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A STUDY OF INTERNAL AND DISTRIBUTED DAMPING FOR VIBRATING TURBOMACHINER BLADES

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The principal purpose of the research was to study internal and distributed damping as possible methods for reducing the vibration response of turbomachinary blades, and to develop theoretical methods for analyzing the damped vibration. Efforts were focussed on continuous, rather than localized, damping. Thus, the widely used damping mechanisms such as rubbing between shrouds, platforms, or in blade roots, which already have received wide-spread attention, were not considered.

This work, which was terminated after the first third of what was to be a more comprehensive study, was limited to the forced vibration analysis of blade-like models subjected to viscous and material (i.e., hysteretic) damping. Both distributed and concentrated exciting forces were considered. It was necessary to make theoretical extensions of the well-known Ritz-Galerkin analytical methods to deal with the out-of-phase response of damped systems. The developed procedure was successfully applied to cantilever beam, plate and shallow shell representation of blades, and is particularly useful for making parametric studies of the effects of damping upon blade vibration response.

The theoretical development and application is summarized in two publications. One is a paper which was presented at the Vibration Damping Workshop in Long Beach, California, February 27-29, 1984 ("Extensions of the Ritz-Galerkin Method for the Forced, Damped Vibrations of Structural Elements," Flight Dynamics

Laboratory Report AFWAL-TR-84-3064, pp. EE-1 to EE-22). Another is the Ph.D. dissertation of Mr. T. H. Young (Ohio State University, Department of Engineering Mechanics) which is currently being completed.

A literature search was made to uncover previously published work dealing with the forced vibration responses of continuous systems with damping. Approximately 150 relevant references were found. A review article summarizing the majority of these references was begun, but was dropped when project funding was discontinued.

EXTENSIONS OF THE RITZ-GALERKIN METHOD FOR THE FORCED, DAMPED VIBRATIONS OF STRUCTURAL ELEMENTS *

of

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ABSTRACT

The classical method for analyzing the forced vibrations of structural elements such as beams, plates and shells is to express the displacements as superpositions of the responses of the free vibration modes. This is only possible for those relatively few problems where exact eigenfunction solutions exist, and often only with considerable difficulty. Ritz-Galerkin methods are widely used to obtain approximate solutions for free undamped, vibration problems. The present paper demonstrates how these same methods may be used straightforwardly to analyze forced vibrations with damping. This is done directly without requiring the free vibration eigenfunctions.

The Galerkin method has been shown previously to be an effective technique for these types of problems. The Ritz method has the further advantage of not needing to satisfy the force-type boundary conditions, which is particularly important for plates and shells. But proper functionals representing the forcing and damping terms must be developed, and this is done in the present work.

The present paper considers two types of damping--viscous and material (hysteretic). Both distributed and concentrated exciting forces are treated. Numerical results are obtained for cantilevered beams and rectangular plates. Studies showing the rates of convergence of the solutions are made. In the case of the cantilever beam, approximate solutions from the present methods are compared with the exact solutions.

* Proceedings of the Vibration Damping Workshop", Long Beach, California, February 27-29, 1984.

I. INTRODUCTION

The Rayleigh-Ritz methods are probably the best known and most widely used methods for analyzing the free, undamped vibrations of more complicated structural elements, such as beams (e.g., variable thickness), plates and shells. The Rayleigh method, which has been in existence for over a century [1], takes advantage of the fact that a system vibrating in one of its normal modes, interchanges its energy completely between potential and kinetic forms. That is, the maximum potential energy (V_{max}) equals the maximum kinetic energy (T_{max}) which occurs in a cycle of vibration. Assuming a trial function for the mode shape, one which satisfies at least the geometric boundary conditions of the structural element, and assuming simple harmonic motion, setting T_{max} equal to V_{max} yields the vibration frequency. Unless one is fortunate enough to have assumed the exact eigenfunction (i.e., mode shape) of free vibration for the trial function, the resulting frequency is too high, and is consequently an upper bound on the exact solution.

In 1908 Ritz [2] improved upon the Rayleigh procedure by assuming a set of admissible trial functions, each having independent amplitude coefficients. He was able to show that a closest upper bound for the frequency could be achieved by minimizing the functional T_{max}-V_{max} with respect to the coefficients. Indeed, he applied this method to the completely free square plate [2] for which no exact solution is possible. Since then literally hundreds of references may be found which apply the method to free vibration problems (cf.[3,4]). In some cases such as shells, an exact solution may exist, but it is so complicated to use that a Ritz procedure may be employed more easily to obtain accurate results (e.g.,[5-9]). But in most cases no exact solution exists, and an approximate, properly convergent technique such as the Ritz method becomes essential.

The classical method for solving forced vibration problems for structural elements is to expand both the forcing function and the displacement response in terms of the free vibration eigenfunctions. This requires first solving the free vibration problem and finding an orthogonal set of eigenfunctions. Further, the forced vibration response typically requires the tedious (and computationally expensive) evaluation of integrals of products of the eigenfunctions with themselves and the loading functions. If the eigenfunctions are complicated (for example, combinations of the regular and modified Bessel-functions in the case of a circular plate) the integrals may have to be evaluated numerically. In many cases (e.g., beams) large roundoff errors are easily generated during the procedure.

