NASA Technical Memorandum 86442

NASA-TM-86442 19850022051

LARGE DEFLECTIONS OF A CANTILEVER BEAM UNDER ARBITRARILY DIRECTED TIP LOAD

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MAY 1985

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Space Administration

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SUMMARY

The nonlinear beam equation is integrated numerically in a direct fashion to obtain results for large deflections of cantilevers under tip loads of arbitrary direction. A short BASIC computer program for performing this integration is presented. Results for selected load cases are presented. The numerical process is performed rapidly on a modern microcomputer, and comparisons in the paper with results from closed form solutions from the literature show that the process is accurate.

INTRODUCTION

The engineering theory of deflection of beams is linear if deflections are small with respect to beam length. Solutions to linear beam equations are presented in textbooks. On the other hand, if the loading and stiffness of the beam are related in such a way as to lead to deflections not small with respect to beam length; the beam equations are nonlinear, and solutions are more difficult. The purpose of this paper is to consider the large deflection of a cantilever beam with an arbitrarily directed tip load. This particular type of beam and loading could be an element of a larger structural or mechanical system, and appropriate solutions could facilitate treatment of the beam element as a nonlinear spring in the context of the large system. It turns out that exact, closed-form solutions to this problem exist in the literature (ref. 1). These solutions, however, are in terms of elliptic functions, and sometimes the desired unknowns are arguments of the elliptic functions. To obtain numerical results, therefore, may require iteration or trial-and-error calculations using tables of elliptic functions. In this paper, a different approach is taken. The nonlinear beam equation is integrated numerically in a direct fashion. The equations for performing this numerical integration are presented, and results are presented for selected cases. Finally, a short BASIC-language computer program is presented which performs the numerical integration of the beam equations. The program is written in Applesoft BASIC for an Apple II+ microcomputer with 64K memory.

N85-30363

SYMBOLS

A	$=FL^2/EI$, Dimensionless load-stiffness parameter
EI	Beam bending stiffness
F	Magnitude of tip load
h	Increment in dimensionless length parameter, $\bar{s}$
L	Length of beam
M	Bending moment in beam
P	Load along neutral axis of beam
Q	Transverse shear load in beam
U	$=d\psi/ds$
V	$=\psi$
s	Coordinate along neutral axis of beam
$\bar{s}$	$= s/L$
X,Y	Beam coordinates with origin at root
X1,Y1	Beam coordinates with origin at tip
XL,YL	Dimensionless coordinates of beam tip in X,Y system
ψ	Angle between tangent to beam neutral axis and horizontal (slope of neutral axis)
ζ	Angle between tip load and vertical
θ	Angle between horizontal and line between root and tip

BEAM EQUATION

The beam is considered horizontal, clamped at the right end and loaded at the left end as shown in figure 1. The coordinate system for the beam is shown in figure 2. The origin of the coordinates X,Y is at the clamped end or root, whereas the origin of coordinates X1,Y1 is at the tip. The beam is considered inextensional, and transverse shear deformations are neglected. Equilibrium equations are written with reference to the element shown in figure 3. Horizontal equilibrium gives

$$(P+dP)\cos(\psi+d\psi) - P\cos\psi + Q\sin\psi - (Q+dQ)\sin(\psi+d\psi) = 0 \qquad (1)$$

By performing algebraic manipulations and neglecting terms containing products of differentials, there results

$$P\cos\psi - Q\sin\psi = \text{constant} = F\sin\zeta \tag{2}$$

Vertical equilibrium gives

$$(P+dP)\sin(\psi+d\psi) + (Q+dQ)\cos(\psi+d\psi) - P\sin\psi - Q\cos\psi = 0 \tag{3}$$

Similar manipulations lead to

$$P\sin\psi + Q\cos\psi = \text{constant} = F\cos\zeta \tag{4}$$

Equilibrium of moments gives

$$M - M - dM - Q\,ds = 0$$

or

$$Q = -\frac{dM}{ds} \tag{5}$$

The load P along the neutral axis of the beam can be eliminated from equations (2) and (4) to give

$$Q = F(\sin\zeta\sin\psi + \cos\zeta\cos\psi) \tag{6}$$

Engineering beam theory gives

$$M = EI\frac{d\psi}{ds} \tag{7}$$

Substituting equations (6) and (7) into equation (5) results in the nonlinear beam differential equation

$$\frac{d^2\psi}{ds^2} = -\frac{F}{EI}(\sin\zeta\sin\psi + \cos\zeta\cos\psi) \tag{8}$$

Equation (8) can be made dimensionless by simply defining

$$\bar{s} = \frac{s}{L}$$

so that equation (8) becomes

$$\frac{d^2\psi}{d\bar{s}^2} = -\frac{FL^2}{EI}(\sin\zeta\sin\psi + \cos\zeta\cos\psi)$$

or

$$\frac{d^2\psi}{d\bar{s}^2} = -A(\sin\zeta\sin\psi + \cos\zeta\cos\psi) \tag{9}$$

where the quantity $A=FL^2/EI$ is termed, in this paper, the load-stiffness parameter.

NUMERICAL INTEGRATION OF BEAM EQUATION

Equation (9) can be integrated numerically in a direct fashion. The second-order differential equation is replaced by two first-order differential equations

$$\left.\begin{aligned} \frac{dU}{d\bar{s}} &= -A(\sin\zeta\sin V + \cos\zeta\cos V) \\ \frac{dV}{d\bar{s}} &= U \end{aligned}\right\} \quad (10)$$

Equations (10) are integrated numerically with the modified Euler method (ref. 3). Let

$$f(\zeta, V) = -A(\sin\zeta\sin V + \cos\zeta\cos V) \quad (11)$$

If the symbol h denotes the increment in the length parameter $\bar{s}$, and the subscripts 0 and 1 denote values at the beginning and end of an increment, respectively, then

$$\left.\begin{aligned} U_1(1) &= U_0 + h\,f(\zeta, V_0) \\ V_1(1) &= V_0 + h\,U_0 \\ U_1(2) &= U_0 + \frac{h}{2}[f(\zeta, V_0) + f(\zeta, V_1(1))] \\ V_1(2) &= V_0 + \frac{h}{2}[U_0 + U_1(1)] \\ U_1(3) &= U_0 + \frac{h}{2}[f(\zeta, V_0) + f(\zeta, V_1(2))] \\ V_1(3) &= V_0 + \frac{h}{2}[U_0 + U_1(2)] \\ &\vdots \\ U_1(N) &= U_0 + \frac{h}{2}[f(\zeta, V_0) + f(\zeta, V_1(N-1))] \\ V_1(N) &= V_0 + \frac{h}{2}[U_0 + U_1(N-1)] \\ &\vdots \end{aligned}\right\} \quad (12)$$

