Excitation of Turbulence by Density Waves

C. M. Tchen

NASA

NASA Contractor Report 3905

Excitation of Turbulence by Density Waves

C. M. Tchen

Universities Space Research Association
Columbia, Maryland

NASA

National Aeronautics
and Space Administration

**Scientific and Technical
Information Branch**

1985

TABLE OF CONTENTS

1. Introduction

Many nonlinear dynamical systems involving density waves can be represented by a system of partial differential equations: a nonlinear equation of wave propagation and an equation of evolution for a certain dynamical variable. The dynamical variable may be a field as in the Schrödinger equation for solitons, or may be a fluid velocity as in the Navier-Stokes equation of motion for compressible turbulence. The general study of coupling between the kinetic energy of turbulence and the wave energy belongs to the topic of acoustic turbulence,[1,2], while the escitations of turbulence by wave motions cause a density-induced turbulence. The latter is treated here, because it occurs more frequently in the atmosphere in view of the general presence of the internal gravity wave and the mean density gradient.

A kinetic method is developed in order to include those collective phenomena which are otherwise not evident. By group-scaling, the master equation is decomposed into a hierarchy of scaled equations for the velocity distributions of decreasing order of correlation times, representative of the macro-evolution, the transport property and the relasation. A cluster of distributions participates into the role of relasation for the approach of the transport property to equilibrium. If the cluster loses the individual identity of velocity distributions by moving together and behaving like a clump, an effective medium can be assumed and a closure is found.

4

By means of propagators that are modified to include the damping from the **effective** medium, the eddy collision is calculated as an operator, and the kinetic equation for the macro-distribution is derived. It contains a memory representative of the collective behavior.

Based upon the kinetic equation. the transport theories for the eddy diffusivity and eddy viscosity are developed. They calculate the transport functions to govern the following processes. A mean density gradient or a large scale gravity wave may escite all the nonlinear mechanisms that control the density waves, i.e. the coupling mechanism for the excitation of turbulence and the cascade transfer from large toward small eddies. By the more efficient coupling mechanism, the density waves feed energy into turbulence that cascades down the velocity spectrum.

The transfer function is found to have two memory-loss functions: one from the eddy collision as an operator, and the other from the Lagrangian correlation between the velocity distribution and the fluid velocity. The two memory-losses formulate a direct cascade toward small eddies and a reverse cascade toward large eddies. respectively. The former is relevant to the coupling subrange.

The spectral intensity in the coupling subrange are found to have a k^{-2} law for the velocity and field (i.e. pressure gradient) fluctuations. and a k^{-4} law for the density fluctuations.

The formation of sawtooths[3] in compressible turbulence can

be seen in numerical computation&. yielding a spectral intensity k^{-2}, in deviation from the Kolmogoroff law[5,6] of incompressible turbulence.

2. Basic equations for the description of the microdynamical state of density-excited turbulence.

The hydrodynamic variables are the fluid velocity $\hat{u}$. the density $\hat{\rho}$ and the pressure $\hat{p}$. We write the density

$$\hat{N}(t,\underline{x}) = \hat{\rho}(t,\underline{x})/\rho_{\infty} \qquad (1)$$

that is normalized **by** means of a constant density ρ_{∞} , and write the thermal speed

$$\hat{c} = (d\hat{p}/d\hat{\rho}) . \qquad (2)$$

A fluctuating variable is denoted by the symbol $(\hat{\ })$, with

$$(\hat{\ }) = (\overline{\ }) + (\tilde{\ }) . \qquad (3)$$

It consists of the superposition of the ensemble average $(\overline{\ }) = \langle\rangle$ and the deviation from the ensemble average, called the fluctuation $(\tilde{\ })$. For such a decomposition, we may use the operators $\overline{A}$, $\tilde{A}$, such that

$$\overset{n}{A} = \overline{A} + \tilde{A}, \qquad (4)$$

where $\hat{A} = 1$ is the unit operator. The decomposition of a fluctuating function into an average and a fluctuation. as by (4), is known as the Reynolds decomposition in turbulent flows.

