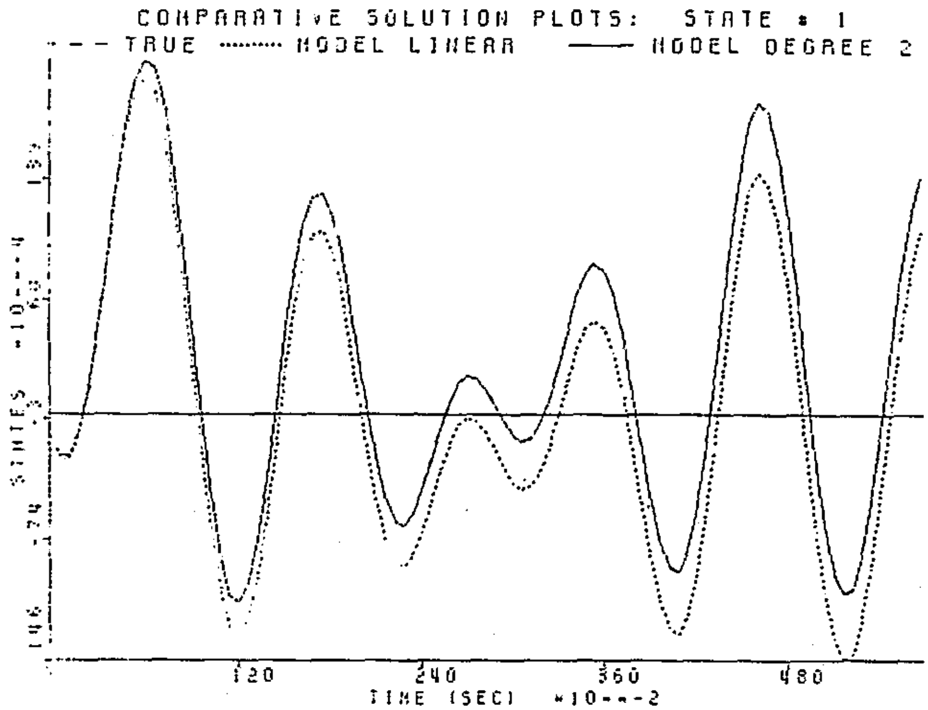


Figure A.2 S#1, Table A.1

ORIGINAL PAGE IS
OF POOR QUALITY



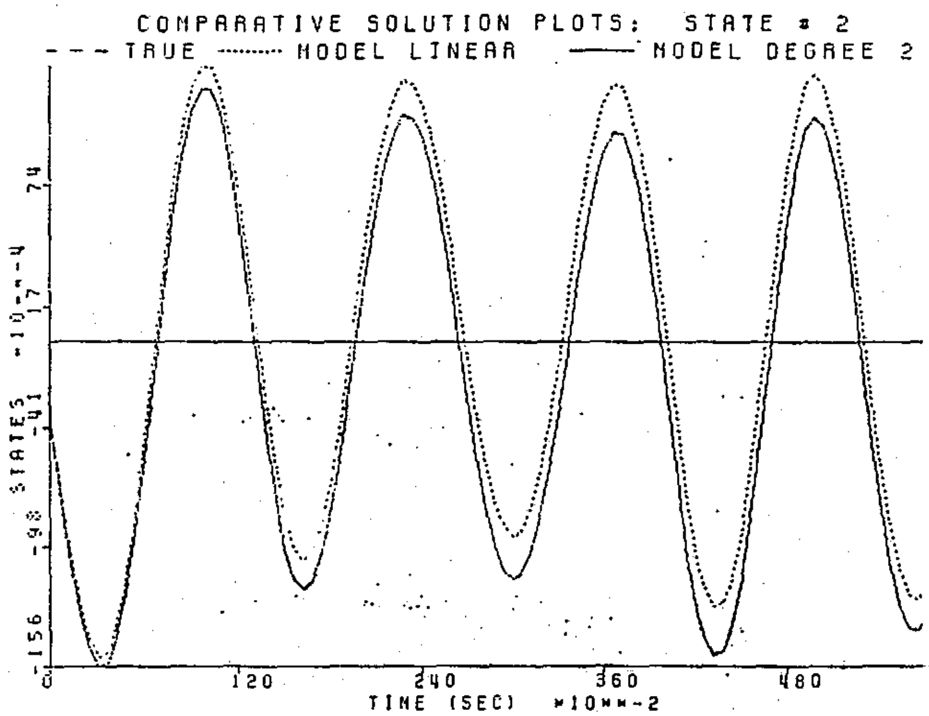
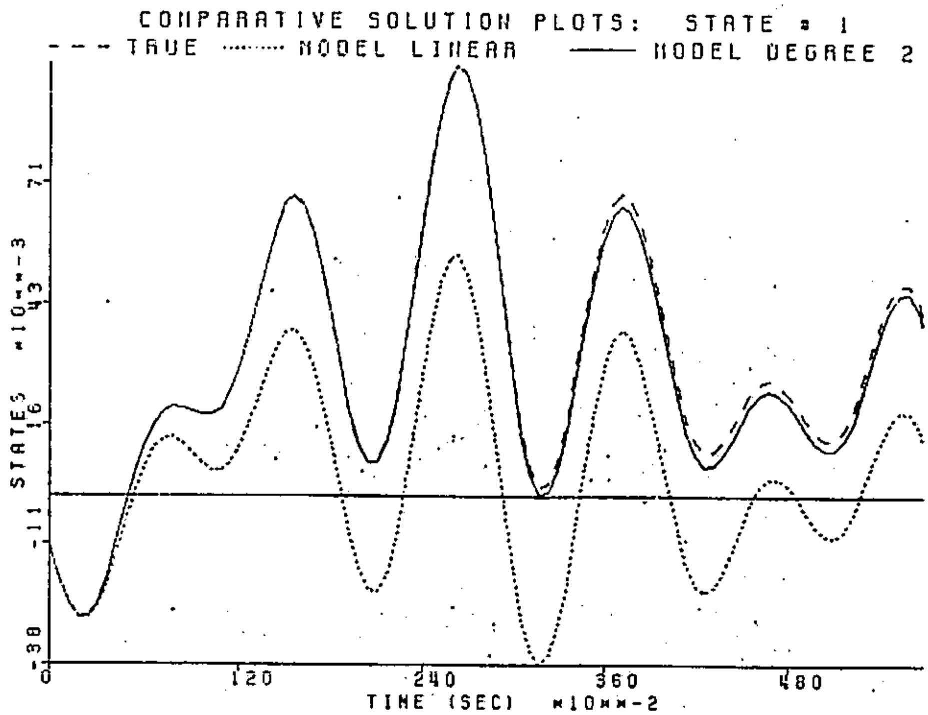


Figure A.3 S#12, Table A.1

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
‡ OF STATES: 2
‡ OF INPUTS: 2
‡ OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S‡	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.005	0.005	0.000	0.000	0.75	1.00	0.172E+04	0.187E+04
2	0.005	0.005	0.050	0.050	0.75	1.00	0.117E-04	0.139E-03
3	0.005	0.005	0.050	-0.050	0.75	1.00	0.133E-04	0.121E-03
4	0.005	0.005	-0.050	-0.050	0.75	1.00	0.102E-04	0.138E-03
5	0.005	0.005	-0.050	0.050	0.75	1.00	0.636E-05	0.515E-04
6	0.005	0.005	0.150	0.150	0.75	1.00	0.320E-03	0.165E-02
7	0.005	0.005	0.150	-0.150	0.75	1.00	0.313E-03	0.263E-02
8	0.005	0.005	-0.150	-0.150	0.75	1.00	0.546E-03	0.317E-02
9	0.005	0.005	-0.150	0.150	0.75	1.00	0.493E-03	0.349E-02
10	0.005	-0.005	0.000	0.000	0.75	1.00	0.446E+04	0.486E+04
11	0.005	-0.005	0.050	0.050	0.75	1.00	0.353E-05	0.257E-04
12	0.005	-0.005	0.050	-0.050	0.75	1.00	0.149E-04	0.166E-03
13	0.005	-0.005	-0.050	-0.050	0.75	1.00	0.519E-05	0.457E-04
14	0.005	-0.005	-0.050	0.050	0.75	1.00	0.301E-05	0.229E-04
15	0.005	-0.005	0.150	0.150	0.75	1.00	0.662E-03	0.299E-02
16	0.005	-0.005	0.150	-0.150	0.75	1.00	0.529E-03	0.369E-02
17	0.005	-0.005	-0.150	-0.150	0.75	1.00	0.109E-02	0.559E-02
18	0.005	-0.005	-0.150	0.150	0.75	1.00	0.653E-03	0.399E-02
19	-0.005	-0.005	0.000	0.000	0.75	1.00	503.	548.
20	-0.005	-0.005	0.050	0.050	0.75	1.00	0.948E-05	0.664E-04
21	-0.005	-0.005	0.050	-0.050	0.75	1.00	0.937E-05	0.116E-03
22	-0.005	-0.005	-0.050	-0.050	0.75	1.00	0.157E-04	0.161E-03
23	-0.005	-0.005	-0.050	0.050	0.75	1.00	0.346E-05	0.242E-04
24	-0.005	-0.005	0.150	0.150	0.75	1.00	0.950E-03	0.408E-02
25	-0.005	-0.005	0.150	-0.150	0.75	1.00	0.580E-03	0.417E-02
26	-0.005	-0.005	-0.150	-0.150	0.75	1.00	0.152E-02	0.739E-02
27	-0.005	-0.005	-0.150	0.150	0.75	1.00	0.890E-03	0.482E-02
28	-0.005	0.005	0.000	0.000	0.75	1.00	0.869E+04	0.944E+04
29	-0.005	0.005	0.050	0.050	0.75	1.00	0.963E-05	0.958E-04
30	-0.005	0.005	0.050	-0.050	0.75	1.00	0.661E-05	0.673E-04
31	-0.005	0.005	-0.050	-0.050	0.75	1.00	0.739E-05	0.772E-04
32	-0.005	0.005	-0.050	0.050	0.75	1.00	0.243E-05	0.314E-04
33	-0.005	0.005	0.150	0.150	0.75	1.00	0.478E-03	0.229E-02
34	-0.005	0.005	0.150	-0.150	0.75	1.00	0.334E-03	0.298E-02
35	-0.005	0.005	-0.150	-0.150	0.75	1.00	0.797E-03	0.434E-02
36	-0.005	0.005	-0.150	0.150	0.75	1.00	0.667E-03	0.414E-02

Table A.2 Linear vs. Degree 2: Constant input frequencies



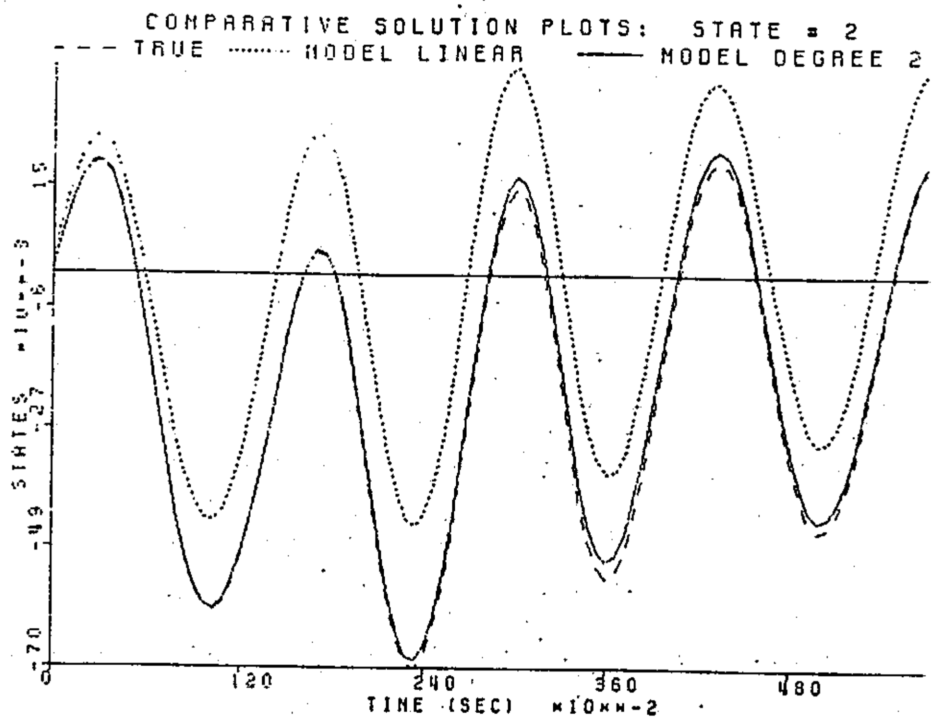
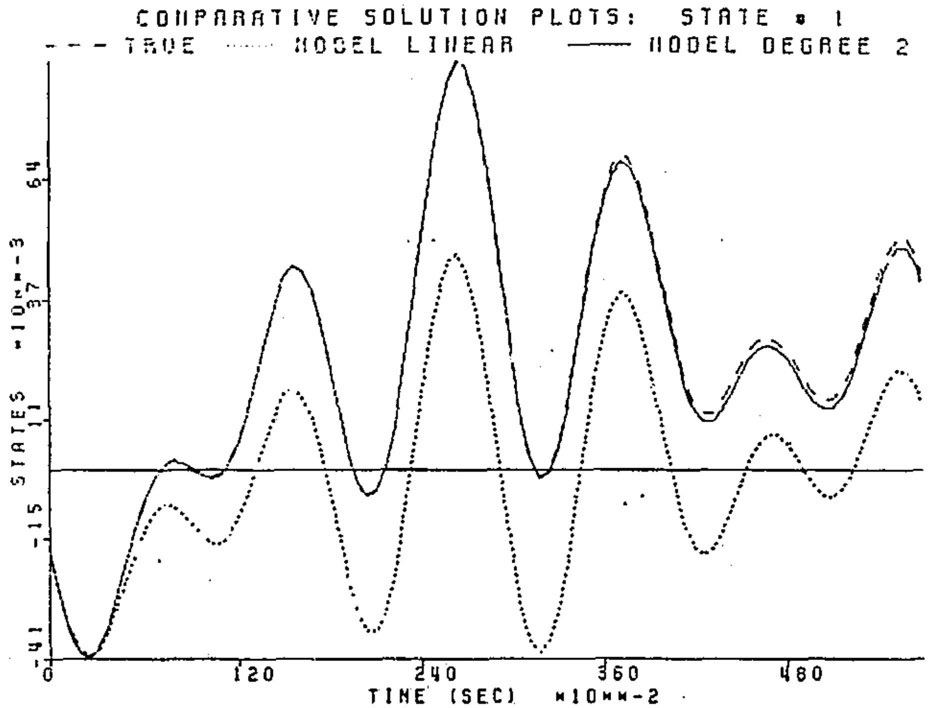


Figure A.4 S#26, Table A.2

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.010	0.010	0.000	0.000	0.75	1.00	142.	150.
2	0.010	0.010	0.050	0.050	0.75	1.00	0.200E-04	0.243E-03
3	0.010	0.010	0.050	-0.050	0.75	1.00	0.212E-04	0.171E-03
4	0.010	0.010	-0.050	-0.050	0.75	1.00	0.205E-04	0.298E-03
5	0.010	0.010	-0.050	0.050	0.75	1.00	0.150E-04	0.974E-04
6	0.010	0.010	0.150	0.150	0.75	1.00	0.244E-03	0.121E-02
7	0.010	0.010	0.150	-0.150	0.75	1.00	0.266E-03	0.225E-02
8	0.010	0.010	-0.150	-0.150	0.75	1.00	0.361E-03	0.215E-02
9	0.010	0.010	-0.150	0.150	0.75	1.00	0.413E-03	0.323E-02
10	0.010	-0.010	0.000	0.000	0.75	1.00	178.	192.
11	0.010	-0.010	0.050	0.050	0.75	1.00	0.541E-05	0.358E-04
12	0.010	-0.010	0.050	-0.050	0.75	1.00	0.266E-04	0.327E-03
13	0.010	-0.010	-0.050	-0.050	0.75	1.00	0.909E-05	0.882E-04
14	0.010	-0.010	-0.050	0.050	0.75	1.00	0.631E-05	0.570E-04
15	0.010	-0.010	0.150	0.150	0.75	1.00	0.813E-03	0.352E-02
16	0.010	-0.010	0.150	-0.150	0.75	1.00	0.678E-03	0.429E-02
17	0.010	-0.010	-0.150	-0.150	0.75	1.00	0.129E-02	0.641E-02
18	0.010	-0.010	-0.150	0.150	0.75	1.00	0.684E-03	0.408E-02
19	-0.010	-0.010	0.000	0.000	0.75	1.00	6.69	6.83
20	-0.010	-0.010	0.050	0.050	0.75	1.00	0.410E-04	0.375E-03
21	-0.010	-0.010	0.050	-0.050	0.75	1.00	0.220E-04	0.336E-03
22	-0.010	-0.010	-0.050	-0.050	0.75	1.00	0.654E-04	0.763E-03
23	-0.010	-0.010	-0.050	0.050	0.75	1.00	0.211E-04	0.188E-03
24	-0.010	-0.010	0.150	0.150	0.75	1.00	0.154E-02	0.616E-02
25	-0.010	-0.010	0.150	-0.150	0.75	1.00	0.821E-03	0.544E-02
26	-0.010	-0.010	-0.150	-0.150	0.75	1.00	0.235E-02	0.107E-01
27	-0.010	-0.010	-0.150	0.150	0.75	1.00	0.124E-02	0.598E-02
28	-0.010	0.010	0.000	0.000	0.75	1.00	732.	784.
29	-0.010	0.010	0.050	0.050	0.75	1.00	0.146E-04	0.145E-03
30	-0.010	0.010	0.050	-0.050	0.75	1.00	0.598E-05	0.702E-04
31	-0.010	0.010	-0.050	-0.050	0.75	1.00	0.113E-04	0.123E-03
32	-0.010	0.010	-0.050	0.050	0.75	1.00	0.331E-05	0.502E-04
33	-0.010	0.010	0.150	0.150	0.75	1.00	0.441E-03	0.211E-02
34	-0.010	0.010	0.150	-0.150	0.75	1.00	0.283E-03	0.285E-02
35	-0.010	0.010	-0.150	-0.150	0.75	1.00	0.704E-03	0.389E-02
36	-0.010	0.010	-0.150	0.150	0.75	1.00	0.710E-03	0.437E-02

Table A.3 Linear vs. Degree 2: Constant input frequencies



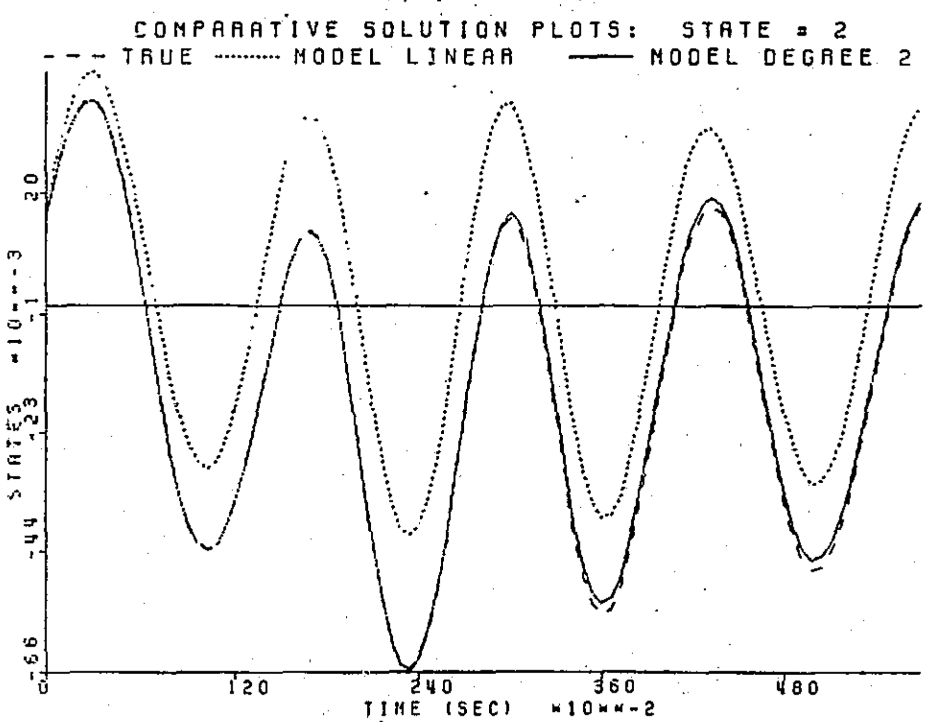
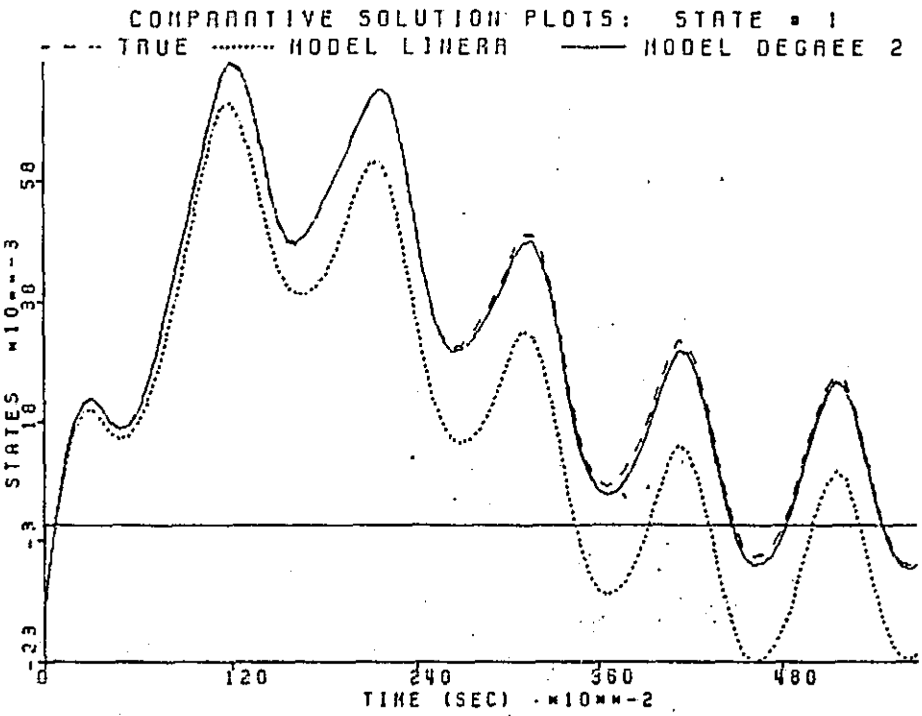