The quantity h is assumed constant over the length of the beam. The accuracy of the calculations depends on h and is improved by reducing the size of h . The numbers (1,2,...,N) in parentheses in equations (12) denote various approximations to the values of U and V at the end of the increment. Continue the iteration until successive values of U_1 or V_1 are equal. Then replace U_0 by U_1 and V_0 by V_1 and advance to the next increment. This integration process is started at the tip of the beam by assuming a value of the slope of the beam V_0 and setting the curvature $U_0=0$. The integration continues toward the root until the slope becomes zero. The whole process requires iteration because the value of the load-stiffness parameter A must be adjusted to make the slope zero at the end of the final integration increment. (See ref. 3 for a related approach to the cantilever beam with uniform distributed load.)

A BASIC computer program which integrates equations (10) using the process laid out in equations (12) is presented in the Appendix. With practice, given starting values at the tip for beam slope V_0 and beam curvature $U_0=0$, the user of this program can converge in just a few iterations to values of A which lead to zero slope at the root of a beam of unit length.

RESULTS

Results are presented for four load cases: (1) tip load vertical (vertical load), (2) tip load normal to a straight line connecting root and tip (slewing load), (3) tip load normal to the beam (follower load), and (4) tip load horizontal (column load). All results presented in the paper are calculated with the increment $h=0.01$.

For cases (1) vertical load, (3) follower load, and (4) column load, the quantity ζ is simply prescribed initially, and the calculation follows directly. For case (2) slewing load, the quantity ζ must be equal to θ. Since θ is not known a priori, an additional trial-and-error operation is required to make ζ equal to θ.

Results are presented in Tables 1 and 2 and in figures 4-9. Values of the load-stiffness parameter A and the angle between the horizontal and the line between root and tip of the beam θ are listed in Table 1 for various values of tip slope. Also in Table 1 are comparisons with results of closed form solutions given in references 1 and 4. The curves in figures 4-7 represent dimensionless deflection shapes of beam neutral axes. Tip load vectors are shown in figures 4-7, and these vector magnitudes are indeed proportional to the tip load magnitudes.

For the case of vertical tip load, the load-stiffness parameter is plotted as a function of dimensionless vertical tip deflection in figure 8. The contrast between linear and nonlinear solutions for this case is striking. For the case of the slewing tip load, the load-stiffness parameter is plotted in figure 9 as a function of θ, the angle between the horizontal and the line between root and tip. For θ up to one radian the beam behavior is nearly linear in this slewing load case.

CONCLUDING REMARKS

Equations are presented for the numerical solution of the nonlinear, large deflection beam equation for cantilever beams under tip loads of various orientations. Results are presented for selected cases, and comparisons are made with closed form solutions from the literature. The numerical solution, implemented in a BASIC-language computer program, is accurate. The computer program can operate on microcomputers, and solutions can be obtained rapidly and cheaply.