It can be shown that the method of Galerkin [10] is equivalent to the Ritz method, and is some ways more general (it may be applied even when no energy functionals exist for the problem). This equivalent method has also been frequently used to analyze the free vibrations of structural elements such as beams, plates and shells (cf.,[3,4]). Several years ago a method of applying the Galerkin method to forced vibration problems of continuous systems having viscous damping was demonstrated [11,12]. The method is a direct one; that is, the forced response is found without first having to solve for the free vibration eigenfunctions. The method was demonstrated for the one-dimensional, second order differential equation of the vibrating string, and the two-dimensional, fourth order differential equation of the vibrating plate.

However, the Galerkin method has one disadvantage in comparison with the Ritz method. Both methods require satisfaction of only the geometric boundary conditions. However, if the force-type boundary conditions are not also satisfied, the Galerkin method requires the use of additional terms. For a plate having free edges, for example, these additional terms consist of two line integrals taken along each free edge which must be added to the orthogonalizing integrals. In the case of a shell, four line integrals would have to be added for each free edge. Thus for a cantilevered plate or shell of rectangular planform, the Galerkin method becomes quite cumbersome.

The Ritz method has been demonstrated in recent years to be an efficient, accurate technique for the analysis of free vibrations of cantilevered shells having rectangular planform [13-16]. Such configurations may be used for turbomachinery blade vibration studies. The method is particularly useful for preliminary design stages, or for making parameter studies, where finite element methods have been found to be very costly and unreliable.

The present paper summarizes recent research in extending the Ritz method to forced vibration problems where damping is present. This eliminates the aforementioned disadvantages of the Galerkin method when free edges are encountered. The crux of the problem is to find a dissipation functional accounting for the damping forces which are 90 degrees out of phase with the exciting, restoring and inertia forces of the system, which may be added to the kinetic, strain and load potential energies in a suitable manner to give the correct solution. The extension is demonstrated in the present work for systems having either viscous or material damping, distributed or concentrated exciting forces, and one-dimensional (beam) or two-dimensional (plate) structural elements. The convergence of the extended method to exact solutions is demonstrated for a set of cantilever beam problems. A similar method was developed by Siu and Bert [17] for laminated composite plates having material damping, subjected to distributed forcing pressure.

II. BEAM ANALYSIS - VISCOUS DAMPING

2.1 Distributed Forcing Function

Consider first a beam of length ℓ subjected to a distributed transverse load $\tilde{q}$ (force/unit length) which varies sinusoidally with time,

$$\tilde{q}(x,t) = q(x)e^{i\Omega t} \tag{2.1}$$

where Ω is the forcing frequency and x is the coordinate measured along the length. The kinetic energy during vibratory motion is

$$T = \frac{1}{2}\int_0^{\ell} \rho A\left(\frac{\partial w}{\partial t}\right)^2 dx \tag{2.2}$$

where w is the transverse displacement, ρ is the mass per unit length and A is the cross-sectional area. The strain energy of the beam due to bending deformation is

$$V = \frac{1}{2}\int_0^{\ell} EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{2.3}$$

where E is the modulus of elasticity and I is the second moment of the cross-sectional area with respect to the neutral axis of bending.

Define a dissipation functional $\mathcal{D}$ by

$$\mathcal{D} = \frac{1}{2}\int_0^{\ell} c\,\frac{\partial w}{\partial t}\,w dx \tag{2.4}$$

where c is the viscous damping coefficient. It should be noted that $\mathcal{D}$ differs from the well-known dissipation function of Rayleigh [1]. The work done by the exciting force is

$$\mathcal{W} = \int_0^{\ell} \bar{q}\; w\; dx \tag{2.5}$$

To apply the Ritz method, assume that the vibratory displacement w(x,t) may be expressed as

$$w(x,t) = W(x)e^{i\Omega t} \tag{2.6}$$

$$= \sum_{j=1}^{J} C_j\phi_j(x)e^{i\Omega t} \tag{2.7}$$

where the $\phi_j(x)$ are trial functions which satisfy at least the geometric boundary conditions (zero displacement and/or zero slope) which exist for the beam, and the C_j are complex coefficients which may be separated into real and imaginary parts as

$$C_j = C_j^R - iC_j^I \tag{2.8}$$

where $i=\sqrt{-1}$. The real part (C_j^R) may be regarded as the vector component of the response in phase with the exciting force, and the imaginary part (C_j^I) is the response component which lags the exciting force by 90 degrees.

Define further a generalization of the functional $T_{max}-V_{max}$ used in free, undamped vibration analysis by the Ritz method. That is, let

$$L_{max} = (T_{max}-\mathcal{D}_{max}) - (V_{max}-\mathcal{W}_{max}) \tag{2.9}$$

where the separate terms on the right-hand-side of Eq.(2.9) are the maximum values of the functionals previously given by Eqs.(2.2) through (2.5), obtained by substituting Eq.(2.6) into them and setting $|e^{2i\Omega t}|=1$. That is,

$$T_{max} = \frac{\Omega^2}{2}\int_0^{\ell} \rho AW^2 dx \tag{2.10a}$$

$$V_{max} = \frac{1}{2}\int_0^{\ell} EI\left(\frac{d^2W}{dx^2}\right)^2 dx \qquad (2.10b)$$

$$\mathcal{D}_{max} = \frac{i\Omega}{2}\int_0^{\ell} cW^2 dx \qquad (2.10c)$$

$$\mathcal{W}_{max} = \int_0^{\ell} q\ W\ dx \qquad (2.10d)$$

The Ritz method is applied by substituting Eq.(7) into Eqs.(2.10) and by using the minimizing equations

$$\frac{\partial L_{max}}{\partial C_j} = 0 \qquad (j=1,2,...J) \qquad (2.11)$$

which results in a set of J linear simultaneous equations in the unknown coefficients C_j. Setting the real and imaginary parts of each equation equal to zero yields a set of 2J equations in the unknowns C_j^R and C_j^I, with coupling between the C_j^R and the C_j^I coefficients arising from the damping. The right-hand sides of the equations evolve from the forcing function (q). Solution of this set of equations completely determines the forced, damped response.