Figure A.5 S#35, Table A.3

```
*****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
*****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	0.466E-03	0.190E-02
2	0.025	0.025	0.100	-0.100	2.00	1.00	0.421E-03	0.206E-02
3	0.025	0.025	-0.100	0.100	2.00	1.00	0.384E-03	0.159E-02
4	0.025	0.025	-0.100	-0.100	2.00	1.00	0.344E-03	0.185E-02
5	0.025	-0.025	0.100	0.100	2.00	1.00	0.207E-03	0.926E-03
6	0.025	-0.025	0.100	-0.100	2.00	1.00	0.238E-03	0.121E-02
7	0.025	-0.025	-0.100	0.100	2.00	1.00	0.276E-03	0.117E-02
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	0.309E-03	0.157E-02
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.210E-03	0.873E-03
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.164E-03	0.852E-03
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.172E-03	0.806E-03
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	0.135E-03	0.828E-03
13	-0.025	-0.025	0.100	0.100	2.00	1.00	0.121E-02	0.484E-02
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	0.128E-02	0.600E-02
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	0.146E-02	0.561E-02
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	0.144E-02	0.687E-02
17	0.025	0.025	0.200	0.200	2.00	1.00	0.123E-02	0.551E-02
18	0.025	0.025	0.200	-0.200	2.00	1.00	0.864E-03	0.510E-02
19	0.025	0.025	-0.200	0.200	2.00	1.00	0.134E-02	0.649E-02
20	0.025	0.025	-0.200	-0.200	2.00	1.00	0.977E-03	0.647E-02
21	0.025	-0.025	0.200	0.200	2.00	1.00	0.318E-02	0.106E-01
22	0.025	-0.025	0.200	-0.200	2.00	1.00	0.318E-02	0.115E-01
23	0.025	-0.025	-0.200	0.200	2.00	1.00	0.429E-02	0.132E-01
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	0.401E-02	0.141E-01
25	-0.025	0.025	0.200	0.200	2.00	1.00	0.208E-02	0.815E-02
26	-0.025	0.025	0.200	-0.200	2.00	1.00	0.171E-02	0.809E-02
27	-0.025	0.025	-0.200	0.200	2.00	1.00	0.259E-02	0.994E-02
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	0.208E-02	0.100E-01
29	-0.025	-0.025	0.200	0.200	2.00	1.00	0.703E-02	0.188E-01
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	0.704E-02	0.205E-01
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	0.883E-02	0.224E-01
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	0.823E-02	0.239E-01
33	0.025	0.025	0.300	0.300	2.00	1.00	0.979E-02	0.379E-01
34	0.025	0.025	0.300	-0.300	2.00	1.00	0.865E-02	0.379E-01
35	0.025	0.025	-0.300	0.300	2.00	1.00	0.122E-01	0.445E-01
36	0.025	0.025	-0.300	-0.300	2.00	1.00	0.103E-01	0.447E-01
37	0.025	-0.025	0.300	0.300	2.00	1.00	0.191E-01	0.541E-01
38	0.025	-0.025	0.300	-0.300	2.00	1.00	0.186E-01	0.561E-01
39	0.025	-0.025	-0.300	0.300	2.00	1.00	0.238E-01	0.633E-01
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	0.217E-01	0.644E-01
41	-0.025	0.025	0.300	0.300	2.00	1.00	0.147E-01	0.473E-01
42	-0.025	0.025	0.300	-0.300	2.00	1.00	0.135E-01	0.481E-01
43	-0.025	0.025	-0.300	0.300	2.00	1.00	0.181E-01	0.553E-01
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	0.157E-01	0.556E-01
45	-0.025	-0.025	0.300	0.300	2.00	1.00	0.278E-01	0.683E-01
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	0.272E-01	0.713E-01
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	0.339E-01	0.791E-01
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	0.310E-01	0.806E-01

Table A.4 Linear vs. Degree 2: Constant input frequencies



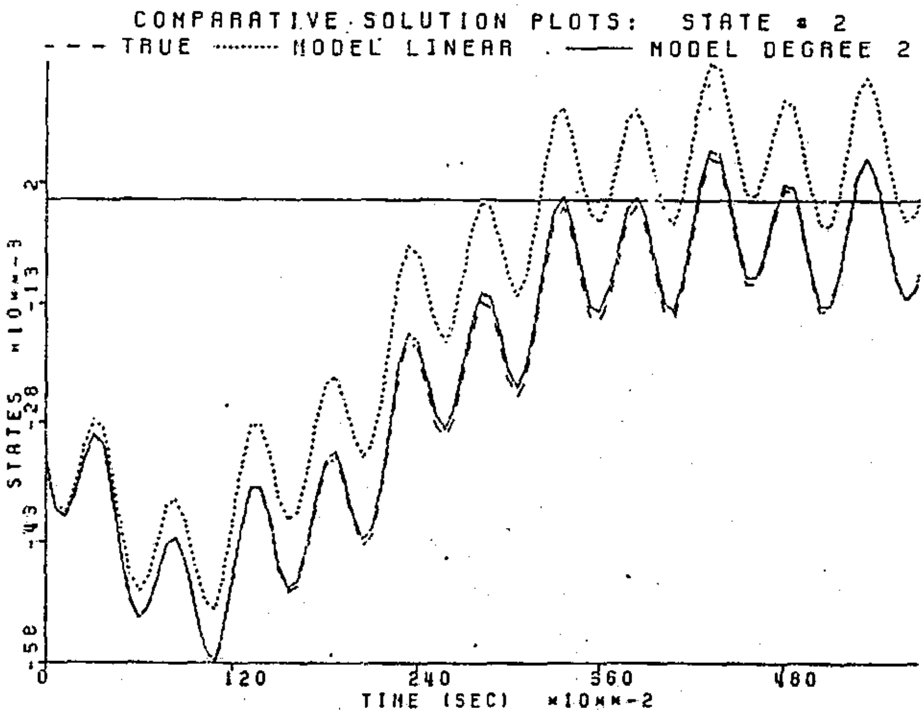
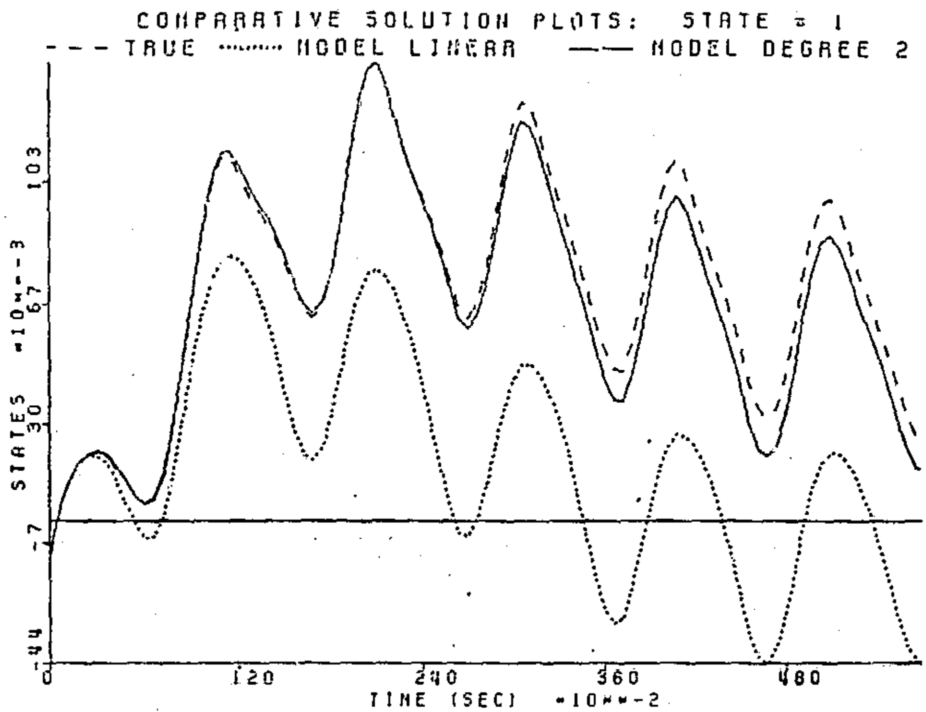


Figure A.6 S#13, Table A.4



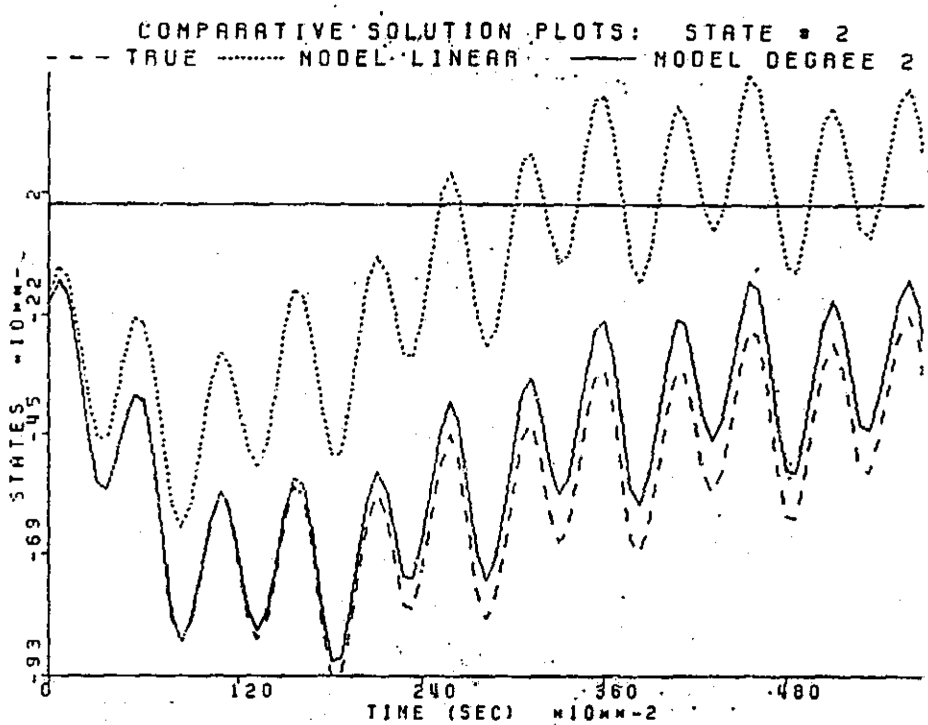


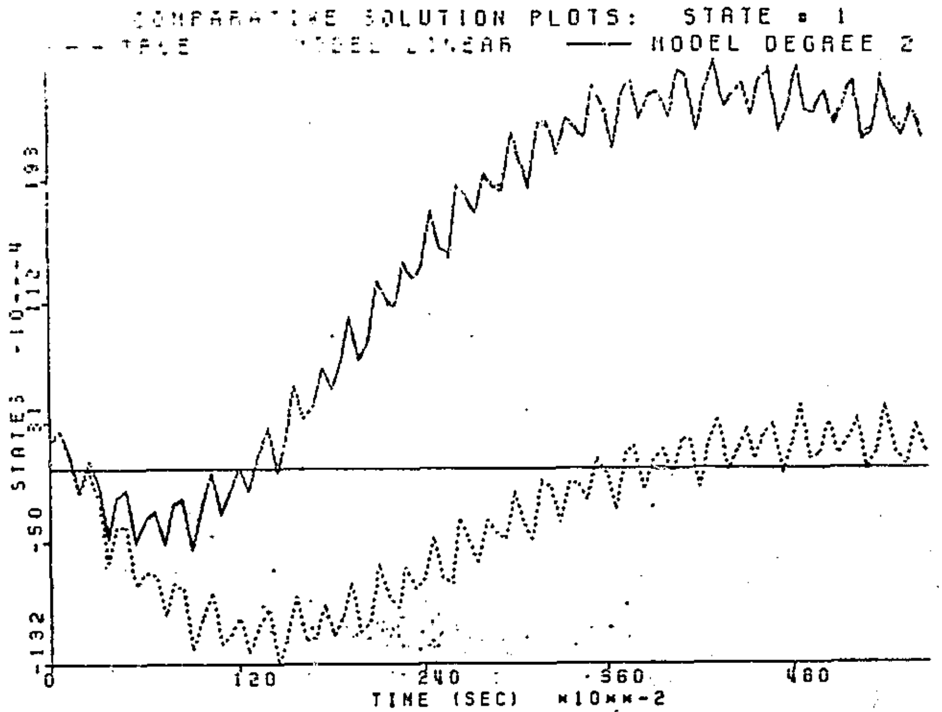
Figure A.7 S#31, Table A.4

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.005	0.005	-0.200	0.168	2.02	2.93	0.276E-02	0.832E-02
2	0.005	0.005	-0.199	0.056	5.09	3.97	0.282E-02	0.790E-02
3	0.005	0.005	0.190	0.141	5.87	5.39	0.218E-02	0.640E-02
4	0.005	0.005	-0.153	0.020	5.39	3.18	0.660E-03	0.196E-02
5	0.005	0.005	0.027	-0.176	4.50	2.94	0.265E-04	0.453E-02
6	0.005	0.005	-0.097	-0.051	2.91	5.18	0.918E-04	0.167E-03
7	0.005	0.005	-0.043	0.022	2.78	3.38	0.825E-04	0.227E-03
8	0.005	0.005	-0.038	0.135	4.95	4.53	0.206E-04	0.396E-03
9	0.005	0.005	-0.119	-0.061	3.66	5.46	0.158E-03	0.449E-03
10	0.005	0.005	-0.099	-0.197	2.65	5.74	0.661E-04	0.257E-03
11	0.005	0.005	0.115	0.162	2.50	3.28	0.103E-03	0.427E-03
12	0.005	0.005	0.002	0.148	5.20	5.97	0.162E-03	0.176
13	0.005	0.005	-0.169	0.085	5.12	4.08	0.116E-02	0.343E-02
14	0.005	0.005	-0.145	-0.029	2.31	3.14	0.482E-03	0.145E-02
15	0.005	0.005	0.088	0.021	5.97	3.23	0.104E-03	0.154E-03
16	0.005	0.005	0.186	-0.080	4.66	5.18	0.189E-02	0.548E-02
17	0.005	0.005	0.124	-0.146	5.96	2.67	0.151E-03	0.634E-03
18	0.005	0.005	-0.151	0.108	5.58	2.46	0.569E-03	0.185E-02
19	0.005	0.005	-0.013	-0.185	2.28	3.54	0.629E-04	0.255E-01
20	0.005	0.005	0.039	-0.024	2.69	3.55	0.568E-04	0.200E-03
21	0.005	0.005	-0.153	-0.114	2.97	4.81	0.674E-03	0.208E-02
22	0.005	0.005	0.008	0.160	4.67	2.23	0.226E-04	0.348E-02
23	0.005	0.005	-0.073	-0.092	3.83	3.55	0.109E-03	0.243E-03
24	0.005	0.005	-0.031	0.125	3.22	2.59	0.832E-05	0.566E-03
25	0.005	0.005	-0.091	0.055	5.22	4.18	0.895E-04	0.148E-03
26	0.005	0.005	-0.003	-0.001	2.28	2.13	0.694E-01	3.31
27	0.005	0.005	0.080	-0.052	2.74	3.92	0.974E-04	0.156E-03
28	0.005	0.005	-0.133	-0.003	5.04	5.51	0.287E-03	0.852E-03
29	0.005	0.005	0.157	-0.042	4.90	2.06	0.746E-03	0.226E-02
30	0.005	0.005	0.166	0.090	5.27	2.60	0.105E-02	0.318E-02
31	0.005	0.005	0.084	0.081	3.38	3.79	0.904E-04	0.171E-03
32	0.005	0.005	-0.107	-0.057	3.66	2.41	0.947E-04	0.238E-03
33	0.005	0.005	-0.040	-0.108	5.29	3.64	0.154E-05	0.136E-04
34	0.005	0.005	0.191	-0.167	5.21	2.55	0.233E-02	0.690E-02
35	0.005	0.005	0.060	-0.073	2.79	2.79	0.151E-03	0.527E-03
36	0.005	0.005	0.013	-0.085	4.16	4.87	0.753E-04	0.239E-01
37	0.005	0.005	-0.013	-0.151	2.55	3.32	0.438E-04	0.146E-01
38	0.005	0.005	0.135	0.160	4.67	5.63	0.328E-03	0.107E-02
39	0.005	0.005	-0.062	-0.098	2.11	2.66	0.208E-04	0.122E-03
40	0.005	0.005	-0.093	0.064	3.10	5.58	0.900E-04	0.154E-03
41	0.005	0.005	-0.099	0.057	5.19	4.11	0.902E-04	0.177E-03
42	0.005	0.005	0.086	0.192	2.36	4.41	0.242E-04	0.136E-03
43	0.005	0.005	-0.172	-0.068	4.42	4.85	0.127E-02	0.367E-02
44	0.005	0.005	-0.148	0.057	2.01	5.06	0.562E-03	0.169E-02
45	0.005	0.005	0.190	0.194	4.35	2.37	0.206E-02	0.654E-02
46	0.005	0.005	-0.002	0.114	4.78	3.67	0.536E-04	0.202E-01
47	0.005	0.005	0.118	-0.161	3.89	2.46	0.102E-03	0.510E-03
48	0.005	0.005	0.040	-0.172	3.71	4.55	0.289E-04	0.991E-03
49	0.005	0.005	0.017	0.147	2.02	2.92	0.293E-04	0.585E-02
50	0.005	0.005	-0.152	0.077	2.88	4.12	0.646E-03	0.195E-02

Table A.5 Linear vs. Degree 2: Constant initial conditions with randomly generated amplitudes and frequencies

ORIGINAL PAGE IS
OF POOR QUALITY



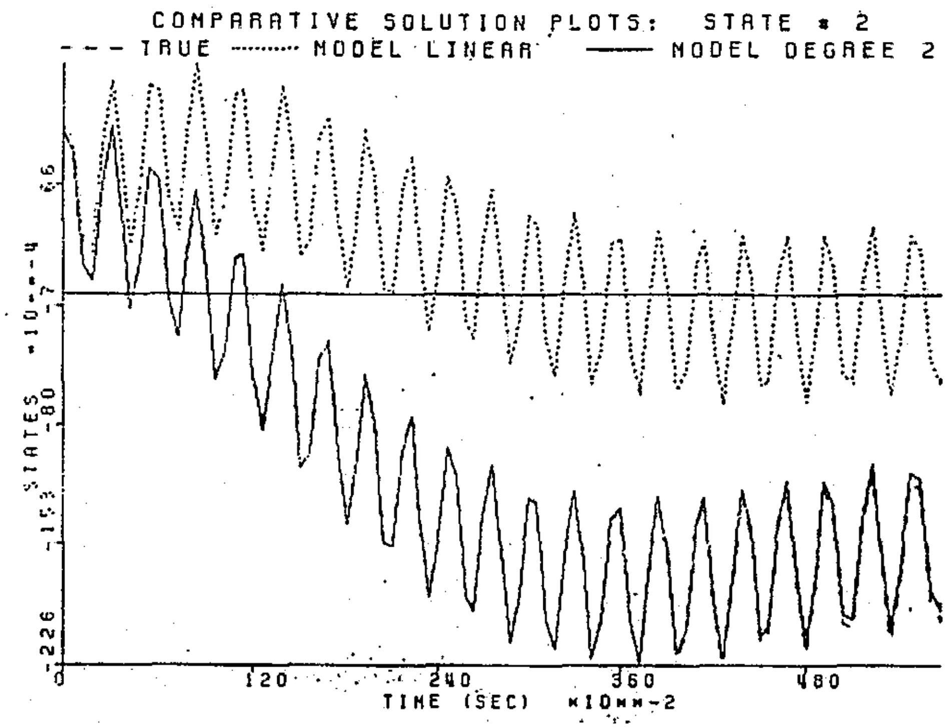
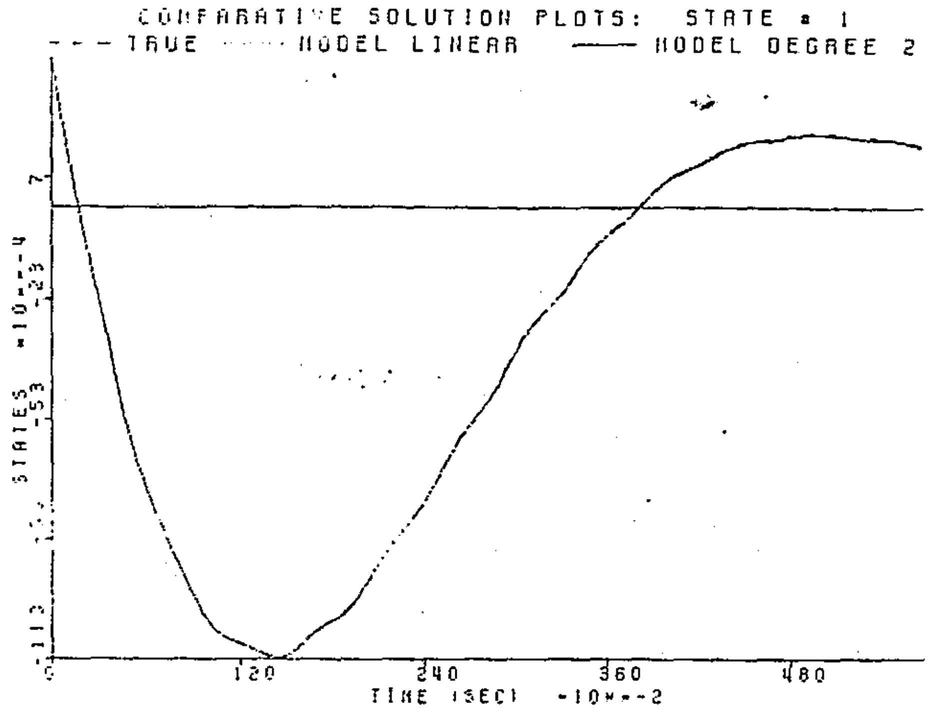


Figure A.8 S#9, Table A.5



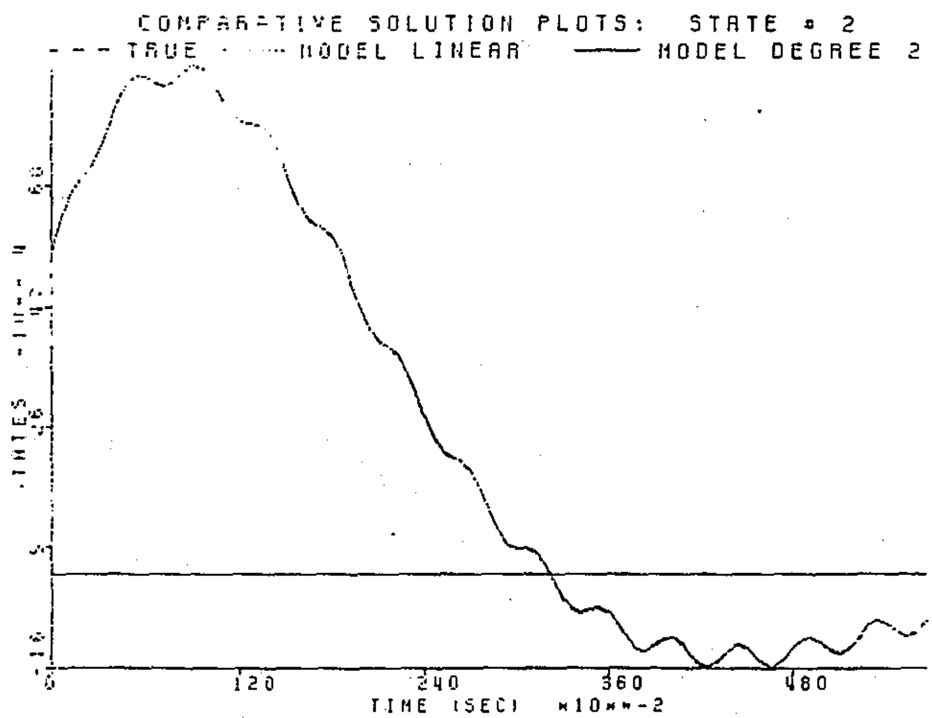


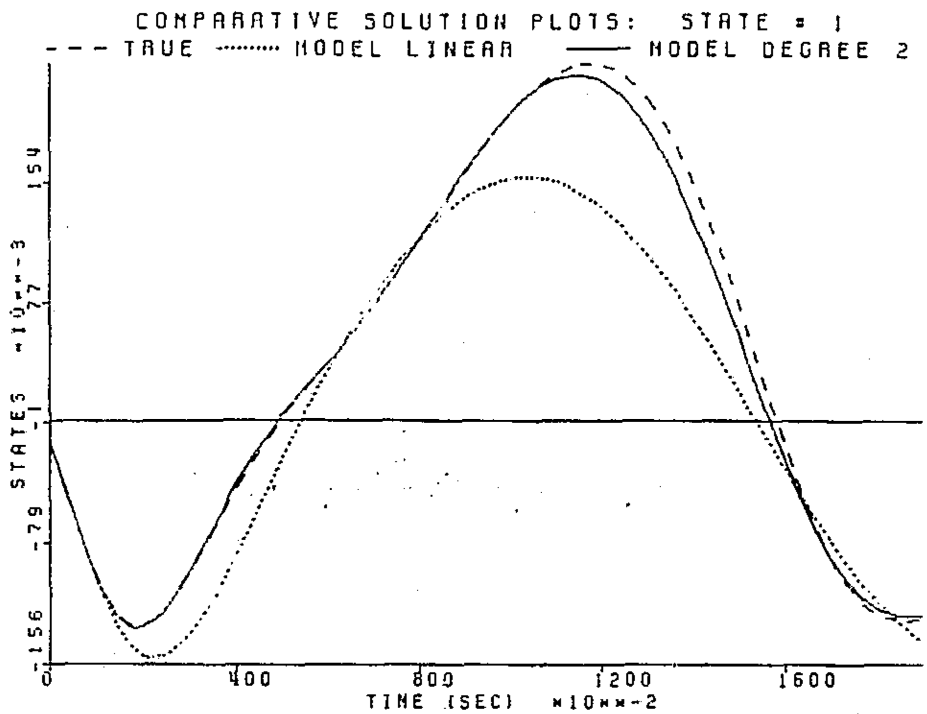
Figure A.9 S#26, Table A.5

```
**************************************
            PROBLEM SUMMARY
  CONFIGURATION:  TRUE,LINEAR,MODEL1
             # OF STATES: 2
             # OF INPUTS: 2
       # OF TERMS IN MODEL 1:  14
      DEGREE OF APPROXIMATION: 2
         SIMULATION WITH COSINE
**************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.265E-03	0.242E-03
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.251E-03	0.226E-03
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.210E-03	0.193E-03
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.104E-03	0.778E-04
5	0.001	-0.001	-0.050	0.050	0.00	0.00	0.408E-02	0.341E-02
6	0.001	-0.001	-0.050	0.050	0.01	0.01	0.372E-02	0.305E-02
7	0.001	-0.001	-0.050	0.050	0.02	0.02	0.272E-02	0.240E-02
8	0.001	-0.001	-0.050	0.050	0.05	0.05	0.630E-03	0.109E-02
9	0.001	-0.001	-0.075	0.010	0.00	0.00	0.298E-01	0.372E-01
10	0.001	-0.001	-0.075	0.010	0.01	0.01	0.320E-01	0.387E-01
11	0.001	-0.001	-0.075	0.010	0.02	0.02	0.420E-01	0.490E-01
12	0.001	-0.001	-0.075	0.010	0.05	0.05	0.227E-01	0.201E-01
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.149E-03	0.169E-03
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.153E-03	0.159E-03
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.180E-03	0.165E-03
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.142E-03	0.234E-03
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.256E-03	0.236E-03
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.243E-03	0.220E-03
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.202E-03	0.187E-03
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.102E-03	0.773E-04
21	0.010	-0.010	-0.050	0.050	0.00	0.00	0.425E-02	0.350E-02
22	0.010	-0.010	-0.050	0.050	0.01	0.01	0.389E-02	0.315E-02
23	0.010	-0.010	-0.050	0.050	0.02	0.02	0.285E-02	0.248E-02
24	0.010	-0.010	-0.050	0.050	0.05	0.05	0.653E-03	0.109E-02
25	0.010	-0.010	-0.075	0.010	0.00	0.00	0.314E-01	0.383E-01
26	0.010	-0.010	-0.075	0.010	0.01	0.01	0.338E-01	0.401E-01
27	0.010	-0.010	-0.075	0.010	0.02	0.02	0.445E-01	0.515E-01
28	0.010	-0.010	-0.075	0.010	0.05	0.05	0.238E-01	0.204E-01
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.147E-03	0.167E-03
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.152E-03	0.157E-03
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.175E-03	0.159E-03
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.127E-03	0.225E-03

Table A.6 Linear vs. Degree 2: Low frequency inputs

ORIGINAL PAGE IS
OF POOR QUALITY



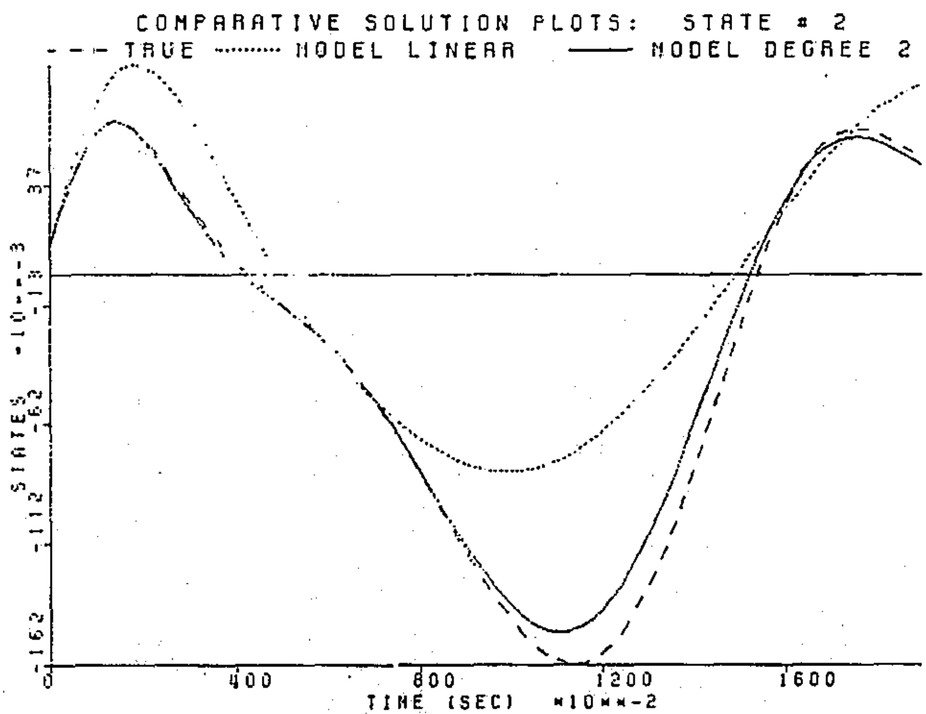
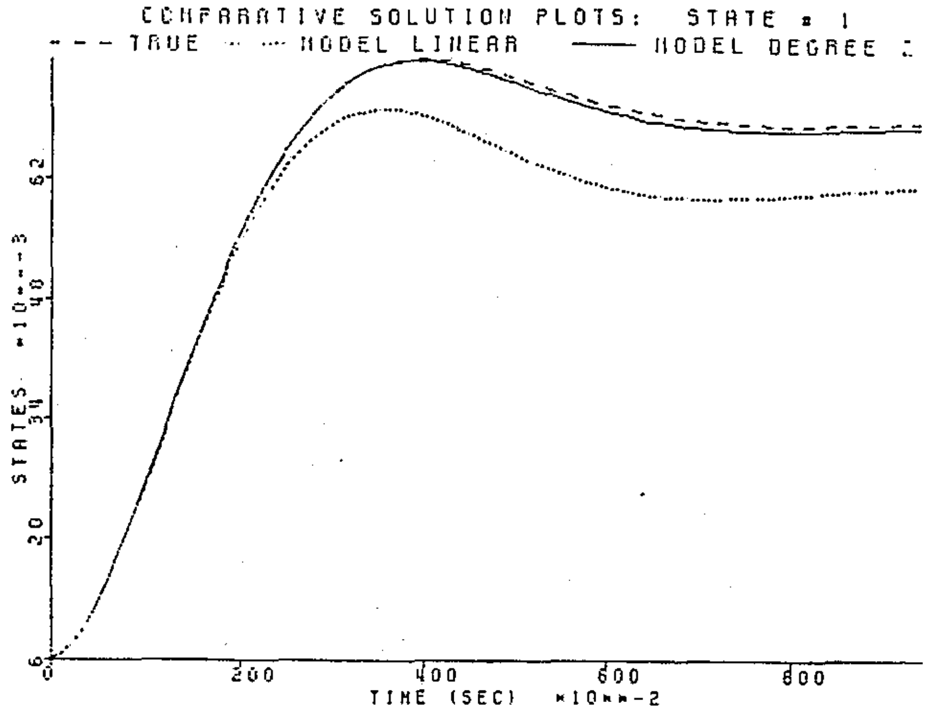