REFERENCES

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APPENDIX

COMPUTER PROGRAM FOR LARGE DEFLECTION OF CANTILEVER BEAM UNDER ARBITRARILY DIRECTED TIP LOAD

SYMBOL IN TEXT	VARIABLE IN COMPUTER PROGRAM
A	A
EI	EI
F	F
h	H
L	L
$\bar{\bar{s}}$	SBAR
U_0, V_0	U0,V0
U_1, V_1	U1,V1
X1,Y1	X1,Y1
XL,YL	XL,YL
ψ	PSI
θ	THETA
ζ	ZETA

```
]LIST

10  REM  INTEGRATION OF BEAM EQUATIONS 021985
20  REM  LARGE DEFLECTION OF CANTILEVER BEAM UNDER ARBITRARILY DIR
ECTED TIP LOAD
30  REM  BASIC DIFFERENTIAL EQUATION:D^2PSI/DSBAR^2=-A*(SIN(ZETA)*
SIN(PSI)+COS(ZETA)*COS(PSI))
40  REM  PSI=SLOPE OF BEAM:SBAR=S/L:S=DISTANCE ALONG BEAM:L=LENGTH
 OF BEAM:A=(F*L^2)/(E*I):ZETA=ANGLE OF TIP LOAD WITH VERTICAL
50  REM  WRITE SECOND ORDER DIFFERENTIAL EQUATION AS TWO FIRST ORD
ER DIFFERENTIAL EQUATIONS
60  REM  DU/DSBAR=-A*(SIN(ZETA)*SIN(V)+COS(ZETA)*COS(V))
70  REM  U=DV/DSBAR
80  REM  U=DPSI/DSBAR:V=PSI
85  INPUT "TODAYS DATE= ";D
90  INPUT "SLOPE AT TIP IN RADIANS= ";VO
100  INPUT "CURVATURE AT TIP= ";UO
110  INPUT "ANGLE OF TIP LOAD WITH VERTICAL IN RADIANS= ";ZETA
120  INPUT "INCREMENT IN LENGTH PARAMETER SBAR= ";H
130  INPUT "LOAD-STIFFNESS COEFFICIENT= ";A
135  INPUT "DO YOU WANT TO PRINT X AND Y?  1 IF YES, 2 IF NO: ";Q1

140  PRINT  CHR$ (4);"PR#1": PRINT  CHR$ (9);"80N"
145  PRINT "RESULTS FROM BEAMINTEGRATIONTIPLOAD021985 CALCULATED O
N ";D
150  PRINT "SLOPE AT TIP IN RADIANS= ";VO
160  PRINT "CURVATURE AT TIP= ";UO
170  PRINT "ANGLE OF TIP LOAD WITH VERTICAL IN RADIANS= ";ZETA
180  PRINT "LOAD-STIFFNESS COEFFICIENT= ";A
190  PRINT "INCREMENT IN LENGTH PARAMETER SBAR= ";H
191  IF Q1 <  > 1 THEN 200
192  PRINT : PRINT : PRINT
194  POKE 36,5: PRINT "X1";
196  POKE 36,25: PRINT "Y1"
200  PRINT  CHR$ (4);"PR#0"
210  REM  U1(1)=UO+H*(-A)*(SIN(ZETA)*SIN(VO)+COS(ZETA)*COS(VO))
220  REM  V1(1)=VO+H*UO
230  REM  FOR N>1,U1(N)=UO+H*(-A)*(SIN(ZETA)*SIN(VO)+COS(ZETA)*COS
(VO)+SIN(ZETA)*SIN(V1(N-1))+COS(ZETA)*COS(V1(N-1)))/2
240  REM  FOR N>1, V1(N)=VO+H*(UO+U1(N-1))/2
250  REM  ITERATE UNTIL SUCCESSIVE VALUES OF V1 ARE EQUAL
260  REM  THEN REPLACE UO AND VO BY U1 AND V1, RESPECTIVELY
270  REM  CONTINUE UNTIL V1<0
280  REM  A1=U1(1):A2=U1(2):B1=V1(1):B2=V1(2)
290 X1 = 0:Y1 = 0:C = 0:J = 1
295  DIM X1(10),Y1(10)
300 A1 = UO + H * ( - A) * ( SIN (ZETA) *  SIN (VO) +  COS (ZETA)
*  COS (VO))
310 B1 = VO + H * UO
320 A2 = UO + H * ( - A) * ( SIN (ZETA) *  SIN (VO) +  COS (ZETA)
*  COS (VO) +  SIN (ZETA) *  SIN (B1) +  COS (ZETA) *  COS (B1)) /
 2
330 B2 = VO + H * (UO + A1) / 2
```

```
335  POKE 36,0: PRINT A2;
336  POKE 36,20: PRINT B2
340  IF B2 = B1 THEN 1000
350 A1 = A2
360 B1 = B2
370  GOTO 320
1000  POKE 36,0: PRINT A2;
1010  POKE 36,20: PRINT B2
1015  PRINT
1020  IF B2 < 0 THEN 1090
1030 X1 = X1 + H *  COS (B2)
1040 Y1 = Y1 + H *  SIN (B2)
1044 C = C + 1
1045  IF C = (.1 / H) * J THEN 3000
1050 U0 = A2
1060 V0 = B2
1080  GOTO 300
1090 R = (V0 * H) / (V0 +  ABS (B2))
1095 X1 = X1 + R *  COS (V0)
1100 Y1 = Y1 + R *  SIN (V0)
2000  PRINT  CHR$ (4);"PR#1": PRINT  CHR$ (9);"80N"
2010  PRINT
2020  POKE 36,0: PRINT "XL= ";X1;
2030  POKE 36,20: PRINT "YL= ";Y1
2040 THETA =  ATN (Y1 / X1)
2050  POKE 36,0: PRINT "THETA= ";THETA;
2060 L = H * C + R
2070  POKE 36,20: PRINT "DISTANCE TO ZERO SLOPE= ";L
2085  IF Q1 <  > 1 THEN 2565
2090  GOTO 2450
2450  PRINT : PRINT : PRINT
2460  POKE 36,5: PRINT "X";
2470  POKE 36,25: PRINT "Y"
2500  PRINT
2502  POKE 36,0: PRINT X1;
2504  POKE 36,20: PRINT Y1
2505  FOR J = 1 TO 10
2510 X = X1(10) - X1(J)
2520 Y = Y1(10) - Y1(J)
2530  PRINT  CHR$ (4);"PR#1": PRINT  CHR$ (9);"80N"
2540  POKE 36,0: PRINT X;
2550  POKE 36,20: PRINT Y
2560  NEXT J
2565  PRINT  CHR$ (4);"PR#0"
2570  END
3000  IF Q1 <  > 1 THEN 1050
3010  PRINT  CHR$ (4);"PR#1": PRINT  CHR$ (9);"80N"
3030  POKE 36,0: PRINT X1;
3040  POKE 36,20: PRINT Y1
3050 X1(J) = X1
3060 Y1(J) = Y1
3070 J = J + 1
3090  PRINT  CHR$ (4);"PR#0"
3100  GOTO 1050
```

TYPICAL PRINTOUT

```
RESULTS FROM BEAMINTEGRATIONTIPLOAD021985 CALCULATED ON 52085
SLOPE AT TIP IN RADIANS= 1.3962634
CURVATURE AT TIP= 0
ANGLE OF TIP LOAD WITH VERTICAL IN RADIANS= 1.57079633
LOAD-STIFFNESS COEFFICIENT= 3.19267
INCREMENT IN LENGTH PARAMETER SBAR= .01
```

X1	Y1
.0179601298	.0983726294
.0391506505	.196091612
.0665560747	.2922358
.102929863	.38533243
.150564535	.473167718
.21096228	.552725781
.284447718	.620330257
.369830612	.672040657
.464276777	.704279598
.563530959	.714550956

```
XL= .563531231          YL= .714550956
THETA= .903014          DISTANCE TO ZERO SLOPE= 1.00000027
```

TYPICAL PRINTOUT (Continued)

X	Y
.563531231	.714550956
.545570829	.616178327
.524380308	.518459344
.496974884	.422315156
.460601095	.329218526
.412966423	.241383238
.352568679	.161825176
.279083241	.0942206995
.193700347	.042510299
.0992541814	.0102713581
0	0

TABLE 1. RESULTS ON LARGE DEFLECTION OF CANTILEVER BEAMS UNDER VARIOUS TIP LOADS

Tip Slope		A		Theta	F/(Buckling Load)	
Degrees	Radians	(This paper)	(Ref. 1)	Radians	(This paper)	(Ref. 4)
VERTICAL TIP LOAD						
10	0.17453293	0.353	--	0.116	0.143	--
20	0.34906585	0.731	0.73	0.232	0.296	--
30	0.52359878	1.163	--	0.349	0.471	--
40	0.69813170	1.692	1.69	0.469	0.686	--
50	0.87266463	2.392	--	0.592	0.970	--
60	1.04719755	3.405	3.405	0.722	1.380	--
70	1.22173048	5.081	--	0.861	2.059	--
80	1.39626340	8.678	8.678	1.024	3.517	--
SLEWING TIP LOAD						
10	0.17453293	0.349	--	0.116	0.142	--
20	0.34906585	0.701	--	0.231	0.284	--
30	0.52359878	1.057	--	0.347	0.428	--
40	0.69813170	1.420	--	0.463	0.575	--
50	0.87266463	1.791	--	0.580	0.726	--
60	1.04719755	2.174	--	0.698	0.881	--
70	1.22173048	2.570	--	0.816	1.042	--
80	1.39626340	2.982	--	0.935	1.209	--
FOLLOWER TIP LOAD						
10	0.17453293	0.349	--	0.115	0.142	--
20	0.34906585	0.701	--	0.231	0.284	--
30	0.52359878	1.057	--	0.346	0.428	--
40	0.69813170	1.420	--	0.461	0.575	--
50	0.87266463	1.791	--	0.575	0.726	--
60	1.04719755	2.175	--	0.689	0.882	--
70	1.22173048	2.575	--	0.801	1.044	--
80	1.39626340	2.994	--	0.913	1.214	--
HORIZONTAL TIP LOAD						
10	0.17453293	2.477	--	0.110	1.004	--
20	0.34906585	2.505	--	0.221	1.015	1.015
30	0.52359878	2.554	--	0.332	1.035	--
40	0.69813170	2.624	--	0.443	1.064	1.063
50	0.87266463	2.719	--	0.556	1.102	--
60	1.04719755	2.842	--	0.670	1.152	1.152
70	1.22173048	2.997	--	0.785	1.215	--
80	1.39626340	3.193	--	0.903	1.294	1.293