For a proper selection, we first examine the following models for the description of the microdynamical state of turbulence.

(a) full Navier-Stokes model

The Navier-Stokes system consists of the equation of momentum

$$\partial_t \hat{N} \hat{u}_i + \nabla_j \hat{N} \hat{u}_j \hat{u}_i = \hat{N} \hat{E}_i \tag{5}$$

and the equation of continuity

$$\partial_t \hat{N} + \nabla_j \hat{N} \hat{u}_j = 0. \tag{6}$$

The driving field $\overset{n}{\underset{\sim}{E}}$ is due to the pressure gradient and is given by the relation

$$\hat{\underset{\sim}{E}} = -\hat{c}^2 \underset{\sim}{\nabla} \ln \hat{N} \tag{7}$$

between $\hat{\underset{\sim}{E}}$ and $\hat{N}$. The differentials are

$$\partial_t \equiv \partial/\partial t , \qquad \underset{\sim}{\nabla} \equiv \partial/\partial \underset{\sim}{x} .$$

and like other operators apply to the functions which follow.

(b) acoustic turbulence model

For describing the interaction between turbulence and finite amplitude density waves, we can transform the full Navier-Stokes system in model (a) into the following simpler

momentum equation

$$\left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla}\right)\hat{\underline{u}} = \hat{\underline{E}} \tag{8}$$

and the equation of propagation of density waves

$$\left(\partial_t^2 - \underline{\nabla} \cdot \hat{c}^2 \cdot \underline{\nabla}\right)\hat{N} = \hat{r} \,, \tag{9}$$

with

$$\hat{r} \equiv \underline{\nabla}\,\underline{\nabla} : \hat{N}\,\hat{\underline{u}}\,\hat{\underline{u}} \,. \tag{10}$$

The relation between $\hat{\underline{E}}$ and $\hat{N}$ remains to be (7).

The equation of propagation is obtained by cross differentiations of (5) and (6) with respect to t and $\underline{x}$ and by a subtraction

It is to be recalled that the simple form of momentum equation (8) is obtained by an elimination of $\hat{N}$ between the two equations (5) and (6) of the full Navier-Stokes system. The same simple form of momentum equation is valid for incompressible turbulence as well with the divergence-free condition

$$\underline{\nabla} \cdot \hat{\underline{u}} = 0 \,. \tag{11}$$

(c) Riemann's model

In order to show the coupling between momentum and density

J

in a concrete manner, we transform the Navier Stokes system into the Riemann system in the form[7] :

$$\left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla}\right)\hat{\underline{u}} = -c \underline{\nabla}\hat{w} \tag{12}$$

$$\left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla}\right)\hat{w} = -c \underline{\nabla} \cdot \hat{\underline{u}} . \tag{13}$$

The Riemann variable

$$\overset{a}{\hat{w}} = c \ln \hat{N} \tag{14}$$

relates $\hat{w}$ and $\hat{N}$, with a constant speed of **sound** c in isothermal gas and a variable speed in adiabatic gas.[8]

Upon multiplying (12) and (13) by $\tilde{\underline{u}}$ and $\tilde{w}$, respectively, and averaging, we find the equations of evolution

$$\left\langle \tilde{\underline{u}} \tilde{A} \left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla}\right)\tilde{\underline{u}} \right\rangle = W \tag{15}$$

$$\left\langle \tilde{w} \tilde{A} \left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla}\right)\tilde{w} \right\rangle = -W, \tag{16}$$

for the kinetic energy and the potential energy in homogenous turbulence. The equations are coupled by the function

$$W = \left\langle \tilde{\underline{u}} \cdot \tilde{\underline{E}} \right\rangle \tag{17}$$

with a net zero balance.

With the supply 'by external source, the wave energy feeds an amount W to the kinetic energy. The latter then undergoes an internal cascade transfer from large into smaller eddies, to be

ultimately dissipated by viscosity.