Figure A.10 S#16, Table A.6

```
****************************************
PROBLEM SUMMARY
CONFIGURATION: TRUE,LINEAR,MODEL1
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.008	0.007	-0.018	0.010	0.00	0.00	0.516E-04	0.226E-04
2	-0.001	0.000	-0.022	0.024	0.00	0.00	0.249E-04	0.148E-04
3	-0.004	-0.008	-0.019	0.012	0.00	0.00	0.359E-04	0.119E-04
4	-0.004	0.009	-0.018	0.015	0.00	0.00	0.229E-04	0.955E-05
5	-0.006	0.002	-0.006	0.014	0.00	0.00	0.182E-04	0.161E-04
6	0.002	-0.002	0.012	-0.006	0.00	0.00	0.896E-04	0.909E-04
7	-0.001	-0.010	-0.023	0.017	0.00	0.00	0.515E-04	0.314E-04
8	-0.005	-0.005	0.023	-0.016	0.00	0.00	0.161E-02	0.166E-02
9	-0.004	-0.000	-0.018	-0.008	0.00	0.00	0.173E-04	0.842E-05
10	0.005	-0.001	-0.014	-0.014	0.00	0.00	0.409E-05	0.234E-05
11	0.007	-0.000	-0.013	0.017	0.00	0.00	0.251E-04	0.180E-04
12	0.000	0.007	-0.006	0.022	0.00	0.00	0.145E-04	0.129E-04
13	0.000	-0.009	-0.010	-0.002	0.00	0.00	0.240E-03	0.187E-03
14	0.006	-0.003	0.021	-0.015	0.00	0.00	0.109E-02	0.113E-02
15	-0.004	0.007	-0.020	-0.008	0.00	0.00	0.135E-04	0.442E-05
16	0.002	-0.008	-0.012	0.023	0.00	0.00	0.160E-04	0.117E-04
17	0.002	-0.001	-0.003	0.001	0.00	0.00	0.174E-03	0.150E-03
18	-0.010	0.003	0.015	-0.024	0.00	0.00	0.222E-03	0.212E-03
19	0.003	-0.003	-0.022	-0.015	0.00	0.00	0.398E-05	0.113E-05
20	-0.008	0.000	-0.004	-0.006	0.00	0.00	0.158E-05	0.124E-05
21	-0.001	0.008	-0.003	-0.015	0.00	0.00	0.899E-06	0.512E-06
22	-0.002	-0.002	0.016	0.000	0.00	0.00	0.352E-02	0.421E-02
23	0.006	0.005	0.025	0.023	0.00	0.00	0.413E-04	0.571E-04
24	0.006	-0.001	-0.005	0.021	0.00	0.00	0.137E-04	0.120E-04
25	-0.006	-0.001	0.014	0.024	0.00	0.00	0.247E-05	0.138E-05
26	-0.005	0.003	0.001	-0.001	0.00	0.00	0.139E-04	0.105E-04
27	0.002	0.005	0.003	-0.004	0.00	0.00	0.393E-05	0.302E-05
28	0.009	0.005	0.005	0.001	0.00	0.00	0.725E-04	0.531E-04
29	0.006	-0.002	0.022	0.005	0.00	0.00	0.379E-01	0.509E-01
30	0.004	0.008	0.010	0.017	0.00	0.00	0.186E-05	0.127E-05
31	-0.009	-0.003	0.006	0.001	0.00	0.00	0.165E-03	0.122E-03
32	0.005	-0.001	-0.003	-0.013	0.00	0.00	0.148E-06	0.111E-06
33	-0.010	0.009	-0.018	-0.020	0.00	0.00	0.159E-05	0.559E-06
34	0.006	0.005	-0.007	-0.012	0.00	0.00	0.337E-05	0.213E-05
35	0.006	-0.006	0.005	0.008	0.00	0.00	0.251E-05	0.177E-05
36	0.009	0.008	-0.003	0.024	0.00	0.00	0.117E-04	0.104E-04
37	-0.009	0.008	0.008	0.012	0.00	0.00	0.243E-05	0.172E-05
38	0.004	-0.003	-0.024	0.019	0.00	0.00	0.569E-04	0.390E-04
39	-0.005	-0.008	-0.010	0.004	0.00	0.00	0.227E-03	0.177E-03
40	-0.002	0.003	0.022	0.012	0.00	0.00	0.436E-03	0.509E-03
41	0.002	-0.003	0.009	-0.016	0.00	0.00	0.299E-04	0.279E-04
42	0.002	0.010	0.024	-0.004	0.00	0.00	0.454E-02	0.498E-02
43	-0.009	-0.004	0.001	0.007	0.00	0.00	0.247E-05	0.156E-05
44	-0.002	-0.002	-0.010	-0.015	0.00	0.00	0.813E-06	0.516E-06
45	0.004	-0.003	-0.002	0.019	0.00	0.00	0.767E-05	0.674E-05
46	0.004	0.006	-0.023	0.013	0.00	0.00	0.854E-04	0.488E-04
47	-0.009	-0.007	0.003	-0.006	0.00	0.00	0.109E-05	0.103E-05
48	-0.001	0.004	0.023	-0.023	0.00	0.00	0.132E-02	0.134E-02
49	0.007	0.007	-0.015	-0.003	0.00	0.00	0.140E-03	0.883E-04
50	0.009	-0.003	-0.018	-0.002	0.00	0.00	0.171E-03	0.813E-04

Table A.7 Linear vs. Degree 2: Step inputs with randomly generated initial conditions and input amplitudes



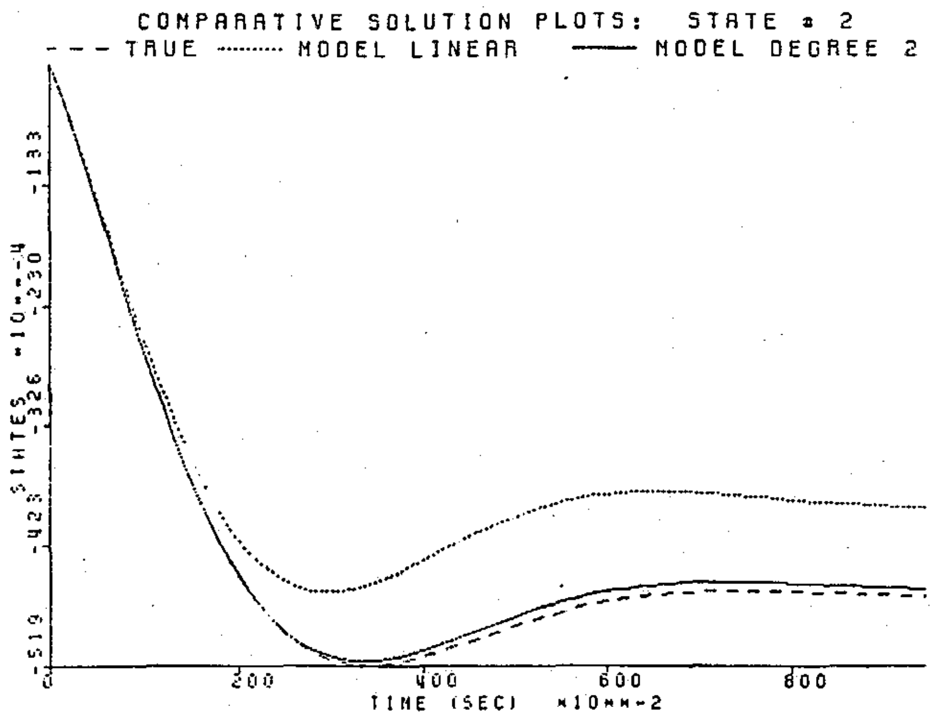
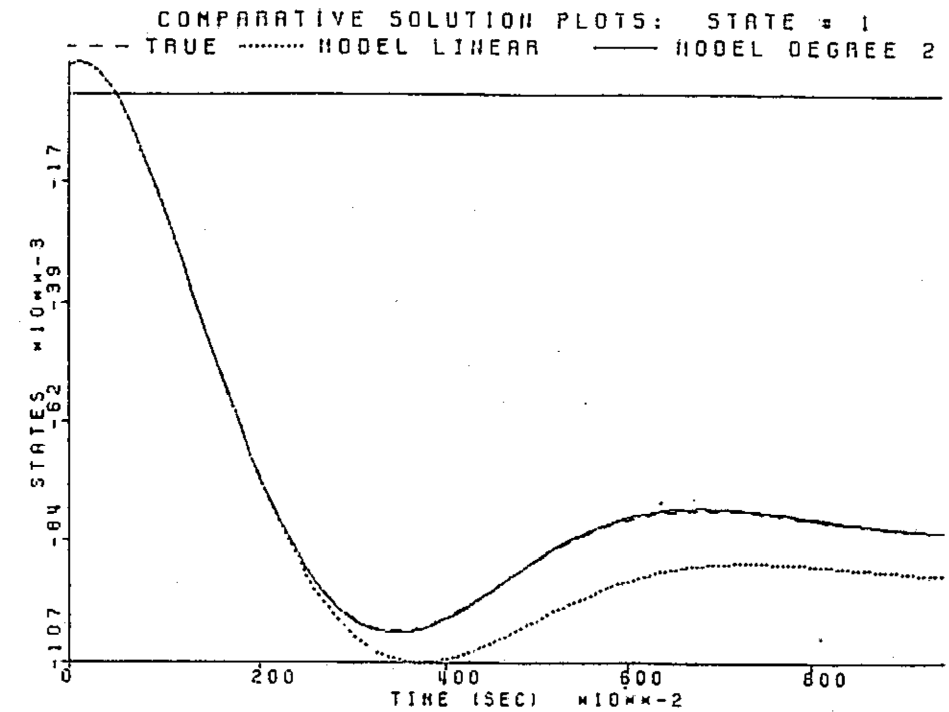


Figure A.11 S#29, Table A.7



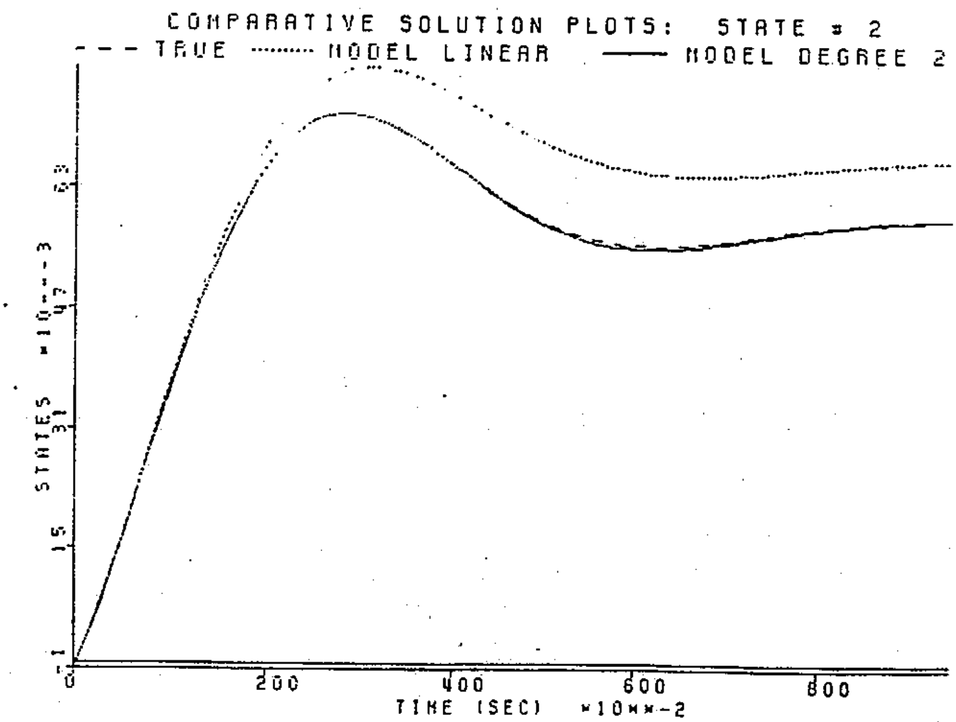


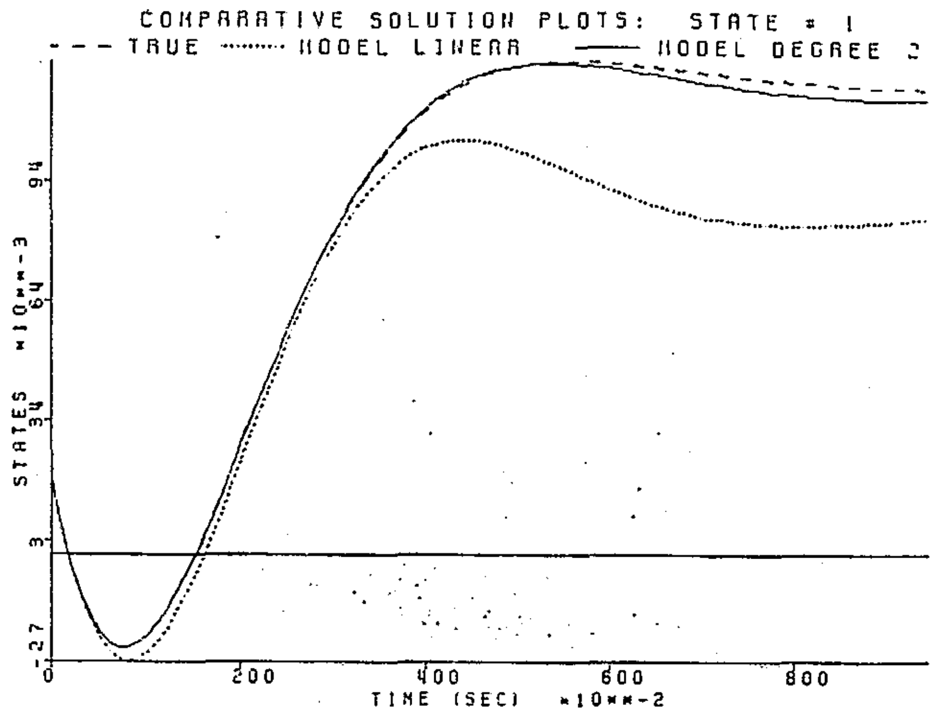
Figure A.12 S#38, Table A.7

ORIGINAL PAGE IS
OF POOR QUALITY

```
****************************************
            PROBLEM SUMMARY
   CONFIGURATION:   TRUE,LINEAR,MODEL1
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:   14
      DEGREE OF APPROXIMATION: 2
        SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	-0.043	-0.033	-0.070	-0.009	0.00	0.00	0.483E-02	0.501E-02
2	0.030	0.034	-0.028	-0.065	0.00	0.00	0.207E-04	0.746E-05
3	0.019	0.024	-0.017	-0.028	0.00	0.00	0.896E-05	0.302E-05
4	-0.046	-0.014	-0.022	-0.010	0.00	0.00	0.145E-04	0.470E-05
5	-0.010	0.025	-0.050	-0.054	0.00	0.00	0.542E-04	0.489E-04
6	-0.017	0.003	0.014	0.023	0.00	0.00	0.223E-05	0.116E-05
7	-0.020	-0.039	0.020	-0.052	0.00	0.00	0.126E-02	0.119E-02
8	-0.030	0.044	-0.048	-0.069	0.00	0.00	0.456E-04	0.287E-04
9	-0.047	-0.033	-0.019	0.073	0.00	0.00	0.420E-04	0.275E-04
10	0.002	0.015	-0.038	-0.069	0.00	0.00	0.731E-05	0.429E-05
11	0.043	-0.001	-0.018	-0.036	0.00	0.00	0.279E-05	0.107E-05
12	0.016	0.034	0.020	-0.074	0.00	0.00	0.127E-02	0.121E-02
13	0.031	0.046	0.048	0.058	0.00	0.00	0.406E-03	0.645E-03
14	-0.033	-0.001	-0.051	-0.071	0.00	0.00	0.640E-05	0.124E-04
15	0.049	0.016	-0.015	-0.031	0.00	0.00	0.104E-04	0.353E-05
16	-0.024	0.011	-0.069	0.052	0.00	0.00	0.289E-01	0.231E-01
17	-0.000	0.048	-0.019	-0.068	0.00	0.00	0.938E-05	0.288E-05
18	0.027	-0.030	0.034	-0.054	0.00	0.00	0.674E-02	0.714E-02
19	0.049	-0.015	-0.019	0.008	0.00	0.00	0.215E-03	0.725E-04
20	0.048	-0.045	0.064	-0.002	0.00	0.00	0.128	0.156
21	0.049	-0.009	-0.025	0.040	0.00	0.00	0.103E-03	0.727E-04
22	0.044	0.021	0.031	-0.036	0.00	0.00	0.432E-02	0.452E-02
23	0.049	-0.007	-0.021	0.009	0.00	0.00	0.238E-03	0.792E-04
24	0.008	-0.037	-0.026	0.067	0.00	0.00	0.146E-03	0.108E-03
25	0.016	-0.014	0.024	0.008	0.00	0.00	0.476E-02	0.524E-02
26	0.039	0.024	-0.054	0.056	0.00	0.00	0.760E-02	0.591E-02
27	-0.036	-0.025	-0.069	0.000	0.00	0.00	0.969E-02	0.108E-01
28	-0.039	-0.049	-0.069	-0.046	0.00	0.00	0.402E-03	0.394E-03
29	-0.040	0.003	-0.040	-0.065	0.00	0.00	0.156E-05	0.315E-05
30	0.033	0.029	0.009	0.041	0.00	0.00	0.733E-05	0.473E-05
31	-0.038	0.040	-0.041	0.031	0.00	0.00	0.183E-02	0.172E-02
32	-0.026	0.043	-0.064	-0.059	0.00	0.00	0.232E-03	0.203E-03
33	0.027	-0.047	-0.040	-0.037	0.00	0.00	0.282E-04	0.259E-04
34	-0.040	0.012	-0.040	-0.045	0.00	0.00	0.546E-05	0.108E-04
35	-0.021	-0.010	0.038	0.051	0.00	0.00	0.115E-03	0.138E-03
36	-0.037	-0.017	-0.023	-0.049	0.00	0.00	0.335E-06	0.271E-06
37	0.014	-0.046	-0.037	-0.058	0.00	0.00	0.253E-05	0.264E-05
38	-0.012	-0.032	-0.040	-0.016	0.00	0.00	0.196E-03	0.199E-03
39	0.019	0.004	-0.008	0.013	0.00	0.00	0.355E-04	0.286E-04
40	-0.030	0.017	-0.061	-0.064	0.00	0.00	0.549E-04	0.719E-04
41	0.029	-0.020	0.070	0.014	0.00	0.00	0.218	0.273
42	0.026	-0.023	0.055	-0.070	0.00	0.00	0.556E-01	0.665E-01
43	0.032	0.047	0.029	0.062	0.00	0.00	0.524E-05	0.259E-05
44	-0.040	0.024	-0.029	0.042	0.00	0.00	0.137E-03	0.126E-03
45	0.042	-0.014	0.048	0.000	0.00	0.00	0.560E-01	0.680E-01
46	-0.030	0.027	0.016	0.067	0.00	0.00	0.696E-06	0.618E-06
47	0.027	0.044	0.050	0.064	0.00	0.00	0.375E-03	0.614E-03
48	0.019	-0.042	-0.040	-0.064	0.00	0.00	0.278E-05	0.343E-05
49	-0.024	-0.034	0.019	-0.063	0.00	0.00	0.108E-02	0.102E-02
50	0.035	-0.018	-0.075	0.069	0.00	0.00	0.332E-01	0.249E-01

Table A.8 Linear vs. Degree 2: Step inputs with randomly generated initial conditions and input amplitudes



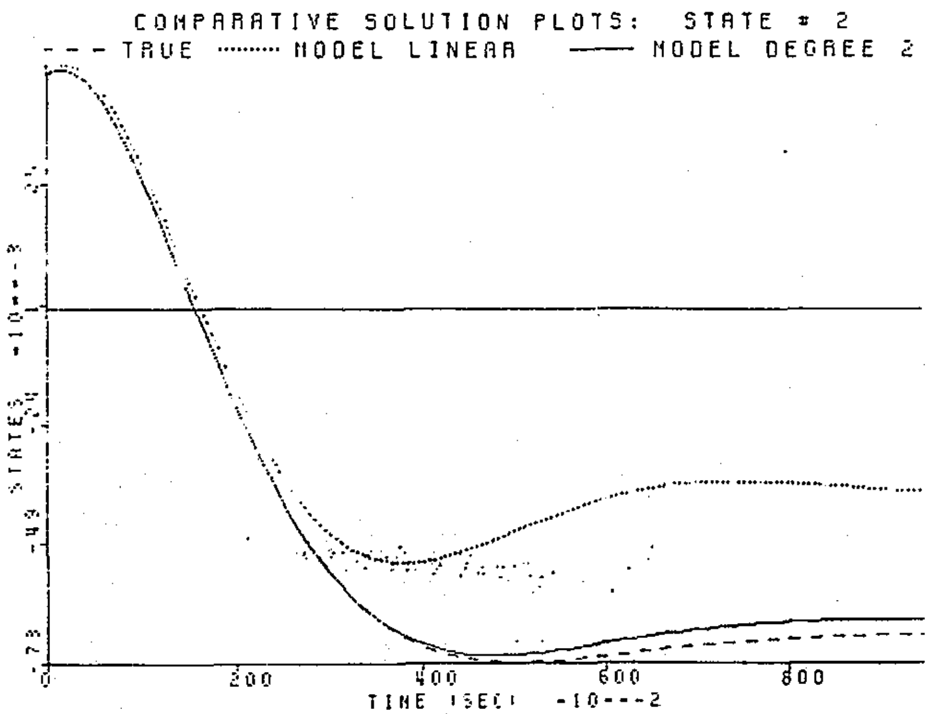


Figure A.13 S#13, Table A.8

A.4 THIRD DEGREE MODELING

The second degree model clearly has a larger region of acceptable performance than the linear approximation, as is evident from the verification studies in the preceding section. In this section we take the identification scheme one step further and construct a third degree truncation approximation using the same procedure.

With an integration timestep of 0.005 seconds the original system is integrated with initial conditions of 0.001 and -0.001 for x_1 and x_2, respectively, and sampled every 0.04 seconds for 100 points. The input for this model identification consists of a sum of two sinusoids whose amplitudes are held constant at 0.055 and 0.0325 for u_1 and u_2, respectively. Optimization is done over the input frequencies with startup values of 1 and 2 hertz. After 53 evaluations of the cost function convergence is achieved with a cost function reduction of approximately 98%. At this point a local optimum is achieved and the final values of the frequencies are 0.509 and 2.010. For this third degree model with 68 parameters (34 for each equation), the result of the least squares identification using the data generated with these inputs is given in Figure A.14. The condition number of the matrix X_s in this case is about 526, still well within the limits of double precision computations. One would expect that this number be higher than in the second degree model case (recall, there it was 19) since for this identification there are roughly 2.5 times the parameters and thus 2.5 times the data to manipulate.

In this case the comparison studies are done for the model of Figure A.14 versus the second degree model of the previous section. All of the tables to follow, then, have as the first model (that is, solution xl(k) in (A.2.9)) the second degree model of Figure A.1, and have as the second model (that is,

$$L_{10} = \begin{bmatrix} -1.998 & -3.001 \\ 0.999 & 1.000 \end{bmatrix} \qquad L_{01} = \begin{bmatrix} 0.002 & 0.998 \\ -1.000 & -0.000 \end{bmatrix}$$

$$L_{20} = \begin{bmatrix} -0.190 & -0.012 & -0.003 \\ 0.036 & 0.002 & 0.000 \end{bmatrix} \qquad L_{02} = \begin{bmatrix} -0.004 & 0.007 & -0.012 \\ -0.998 & 0.001 & 0.001 \end{bmatrix}$$

$$L_{11} = \begin{bmatrix} -4.069 & 0.026 & 0.028 & -0.008 \\ 0.021 & 0.003 & 0.003 & 0.004 \end{bmatrix}$$

$$L_{30} = \begin{bmatrix} 0.705 & 1.663 & 3.316 & 0.781 \\ -0.222 & -0.668 & -1.262 & -0.339 \end{bmatrix}$$

$$L_{21} = \begin{bmatrix} 0.621 & -0.614 & -0.252 & -0.765 & 0.447 & -0.675 \\ -0.289 & -0.025 & -0.055 & -0.058 & -0.049 & -0.006 \end{bmatrix}$$

$$L_{12} = \begin{bmatrix} -3.549 & -0.102 & 0.414 & 0.165 & 0.037 & 0.076 \\ -0.172 & 0.960 & -0.044 & -0.056 & -0.035 & -0.013 \end{bmatrix}$$

$$L_{03} = \begin{bmatrix} -0.187 & -0.032 & -0.025 & -0.020 \\ -0.517 & -0.016 & -0.019 & -0.001 \end{bmatrix}$$

Figure A.14 Third Degree Model

solution x2(k)) the third degree model of Figure A.14. Thus, an error ratio less than one indicates that we have succeeded in expanding the region of acceptable behavior by adding more terms in the higher degree approximation.