TABLE 2. TIP COORDINATES FROM LARGE DEFLECTIONS OF CANTILEVER BEAMS UNDER VARIOUS TIP LOADS

Tip Slope	Dimensionless Tip Coordinates			
	(This paper)		(Ref. 4)	
Degrees	XL	YL	XL	YL
VERTICAL TIP LOAD				
10	0.992	0.115	--	--
20	0.968	0.228	--	--
30	0.928	0.338	--	--
40	0.873	0.442	--	--
50	0.802	0.540	--	--
60	0.716	0.630	--	--
70	0.611	0.712	--	--
80	0.481	0.790	--	--
SLEWING TIP LOAD				
10	0.992	0.115	--	--
20	0.968	0.228	--	--
30	0.929	0.336	--	--
40	0.875	0.437	--	--
50	0.808	0.529	--	--
60	0.729	0.611	--	--
70	0.640	0.680	--	--
80	0.543	0.735	--	--
FOLLOWER TIP LOAD				
10	0.992	0.115	--	--
20	0.968	0.228	--	--
30	0.929	0.335	--	--
40	0.876	0.435	--	--
50	0.810	0.525	--	--
60	0.734	0.604	--	--
70	0.647	0.672	--	--
80	0.557	0.721	--	--
HORIZONTAL TIP LOAD				
10	0.992	0.110	--	--
20	0.970	0.218	0.970	0.220
30	0.933	0.321	--	--
40	0.882	0.419	0.881	0.422
50	0.819	0.509	--	--
60	0.744	0.589	0.741	0.593
70	0.658	0.658	--	--
80	0.564	0.715	0.560	0.719

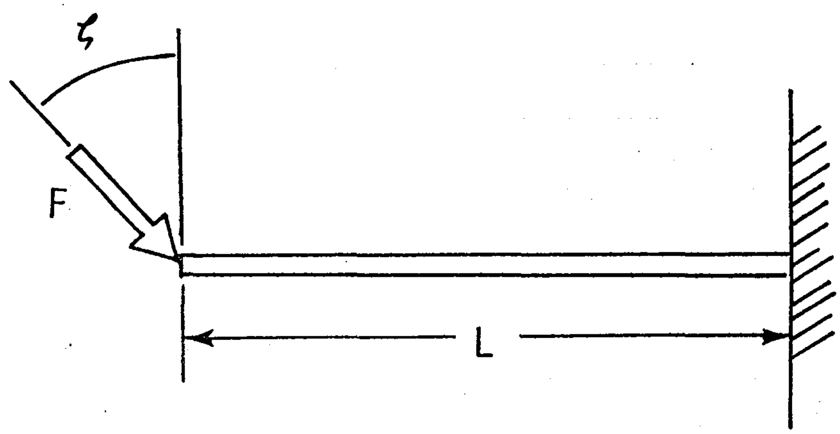

Figure 1. Cantilever beam with arbitrarily directed tip load.

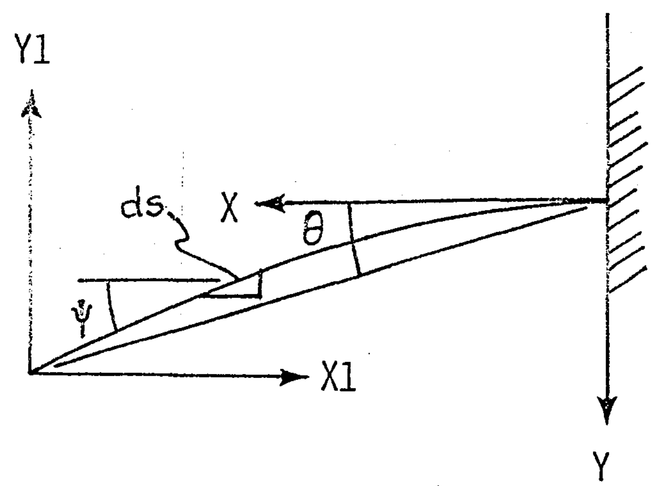

Figure 2. Coordinate systems and slope of deflected beam.

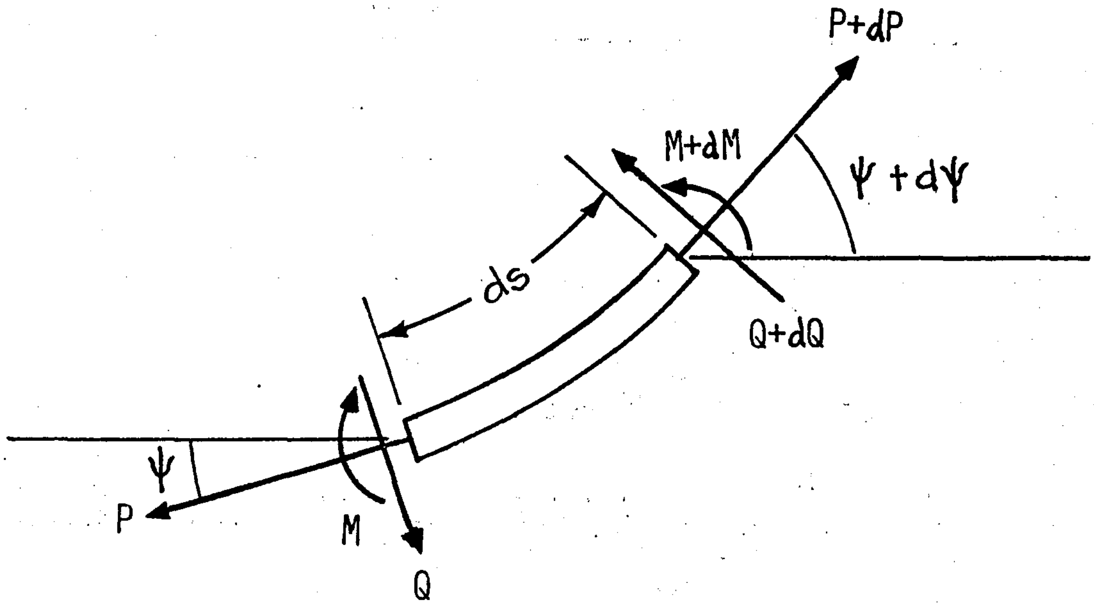

Figure 3. Equilibrium of beam element.