2.2 Example. Cantilever Beam with Uniform Pressure

Consider the uniform, homogeneous, cantilever beam of length ℓ clamped at the end x=0 and free at the end x=ℓ (see Fig. 1), subjected to a uniformly distributed pressure

$$q(x,t) = q_o e^{i\Omega t} \qquad (2.12)$$

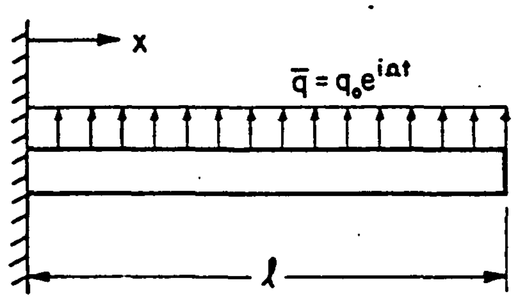

Figure 1. Cantilever beam with uniformly distributed forcing pressure.

where q_o is a constant. Let the trial functions ϕ_j to be used in Eq.(2.7) be expressed as simple algebraic polynomials,

$$\phi_j(x) = x^j \qquad (j=2,3,\ldots,J) \tag{2.13}$$

which, by beginning with j=2, guarantees that the boundary conditions at the clamped edge

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0 \tag{2.14}$$

are satisfied exactly. Substituting Eqs.(2.12) and (2.13) into Eqs.(2.7), (2.9) and (2.10), carrying out the minimization indicated by Eqs.(2.11) and the necessary integrations over x, and separating the real and imaginary parts, yields the following set of equations:

$$\sum_{j=2} \left\{ \left[\frac{EI}{\ell^4} \frac{j(j-1)k(k-1)}{j+k-3} - \frac{\rho A\Omega^2}{j+k+1}\right] C_j^R + \left[\frac{c\Omega}{j+k+1}\right] C_j^I \right\} = \frac{q_o}{k+1} \tag{2.15a}$$

$$\sum_{j=2} \left\{ -\left[\frac{c\Omega}{j+k+1}\right] C_j^R + \left[\frac{EI}{\ell^4} \frac{j(j-1)k(k-1)}{j+k-3} - \frac{\rho A\Omega^2}{j+k+1}\right] C_j^I \right\} = 0 \quad (k=2,3,\ldots,J) \tag{2.15b}$$

These 2(J-1) equations may be solved for the 2(J-1) unknowns C_j^R and C_j^I. The Galerkin method [11,12] was applied in the present work for purposes of comparison, and yielded Eqs.(2.15) identically.

The response of the beam at a typical point may now be obtained by adding the in-phase and out-of-phase components separately, and combining them vectorially. That is,

$$w(x,t) = \bar{W}(x) e^{i(\Omega t-\phi)} \tag{2.16}$$

with the amplitude of the response given by

$$\bar{W}(x) = \sqrt{\left[\sum_{j=2}^{J} C_j^R \phi_j(x)\right]^2 + \left[\sum_{j=2}^{J} C_j^I \phi_j(x)\right]^2} \tag{2.17}$$

and the phase angle lag by

$$\phi(x) = \tan^{-1} \frac{\sum_{j=2}^{J} C_j^I \phi_j(x)}{\sum_{j=2}^{J} C_j^R \phi_j(x)} \quad \text{(in radians)} \tag{2.18}$$

2.3 Exact Solution of the Previous Problem

The preceding problem of the cantilever beam subjected to uniformly distributed, sinusoidally oscillating exciting pressure has an exact solution. The equation of motion for the problem is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = q(x)e^{i\Omega t} \tag{2.19}$$

Assume a solution to Eq.(2.19) in the form

$$w(x,t) = \sum_{m=1}^{\infty} X_m(x)T_m(t) \tag{2.20}$$

where $X_m(x)$ is a typical eigenfunction of the free, undamped vibration problem. That is,

$$X_m(x) = \cosh\alpha_m\xi - \cos\alpha_m\xi - \gamma_m(\sinh\alpha_m\xi - \sin\alpha_m\xi) \tag{2.21}$$

where $\xi = x/\ell$ and $\alpha_m\ell$ is the nondimensional frequency parameter given by

$$(\alpha_m\ell)^2 = \omega_m\ell^2\sqrt{\frac{\rho A}{EI}} \tag{2.22}$$

which are the eigenvalues of the equation

$$\cos\alpha_m\ell \cdot \cosh\alpha_m\ell = -1 \tag{2.23}$$

The coefficient γ_m is determined from

$$\gamma_m = \frac{\cosh\alpha_m\ell + \cos\alpha_m\ell}{\sin\alpha_m\ell + \sinh\alpha_m\ell} \tag{2.24}$$

Values of $\alpha_m\ell$ and γ_m are given in Table 1.