In the first three tables of this section--A.9, A.10, and A.11 (repeated simulation parameters of Tables A.1, A.2, and A.3)--we find that the third degree model outperforms the second degree model almost uniformly. This of course is an added bonus since these simulation parameters are close to the point of expansion. That is, with these relatively small amplitudes we would hope that the third degree model at least match the performance of the second degree model. Notice, though, that as the input amplitudes get larger the comparative performance for the third degree model improves. One simulation is plotted from each table, revealing that the two models are very close in performance.

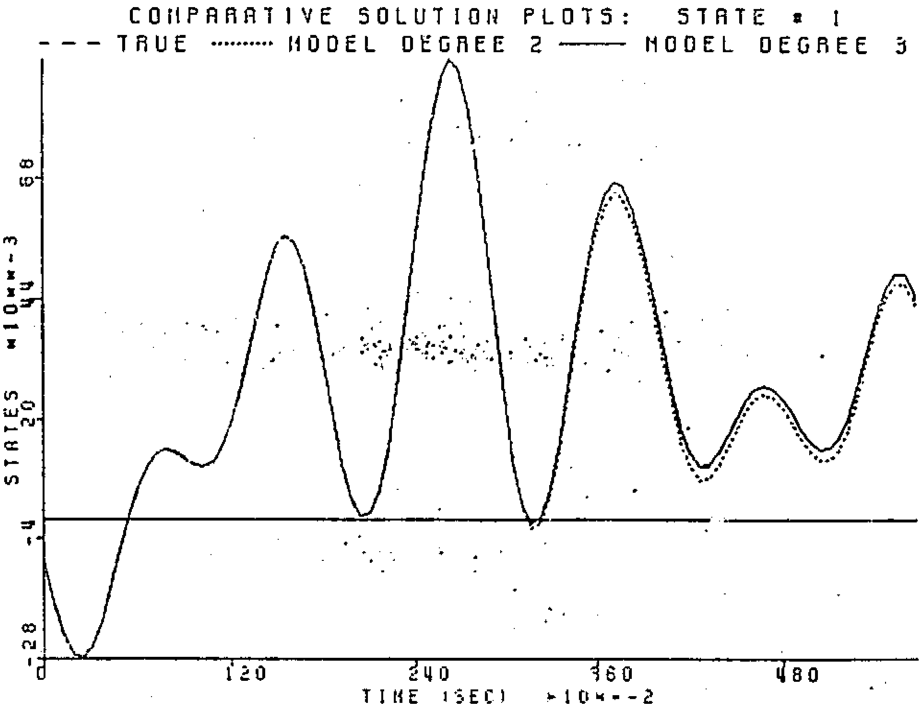
The next three tables--A.12, A.13, and A.14--offer simulations with fixed frequencies which progressively stretch the region of operation in terms of the input amplitudes and initial state conditions. While it is clear from the plots which follow that these simulation parameter sets push the second degree model beyond an acceptable region, we note that at the upper end (that is, for input amplitudes of 0.5 and 0.6) we have reached the practical limits of validity for the third degree model performance. Table A.15 offers randomly generated input amplitudes, between -0.3 and 0.3, and input frequencies between 2 and 6 hertz. More of the same type of behavior is observed.

The simulations represented in these first seven tables of this section were all carried out from zero to six seconds, even though some of the plots extend much further. For the final two tables--A.16 and A.17--simulations are carried out to ten seconds. These tables represent simulations for low frequency inputs (including the DC case) and contain identical parameter sets as those in Tables A.6 and A.7 of the last section. These simulations point out the fact that in general, for low frequencies, the third degree model performance is about equal with that of the second degree model. We include plots of two of the worst case results (in terms of the error ratios) in Figure A.30 and Figure A.32. Notice that while the second degree model outperforms the third degree model in these plots, each model exhibits fairly good performance for the worst case. This fact is brought out in the individual error normalization factors as well. In the next section we address this minor difficulty which arises for low frequency inputs.

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	0.001	0.000	0.000	0.75	1.00	0.618	0.552
2	0.001	0.001	0.050	0.050	0.75	1.00	0.465	0.212
3	0.001	0.001	0.050	-0.050	0.75	1.00	0.229	0.185
4	0.001	0.001	-0.050	-0.050	0.75	1.00	0.562	0.279
5	0.001	0.001	-0.050	0.050	0.75	1.00	0.863	0.605
6	0.001	0.001	0.150	0.150	0.75	1.00	0.134E-01	0.343E-02
7	0.001	0.001	0.150	-0.150	0.75	1.00	0.113E-01	0.176E-02
8	0.001	0.001	-0.150	-0.150	0.75	1.00	0.837E-02	0.211E-02
9	0.001	0.001	-0.150	0.150	0.75	1.00	0.221E-01	0.659E-02
10	0.001	-0.001	0.000	0.000	0.75	1.00	0.624	0.552
11	0.001	-0.001	0.050	0.050	0.75	1.00	0.423	0.144
12	0.001	-0.001	0.050	-0.050	0.75	1.00	0.193	0.112
13	0.001	-0.001	-0.050	-0.050	0.75	1.00	0.478	0.168
14	0.001	-0.001	-0.050	0.050	0.75	1.00	0.902	0.469
15	0.001	-0.001	0.150	0.150	0.75	1.00	0.108E-01	0.286E-02
16	0.001	-0.001	0.150	-0.150	0.75	1.00	0.106E-01	0.176E-02
17	0.001	-0.001	-0.150	-0.150	0.75	1.00	0.669E-02	0.174E-02
18	0.001	-0.001	-0.150	0.150	0.75	1.00	0.199E-01	0.582E-02
19	-0.001	-0.001	0.000	0.000	0.75	1.00	0.495	0.440
20	-0.001	-0.001	0.050	0.050	0.75	1.00	0.372	0.108
21	-0.001	-0.001	0.050	-0.050	0.75	1.00	0.208	0.103
22	-0.001	-0.001	-0.050	-0.050	0.75	1.00	0.444	0.150
23	-0.001	-0.001	-0.050	0.050	0.75	1.00	1.17	0.585
24	-0.001	-0.001	0.150	0.150	0.75	1.00	0.969E-02	0.263E-02
25	-0.001	-0.001	0.150	-0.150	0.75	1.00	0.107E-01	0.182E-02
26	-0.001	-0.001	-0.150	-0.150	0.75	1.00	0.631E-02	0.166E-02
27	-0.001	-0.001	-0.150	0.150	0.75	1.00	0.196E-01	0.590E-02
28	-0.001	0.001	0.000	0.000	0.75	1.00	0.717	0.634
29	-0.001	0.001	0.050	0.050	0.75	1.00	0.413	0.171
30	-0.001	0.001	0.050	-0.050	0.75	1.00	0.237	0.164
31	-0.001	0.001	-0.050	-0.050	0.75	1.00	0.557	0.269
32	-0.001	0.001	-0.050	0.050	0.75	1.00	1.12	0.690
33	-0.001	0.001	0.150	0.150	0.75	1.00	0.120E-01	0.313E-02
34	-0.001	0.001	0.150	-0.150	0.75	1.00	0.114E-01	0.181E-02
35	-0.001	0.001	-0.150	-0.150	0.75	1.00	0.787E-02	0.200E-02
36	-0.001	0.001	-0.150	0.150	0.75	1.00	0.218E-01	0.666E-02

Table A.9 Degree 2 vs. Degree 3: Constant input frequencies



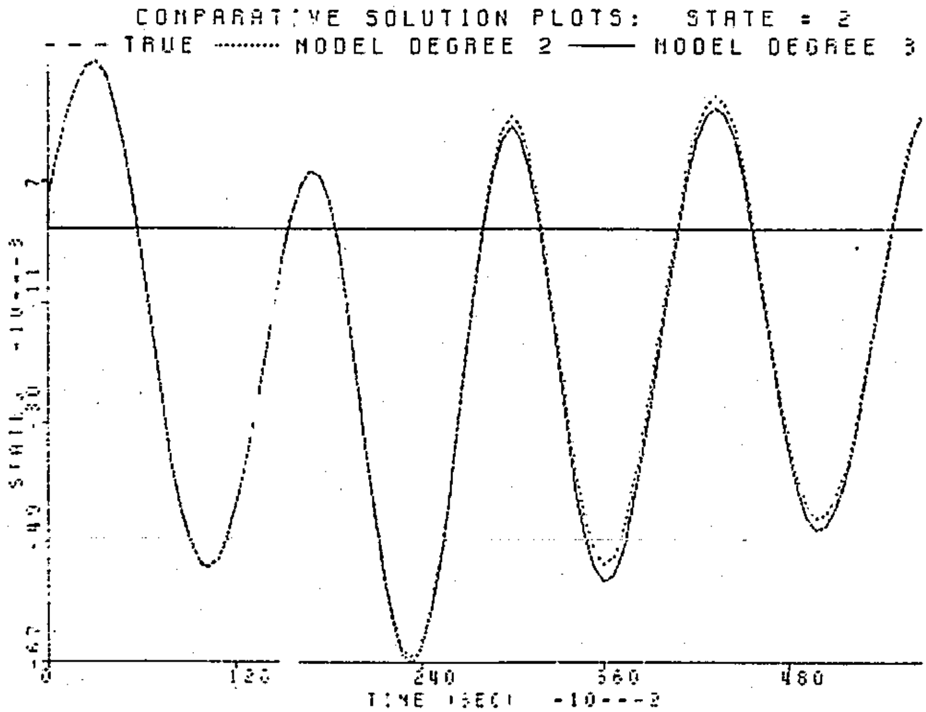


Figure A.15 S#26, Table A.9

ORIGINAL PAGE IS
OF POOR QUALITY

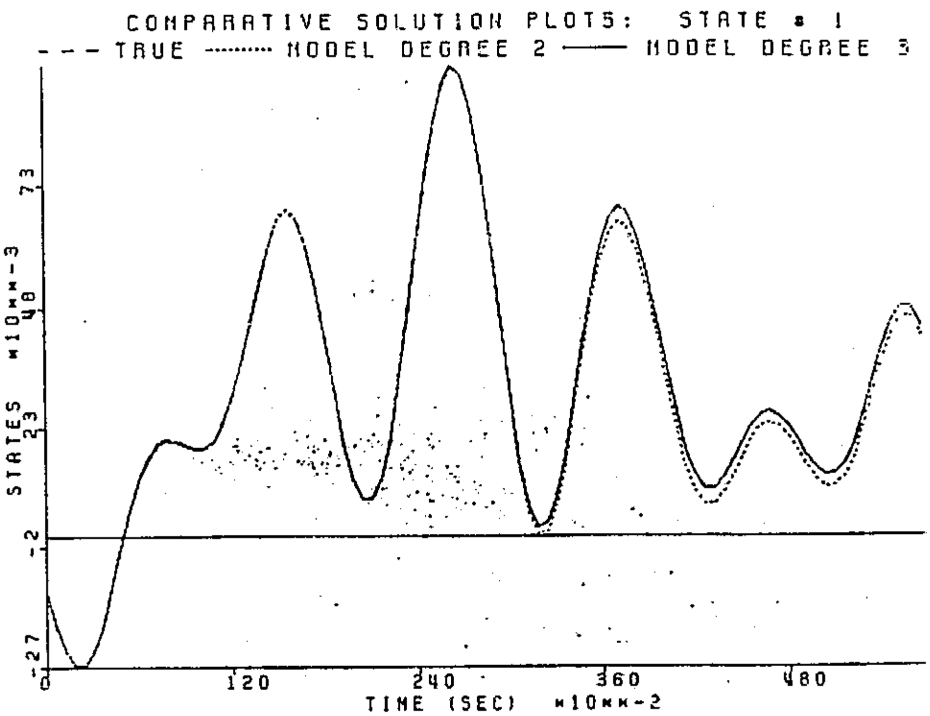
****

PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE

****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.005	0.005	0.000	0.000	0.75	1.00	0.923	0.826
2	0.005	0.005	0.050	0.050	0.75	1.00	0.898	0.602
3	0.005	0.005	0.050	-0.050	0.75	1.00	0.504	0.730
4	0.005	0.005	-0.050	-0.050	0.75	1.00	0.788	0.596
5	0.005	0.005	-0.050	0.050	0.75	1.00	0.961	1.33
6	0.005	0.005	0.150	0.150	0.75	1.00	0.267E-01	0.670E-02
7	0.005	0.005	0.150	-0.150	0.75	1.00	0.126E-01	0.173E-02
8	0.005	0.005	-0.150	-0.150	0.75	1.00	0.157E-01	0.405E-02
9	0.005	0.005	-0.150	0.150	0.75	1.00	0.279E-01	0.846E-02
10	0.005	-0.005	0.000	0.000	0.75	1.00	0.467	0.414
11	0.005	-0.005	0.050	0.050	0.75	1.00	0.475	0.129
12	0.005	-0.005	0.050	-0.050	0.75	1.00	0.132	0.634E-01
13	0.005	-0.005	-0.050	-0.050	0.75	1.00	0.335	0.573E-01
14	0.005	-0.005	-0.050	0.050	0.75	1.00	0.586	0.208
15	0.005	-0.005	0.150	0.150	0.75	1.00	0.866E-02	0.246E-02
16	0.005	-0.005	0.150	-0.150	0.75	1.00	0.912E-02	0.171E-02
17	0.005	-0.005	-0.150	-0.150	0.75	1.00	0.495E-02	0.143E-02
18	0.005	-0.005	-0.150	0.150	0.75	1.00	0.168E-01	0.448E-02
19	-0.005	-0.005	0.000	0.000	0.75	1.00	0.302	0.260
20	-0.005	-0.005	0.050	0.050	0.75	1.00	0.160	0.196E-01
21	-0.005	-0.005	0.050	-0.050	0.75	1.00	0.180	0.559E-01
22	-0.005	-0.005	-0.050	-0.050	0.75	1.00	0.139	0.217E-01
23	-0.005	-0.005	-0.050	0.050	0.75	1.00	0.600	0.274
24	-0.005	-0.005	0.150	0.150	0.75	1.00	0.549E-02	0.180E-02
25	-0.005	-0.005	0.150	-0.150	0.75	1.00	0.935E-02	0.190E-02
26	-0.005	-0.005	-0.150	-0.150	0.75	1.00	0.389E-02	0.124E-02
27	-0.005	-0.005	-0.150	0.150	0.75	1.00	0.155E-01	0.480E-02
28	-0.005	0.005	0.000	0.000	0.75	1.00	0.933	0.822
29	-0.005	0.005	0.050	0.050	0.75	1.00	0.485	0.280
30	-0.005	0.005	0.050	-0.050	0.75	1.00	0.408	0.406
31	-0.005	0.005	-0.050	-0.050	0.75	1.00	0.775	0.555
32	-0.005	0.005	-0.050	0.050	0.75	1.00	1.91	1.34
33	-0.005	0.005	0.150	0.150	0.75	1.00	0.151E-01	0.387E-02
34	-0.005	0.005	0.150	-0.150	0.75	1.00	0.133E-01	0.203E-02
35	-0.005	0.005	-0.150	-0.150	0.75	1.00	0.111E-01	0.284E-02
36	-0.005	0.005	-0.150	0.150	0.75	1.00	0.259E-01	0.873E-02

Table A.10 Degree 2 vs. Degree 3: Constant input frequencies



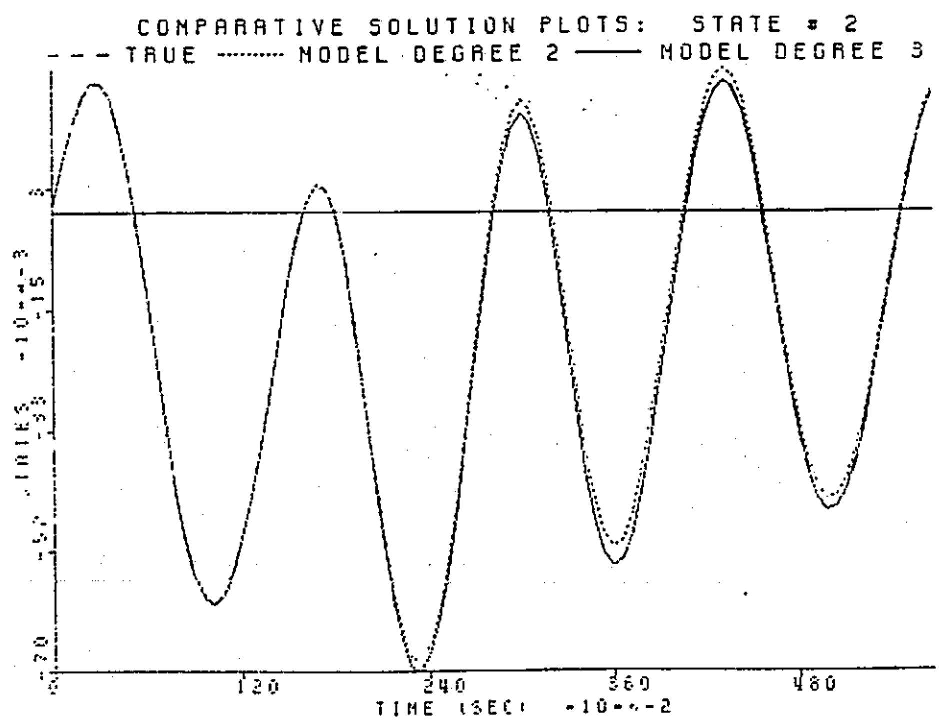
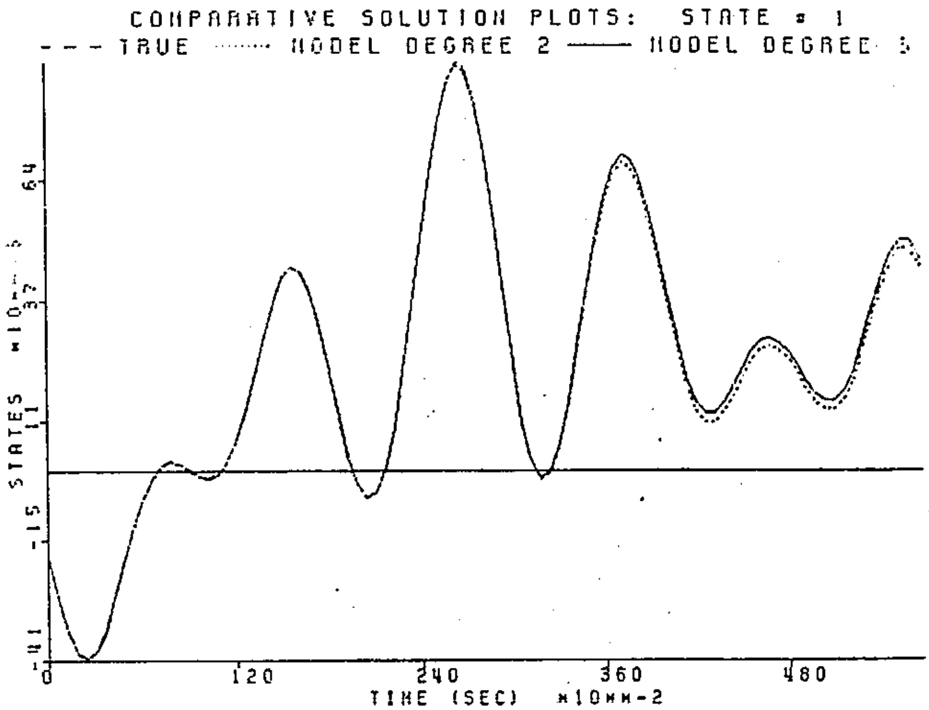


Figure A.16 S#26, Table A.10: Compare to Figure A.4

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.010	0.010	0.000	0.000	0.75	1.00	1.46	1.31
2	0.010	0.010	0.050	0.050	0.75	1.00	2.37	1.76
3	0.010	0.010	0.050	-0.050	0.75	1.00	1.61	2.76
4	0.010	0.010	-0.050	-0.050	0.75	1.00	1.81	1.55
5	0.010	0.010	-0.050	0.050	0.75	1.00	2.20	3.96
6	0.010	0.010	0.150	0.150	0.75	1.00	0.549E-01	0.170E-01
7	0.010	0.010	0.150	-0.150	0.75	1.00	0.151E-01	0.281E-02
8	0.010	0.010	-0.150	-0.150	0.75	1.00	0.347E-01	0.109E-01
9	0.010	0.010	-0.150	0.150	0.75	1.00	0.371E-01	0.125E-01
10	0.010	-0.010	0.000	0.000	0.75	1.00	0.303	0.266
11	0.010	-0.010	0.050	0.050	0.75	1.00	0.280	0.646E-01
12	0.010	-0.010	0.050	-0.050	0.75	1.00	0.764E-01	0.352E-01
13	0.010	-0.010	-0.050	-0.050	0.75	1.00	0.172	0.229E-01
14	0.010	-0.010	-0.050	0.050	0.75	1.00	0.295	0.895E-01
15	0.010	-0.010	0.150	0.150	0.75	1.00	0.672E-02	0.213E-02
16	0.010	-0.010	0.150	-0.150	0.75	1.00	0.760E-02	0.169E-02
17	0.010	-0.010	-0.150	-0.150	0.75	1.00	0.358E-02	0.127E-02
18	0.010	-0.010	-0.150	0.150	0.75	1.00	0.137E-01	0.330E-02
19	-0.010	-0.010	0.000	0.000	0.75	1.00	0.174	0.814E-01
20	-0.010	-0.010	0.050	0.050	0.75	1.00	0.389E-01	0.406E-02
21	-0.010	-0.010	0.050	-0.050	0.75	1.00	0.658E-01	0.195E-01
22	-0.010	-0.010	-0.050	-0.050	0.75	1.00	0.348E-01	0.680E-02
23	-0.010	-0.010	-0.050	0.050	0.75	1.00	0.873E-01	0.363E-01
24	-0.010	-0.010	0.150	0.150	0.75	1.00	0.318E-02	0.141E-02
25	-0.010	-0.010	0.150	-0.150	0.75	1.00	0.798E-02	0.193E-02
26	-0.010	-0.010	-0.150	-0.150	0.75	1.00	0.254E-02	0.120E-02
27	-0.010	-0.010	-0.150	0.150	0.75	1.00	0.118E-01	0.380E-02
28	-0.010	0.010	0.000	0.000	0.75	1.00	1.27	1.11
29	-0.010	0.010	0.050	0.050	0.75	1.00	0.777	0.559
30	-0.010	0.010	0.050	-0.050	0.75	1.00	1.04	1.16
31	-0.010	0.010	-0.050	-0.050	0.75	1.00	1.24	1.11
32	-0.010	0.010	-0.050	0.050	0.75	1.00	3.88	2.66
33	-0.010	0.010	0.150	0.150	0.75	1.00	0.200E-01	0.536E-02
34	-0.010	0.010	0.150	-0.150	0.75	1.00	0.160E-01	0.264E-02
35	-0.010	0.010	-0.150	-0.150	0.75	1.00	0.176E-01	0.488E-02
36	-0.010	0.010	-0.150	0.150	0.75	1.00	0.325E-01	0.122E-01