(d) kinetic model

It is not difficult to verify that the two equations (5) and (6) of the Navier-Stokes system are the first two moments of the master equation, written as[9]

$$\left(\frac{\partial}{\partial t} + \hat{L}\right) \hat{f}(t, \underline{x}, \underline{v}) = 0 \tag{18}$$

for the distribution function in the form

$$\hat{f}(t, \underline{x}, \underline{v}) = \hat{N}(t, \underline{x}) \, \delta\left[\underline{v} - \hat{\underline{u}}(t, \underline{x})\right] \tag{19}$$

as the condition of the normalization. i.e.

$$\int d\underline{v} \, \hat{f}(t, \underline{x}, \underline{v}) = \hat{N}(t, \underline{x}) \ . \tag{20}$$

The differential operator in the phase space $t, \underline{x}, \underline{v}$ is:

$$\hat{L}(t, \underline{x}, \underline{v}) \equiv \underline{v} \cdot \underline{\nabla} + \hat{\underline{E}}(t, \underline{x}) \cdot \underline{\partial} \tag{21}$$

with $\underline{\partial} = \partial/\partial \underline{v}$.

(e) wave-kinetic model

The present model uses the master equation (18) with the condition

$$\hat{f}(t, \underline{x}, \underline{v}) = \delta\left[\underline{v} - \hat{\underline{u}}(t, \underline{x})\right] \ , \tag{22}$$

to replace the momentum equation (8) of model (b), while retaining the equation of propagation (9) and the relation (7) for defining $\hat{\underset{\sim}{E}}$-field.

Being homogenous and having the velocity $\underset{\sim}{v}$ as an independent phase variable, the master equation has lesser nonlinearity. It is most equipped to treat the collisionless transport properties for the wave-induced turbulence. It provides a mechanism of collisionless damping as the result of the interaction between the velocity distribution and waves. This property cannot be derived from the hydrodynamic approach. Among the kinetic methods, the wave-kinetic model is preferred, because it describes more explicitly the interaction between finite amplitude density waves and turbulence.

The transformation of the Navier-Stokes equation into the master equation and the subsequent derivation of the kinetic equation of turbulence have been attempted by Monin[10] and Lundgren.[11] But thes encountered the difficulty of kinetic hierarchy like in Bogoliubov's theory.[12] To break the hierarchy, we introduce a method of decomposition into many scales.

3. Multi-scale group decomposition

The hydrodynamic equations and the master equation discussed above describe the microdynamical state of

turbulence. They contain too many minute details which are not suitable and necessary for the statistical treatment. We introduce a course-graining procedure called group-scaling by writing

$$\tilde{A} = A^o + A' \tag{23a}$$

$$A' = A^1 + A'' \quad , \tag{23b}$$

as an extension of the Reynolds decomposition (4). The three groups

$$A^o , A' , A'' \tag{24a}$$

are called the macro-group, the micro-group and the submicro-group, respectively. while A^1 is called the submacro-group. The groups (24a) and the following groups

$$A^o , A^1 , A^{(2)} , \ldots \tag{24b}$$

must conform to the scaling differential

$$\tau_c^{[o]} \qquad \tau^{[\prime]} \qquad \tau^{[} \tag{25}$$

of decreasing duration of correlations, indicating a decreasing coherence.

The three groups (24a) represent the three processes of spectral evolution, transport property and relasation. The latter governs the approach of the transport property to equilibrium. The decomposition gives the possibility of exploiting the property of local quasi-homogeneity between any two groups. This assumption implies that the interaction among the groups are restricted to nearest neighbor group-pairs.

The fluctuating groups. as formed by the scaling operators

(24), are denoted bv the superscripts

$$( )^{o}, ( )^{1}, ( )^{(2)}, \ldots \tag{26a}$$

and

$$( )^{\overset{o}{.}}, ( )^{.}, ( )^{"} \tag{26b}$$

in the open sequence and the closed sequence, respectively.