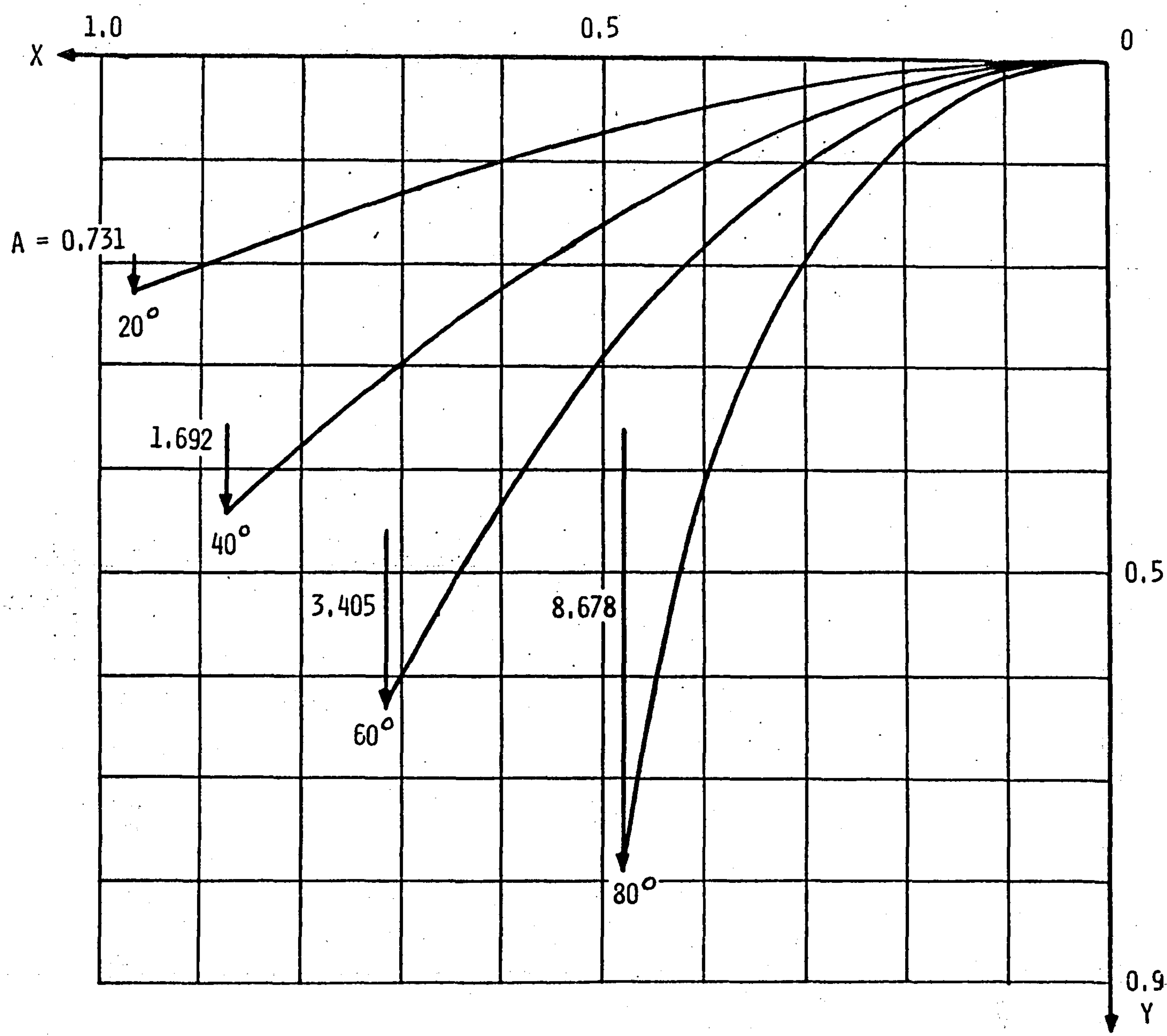

Figure 4. Deflection curves for cantilever beam under vertical tip load.