Table A.11 Degree 2 vs. Degree 3: Constant input frequencies



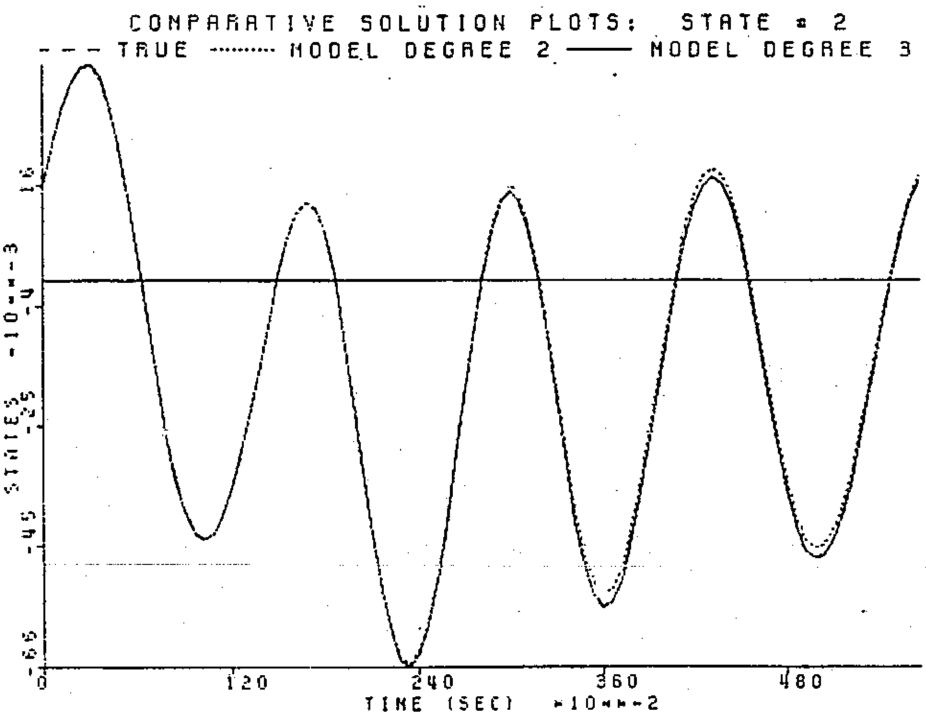


Figure A.17 S#35, Table A.11

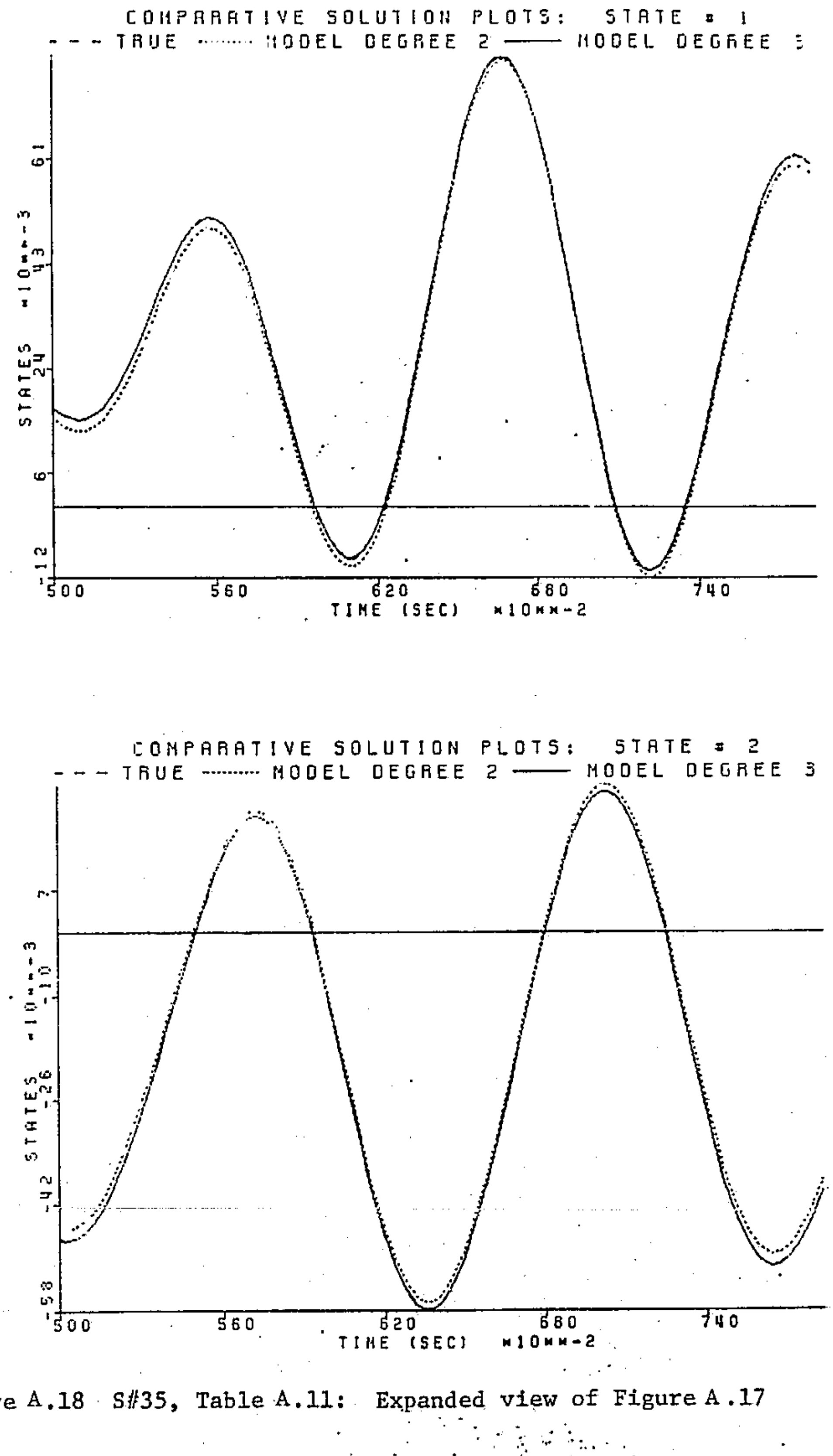
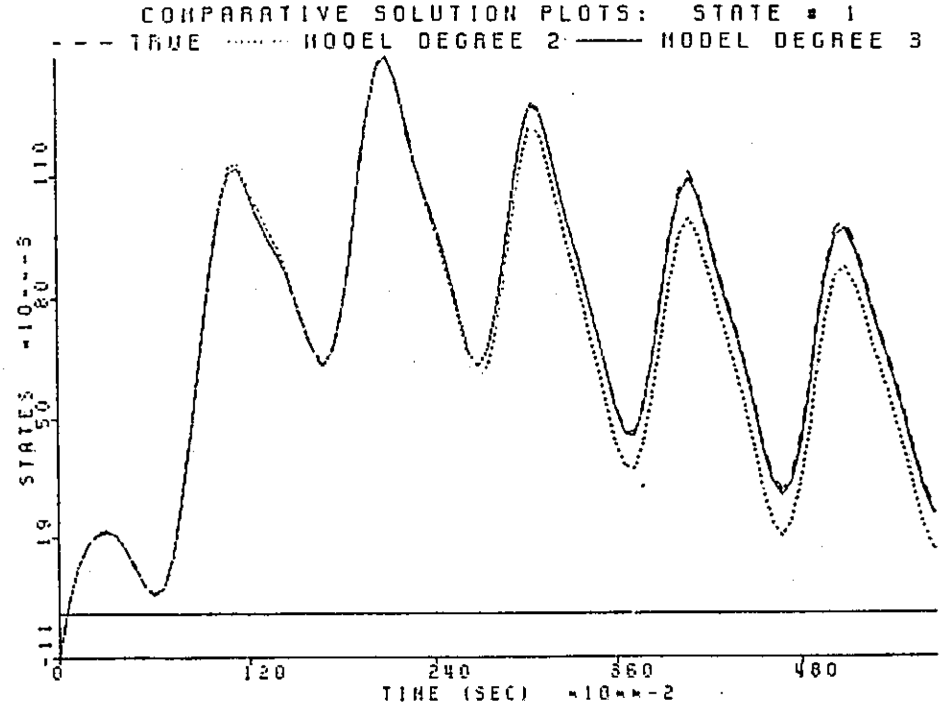


Figure A.18 S#35, Table A.11: Expanded view of Figure A.17

```
*************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.025	0.025	0.100	0.100	2.00	1.00	0.446	0.411
2	0.025	0.025	0.100	-0.100	2.00	1.00	0.402	0.375
3	0.025	0.025	-0.100	0.100	2.00	1.00	0.458	0.391
4	0.025	0.025	-0.100	-0.100	2.00	1.00	0.445	0.358
5	0.025	-0.025	0.100	0.100	2.00	1.00	0.380E-02	0.285E-02
6	0.025	-0.025	0.100	-0.100	2.00	1.00	0.296E-02	0.285E-02
7	0.025	-0.025	-0.100	0.100	2.00	1.00	0.759E-02	0.246E-02
8	0.025	-0.025	-0.100	-0.100	2.00	1.00	0.677E-02	0.196E-02
9	-0.025	0.025	0.100	0.100	2.00	1.00	0.209	0.172
10	-0.025	0.025	0.100	-0.100	2.00	1.00	0.203	0.168
11	-0.025	0.025	-0.100	0.100	2.00	1.00	0.210	0.135
12	-0.025	0.025	-0.100	-0.100	2.00	1.00	0.191	0.122
13	-0.025	-0.025	0.100	0.100	2.00	1.00	0.245E-02	0.689E-03
14	-0.025	-0.025	0.100	-0.100	2.00	1.00	0.231E-02	0.559E-03
15	-0.025	-0.025	-0.100	0.100	2.00	1.00	0.482E-02	0.271E-02
16	-0.025	-0.025	-0.100	-0.100	2.00	1.00	0.479E-02	0.226E-02
17	0.025	0.025	0.200	0.200	2.00	1.00	0.316E-01	0.795E-02
18	0.025	0.025	0.200	-0.200	2.00	1.00	0.287E-01	0.560E-02
19	0.025	0.025	-0.200	0.200	2.00	1.00	0.203E-01	0.963E-02
20	0.025	0.025	-0.200	-0.200	2.00	1.00	0.212E-01	0.994E-02
21	0.025	-0.025	0.200	0.200	2.00	1.00	0.243E-02	0.427E-02
22	0.025	-0.025	0.200	-0.200	2.00	1.00	0.250E-02	0.427E-02
23	0.025	-0.025	-0.200	0.200	2.00	1.00	0.112E-01	0.141E-01
24	0.025	-0.025	-0.200	-0.200	2.00	1.00	0.114E-01	0.135E-01
25	-0.025	0.025	0.200	0.200	2.00	1.00	0.121E-01	0.502E-02
26	-0.025	0.025	0.200	-0.200	2.00	1.00	0.103E-01	0.429E-02
27	-0.025	0.025	-0.200	0.200	2.00	1.00	0.121E-01	0.111E-01
28	-0.025	0.025	-0.200	-0.200	2.00	1.00	0.115E-01	0.109E-01
29	-0.025	-0.025	0.200	0.200	2.00	1.00	0.841E-02	0.842E-02
30	-0.025	-0.025	0.200	-0.200	2.00	1.00	0.816E-02	0.826E-02
31	-0.025	-0.025	-0.200	0.200	2.00	1.00	0.157E-01	0.177E-01
32	-0.025	-0.025	-0.200	-0.200	2.00	1.00	0.154E-01	0.167E-01
33	0.025	0.025	0.300	0.300	2.00	1.00	0.104E-01	0.181E-01
34	0.025	0.025	0.300	-0.300	2.00	1.00	0.835E-02	0.179E-01
35	0.025	0.025	-0.300	0.300	2.00	1.00	0.137E-01	0.320E-01
36	0.025	0.025	-0.300	-0.300	2.00	1.00	0.127E-01	0.302E-01
37	0.025	-0.025	0.300	0.300	2.00	1.00	0.109E-01	0.249E-01
38	0.025	-0.025	0.300	-0.300	2.00	1.00	0.102E-01	0.247E-01
39	0.025	-0.025	-0.300	0.300	2.00	1.00	0.200E-01	0.398E-01
40	0.025	-0.025	-0.300	-0.300	2.00	1.00	0.184E-01	0.371E-01
41	-0.025	0.025	0.300	0.300	2.00	1.00	0.111E-01	0.217E-01
42	-0.025	0.025	0.300	-0.300	2.00	1.00	0.977E-02	0.213E-01
43	-0.025	0.025	-0.300	0.300	2.00	1.00	0.166E-01	0.356E-01
44	-0.025	0.025	-0.300	-0.300	2.00	1.00	0.151E-01	0.334E-01
45	-0.025	-0.025	0.300	0.300	2.00	1.00	0.191E-01	0.322E-01
46	-0.025	-0.025	0.300	-0.300	2.00	1.00	0.185E-01	0.319E-01
47	-0.025	-0.025	-0.300	0.300	2.00	1.00	0.291E-01	0.478E-01
48	-0.025	-0.025	-0.300	-0.300	2.00	1.00	0.271E-01	0.447E-01

Table A.12 Degree 2 vs. Degree 3: Constant input frequencies



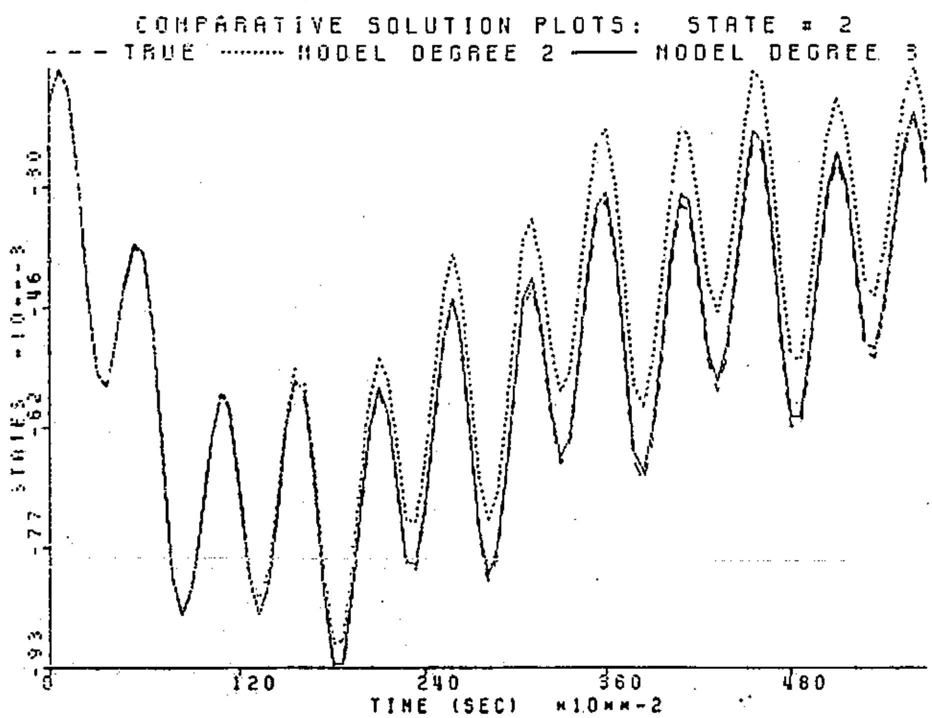
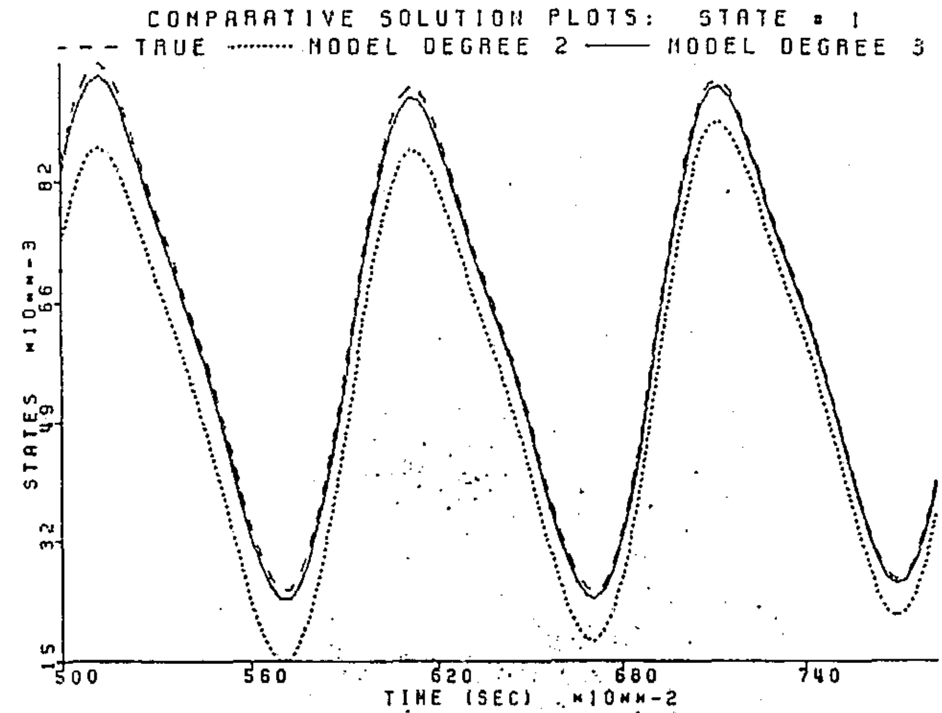


Figure A.19 S#31, Table A.12



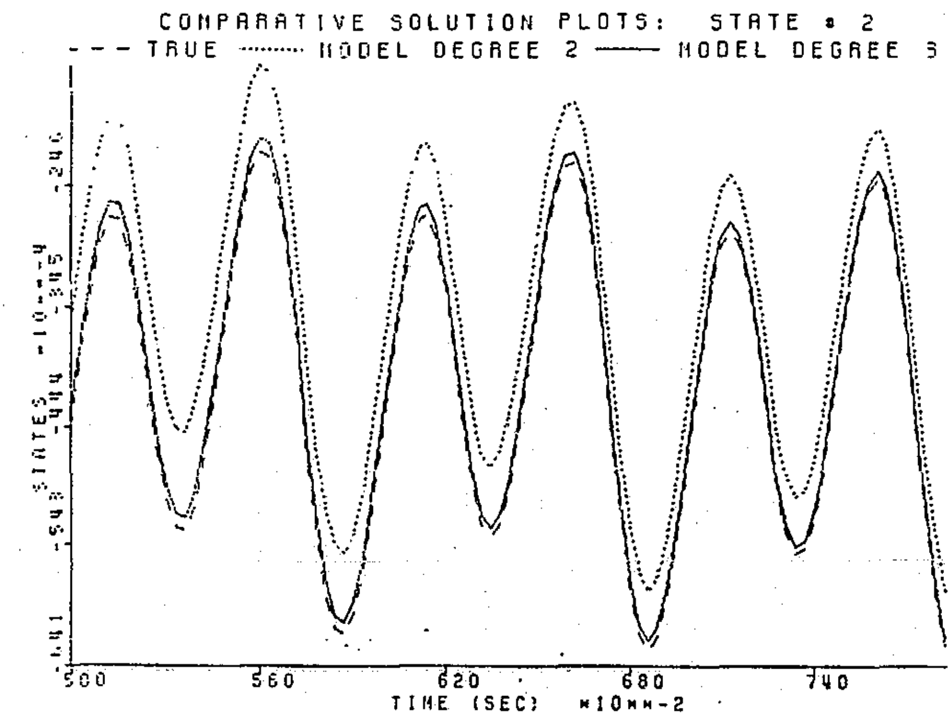
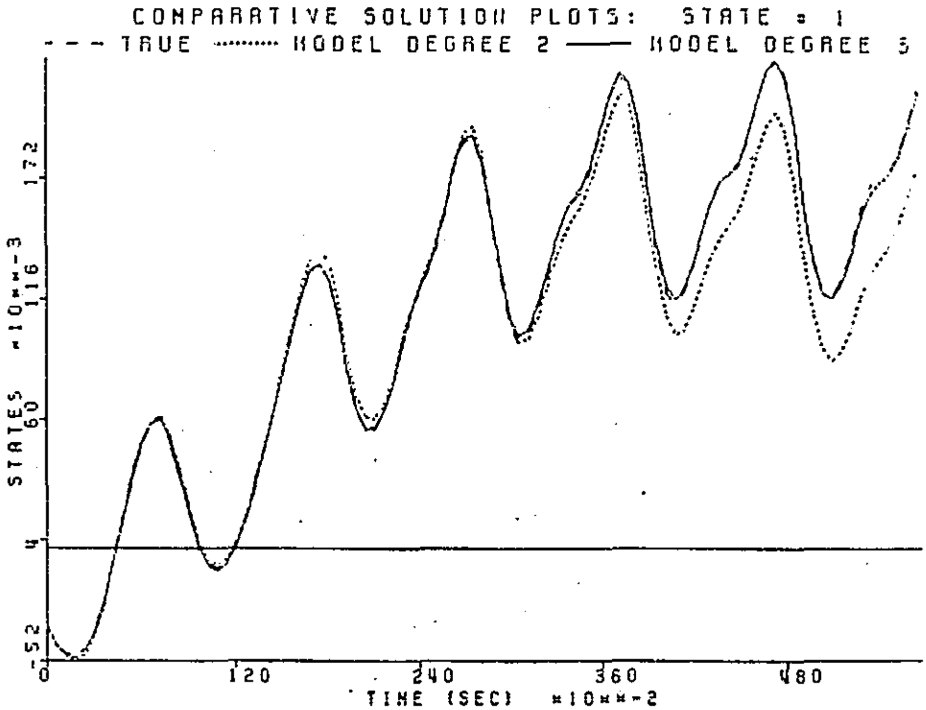


Figure A.20 S#31, Table A.12: Expanded view of Figure A.19

ORIGINAL PAGE IS
OF POOR QUALITY



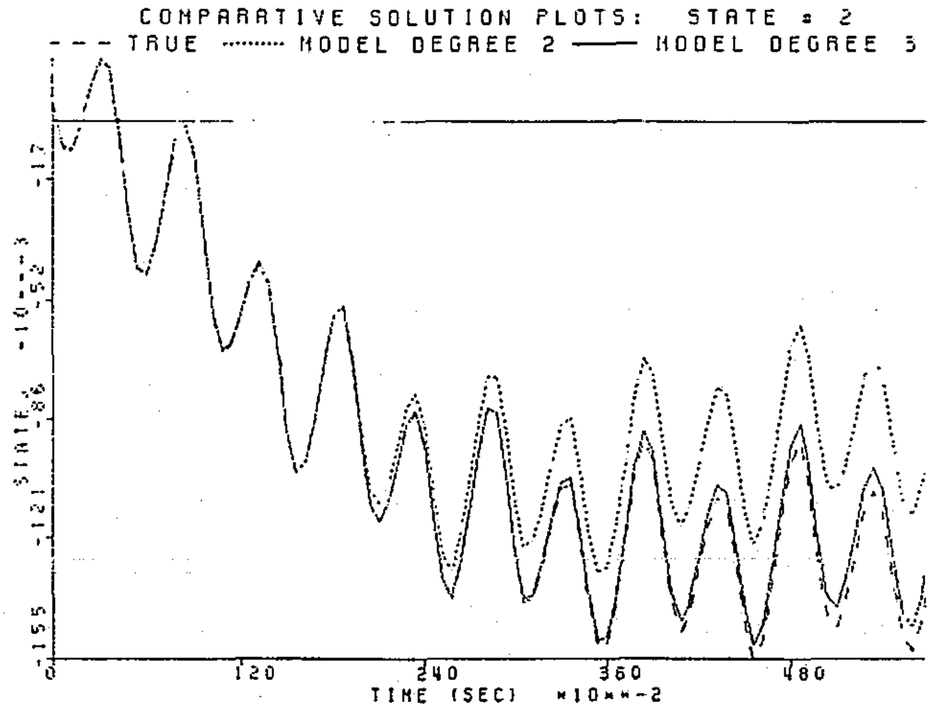


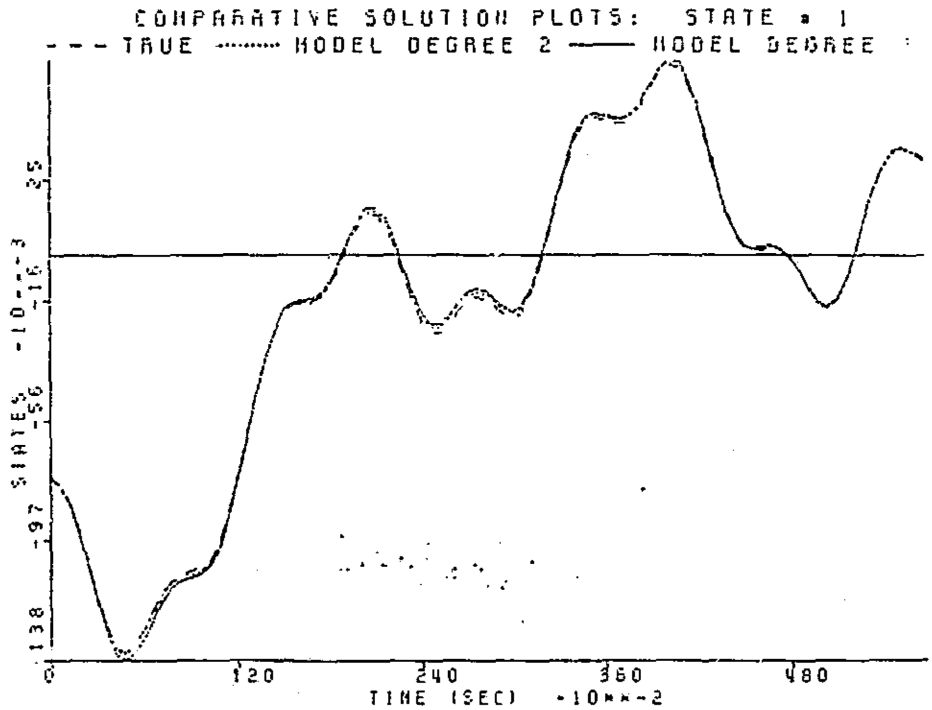
Figure A.21 S#42, Table A.12

****
PROBLEM SUMMARY
CONFIGURATION: TRUE,MODEL1,MODEL2
OF STATES: 2
OF INPUTS: 2
OF TERMS IN MODEL 1: 14
DEGREE OF APPROXIMATION: 2
OF TERMS IN MODEL 2: 34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.075	0.075	0.100	0.100	0.50	1.50	4.42	3.69
2	0.075	0.075	0.100	-0.100	0.50	1.50	4.32	3.63
3	0.075	0.075	-0.100	0.100	0.50	1.50	2.92	3.25
4	0.075	0.075	-0.100	-0.100	0.50	1.50	3.04	3.24
5	0.075	-0.075	0.100	0.100	0.50	1.50	0.393E-01	0.319E-01
6	0.075	-0.075	0.100	-0.100	0.50	1.50	0.424E-01	0.346E-01
7	0.075	-0.075	-0.100	0.100	0.50	1.50	0.755E-02	0.559E-02
8	0.075	-0.075	-0.100	-0.100	0.50	1.50	0.798E-02	0.624E-02
9	-0.075	0.075	0.100	0.100	0.50	1.50	1.35	1.11
10	-0.075	0.075	0.100	-0.100	0.50	1.50	1.35	1.11
11	-0.075	0.075	-0.100	0.100	0.50	1.50	2.70	2.14
12	-0.075	0.075	-0.100	-0.100	0.50	1.50	2.54	1.91
13	-0.075	-0.075	0.100	0.100	0.50	1.50	0.852E-01	0.648E-01
14	-0.075	-0.075	0.100	-0.100	0.50	1.50	0.856E-01	0.657E-01
15	-0.075	-0.075	-0.100	0.100	0.50	1.50	0.202	0.162
16	-0.075	-0.075	-0.100	-0.100	0.50	1.50	0.203	0.164
17	0.075	0.075	0.200	0.200	0.50	1.50	0.458	0.330
18	0.075	0.075	0.200	-0.200	0.50	1.50	0.464	0.339
19	0.075	0.075	-0.200	0.200	0.50	1.50	0.186	0.212
20	0.075	0.075	-0.200	-0.200	0.50	1.50	0.186	0.195
21	0.075	-0.075	0.200	0.200	0.50	1.50	0.216E-01	0.167E-01
22	0.075	-0.075	0.200	-0.200	0.50	1.50	0.255E-01	0.198E-01
23	0.075	-0.075	-0.200	0.200	0.50	1.50	0.119E-01	0.778E-02
24	0.075	-0.075	-0.200	-0.200	0.50	1.50	0.114E-01	0.842E-02
25	-0.075	0.075	0.200	0.200	0.50	1.50	0.602E-01	0.364E-01
26	-0.075	0.075	0.200	-0.200	0.50	1.50	0.699E-01	0.429E-01
27	-0.075	0.075	-0.200	0.200	0.50	1.50	0.329	0.192
28	-0.075	0.075	-0.200	-0.200	0.50	1.50	0.268	0.148
29	-0.075	-0.075	0.200	0.200	0.50	1.50	0.312E-01	0.224E-01
30	-0.075	-0.075	0.200	-0.200	0.50	1.50	0.332E-01	0.237E-01
31	-0.075	-0.075	-0.200	0.200	0.50	1.50	0.130	0.106
32	-0.075	-0.075	-0.200	-0.200	0.50	1.50	0.131	0.109
33	0.075	0.075	0.300	0.300	0.50	1.50	0.805E-01	0.254E-01
34	0.075	0.075	0.300	-0.300	0.50	1.50	0.987E-01	0.297E-01
35	0.075	0.075	-0.300	0.300	0.50	1.50	0.549E-01	0.261E-01
36	0.075	0.075	-0.300	-0.300	0.50	1.50	0.536E-01	0.222E-01
37	0.075	-0.075	0.300	0.300	0.50	1.50	0.988E-02	0.605E-02
38	0.075	-0.075	0.300	-0.300	0.50	1.50	0.139E-01	0.885E-02
39	0.075	-0.075	-0.300	0.300	0.50	1.50	0.136E-01	0.802E-02
40	0.075	-0.075	-0.300	-0.300	0.50	1.50	0.123E-01	0.805E-02
41	-0.075	0.075	0.300	0.300	0.50	1.50	0.138E-01	0.297E-02
42	-0.075	0.075	0.300	-0.300	0.50	1.50	0.197E-01	0.368E-02
43	-0.075	0.075	-0.300	0.300	0.50	1.50	0.561E-01	0.213E-01
44	-0.075	0.075	-0.300	-0.300	0.50	1.50	0.409E-01	0.163E-01
45	-0.075	-0.075	0.300	0.300	0.50	1.50	0.148E-01	0.742E-02
46	-0.075	-0.075	0.300	-0.300	0.50	1.50	0.172E-01	0.892E-02
47	-0.075	-0.075	-0.300	0.300	0.50	1.50	0.105	0.814E-01
48	-0.075	-0.075	-0.300	-0.300	0.50	1.50	0.104	0.818E-01

Table A.13 Degree 2 vs. Degree 3: Constant input frequencies

ORIGINAL PAGE IS
OF POOR QUALITY



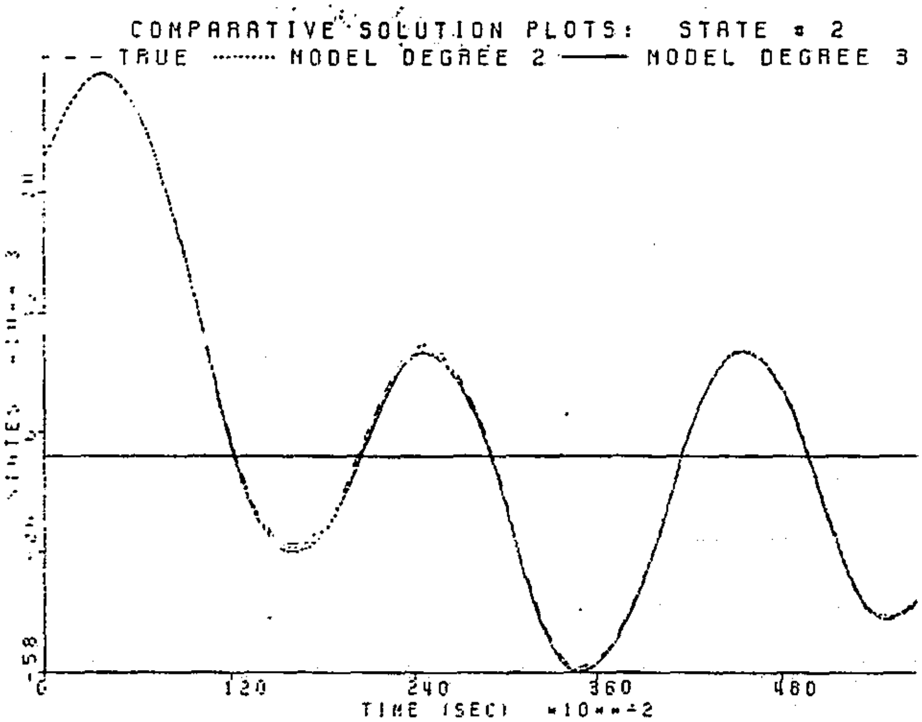
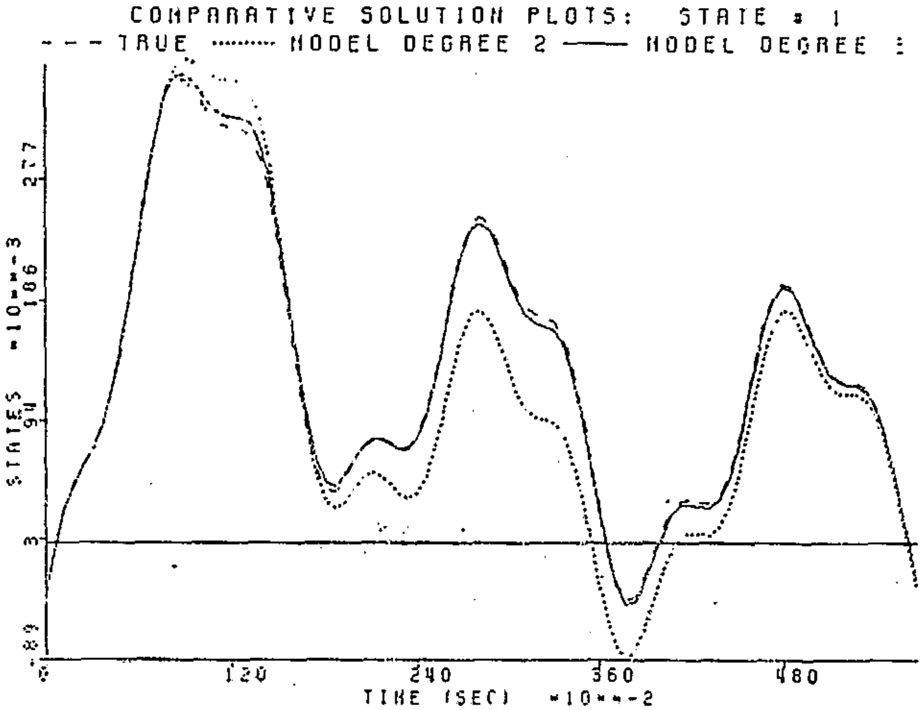


Figure A.22 S#11, Table A.13

ORIGINAL PAGE IS
OF POOR QUALITY



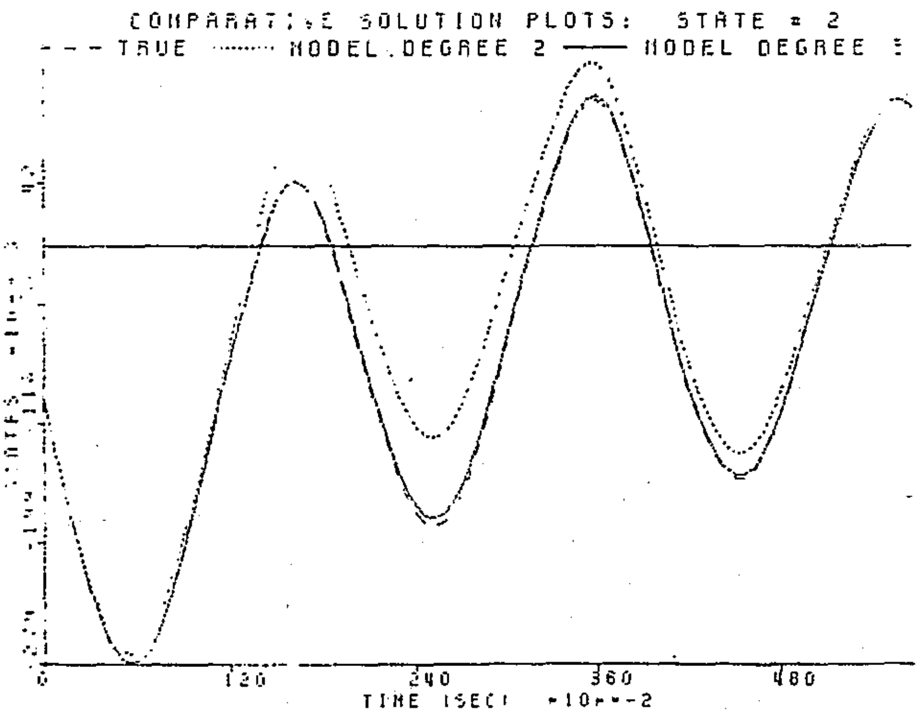
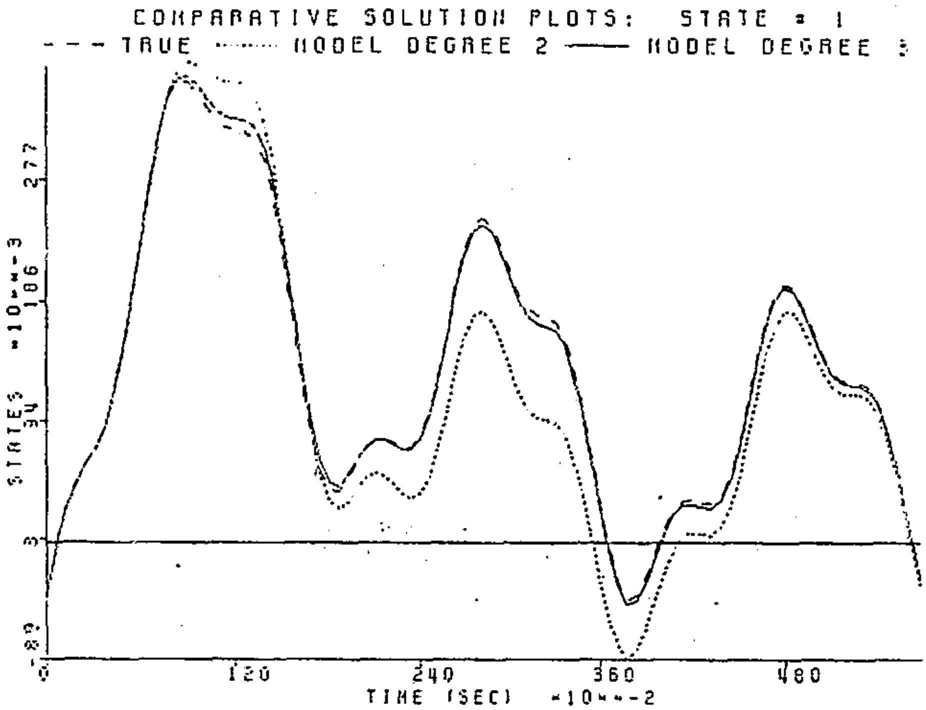


Figure A.23 S#45, Table A.13

ORIGINAL PAGE IS
OF POOR QUALITY



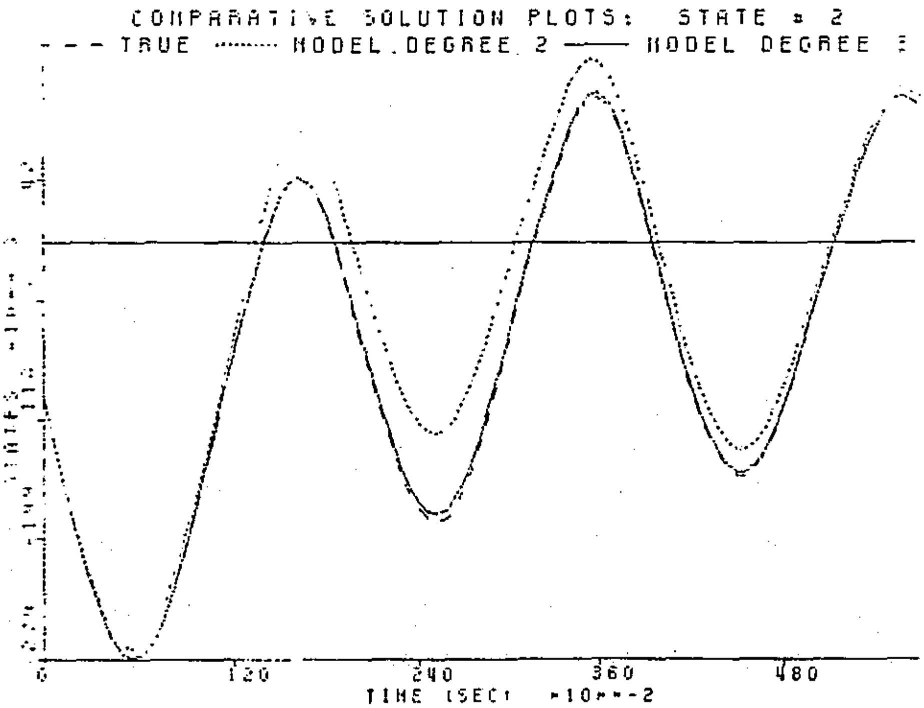
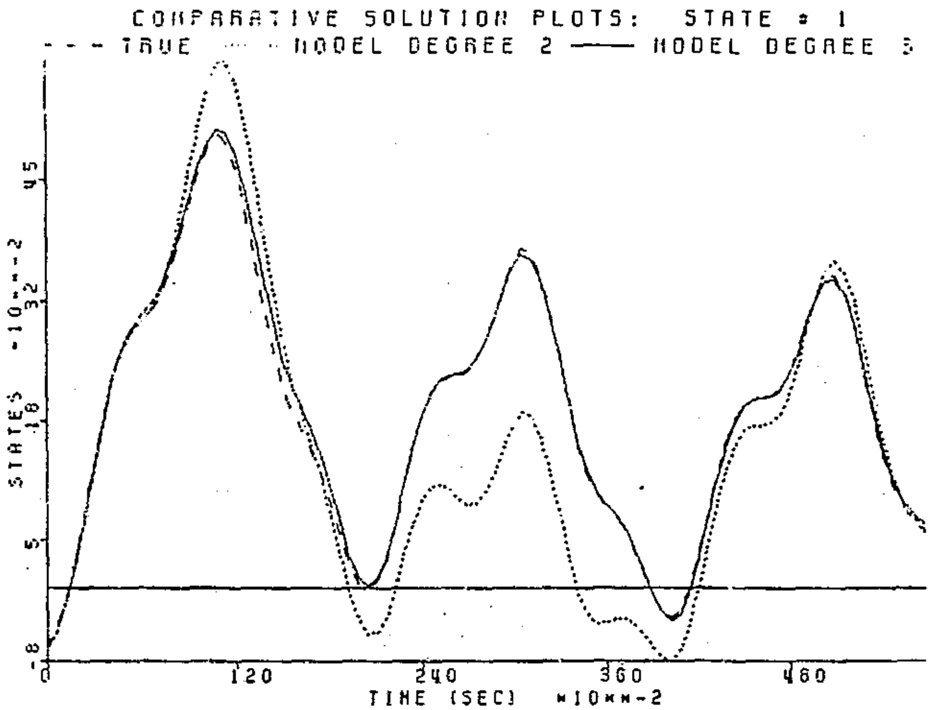


Figure A.23 S#45, Table A.13