Substituting Eq.(2.20) into Eq.(2.19), multiplying through by a typical eigenfunction χ_n, integrating over the length, using the orthogonality relationships

$$\int_0^{\ell} \chi_m\chi_n\, dx = 0 = \int_0 \frac{d^4\chi_m}{dx^4}\chi_n\, dx \tag{2.25}$$

(when $m \neq n$) and the equation of motion for free vibrations

$$EI\frac{d^4\chi_m}{dx^4} - \rho A\omega_m^2\chi_m = 0 \qquad (m=1,2,\ldots) \tag{2.26}$$

Table 1. Eigenfunction parameters for clamped-free beams.

m	$\alpha_m \ell$	γ_m
1	1.8751041	0.7340955
2	4.6940911	1.01846644
3	7.8547574	0.99922450
4	10.9955407	1.00003355
5	14.1371684	0.99999855
...	...	...
∞	$(2m-1)\pi/2$	1

yields the equation for the response of the mth mode:

$$\rho A \ddot{T}_m + c\dot{T}_m + \rho A \omega_m^2 T_m = q_m e^{i\Omega t} \tag{2.27}$$

where the dots represent time derivatives, and where

$$q_m = \frac{\int_0^{\ell} q_o \chi_m \, dx}{\int_0^{\ell} \chi_m^2 \, dx}, \qquad (m=1,2\ldots) \tag{2.28}$$

The solution to Eq.(2.27) is

$$T_m(t) = (A_m - iB_m) e^{i\Omega t} \qquad (m=1,2\ldots) \tag{2.29}$$

where A_m and B_m are given in nondimensional form as

$$A_m = \frac{\delta_m}{\Delta}\left(1 - \frac{\Omega^2}{\omega_m^2}\right)$$

$$B_m = \frac{\delta_m}{\Delta}\left(2 \frac{c}{c_{cm}} \frac{\Omega}{\omega_m}\right)^2 \tag{2.30}$$

$$\Delta = \left(1 - \frac{\Omega^2}{\omega_m^2}\right)^2 + \left(2 \frac{c}{c_{cm}} \frac{\Omega}{\omega_m}\right)^2$$

Here δ_m is the static amplitude of the mth mode (i.e., A_m when $\Omega/\omega_m=0$) and c_{cm} is the critical damping constant of the mth mode. In detail,

$$\delta_m = \frac{q_m}{\rho A \omega_m^2} , \quad c_{cm} = 2\rho A \omega_m \tag{2.31}$$

The total response at a typical point on the beam is given by

$$w(x,t) = Ce^{i(\Omega t-\phi)} \tag{2.32}$$

where
$$C^2 = \left[\sum_{m=1}^{\infty} A_m X_m(x)\right]^2 + \left[\sum_{m=1}^{\infty} B_m X_m(x)\right]^2 \tag{2.33}$$

and
$$\phi = \tan^{-1} \frac{\sum_{m=1}^{\infty} B_m X_m(x)}{\sum_{m=1}^{\infty} A_m X_m(x)} \quad \text{(in radians)} \tag{2.34}$$

It is observed that C and ϕ are both functions of x.

The classical solution procedure shown above is exact in that explicit expressions for responses of the free vibration modes (i.e, Eqs.(2.30)) may be written, even though the total response requires infinite summations of terms. The crux of the problem is the evaluation of the integrals of Eq.(2.28). For this purpose the tables calculated by Young and Felgar [18] are useful.

2.4 Numerical Results for the Uniform Pressure Loading

Table 2 presents numerical results for the amplitude response C/δ_o of a cantilever beam subjected to uniform pressure varying sinusoidally with time, where δ_o is the static displacement ($\Omega=0$) of the point considered. In this case the viscous damping is small ($c/c_{c1}=0.01$), and response at the free end ($x=\ell$) is evaluated. The frequency ratio (Ω/ω_1) is varied from 0 to 2, and data are given also for excitation at the second and third natural frequencies (Ω/ω_1 = 6.267 and 17.547). Solutions using the Ritz method are obtained using 2, 4 and 7 polynomial terms of the type given by Eq.(2.13), yielding 4, 8 and 14 simultaneous equations from Eqs.(2.15). Comparison is made with the results of the exact solution, described in Section 2.3.

It was found that, to at least five significant figure accuracy, the exact solution is the same as the 7-term Ritz solution for all Ω/ω_1 except at the third resonance and that (again, except near the third resonance) a 4-term solution would be sufficiently accurate for engineering purposes. If 9 terms are used, five significant figure agreement with the amplitude at the third resonant point is also found. Results for the response at the middle of the beam ($x=0.5\ell$) yielded nearly identical comparisons.

Table 2. Amplitude response C/δ_o at the end ($x=\ell$) of a cantilever beam; uniformly distributed load, viscous damping ($c/c_{c1}=0.01$).