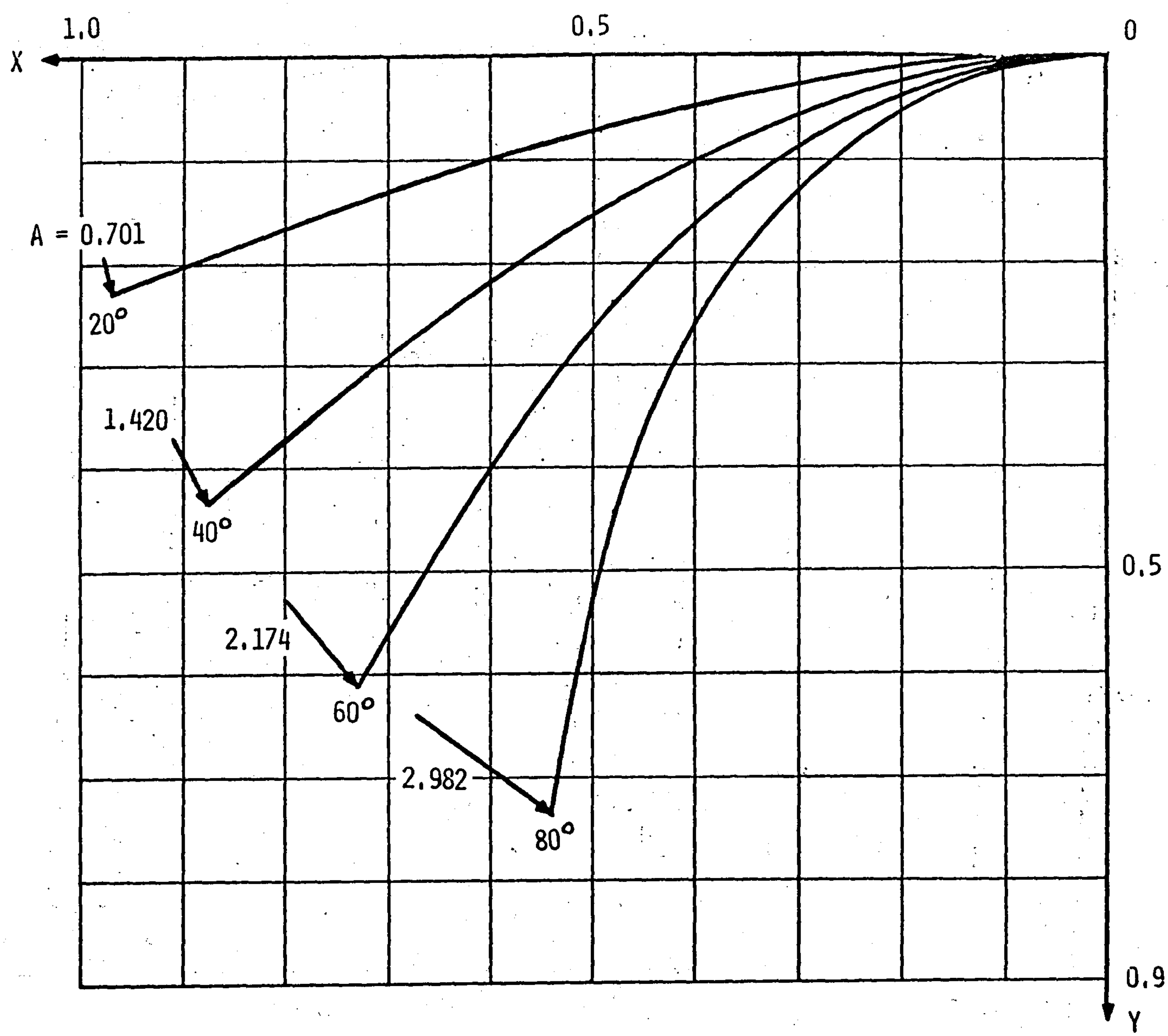

Figure 5. Deflection curves for cantilever beam under slewing tip load.

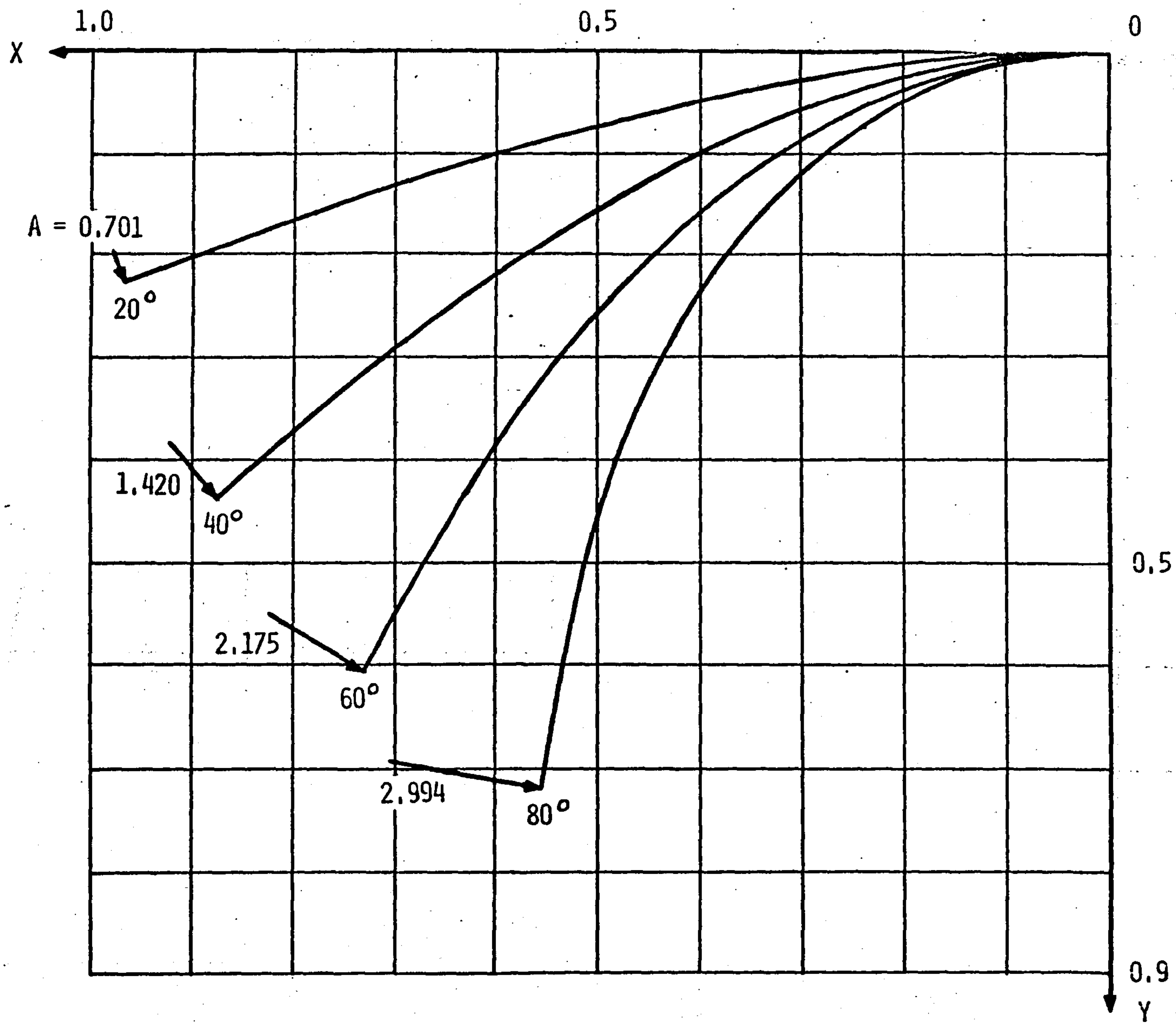

Figure 6. Deflection curves for cantilever beam under follower tip load.