```
*****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.075	0.075	0.400	0.400	0.50	1.50	0.371E-01	0.118E-01
2	0.075	0.075	0.400	-0.400	0.50	1.50	0.413E-01	0.985E-02
3	0.075	0.075	-0.400	0.400	0.50	1.50	0.530E-01	0.101E-01
4	0.075	0.075	-0.400	-0.400	0.50	1.50	0.485E-01	0.112E-01
5	0.075	-0.075	0.400	0.400	0.50	1.50	0.665E-02	0.145E-02
6	0.075	-0.075	0.400	-0.400	0.50	1.50	0.815E-02	0.223E-02
7	0.075	-0.075	-0.400	0.400	0.50	1.50	0.180E-01	0.650E-02
8	0.075	-0.075	-0.400	-0.400	0.50	1.50	0.161E-01	0.533E-02
9	-0.075	0.075	0.400	0.400	0.50	1.50	0.190E-01	0.518E-02
10	-0.075	0.075	0.400	-0.400	0.50	1.50	0.194E-01	0.337E-02
11	-0.075	0.075	-0.400	0.400	0.50	1.50	0.184E-01	0.516E-02
12	-0.075	0.075	-0.400	-0.400	0.50	1.50	0.151E-01	0.594E-02
13	-0.075	-0.075	0.400	0.400	0.50	1.50	0.122E-01	0.202E-02
14	-0.075	-0.075	0.400	-0.400	0.50	1.50	0.121E-01	0.214E-02
15	-0.075	-0.075	-0.400	0.400	0.50	1.50	0.958E-01	0.603E-01
16	-0.075	-0.075	-0.400	-0.400	0.50	1.50	0.933E-01	0.560E-01
17	0.075	0.075	0.500	0.500	0.50	1.50	0.895E-01	0.321E-01
18	0.075	0.075	0.500	-0.500	0.50	1.50	0.692E-01	0.176E-01
19	0.075	0.075	-0.500	0.500	0.50	1.50	0.854E-01	0.228E-01
20	0.075	0.075	-0.500	-0.500	0.50	1.50	0.748E-01	0.345E-01
21	0.075	-0.075	0.500	0.500	0.50	1.50	0.281E-01	0.164E-01
22	0.075	-0.075	0.500	-0.500	0.50	1.50	0.174E-01	0.602E-02
23	0.075	-0.075	-0.500	0.500	0.50	1.50	0.316E-01	0.490E-02
24	0.075	-0.075	-0.500	-0.500	0.50	1.50	0.330E-01	0.607E-02
25	-0.075	0.075	0.500	0.500	0.50	1.50	0.915E-01	0.301E-01
26	-0.075	0.075	0.500	-0.500	0.50	1.50	0.653E-01	0.150E-01
27	-0.075	0.075	-0.500	0.500	0.50	1.50	0.201E-01	0.775E-02
28	-0.075	0.075	-0.500	-0.500	0.50	1.50	0.214E-01	0.144E-01
29	-0.075	-0.075	0.500	0.500	0.50	1.50	0.364E-01	0.191E-01
30	-0.075	-0.075	0.500	-0.500	0.50	1.50	0.229E-01	0.768E-02
31	-0.075	-0.075	-0.500	0.500	0.50	1.50	0.107	0.346E-01
32	-0.075	-0.075	-0.500	-0.500	0.50	1.50	0.112	0.272E-01
33	0.075	0.075	0.600	0.600	0.50	1.50	0.555	0.168
34	0.075	0.075	0.600	-0.600	0.50	1.50	0.335	0.796E-01
35	0.075	0.075	-0.600	0.600	0.50	1.50	0.106	0.620E-01
36	0.075	0.075	-0.600	-0.600	0.50	1.50	0.102	0.885E-01
37	0.075	-0.075	0.600	0.600	0.50	1.50	0.771E-01	0.730E-01
38	0.075	-0.075	0.600	-0.600	0.50	1.50	0.453E-01	0.316E-01
39	0.075	-0.075	-0.600	0.600	0.50	1.50	0.708E-01	0.172E-01
40	0.075	-0.075	-0.600	-0.600	0.50	1.50	0.908E-01	0.384E-01
41	-0.075	0.075	0.600	0.600	0.50	1.50	0.595	0.173
42	-0.075	0.075	0.600	-0.600	0.50	1.50	0.351	0.810E-01
43	-0.075	0.075	-0.600	0.600	0.50	1.50	0.394E-01	0.236E-01
44	-0.075	0.075	-0.600	-0.600	0.50	1.50	0.475E-01	0.438E-01
45	-0.075	-0.075	0.600	0.600	0.50	1.50	0.829E-01	0.788E-01
46	-0.075	-0.075	0.600	-0.600	0.50	1.50	0.483E-01	0.346E-01

Table A.14 Degree 2 vs. Degree 3: Constant input frequencies



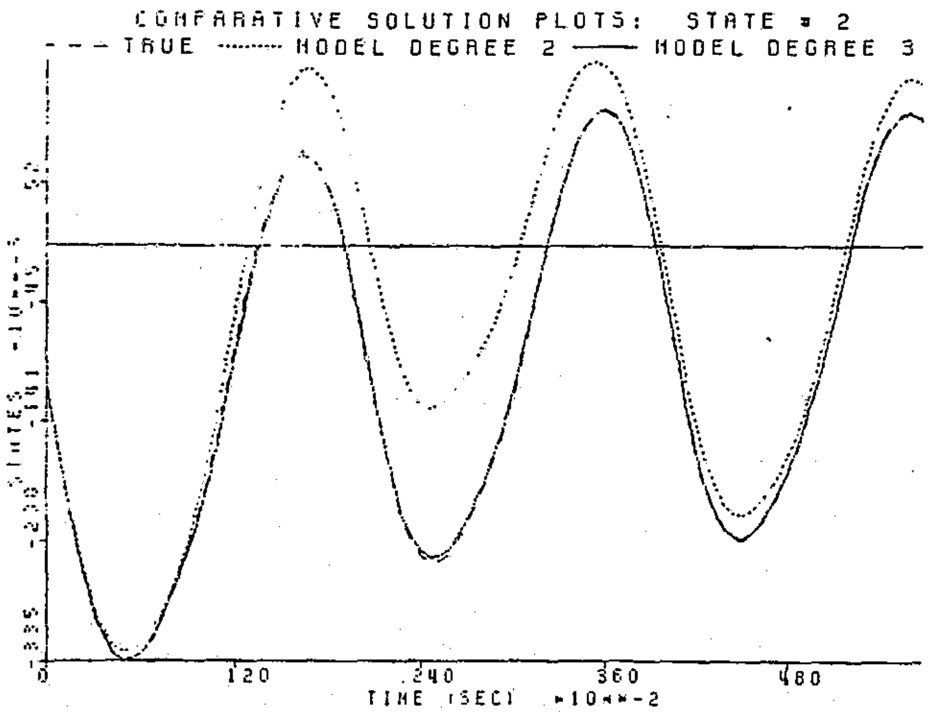
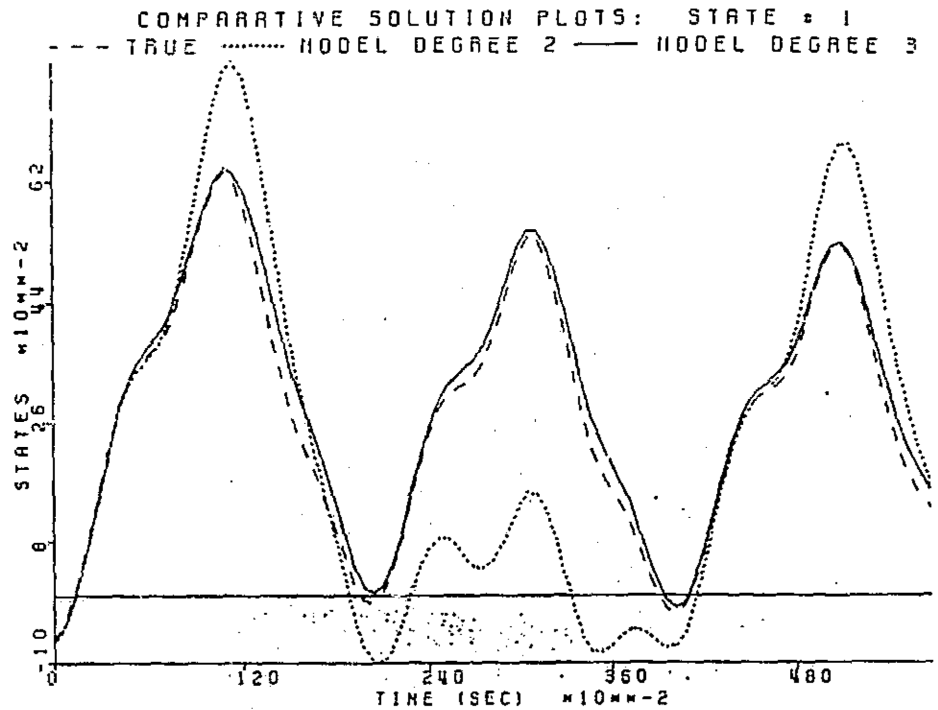


Figure A.24 S#14, Table A.14



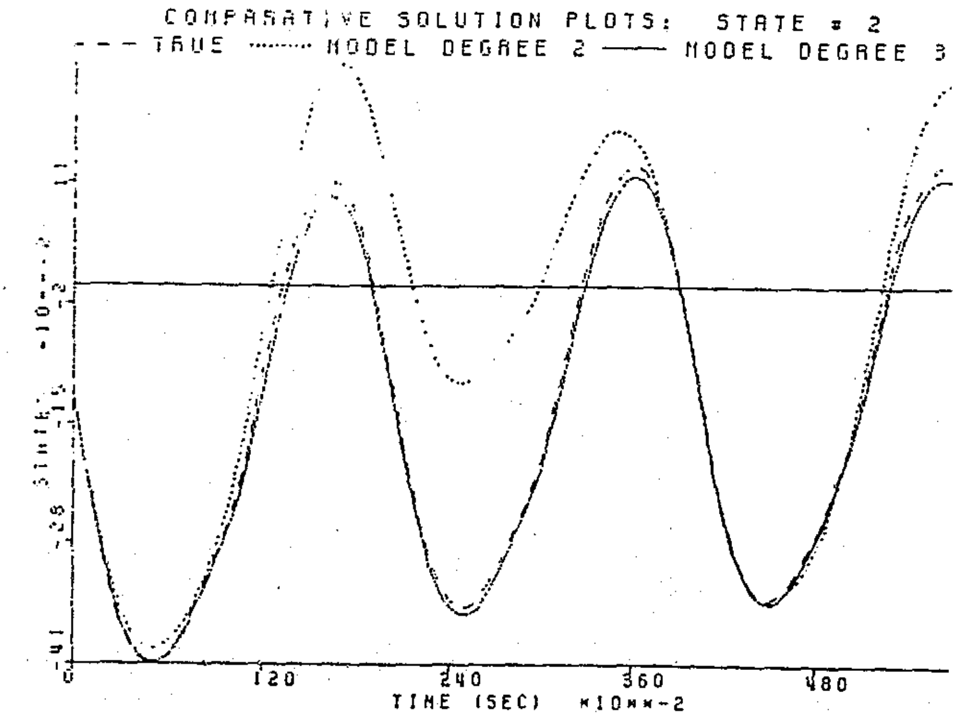
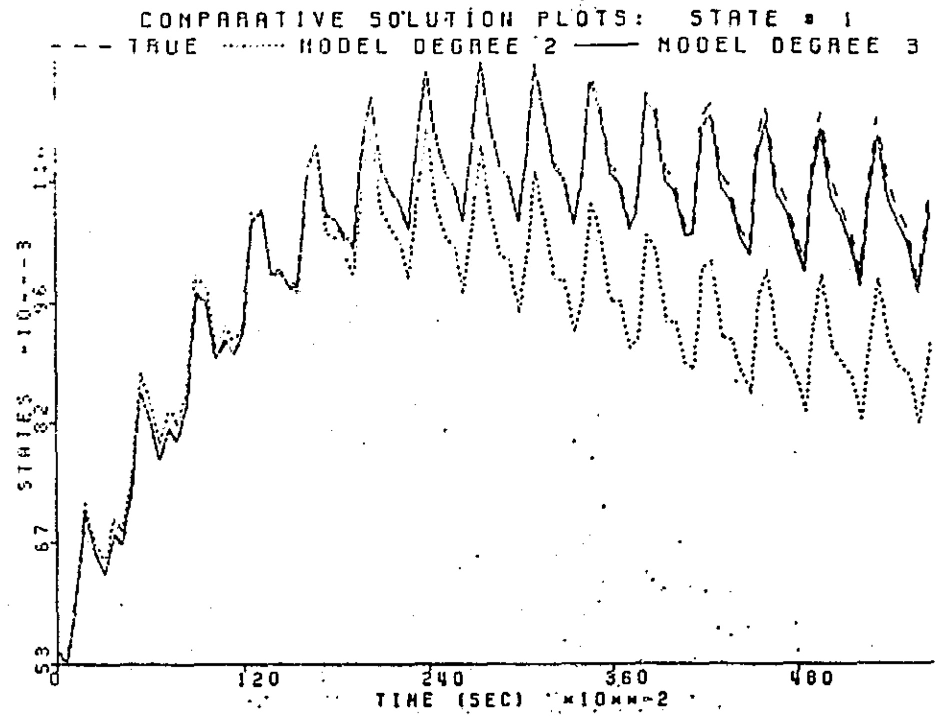