$\frac{\Omega}{\omega_1}$	Number of trial functions in Ritz method			Exact Solution
	2	4	7	
0.0	1.00000	1.00000	1.00000	same
0.5	1.33134	1.33759	1.33759	"
0.9	5.06759	5.29616	5.29622	"
0.99	28.6556	36.0870	36.0896	"
1.0	45.8860	50.6697	50.6694	"
1.01	44.5984	35.5744	35.5715	"
1.1	5.04832	4.81325	4.81315	"
1.5	0.82658	0.82473	0.82472	"
2.0	0.34731	0.35276	0.35276	"
6.267	0.03837	1.27647	4.48085	"
17.547	0.00001	0.02056	0.83682*	0.93829

*9-term solution yields 0.93829

The static solution ($\Omega/\omega_1=0$) in this case is exactly expressible as a polynomial of 4th degree. Thus a 3-term Ritz solution would be exact at this frequency, while the "exact" method must laboriously sum up an infinite set of eigenfunctions. Table 2 shows the 2-term solution to be also exact at the particular point $x=\ell$. At $x=0.5\ell$, it would yield 0.941176, instead of unity.

Interestingly enough, the exact solution procedure gave numerical difficulty in obtaining accurately converged results for Table 2, even with the use of double-precision (i.e., 16 significant figure) arithmetic. This was largely for large $\alpha_m\ell$, the γ_m are very close to unity, and $\cosh \alpha_m\xi$ and $\sinh\alpha_m\xi$ are nearly identical. However, the Ritz method proceeded straightforwardly without difficulties.

2.5 Point Loading

Consider next the case of excitation by a concentrated transverse force $\tilde{F}_1 = F_1 e^{i\Omega t}$ acting at an arbitrary point $x=x_1$ along the beam. Solution of the problem by the Ritz method is simple and straightforward. Instead of Eq.(2.5), the work done by the exciting force is

$$\mathcal{W} = \tilde{F}_1 w(x_1) \tag{2.35}$$

and the appropriate functional for use with the Ritz method is

$$\mathcal{W}_{max} = F_1 W(x_1) \tag{2.36}$$

instead of Eq.(2.10d). When two or more point loads are present, or if a point load is acting in addition to a distributed load, then superposition applies.

In the case of a point load acting at the free end of a cantilever beam, the right-hand-side of Eq.(2.15a) is simply replaced by F_1.

To obtain an "exact" solution to the problem in terms of free vibration modes when a concentrated force is present, either of two methods may be used. One method would represent the concentrated force as a Dirac-delta function for q(x,t), and expand this function in terms of the eigenfunctions as in Eq. (2.28). However, this procedure would be extremely tedious, for it would require many terms in the series (Eq.(2.20)) to determine a reasonably accurate representation.

Another exact procedure introduces a change of variables. For example, consider the end loaded beam. The equation of motion is homogeneous:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} = 0 \tag{2.37}$$

But one of the boundary conditions is nonhomogeneous; i.e.,

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = \frac{\partial^2 w}{\partial x^2}(\ell,t) = 0 \tag{2.38a}$$

$$EI\frac{\partial^3 w}{\partial x^3}(\ell,t) = F_1 e^{i\Omega t} \tag{2.38b}$$

Let w(x,t) be replaced by

$$w(x,t) = v(x,t) + g(x)e^{i\Omega t} \tag{2.39}$$

where g(x) is chosen so that

$$g(0) = \frac{\partial g}{\partial x}(0) = \frac{\partial^2 g}{\partial x^2}(\ell) = 0 \tag{2.40a}$$

$$EI\frac{\partial^3 g}{\partial x^3}(\ell) = F_1 \tag{2.40b}$$

Then g(x) is found to be

$$g(x) = -\frac{F_1}{2EI}x^2 + \frac{F_1}{6EI}x^3 \tag{2.41}$$

Substituting Eqs.(2.39) and (2.41) into (2.37) transforms the problem into one having a nonhomogeneous equation of motion, with homogeneous boundary conditions; viz.

$$EI\frac{\partial^4 v}{\partial x^4} + \rho A\frac{\partial^2 v}{\partial t^2} + c\frac{\partial v}{\partial t} = (\rho A\Omega^2 - iC\Omega)g(x)e^{i\Omega t} \tag{2.42a}$$

$$v(0,t) = \frac{\partial v}{\partial x}(0,t) = \frac{\partial^2 v}{\partial x^2}(\ell,t) = \frac{\partial^3 v}{\partial x^3}(\ell,t) = 0 \qquad (2.42b)$$

and the solution proceeds as in Section 2.3.

III. BEAM ANALYSIS - MATERIAL DAMPING

3.1 Incorporation of Complex Stiffness

In the case of material damping (also called "structural" or "hysteretic" damping), it is possible to use an equivalent viscous damping representation under certain conditions, the viscous damping coefficient being chosen so as to dissipate an equivalent amount of energy per vibratory cycle. The most important condition to be met is that, for the given forcing function and the range of frequency ratio under consideration, a single normal mode is strongly dominant among the modes excited. While it is possible to represent any single mode response adequately by equivalent viscous damping, especially in the vicinity of resonance and if the damping is reasonably small, the individual modes respond differently. Therefore, strong normal mode coupling makes the representation less accurate.

The preferred method of treating material damping is by means of a complex modulus of elasticity (cf. [19]). That is, let the modulus be

$$E^* = E(1 + i\eta) \qquad (3.1)$$

where $i=\sqrt{-1}$ and η is the loss factor.