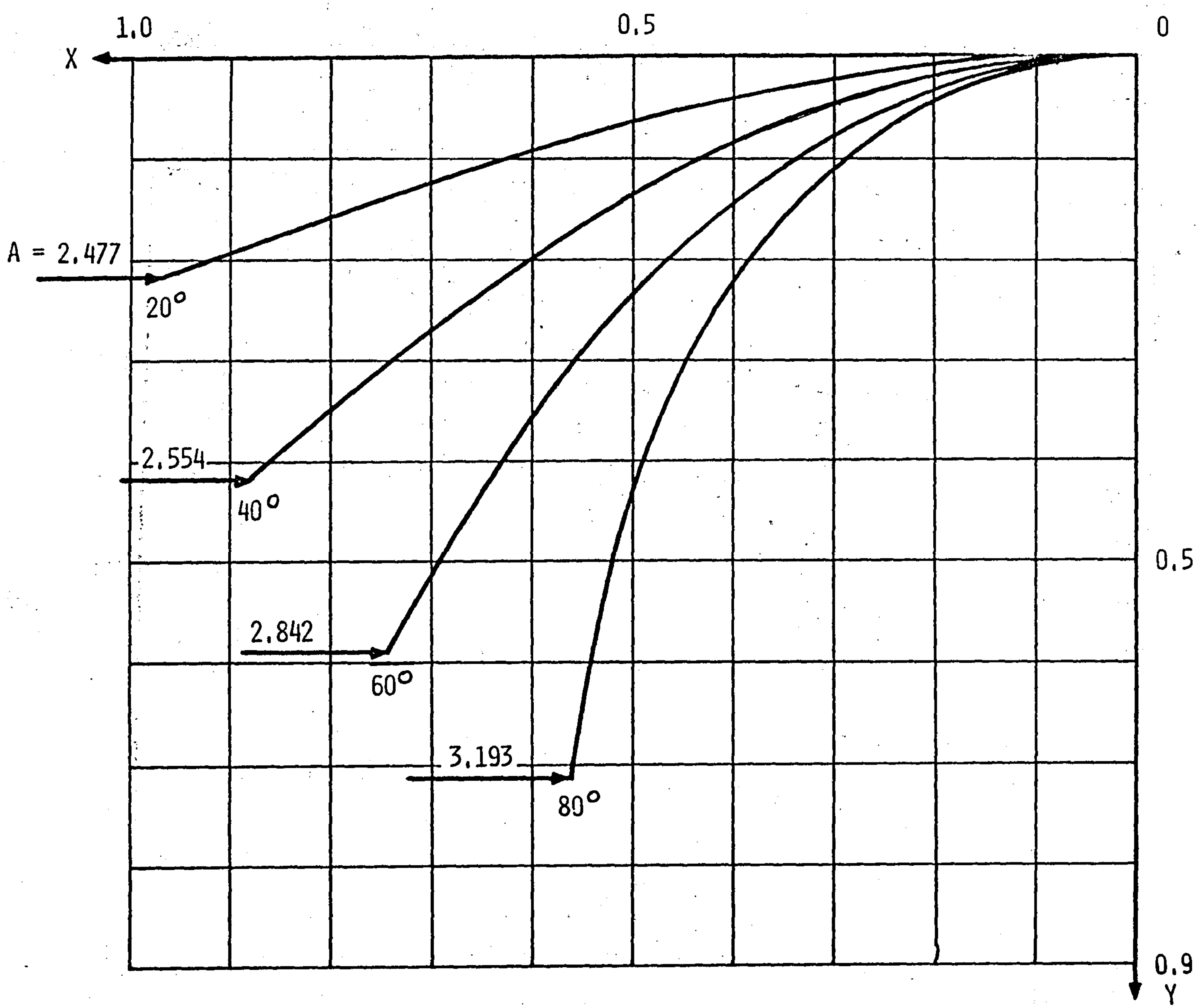

Figure 7. Deflection curves for cantilever beam under horizontal tip load.