Figure A.25 S#30, Table A.14

```
****************************************
            PROBLEM SUMMARY
  CONFIGURATION:  TRUE,MODEL1,MODEL2
           # OF STATES: 2
           # OF INPUTS: 2
      # OF TERMS IN MODEL 1:  14
     DEGREE OF APPROXIMATION: 2
      # OF TERMS IN MODEL 2:  34
     DEGREE OF APPROXIMATION: 3
       SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.050	-0.050	0.017	0.277	2.00	3.05	0.558	1.02
2	0.050	-0.050	-0.087	0.063	4.04	5.67	0.110E-02	0.521E-03
3	0.050	-0.050	0.204	-0.277	2.26	3.75	0.124E-01	0.154E-01
4	0.050	-0.050	-0.133	-0.138	2.38	3.43	0.456E-02	0.363E-02
5	0.050	-0.050	-0.078	0.089	3.43	5.48	0.196E-02	0.188E-02
6	0.050	-0.050	0.225	0.215	5.07	2.16	0.103E-01	0.145E-01
7	0.050	-0.050	-0.076	0.004	4.65	2.28	0.696E-03	0.200E-03
8	0.050	-0.050	0.209	-0.246	2.43	2.24	0.225E-01	0.187E-01
9	0.050	-0.050	-0.192	0.261	3.37	2.06	0.126E-01	0.149E-01
10	0.050	-0.050	-0.271	0.201	5.07	2.53	0.146E-01	0.251E-01
11	0.050	-0.050	0.138	-0.091	5.34	3.49	0.357E-02	0.304E-02
12	0.050	-0.050	0.249	0.134	2.71	5.44	0.636E-02	0.107E-01
13	0.050	-0.050	-0.234	-0.139	4.77	2.22	0.806E-02	0.117E-01
14	0.050	-0.050	-0.088	0.106	2.74	3.23	0.267E-02	0.204E-02
15	0.050	-0.050	0.010	-0.145	5.33	3.61	0.122	0.291
16	0.050	-0.050	0.136	0.162	5.28	4.10	0.619E-02	0.486E-02
17	0.050	-0.050	0.274	0.050	4.10	2.38	0.928E-02	0.172E-01
18	0.050	-0.050	-0.132	-0.138	3.89	4.63	0.487E-02	0.385E-02
19	0.050	-0.050	-0.200	0.079	3.06	4.05	0.515E-02	0.592E-02
20	0.050	-0.050	0.198	-0.012	5.93	4.92	0.490E-02	0.553E-02
21	0.050	-0.050	0.235	0.129	3.98	5.86	0.773E-02	0.112E-01
22	0.050	-0.050	0.023	0.130	3.78	4.03	0.470E-01	0.140
23	0.050	-0.050	0.220	0.153	3.21	2.58	0.751E-02	0.984E-02
24	0.050	-0.050	-0.002	-0.235	3.05	5.08	0.457	0.885
25	0.050	-0.050	-0.073	-0.004	4.18	5.49	0.740E-03	0.185E-03
26	0.050	-0.050	-0.052	0.239	4.13	5.82	0.143	0.282
27	0.050	-0.050	-0.075	-0.018	3.90	4.20	0.748E-03	0.194E-03
28	0.050	-0.050	-0.269	-0.150	4.87	3.33	0.105E-01	0.189E-01
29	0.050	-0.050	-0.218	-0.294	5.89	5.18	0.144E-01	0.200E-01
30	0.050	-0.050	0.131	0.085	5.27	2.48	0.325E-02	0.265E-02
31	0.050	-0.050	0.099	0.233	5.24	2.04	0.246E-01	0.249E-01
32	0.050	-0.050	-0.155	-0.294	2.13	4.55	0.155E-01	0.141E-01
33	0.050	-0.050	0.027	0.281	5.89	5.95	1.04	2.42
34	0.050	-0.050	0.181	-0.117	4.13	2.73	0.554E-02	0.580E-02
35	0.050	-0.050	0.173	-0.045	5.58	2.59	0.412E-02	0.410E-02
36	0.050	-0.050	0.099	-0.287	2.01	5.07	0.465E-01	0.537E-01
37	0.050	-0.050	-0.250	-0.040	4.22	5.31	0.758E-02	0.119E-01
38	0.050	-0.050	-0.137	-0.124	2.24	5.72	0.390E-02	0.324E-02
39	0.050	-0.050	-0.025	-0.203	5.53	2.40	0.235	0.477
40	0.050	-0.050	0.106	-0.007	4.09	4.16	0.117E-02	0.101E-02
41	0.050	-0.050	-0.201	0.228	4.62	3.00	0.104E-01	0.129E-01
42	0.050	-0.050	0.085	0.163	2.13	3.51	0.115E-01	0.144E-01
43	0.050	-0.050	0.023	0.013	4.95	3.04	0.134E-01	0.120E-01
44	0.050	-0.050	0.237	-0.286	3.89	3.80	0.191E-01	0.284E-01
45	0.050	-0.050	-0.027	0.092	6.00	5.11	0.230E-01	0.573E-01
46	0.050	-0.050	0.100	-0.207	3.87	4.45	0.163E-01	0.167E-01
47	0.050	-0.050	0.263	0.168	4.81	2.11	0.106E-01	0.183E-01
48	0.050	-0.050	0.280	0.070	2.73	4.77	0.923E-02	0.175E-01
49	0.050	-0.050	-0.081	-0.161	5.30	4.73	0.113E-01	0.150E-01
50	0.050	-0.050	0.106	0.151	4.15	5.32	0.587E-02	0.450E-02

Table A.15 Degree 2 vs. Degree 3: Constant initial conditions with randomly generated amplitudes and frequencies



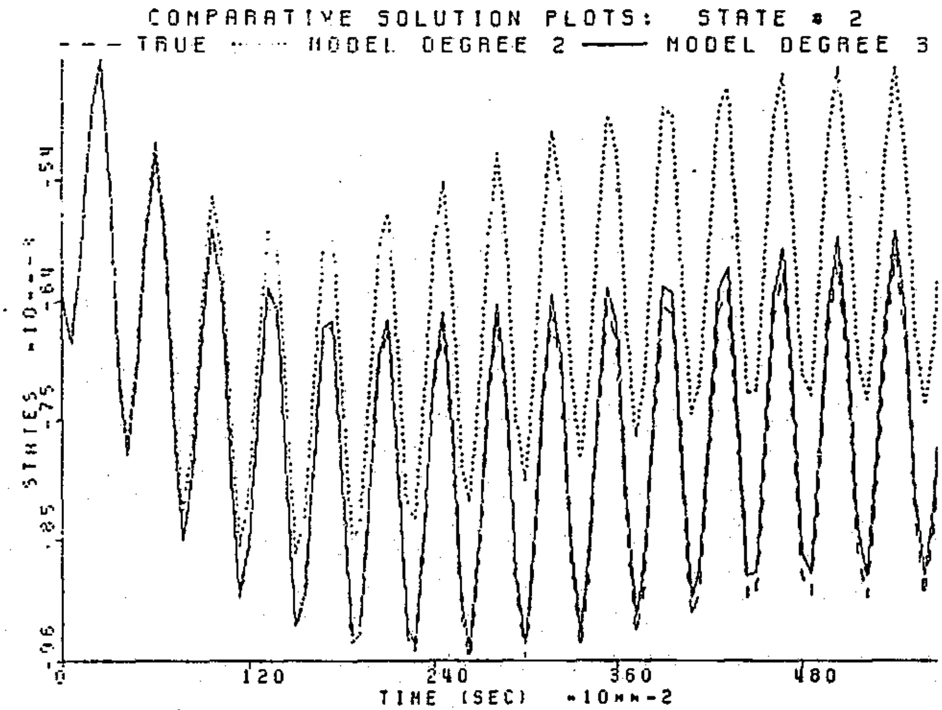
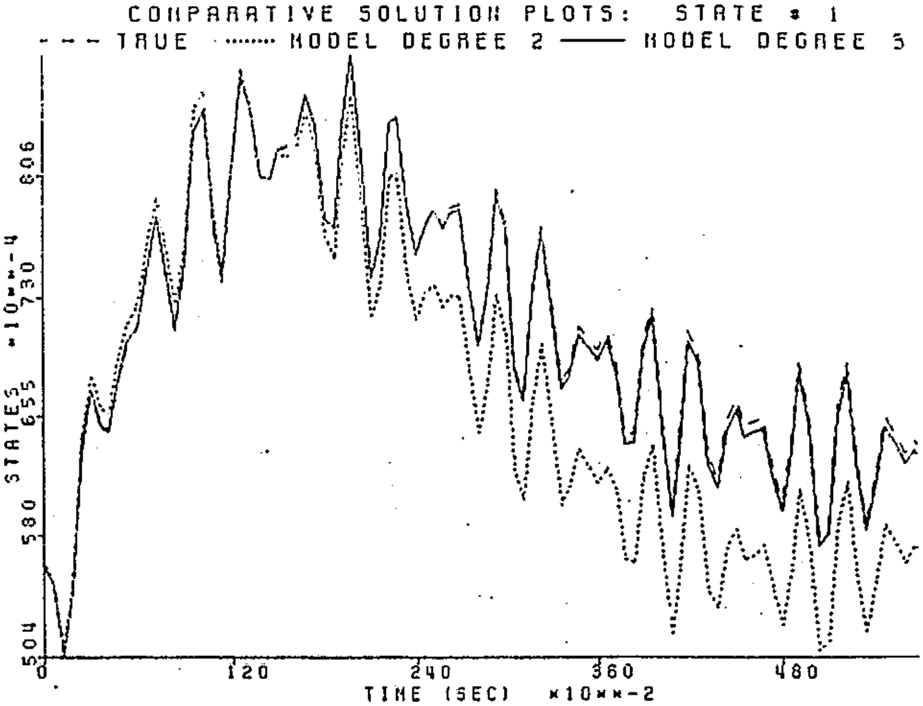


Figure A.26 S#12, Table A.15



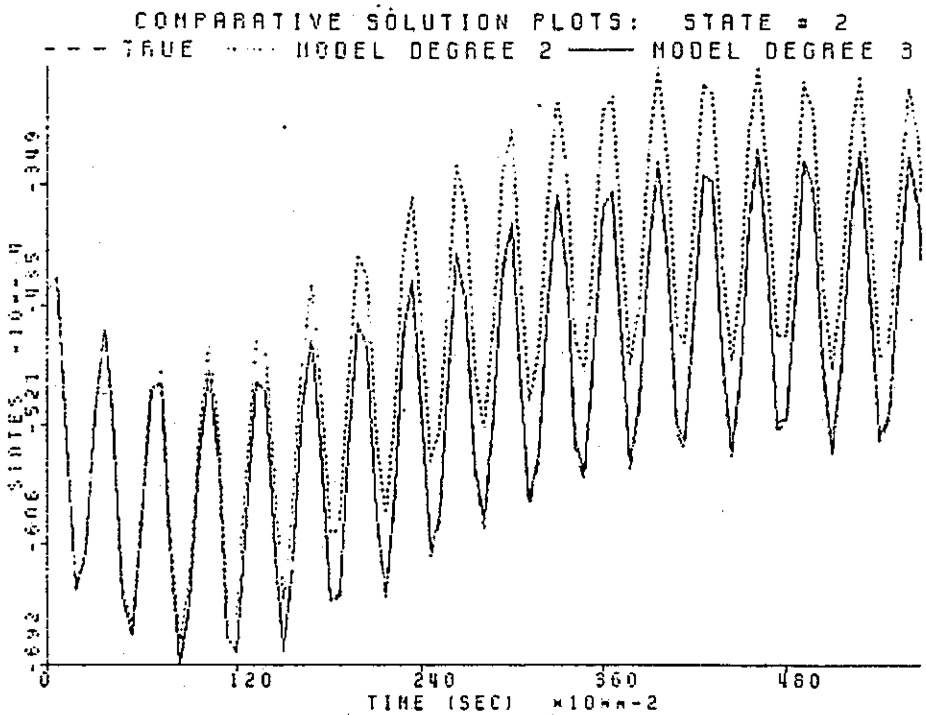
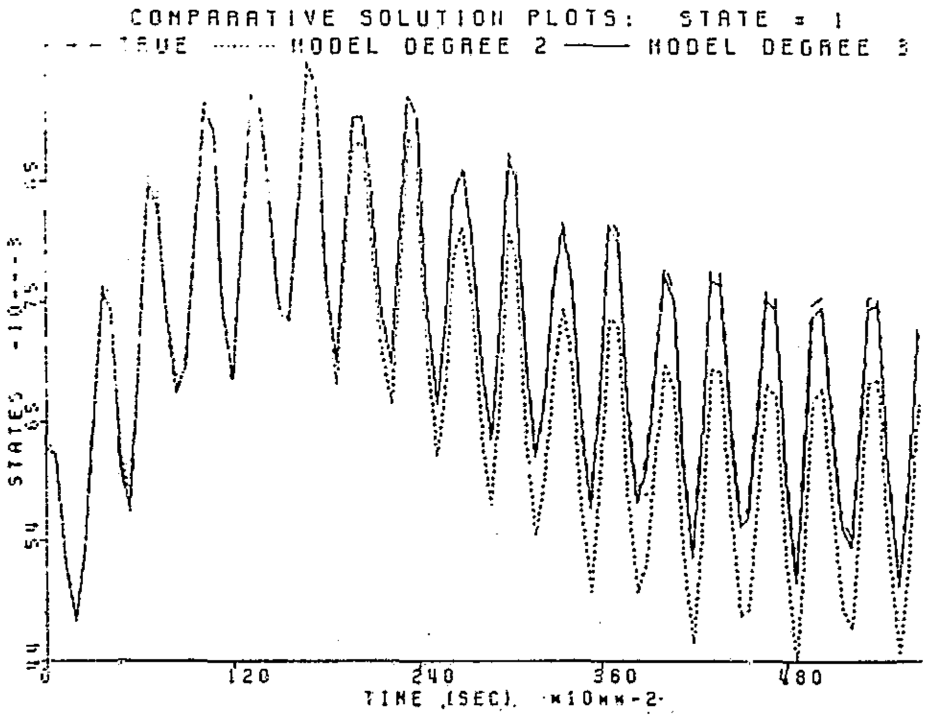


Figure A.27 S#19, Table A.15

ORIGINAL PAGE IS
OF POOR QUALITY



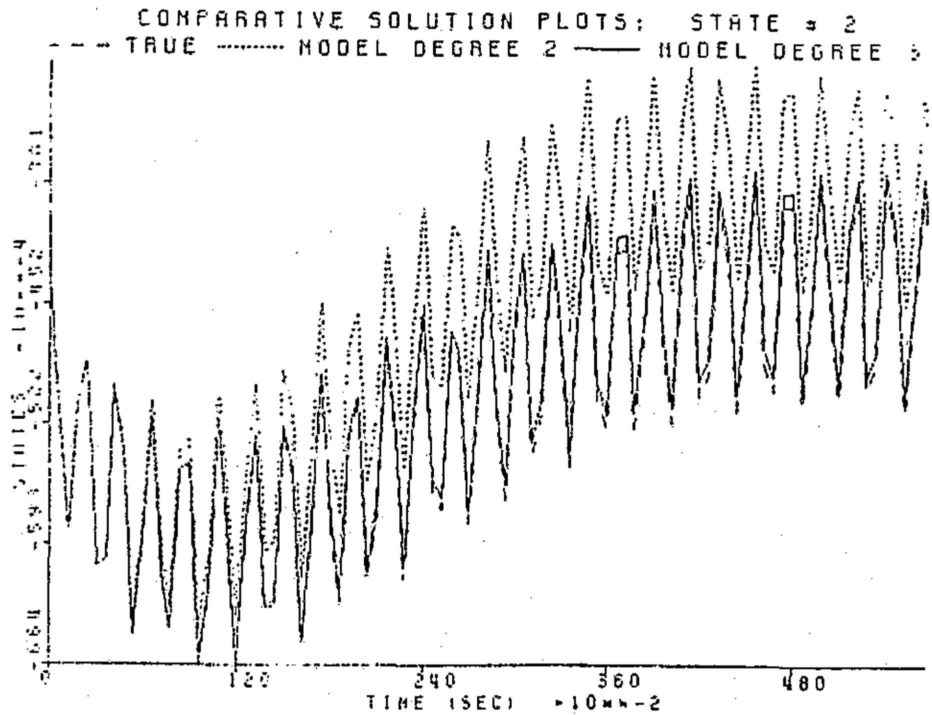
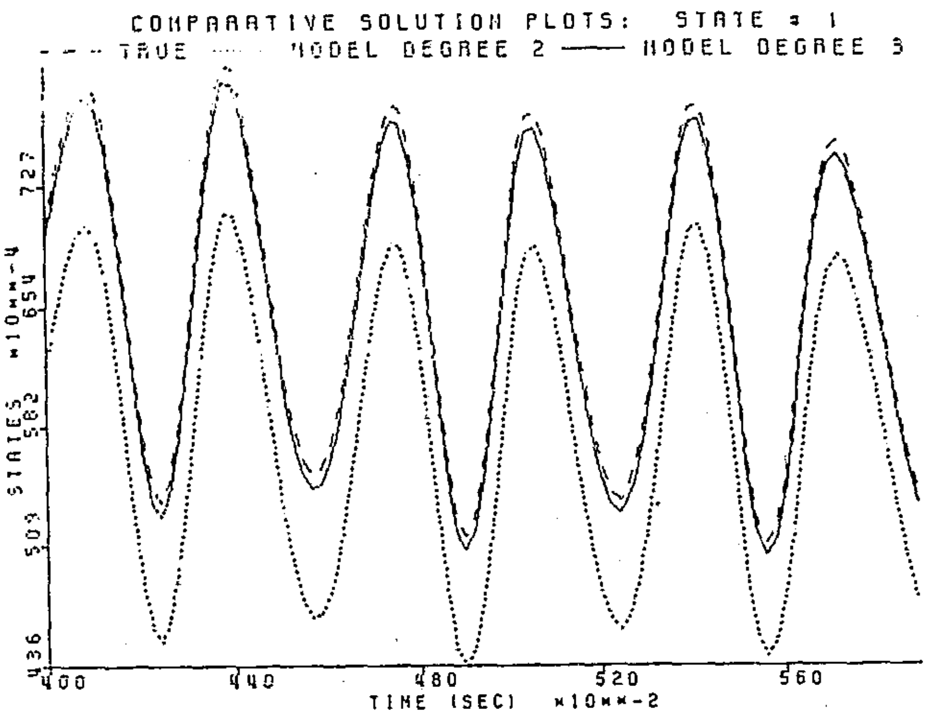


Figure A.28 S#41, Table A.15



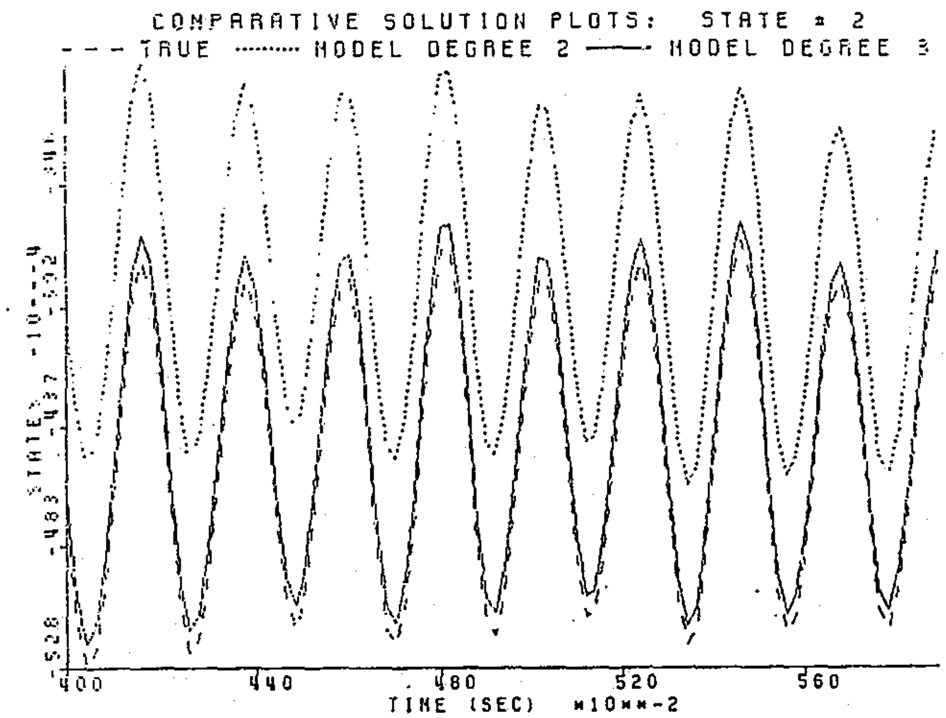
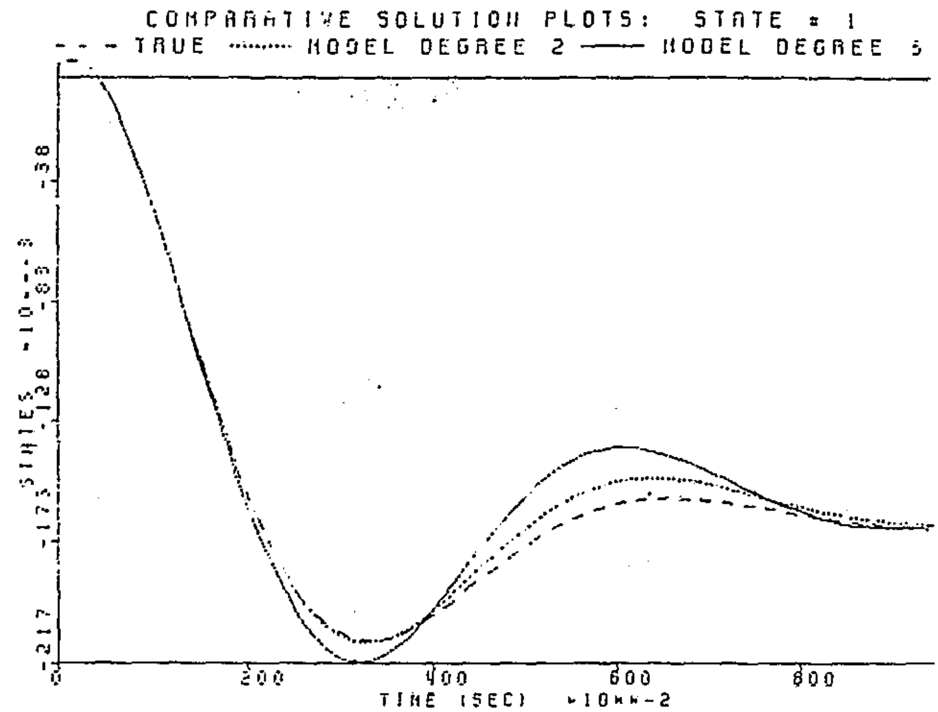


Figure A.29 S#41, Table A.15: Expanded view of Figure A.28

```
*****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.214	0.301
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.224	0.285
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.234	0.238
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.200	0.367
5	0.001	-0.001	-0.050	0.050	0.00	0.00	6.11	7.07
6	0.001	-0.001	-0.050	0.050	0.01	0.01	6.85	7.74
7	0.001	-0.001	-0.050	0.050	0.02	0.02	10.3	10.3
8	0.001	-0.001	-0.050	0.050	0.05	0.05	24.0	7.80
9	0.001	-0.001	-0.075	0.010	0.00	0.00	3.59	3.37
10	0.001	-0.001	-0.075	0.010	0.01	0.01	3.87	3.67
11	0.001	-0.001	-0.075	0.010	0.02	0.02	5.10	4.68
12	0.001	-0.001	-0.075	0.010	0.05	0.05	10.5	3.07
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	2.19	1.58
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	2.28	1.76
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	2.68	2.24
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	4.12	1.74
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.207	0.297
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.217	0.281
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.228	0.235
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.199	0.365
21	0.010	-0.010	-0.050	0.050	0.00	0.00	6.26	7.08
22	0.010	-0.010	-0.050	0.050	0.01	0.01	6.97	7.73
23	0.010	-0.010	-0.050	0.050	0.02	0.02	10.4	10.3
24	0.010	-0.010	-0.050	0.050	0.05	0.05	23.6	7.88
25	0.010	-0.010	-0.075	0.010	0.00	0.00	3.71	3.42
26	0.010	-0.010	-0.075	0.010	0.01	0.01	3.98	3.72
27	0.010	-0.010	-0.075	0.010	0.02	0.02	5.20	4.75
28	0.010	-0.010	-0.075	0.010	0.05	0.05	10.8	3.13
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	2.23	1.59
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	2.32	1.77
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	2.76	2.30
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	4.37	1.71