To apply the Ritz method to a forced vibration problem with material damping, the energy dissipation due to friction is combined with the elastic strain energy in complex form, and the functional to be minimized is

$$L_{max} = T_{max} - V^*_{max} + \mathcal{W}_{max} \qquad (3.2)$$

In the case of a beam, for example, V^*_{max} is given by

$$V^*_{max} = \frac{1}{2}\int_0^\ell E(1+i\eta)I\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \qquad (3.3)$$

whereas T_{max} and $\mathcal{W}_{max}$ remain as given previously by Eqs.(2.10a) and (2.10d), respectively.

The exact solution uses the equation of motion expressed in complex form, which is

$$E(1+i\eta)I\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = q(x)e^{i\Omega t} \qquad (3.4)$$

Assuming that the displacement (w) and the transverse loading (q) can both be represented as summations of the eigenfunctions of free vibration, a procedure similar to that employed in Section 2.3 may be followed, leading to the solution forms given by Eqs.(2.29) and (2.32), except that in the case of material damping the coefficients A_m and B_m become

$$A_m = \frac{\delta_m}{\Delta_s}\left(1 - \frac{\Omega^2}{\omega_m{}^2}\right)$$

$$B_m = \frac{\delta_m}{\Delta_s}\,\eta \tag{3.5}$$

$$\Delta_s = \left(1 - \frac{\Omega^2}{\omega_m{}^2}\right)^2 + \eta^2$$

with δ_m given by Eq.(2.31), and the amplitude and phase angle again given by Eqs.(2.33) and (2.34), respectively.

3.2 Example. Cantilever Beam with Uniform Pressure

The uniform, homogeneous, cantilever beam subjected to sinusoidally varying, uniform pressure, as described previously in Section 2.2, is again taken as an example; however, in this case the damping is hysteretic. Using algebraic polynomial trial functions as in Eq.(2.13) with the Ritz method, the following set of equations arises for the determination of the coefficients C_j^R and C_j^I associated with the real and imaginary parts of the solution:

$$\sum_{j=2}^{J}\left\{\left[\frac{EI}{\ell^4}\,\frac{j(j-1)k(k-1)}{j+k-3} - \frac{\rho A\Omega^2}{j+k+1}\right]C_j^R + \left[\eta\frac{EI}{\ell^4}\,\frac{j(j-1)k(k-1)}{j+k-3}\right]C_j^I\right\} = \frac{q_o}{k+1} \tag{3.6a}$$

$$\sum_{j=2}^{J}\left\{-\left[\eta\frac{EI}{\ell^4}\,\frac{j(j-1)k(k-1)}{j+k-3}\right]C_j^R + \left[\frac{EI}{4}\,\frac{j(j-1)k(k-1)}{j+k-3} - \frac{\rho A\Omega^2}{j+k+1}\right]C_j^I\right\} = 0 \tag{3.6b}$$

Application of the Galerkin method to the problem yielded Eqs.(3.6) identically. The displacement w(x,t) of an arbitrary point is again given by Eqs. (2.16), (2.17) and (2.18).

Numerical results for the nondimensional amplitude ratio C/δ_o at the free end of the beam as a function of the frequency ratio Ω/ω_1 are presented in Table 3 for a loss factor representative of many metals (η=0.001), and in Table 4 for a large loss factor (η=0.1). Once again it is seen that the Ritz method converges rapidly to the exact solution as polynomial terms are added to the displacements. A 7-term solution is accurate to 6 significant figure accuracy except in the vicinity of the third resonance.

In studying Tables 3 and 4, the data is seen to exhibit certain characteristics similar to those for a single degree of free system having material damping (e.g., [20], p. 230). That is,

Table 3. Amplitude response C/δ_o at the end $(x=\ell)$ of a cantilever beam; uniformly distributed load, material damping (η=0.001).

$\frac{\Omega}{\omega_1}$	Trial functions-Ritz method		Exact Solution
	4	7	
0.0	0.999999	0.999999	same
0.5	1.33770	1.33770	"
0.9	5.31980	5.31986	"
0.99	50.8387	50.8463	"
1.0	1013.39	1013.39	"
1.01	50.3766	50.3686	"
1.1	4.83954	4.83944	"
1.5	0.834967	0.824959	"
2.0	0.352793	0.352792	"
6.267	1.32170	14.3001	"
17.547	0.025578	0.925293*	1.06942

*9-term solution yields 1.06942

Table 4. Amplitude response C/δ_o at the end $(x=\ell)$ of a cantilever beam; uniformly distributed load, material damping (η=0.1).

$\frac{\Omega}{\omega_1}$	Trial functions-Ritz method		Exact Solution
	4	7	
0.0	0.995037	0.995037	same
0.5	1.32593	1.32593	"
0.9	4.70715	4.70719	"
0.99	9.93494	9.93498	"
1.0	10.13252	10.13249	"
1.01	9.93658	9.93649	"
1.1	4.36873	4.36865	"
1.5	0.822066	0.822058	"
2.0	0.352364	0.352362	"
6.267	0.148371	0.145056	"
17.547	0.009427	0.010733	0.010730

(1) At $\Omega/\omega_1=0$, C/δ_o is not unity (as in viscous damping), but $C/\delta_o = [1/(1+\eta^2)]^{1/2}$.

(2) Damping does not shift the first resonant peak - it remains at $\Omega/\omega_1 = 1$.

(3) The first resonant amplitude ratio (C/δ_o) is very nearly equal to $1/\eta$.