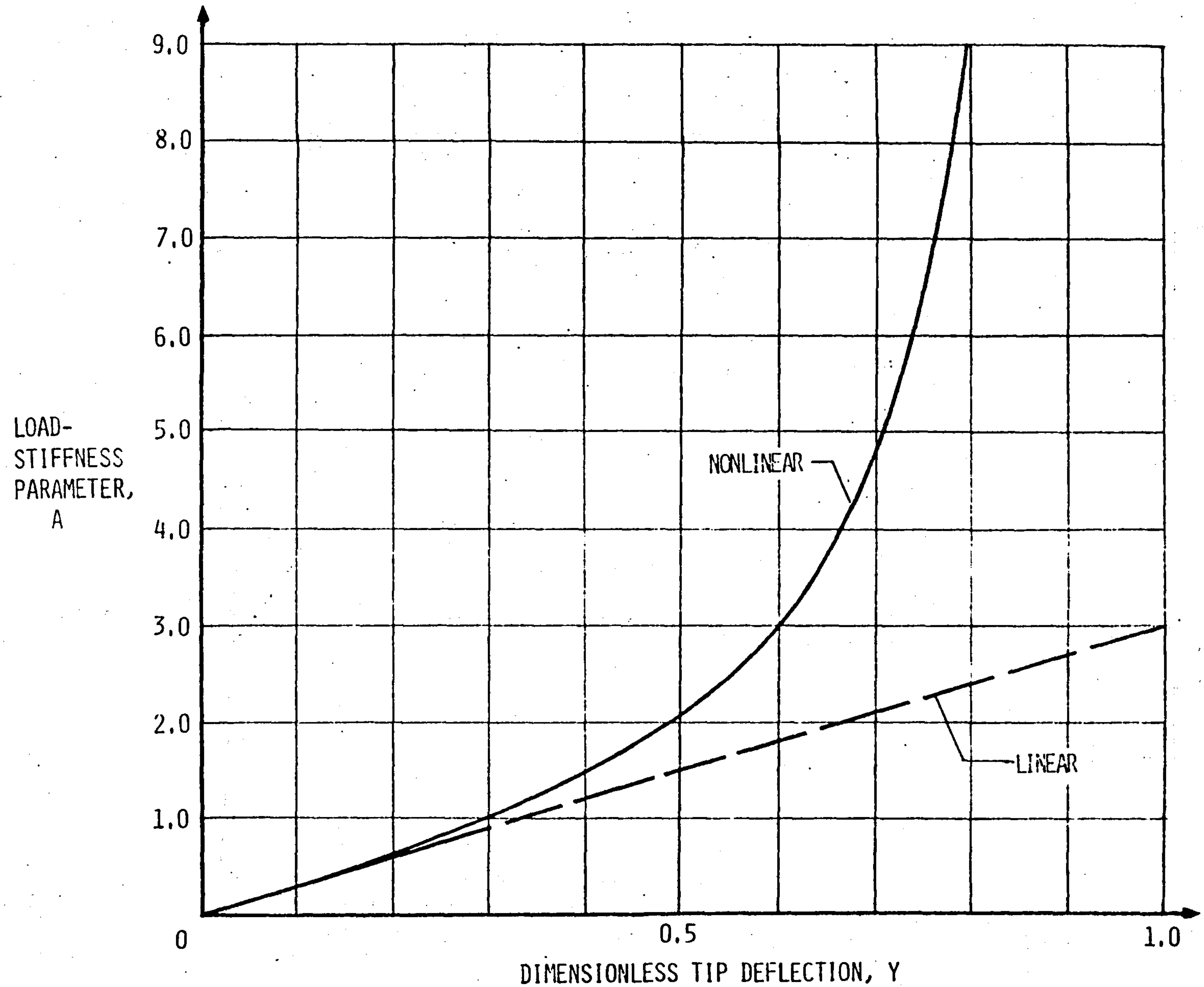

Figure 8. Load-stiffness parameter as a function of vertical tip deflection for cantilever beam under vertical tip load.

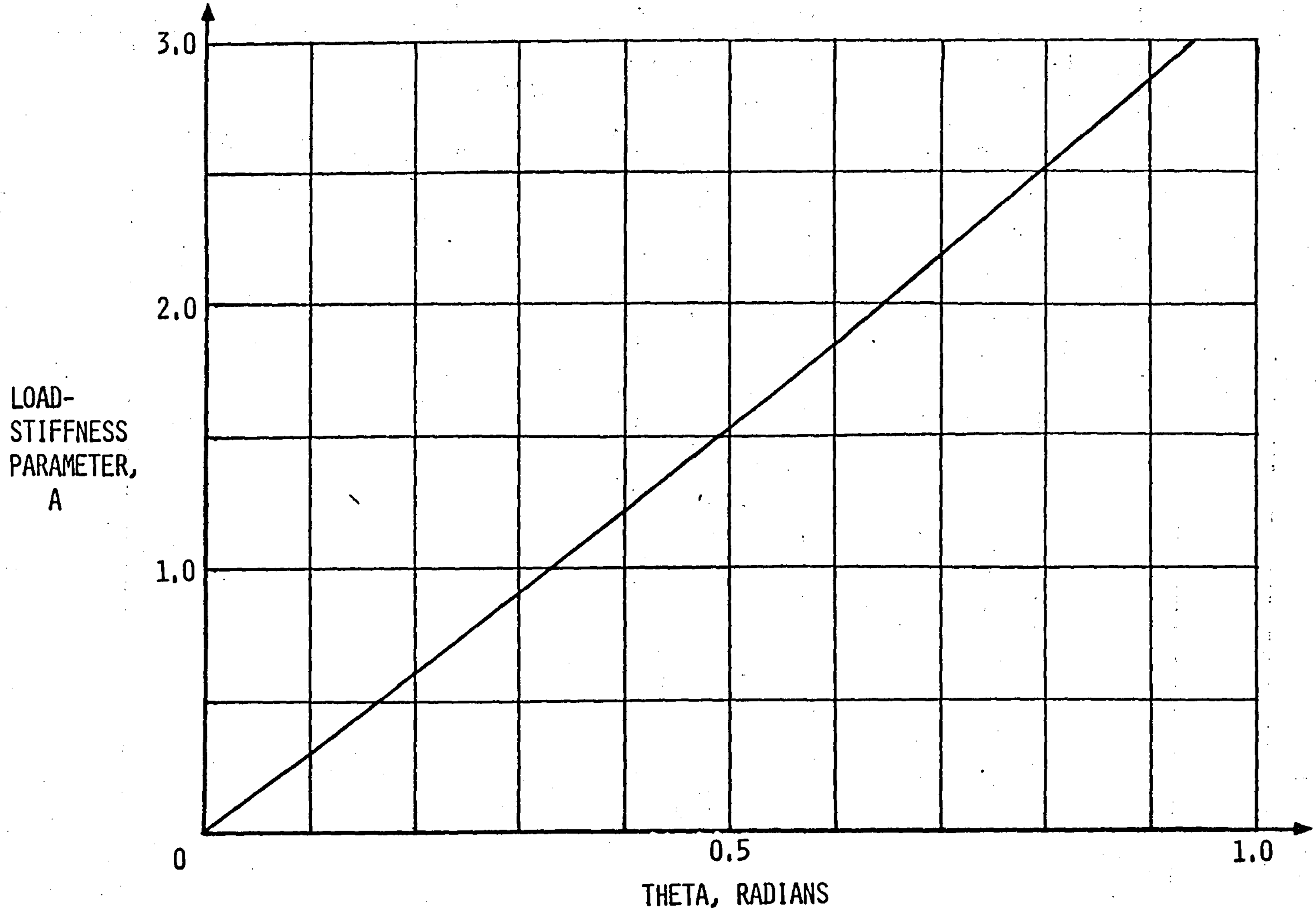

Figure 9. Load-stiffness parameter as a function of angle θ (between horizontal and line from root to tip) for cantilever beam under slewing tip load.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-86442		

4. Title and Subtitle	5. Report Date
Large Deflections of a Cantilever Beam Under Arbitrarily Directed Tip Load	May 1985
	6. Performing Organization Code 506-53-43-15
7. Author(s) Harvey G. McComb, Jr.	8. Performing Organization Report No.
	10. Work Unit No.
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665	11. Contract or Grant No.
	13. Type of Report and Period Covered Technical Memorandum
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The nonlinear beam equation is integrated numerically in a direct fashion to obtain results for large deflections of cantilevers under tip loads of arbitrary direction. A short BASIC computer program for performing this integration is presented. Results for selected load cases are presented. The numerical process is performed rapidly on a modern microcomputer, and comparisons in the paper with results from closed form solutions from the literature show that the process is accurate.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Cantilever beams Nonlinear structural analysis Large deflections	Unclassified-Unlimited Subject Category 39

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	22	A02

N-305

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