Table A.16 Degree 2 vs. Degree 3: Low frequency inputs



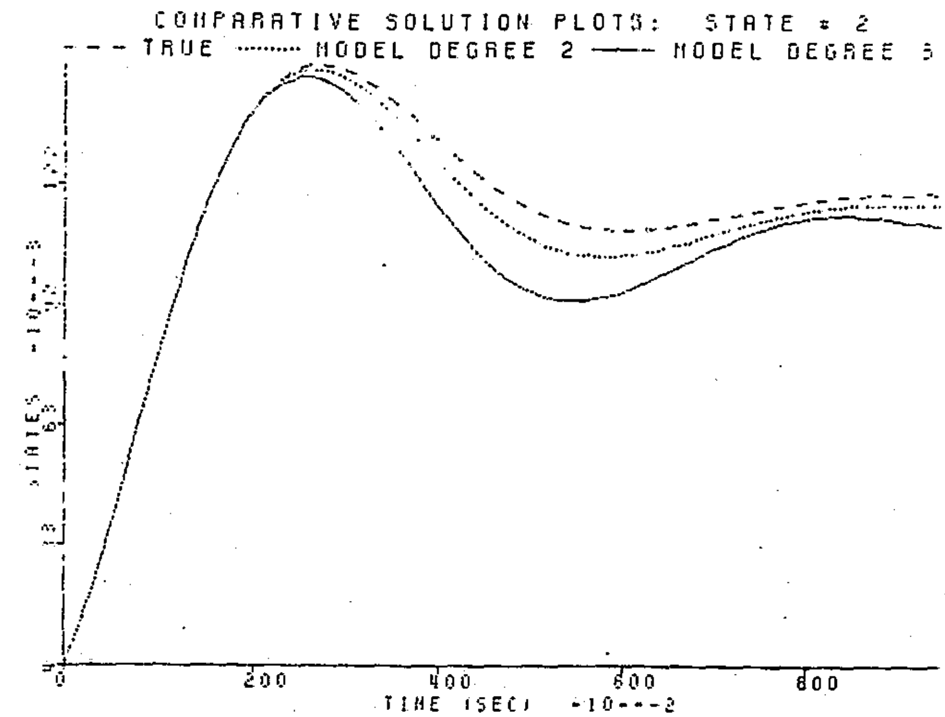


Figure A.30 S#5, Table A.16

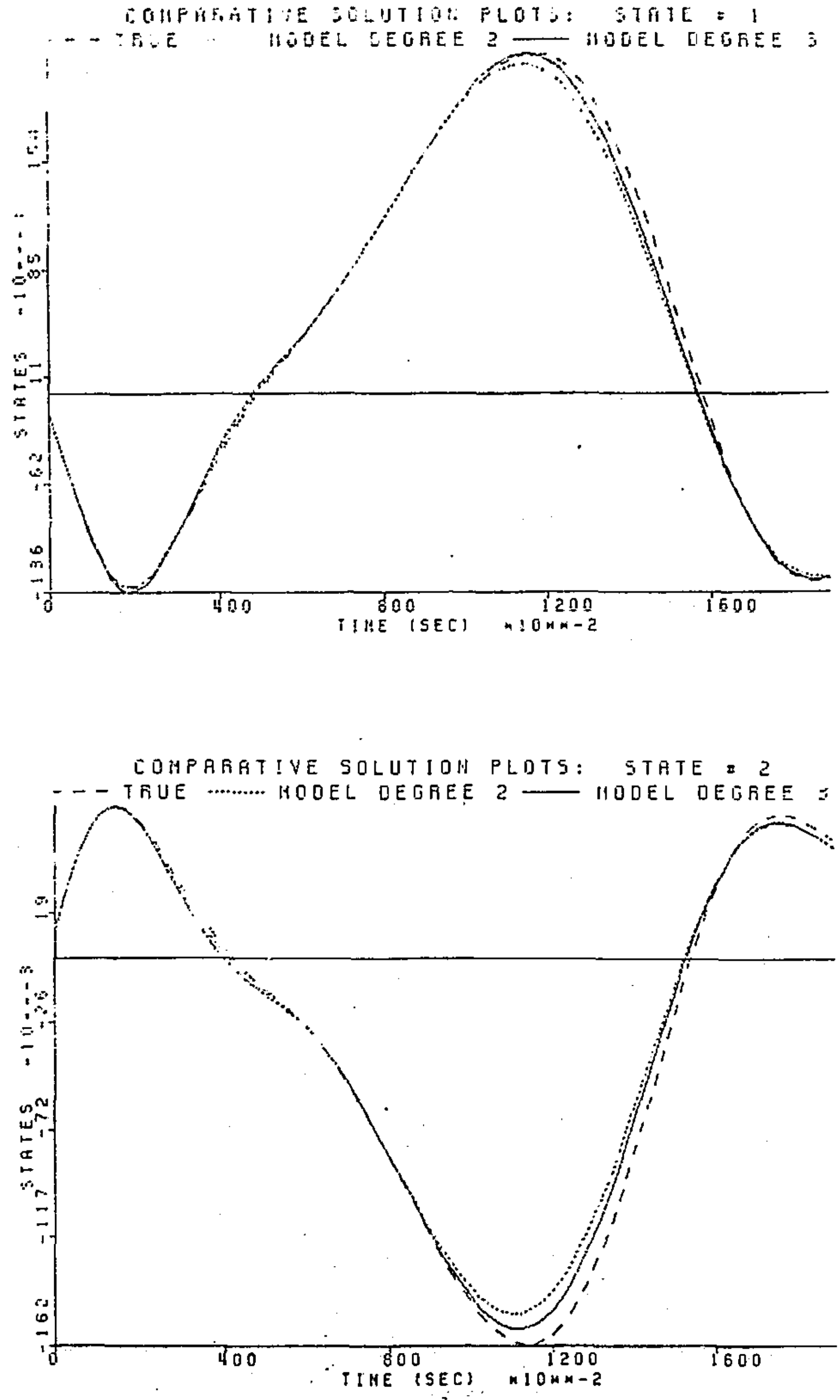
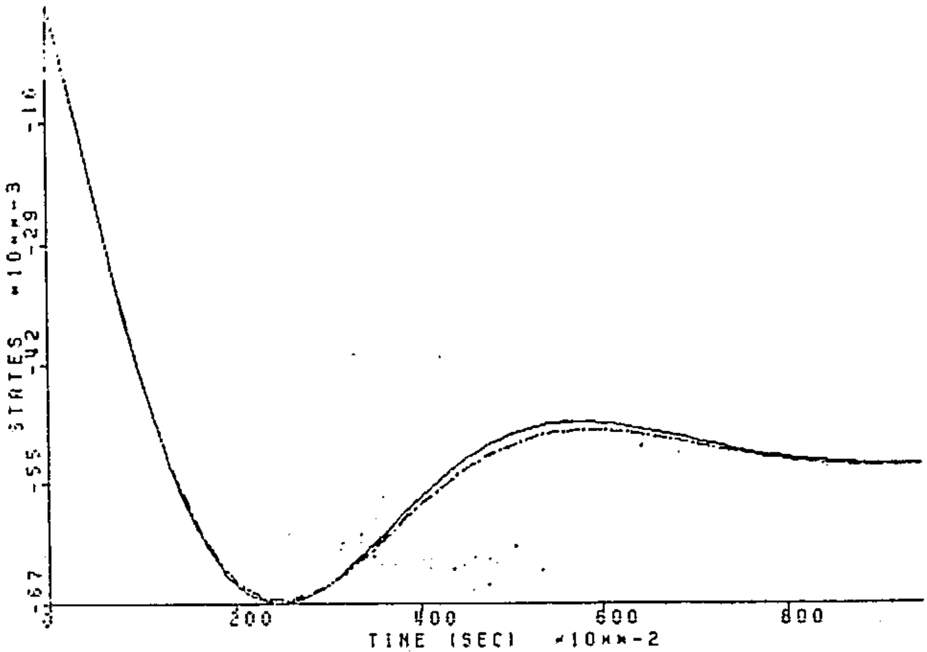


Figure A.31 S#16, Table A.16

```
*****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
*****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.000	0.010	-0.022	-0.008	0.00	0.00	40.8	185.
2	-0.006	0.004	0.008	-0.011	0.00	0.00	0.108	0.462E-01
3	0.009	0.006	-0.010	0.017	0.00	0.00	9.61	16.5
4	0.006	0.010	0.022	-0.010	0.00	0.00	0.557E-01	0.629E-01
5	0.006	-0.007	-0.018	-0.003	0.00	0.00	26.2	84.1
6	-0.004	0.001	-0.021	-0.005	0.00	0.00	44.8	233.
7	0.006	0.003	0.012	0.018	0.00	0.00	0.463E-01	0.171
8	-0.009	-0.003	-0.021	0.024	0.00	0.00	87.7	237.
9	-0.000	0.003	0.004	0.003	0.00	0.00	0.125	0.182
10	-0.000	0.002	-0.009	-0.013	0.00	0.00	1.25	2.18
11	0.009	0.001	0.011	0.011	0.00	0.00	0.406E-01	0.111
12	0.002	-0.008	0.017	0.002	0.00	0.00	0.836E-02	0.212E-02
13	0.005	-0.008	-0.019	-0.010	0.00	0.00	23.7	94.1
14	-0.004	-0.005	0.022	-0.018	0.00	0.00	0.822E-01	0.102
15	-0.007	0.007	-0.001	-0.020	0.00	0.00	4.30	10.1
16	-0.010	0.005	-0.005	-0.006	0.00	0.00	0.371	0.773
17	-0.007	0.006	-0.018	0.009	0.00	0.00	39.4	123.
18	-0.001	-0.004	0.015	0.021	0.00	0.00	0.235	1.07
19	0.008	-0.005	0.007	0.017	0.00	0.00	0.261	0.434
20	0.003	-0.003	-0.016	-0.008	0.00	0.00	9.66	25.0
21	0.004	-0.003	0.019	0.006	0.00	0.00	0.708E-02	0.103E-02
22	-0.003	0.006	-0.010	0.007	0.00	0.00	5.52	10.5
23	0.007	-0.002	0.014	-0.012	0.00	0.00	0.197E-01	0.108E-01
24	-0.003	-0.004	0.010	-0.009	0.00	0.00	0.453E-01	0.146E-01
25	-0.000	-0.009	-0.001	-0.013	0.00	0.00	8.89	8.75
26	-0.009	0.007	0.007	-0.015	0.00	0.00	0.113	0.595E-01
27	-0.002	0.002	-0.006	-0.023	0.00	0.00	5.75	10.0
28	0.009	0.002	-0.012	0.012	0.00	0.00	12.0	22.0
29	-0.006	-0.007	-0.003	0.006	0.00	0.00	1.62	2.66
30	0.004	-0.000	-0.000	-0.009	0.00	0.00	3.27	4.57
31	0.001	-0.007	0.022	0.014	0.00	0.00	0.420E-02	0.274E-02
32	0.008	-0.002	0.004	0.024	0.00	0.00	2.27	3.57
33	0.001	-0.007	-0.023	0.018	0.00	0.00	73.3	168.
34	0.003	-0.005	-0.024	0.024	0.00	0.00	51.6	90.6
35	-0.001	-0.007	0.017	0.012	0.00	0.00	0.480E-01	0.919E-01
36	-0.010	-0.006	0.015	0.014	0.00	0.00	0.255	0.622
37	-0.009	-0.006	-0.004	0.002	0.00	0.00	1.48	2.49
38	0.001	0.008	-0.005	0.020	0.00	0.00	3.69	6.08
39	0.000	-0.001	-0.004	0.025	0.00	0.00	3.81	6.13
40	0.003	-0.009	-0.011	0.009	0.00	0.00	7.25	12.9
41	-0.000	0.004	0.011	-0.006	0.00	0.00	0.741E-01	0.281E-01
42	0.008	-0.005	-0.010	0.010	0.00	0.00	6.26	10.9
43	-0.009	-0.008	0.012	-0.021	0.00	0.00	0.286E-01	0.262E-01
44	0.003	-0.007	-0.021	0.015	0.00	0.00	85.1	295.
45	0.001	0.007	-0.016	0.012	0.00	0.00	32.0	80.8
46	-0.008	0.004	0.005	0.015	0.00	0.00	1.80	3.93
47	-0.001	-0.002	0.015	0.010	0.00	0.00	0.270	0.437
48	-0.007	0.009	0.018	-0.002	0.00	0.00	0.159E-01	0.825E-02
49	0.006	-0.005	0.004	0.008	0.00	0.00	0.515E-01	0.589E-01
50	-0.006	0.003	-0.011	-0.021	0.00	0.00	1.57	2.48

Table A.17 Degree 2 vs. Degree 3: Step inputs with randomly generated initial conditions and input amplitudes



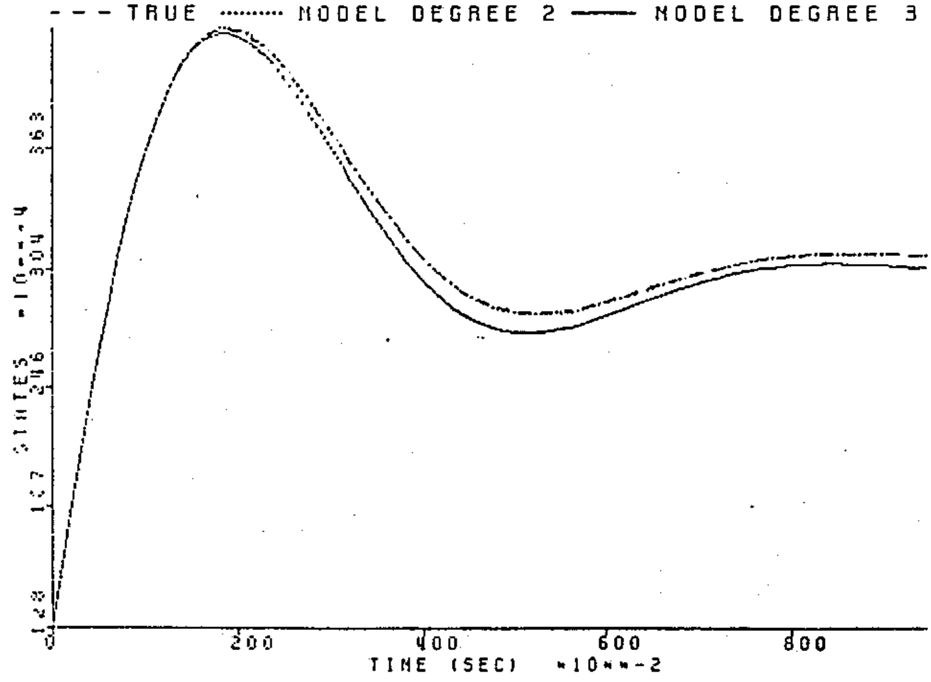


Figure A.32 S#1, Table A.17

A.5 CONCLUSIONS

In this case study we have demonstrated the nonlinear multilinear modeling scheme by way of a two state, two input example system. We began by describing the ways in which the tensor parameterization transforms the nonlinear system into a least squares problem through optimal design of the experimental, measured data. Verification studies were performed with various error analyses to test the region of acceptable performance for the resulting model, usually in reference to a lower degree approximation model. While no strict definition of "region" has been given to this point, it is clear that for this example the range of permissible inputs (that is, those which result in acceptable state trajectories) increases as the degree of the approximation model increases. This concept is of course intuitively appealing and in harmony with the implications of Taylor's theorem. The actual implementation is not, however, a trivial exercise when one considers the various software pieces which make up the overall package.

There remain in general several issues concerning the current limitations of the software and numerical routines involved in this procedure. And there remains in particular, for this example, an issue of some concern for model behavior for low frequency inputs. We address this latter issue first by offering a simple means by which the results may be improved.

Recall that the third degree model of the preceding section represents an identification over a four second timespan; 100 samples were taken, spaced evenly in time at 0.04 second intervals. A look at Figure A.30 indicates that the model solution breaks away from the true solution trajectory in each state early in the transient portion of the curve, around the three to four second mark. With this in mind, a re-identification is attempted, this time for data sampled at 0.06 intervals so that a six second timespan is utilized in an effort to capture more transient information. The final values for the input frequencies from the previous optimization (0.509 and 2.01 hertz) are used now as startup values, everything else being the same. After 60 evaluations of the cost function the final values for the frequencies are 0.351 and 1.33 hertz, the cost function undergoing a 57% reduction. We offer one final table, Table A.18, to illustrate the results for this re-identification, a repeat of Table A.16. For purposes of comparison the identical procedure described above for re-optimization, re-identification is carried out on the second degree model of Section A.3. Both models show modest improvement over their predecessors[2], but, as evidenced by Table A.18 compared now against Table A.16, we have improved the low frequency performance of the third degree model a considerable amount. Figure A.33 displays the results of a simulation parameter set which may be compared with Figure A.30. We remind the reader that this simulation represents one of the worst case parameter sets from the table; in the majority of the plots for this table all three curves practically lie atop one another.

As alluded to earlier, the versatility of the software is not currently being exercised. For instance, all of the optimizations for the design of the input are done over the input frequencies, basically because the software package is limited to unconstrained optimizations. Attempting unconstrained optimizations over input amplitudes often results in overly large amplitudes in one or both input channels. Currently, the choice of amplitudes for the identifications is arrived at after a few trial identifications. One method for determining how large these amplitudes should be is to observe the re-

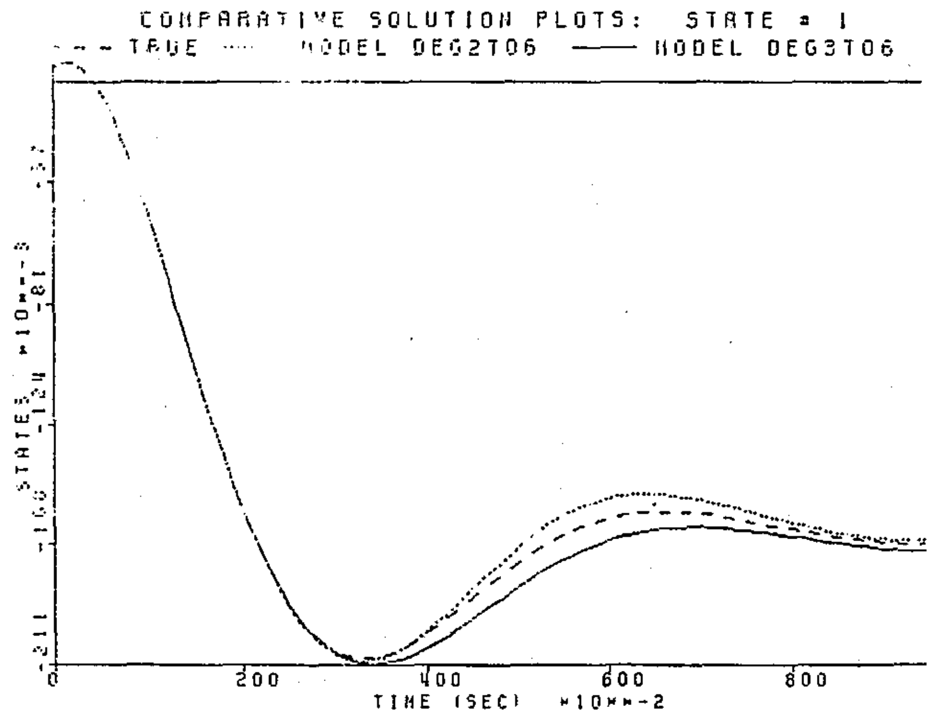
[2]Resimulation of several tables from Section A.4 illustrate this fact.

```
****************************************
PROBLEM SUMMARY
CONFIGURATION:  TRUE,MODEL1,MODEL2
# OF STATES: 2
# OF INPUTS: 2
# OF TERMS IN MODEL 1:  14
DEGREE OF APPROXIMATION: 2
# OF TERMS IN MODEL 2:  34
DEGREE OF APPROXIMATION: 3
SIMULATION WITH COSINE
****************************************
```

S#	INITIAL CONDITIONS		AMPLITUDES		FREQUENCIES		ERROR RATIOS	
1	0.001	-0.001	0.010	-0.075	0.00	0.00	0.757E-02	0.498E-02
2	0.001	-0.001	0.010	-0.075	0.01	0.01	0.729E-02	0.560E-02
3	0.001	-0.001	0.010	-0.075	0.02	0.02	0.877E-02	0.897E-02
4	0.001	-0.001	0.010	-0.075	0.05	0.05	0.177E-01	0.197E-01
5	0.001	-0.001	-0.050	0.050	0.00	0.00	1.69	1.34
6	0.001	-0.001	-0.050	0.050	0.01	0.01	1.77	1.42
7	0.001	-0.001	-0.050	0.050	0.02	0.02	2.15	1.76
8	0.001	-0.001	-0.050	0.050	0.05	0.05	4.21	1.03
9	0.001	-0.001	-0.075	0.010	0.00	0.00	0.632	0.379
10	0.001	-0.001	-0.075	0.010	0.01	0.01	0.650	0.415
11	0.001	-0.001	-0.075	0.010	0.02	0.02	0.753	0.536
12	0.001	-0.001	-0.075	0.010	0.05	0.05	1.39	0.331
13	0.001	-0.001	-0.075	-0.075	0.00	0.00	0.379	0.281
14	0.001	-0.001	-0.075	-0.075	0.01	0.01	0.368	0.277
15	0.001	-0.001	-0.075	-0.075	0.02	0.02	0.299	0.226
16	0.001	-0.001	-0.075	-0.075	0.05	0.05	0.154	0.300E-01
17	0.010	-0.010	0.010	-0.075	0.00	0.00	0.745E-02	0.501E-02
18	0.010	-0.010	0.010	-0.075	0.01	0.01	0.724E-02	0.569E-02
19	0.010	-0.010	0.010	-0.075	0.02	0.02	0.919E-02	0.949E-02
20	0.010	-0.010	0.010	-0.075	0.05	0.05	0.192E-01	0.211E-01
21	0.010	-0.010	-0.050	0.050	0.00	0.00	1.66	1.33
22	0.010	-0.010	-0.050	0.050	0.01	0.01	1.73	1.41
23	0.010	-0.010	-0.050	0.050	0.02	0.02	2.10	1.74
24	0.010	-0.010	-0.050	0.050	0.05	0.05	4.03	1.03
25	0.010	-0.010	-0.075	0.010	0.00	0.00	0.633	0.388
26	0.010	-0.010	-0.075	0.010	0.01	0.01	0.651	0.423
27	0.010	-0.010	-0.075	0.010	0.02	0.02	0.756	0.547
28	0.010	-0.010	-0.075	0.010	0.05	0.05	1.41	0.339
29	0.010	-0.010	-0.075	-0.075	0.00	0.00	0.383	0.283
30	0.010	-0.010	-0.075	-0.075	0.01	0.01	0.370	0.278
31	0.010	-0.010	-0.075	-0.075	0.02	0.02	0.298	0.223
32	0.010	-0.010	-0.075	-0.075	0.05	0.05	0.153	0.257E-01

Table A.18 A repeat of Table A.16 for improved models: Degree 2 vs. Degree 3

ORIGINAL PAGE IS
OF POOR QUALITY



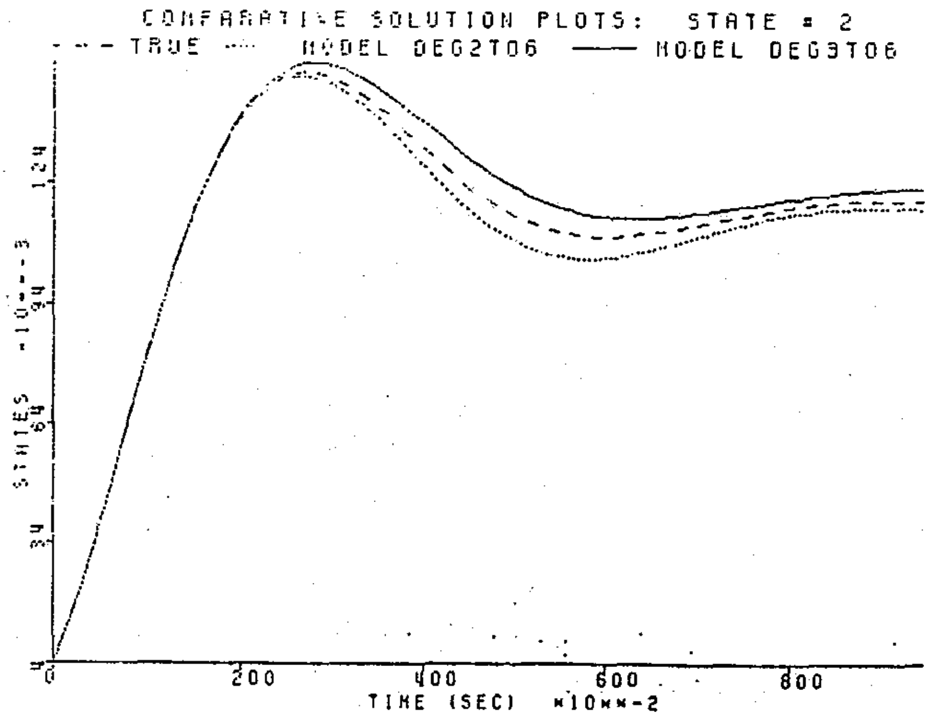


Figure A.33 S#5, Table A.18: Compare to Figure A.30

sulting parameters in the matrices L_{10} and L_{01}; if these parameters stray too far from the values found for a linear model, local performance will deteriorate.

Another limiting factor of the software currently being used is inherent to the IMSL optimization routine. This quasi-Newton method for finding the optimum parameters in the nonlinear programming problem usually results in local optimum values. Therefore, the choice of startup values for the identification has some influence on the final answer. In the example of this case study, however, the choice of startup frequencies is governed by two factors: 1) knowledge of the speed of response of the system being sampled, and 2) sampling parameters in the identification. We make note of the fact that choice of initial conditions for the identification in this example has a relatively small effect on the result.

We close this case study by commenting on the results of the models discussed in Sections A.3 and A.4 in terms of the so-called region of validity. For simplicity, let us define a "region" in the u_1-u_2 plane as a square whose area is given as the square of some input amplitude. Thus, we speak of regions in terms of maximum input amplitudes.

At low frequencies (that is, up to about 0.1 hertz) both models have roughly the same region of validity, since both perform about the same for input amplitudes out to about 0.075 in magnitude. For higher frequencies, however, we can give some idea for the difference in size of the region in the u_1-u_2 plane, in terms of amplitudes, in which each model has acceptable behavior. The performance of the second degree model begins to deteriorate for amplitudes out to about 0.2 in magnitude; recall, for example, Figure A.7. The performance of the third degree model is still very good in this region;

compare Figure A.16 to Figure A.4 for amplitudes of magnitude 0.15. In fact, as the plots and simulations from Tables A.14 and A.15 reveal, the third degree model performs well for regions out to amplitude magnitudes of 0.4 and greater. Simply put, then, the third degree model has a region of validity which is at least four times larger than that of the second degree model. A similar measure could be made for the second degree model in terms of the linear approximation. Future software considerations involve a way of graphically displaying these regions as a function of the mean square errors and the error normalization factors. This of course would require a strict definition of when a model's performance is "acceptable".

REFERENCES

1. G.F. Franklin and J.B. Powell, Digital Control of Dynamic Systems. Reading, Massachusetts: Addison-Wesley, 1980.

2. S. Yurkovich, "Application of Tensor Ideas to Nonlinear Modeling and Control", M.S. Thesis, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, January 1981.

3. T.A. Klingler, "Nonlinear Modeling of a Turbofan Jet Engine: A Tensor Method Application", M.S. Thesis, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, August 1982.

4. W.H. Greub, Multilinear Algebra. New York: Springer-Verlag, 1967.

5. T.A. Klingler, S. Yurkovich, and M.K. Sain, "An Application of Tensor Ideas to Nonlinear Modeling of a Turbofan Jet Engine", Proceedings Thirteenth Pittsburgh Conference on Modeling and Simulation, April 1982.

6. S. Cohen and S. Pieper, "The Speakeasy III Reference Manual", Speakeasy Computing Corporation, Chicago, Illinois, 1979.

7. S. Yurkovich, "Multilinear Modeling and Linear Analysis of Nonlinear Systems", Ph.D Dissertation, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, May 1984.

8. G.C. Goodwin and R.L. Payne, Dynamic System Identification: Experiment Design and Data Analysis. New York: Academic Press, 1977.

9. R. Bellman, Introduction to Matrix Analysis. New York: McGraw-Hill, 1960.

10. C.L. Lawson and R.J. Hanson, Solving Least Squares Problems. Englewood Cliffs, N.J.: Prentice-Hall, 1974.

11. A.J. Laub, "Linear Multivariable Control. Numerical Considerations", Presented at the American Mathematical Society Short Course on Control Theory, Providence, R.I., August 1978.

12. V.C. Klema and A.J. Laub, "The Singular Value Decomposition: Its Computation and Some Applications", IEEE Transactions on Automatic Control, Vol. AC-25, No. 2, April 1980.

13. A.A. Desrochers, "Optimal Model Reduction for Nonlinear Systems", Proceedings of the Joint Automatic Control Conference, June 1980.

14. L.H. McWilliams, "Nonlinear Models by Layered Tensors", M.S. Thesis, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, May 1984.

15. "IBM 370/3033 Nonlinear Modelling/Simulation Software", S. Yurkovich, D.J. Bugajski, and T.A. Klingler, Control Systems Technical Note #7, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, March 1984.

16. "DEC PDP 11/44 Nonlinear Model Simulation Software", D.J. Bugajski, S. Yurkovich, and T.A. Klingler, Control Systems Technical Note #8, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN, March 1984.