That C/δ_o at $\Omega/\omega_1=0$ is not exactly $1/\eta$ is due to the slight influence of the higher modes. Comparing Tables 3 and 4, it is seen that increased material damping causes greater contribution from the first mode at the second and third resonant frequencies, as might be expected.

IV. PLATE ANALYSIS

4.1 Ritz Method

Except for the fortuitous case of a plate having at least its two opposite sides simply supported, no exact solution for the free vibration eigenfunctions is possible [3]. Nevertheless, the Ritz method may be generalized from the preceding beam analysis to deal with the forced vibrations of plates having arbitrary edge conditions, including point supports.

To be somewhat specific, let us consider a rectangular plate having dimensions a x b, such as the cantilever shown in Fig. 2. The exciting force

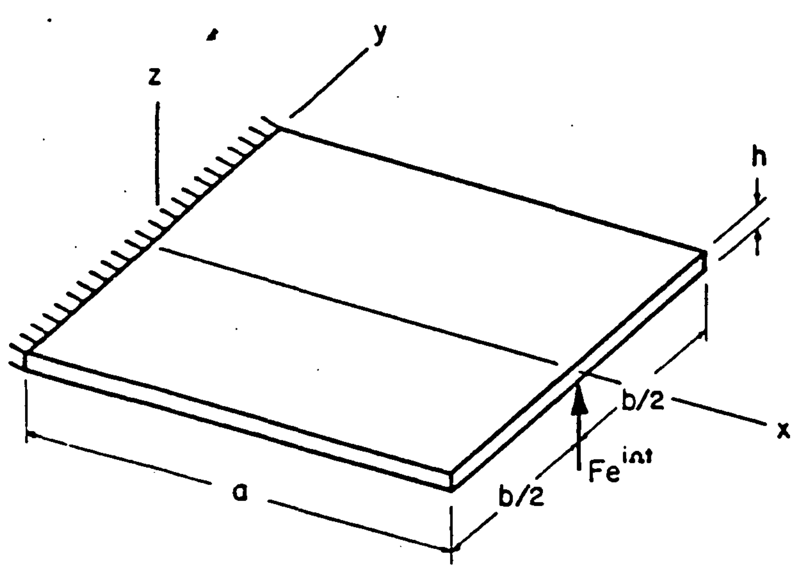

Figure 2. Cantilever plate with concentrated exciting force.

may be either distributed or concentrated. A sinusoidally varying distributed load will, in general, be of the form

$$\bar{q}(x,y,t) = q(x,y)e^{i\Omega t} \tag{4.1}$$

where now q is force per unit area, whereas a point load is of the form

$$\tilde{F} = Fe^{i\Omega t} \tag{4.2}$$

The transverse displacement is taken as

$$w(x,y,t) = W(x,y)e^{i\Omega t} \tag{4.3}$$

$$= \sum_{j=1}^{J} \sum_{k=1}^{K} C_{jk}\phi_{jk}(x,y)e^{i\Omega t} \tag{4.4}$$

where the ϕ_{jk} satisfy the geometric boundary conditions exactly, and the C_{jk} are complex coefficients given by

$$C_{jk} = C_{jk}^{R} - iC_{jk}^{I} \tag{4.5}$$

The functionals for the plate, which are the generalizations of the beam functionals given by Eqs.(2.10), are:

$$T_{max} = \frac{\Omega^2}{2}\int_{o}^{a}\int_{-b/2}^{b/2} \rho h W^2 dxdy \tag{4.6a}$$

$$V_{max} = \frac{1}{2}\int_{o}^{a}\int_{-b/2}^{b/2} D\{(\nabla^2 W)^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - (\frac{\partial^2 W}{\partial x \partial y})^2\right]dxdy \tag{4.6b}$$

$$\mathcal{D}_{max} = \frac{i\Omega}{2}\int_{o}^{a}\int_{-b/2}^{b/2} c\, W^2\, dxdy \tag{4.6c}$$

$$\mathcal{W}_{max} = \int_{o}^{a}\int_{-b/2}^{b/2} q\, W\, dxdy \tag{4.6d}$$

In these expressions the flexural rigidity (D) is

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{4.7}$$

with h being the plate thickness and ν being Poisson's ratio. In the case of material damping, the integrand for V_{max} is simply multiplied by $(1+i\eta)$.

The functional L_{max} is constructed as in Eq.(2.9) and minimized as

$$\frac{\partial L_{max}}{\partial C_{jk}} = 0 \qquad (j=1,\ldots,J;\ k=1,\ldots,K) \tag{4.8}$$

yielding a set of 2JK simultaneous equations in the coefficients C^R_{jk} and C^I_{jk}. Solution of these equations determines the forced, damped response.

4.2 Example. Cantilever Square Plate with Point Load

To demonstrate the Ritz method, a cantilever plate of square planform (a/b=1) is chosen, and is excited by a concentrated force F acting at the center of the outer edge, the force varying sinusoidally with time (Fig. 2). Assume viscous damping affects the plate uniformly (i.e., c is a constant). Let Poisson's ratio (ν) be 0.3.

A set of polynomials for use in Eq. (4.4) may be chosen as

$$\phi_{jk}(x,y) = x^j y^k \qquad (j=2,3\ldots J;\ k=0,1\ldots K) \tag{4.9}$$

which satisfy the geometric boundary conditions

$$w(0,y,t) = \frac{\partial w}{\partial x}(0,y,t) = 0 \tag{4.10}$$

exactly. Moreover, for the present example problem where the exciting force preserves the geometric symmetry already present, the antisymmetric modes are not excited, and the index k may be taken as k=0,2,4... with no loss of generality.

Accurate numerical results were obtained by taking 6 terms in the x-direction (i.e., j=2,3,...,7) and 4 terms in the y-direction (i.e., k=0,2,4,6). This yields 6 x 4 = 24 polynomial terms ϕ_{jk} and, hence, 24 modal degrees of freedom to the problem. Considering the in-phase and out-of-phase components of motions, the resulting problem consists of solving 2 x 24 = 48 simultaneous equations for the C^R_{jk} and C^I_{jk}.

Numerical results for a small damping ratio (c/c_{c1}=0.01) are shown in Table 5. Here, as in the beam analysis c_{c1} ($=2\rho h\omega_1$) is the critical damping coefficient for the first free vibration mode of the square cantilever plate. In Table 5 the nondimensional displacements WD/Fa^2 of three significant points are given, as functions of Ω/ω_1. These points are the plate center (x=0.5a, y=0), the point of loading (x=a, y=0) and the corner points (x=a, y=$\pm$0.5b). In addition to the first resonant frequency (Ω/ω_1=1), data is also given for the next three frequencies of symmetric modes (Ω/ω_1=6.1317, 7.8307 and 15.6165). ω_3/ω_1=7.8307 corresponds to a chordwise bending free vibration mode, whereas the other three are spanwise (i.e., flapwise) bending modes (cf., [3,13]).

Comparing Table 5 for the plate with Table 2 for the beam, both having the same damping ratio (c/c_{c1}=0.01), one observes that the amplitude responses at the second and third spanwise bending frequencies is considerably greater for the plate than the beam. This is mainly due to two differences:

(1) The concentrated end force excites the higher spanwise bending modes more than a uniformly distributed force.

Table 5. Displacements WD/Fa^2 of a cantilever plate with a sinusoidally varying point load (P) at the center of the outer edge; viscous damping (c/c_{c1}=0.01).

$\frac{\Omega}{\omega_1}$	Plate center $\frac{x}{a} = 0.5,\ \frac{y}{b} = 0$	Load point $\frac{x}{a} = 1,\ \frac{y}{b} = 0$	Corner point $\frac{x}{a} = 1,\ \frac{y}{b} = 0.5$
0.0	0.111	0.360	0.329
0.5	.150	.473	.440
0.9	.601	1.798	1.744
0.99	4.153	12.209	11.982
1.0	5.755	16.887	16.597
1.01	3.992	11.690	11.507
1.1	.548	1.574	1.572
1.5	.096	.247	.268
2.0	.043	.089	.114
6.1317	1.714	3.997	1.843
7.8307	1.193	1.254	2.865
15.6165	.736	1.258	1.307

(2) The concentrated force is not along the entire free end of the plate, but only at its center, thereby causing a significant contribution of the first chordwise bending mode.

It is interesting to observe the change in chordwise bending effects in Table 5 as Ω/ω_1 is swept from zero to the fourth natural frequency. At Ω/ω_1 = 0 one has static loading, and the plate exhibits significant anticlastic curvature; that is, the center of the free edge deflects more than the corner points. The anticlastic curvature is greater here than for a uniformly loaded square cantilever [22], as might be expected. As Ω/ω_1 is increased, the curvature remains anticlastic until slightly past the first resonance ($\Omega/\omega_1 \approx 1.1$). The curvature is strongly anticlastic at the second resonance, but reverses itself for the third and fourth resonances.

Table 6 gives the results for the same problem when the damping coefficient is ten times greater (c/c_{c1}=0.1). The responses at the resonant peaks are all very similar to those with small damping, except with amplitudes essentially one-tenth as large, except for the fourth resonance.

V. CONCLUDING REMARKS

An extension of the Ritz method has been developed for analyzing the forced vibrations of structural elements such as beams, plates and shells in a straightforward and efficient manner. Both viscous and material damping may be accounted for. The exciting forces may be distributed and/or concentrated. Convergence to the exact solutions for a set of cantilever beam problems was demonstrated.

Table 6. Displacements WD/Fa^2 of a cantilever plate with a sinusoidally varying point load (P) at the center of the outer edge; viscous damping (c/c_{cl}=0.1).

$\frac{\Omega}{\omega_1}$	Plate center $\frac{x}{a} = 0.5, \frac{y}{b} = 0$	Load point $\frac{x}{a} = 1, \frac{y}{b} = 0$	Corner point $\frac{x}{a} = 1, \frac{y}{b} = 0.5$
0.0	0.111	0.360	0.329
0.5	.148	.469	.436
0.9	.438	1.309	1.268
0.99	.578	1.700	1.669
1.0	.576	1.690	1.660
1.01	.567	1.662	1.636
1.1	.381	1.094	1.093
1.5	.093	.240	.261
2.0	.042	.088	.113
6.1317	.172	.404	.187
7.8307	.119	.129	.287
15.6165	.117	.200	.207

Although the method was demonstrated for homogeneous beams and plates of uniform thickness, the energy functionals presented are in more general form, capable of accounting for variable cross-sections (i.e., A, I, h) and nonhomogeneous material (E, ρ). Work is currently progressing to extend the method further to shell problems. The method will also be developed for analyzing the free, damped response of structural elements, and other types of damping will also be considered.

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