For completeness we should add the cumulative operators

$$A_{o} = \overline{A} + \overset{o}{A} \tag{27a}$$

$$A_{1} = \overline{A} + \overset{o}{A} + A' \tag{27b}$$

$$A_{2} = \overline{A} + \overset{o}{A} + A' + A'' \; , \tag{27c}$$

giving the cumulative groups

$$( )_{o}, ( )_{1}, ( )_{2} . \tag{28}$$

The deterministic transport properties and transport functions, as shaped by these fluctuating groups, are denoted **by** the superscripts

$$( )^{[o]}, ( )^{[1]}, ( )^{[']}, ( )^{[''}\tag{29}$$

with square brackets.

The fluctuating groups may be Fourier transformed to find their **cumulative** spectral distributions, as follows

$$\frac{1}{2} \langle \underset{\sim}{E}^{o2} \rangle = \int_{0}^{k} dk' \; F_{E}(k') \tag{30a}$$

$$\frac{1}{2}\langle \underline{E}'^2 \rangle = \int_{k}^{\infty} dk'' \, F_E(k'') \tag{30b}$$

$$\frac{1}{2}\langle \underline{E}''^2 \rangle = \int_{k''}^{\infty} dk''' \, F_E(k''') \, , \tag{30c}$$

with the inequalities

$$k''' \geqslant k'' \geqslant k \geqslant k' \tag{31}$$

among the wavenumbers. They are independent variables or variables of integrations. The limits of integrations are the demarcation wavenumbers of the three groups. The cumulative spectral distribution obviously yields the spectral density $F_E(k)$ of field fluctuations by a simple differentiation with respect to k.

4. Kinetic treatment of turbulence

4.1 Effective medium approximation for the closure of kinetic hierarchy

In the statistical mechanics of many particles, the master equation generates a hierarchy of many-particle distribution functions. Bogoliubov closed it at the triplet distribution in an arbitrary manner. [**] Analogously, our master equation is expected to generate a hierarchy of scaled

1

equations by the group decomposition. We may choose a closed sequence (24a) or an open sequence (24b). The latter contains more information and is investigated here.

The scaling of our master equation by the operators of open sequence yields the scaled equations

$$\left(\tfrac{\partial}{\partial t} + L_0\right) f^\circ = -L^\bullet \bar{f} + Z^\circ - \bar{Z} \tag{32}$$

$$\left(\tfrac{\partial}{\partial t} + L_1\right) f^1 = -L^1(\bar{f} + f^\circ) + Z^1 - Z^\circ, \, ... \tag{33}$$

for the distribution of open groups, with the differential operators

$$L_0, \; L_1, \cdots \tag{34}$$

from (21). The scaled equations have their eddy collisions

$$\bar{Z} \equiv -\bar{A} L^\circ f^\circ = \mathcal{C}^{[\circ]}\{\bar{f}\} \tag{35a}$$

$$Z^\bullet \equiv -A^\circ L^1 f^1 = \mathcal{C}^{[1]}\{f^\circ\} \tag{35b}$$

$$Z^1 \equiv -A^1 L^{(2)} f^{(2)} = \mathcal{C}^{[2]}\{f^1\}, \; \cdots , \tag{35c}$$

with the collision operators

$$\mathcal{C}^{[\circ]}, \; \mathcal{C}^{[1]}, \; \mathcal{C}^{[2]}, \cdots . \tag{36}$$

Through collisions each group is coupled to the next one by developing a hierarchy. Physically speaking, (32) determines

the evolution of the velocity $\underline{u}^o$ and of its kinetic energy, and
(33) determines the transport property by (35b). Finally, the
cluster

$$f'' = f^{[2]} + f^{[3]} + \ldots \tag{37}$$

organizes a relaxation process for the transport property to
approach its equilibrium and is called the relaxational
cluster.

Two methods are available for the closure. Firstly, the
equation

$$\left(\partial_t + A'' L_2\right) f'' = -L'' f_2 \tag{38}$$

without collision may serve as a closing equation. It will
calculate the collision by writing

$$Z^1 \cong Z' \tag{39}$$

and

$$Z' = -A' L'' f'' = \mathcal{C}''\{f^1\} . \tag{40}$$

The early cutoff neglects the collective behavior of
long range. Secondly, we realize that the problem of hierarchy
can be found in many nonlinear dynamical systems. The familiar
ones are the hydrodynamics of long chain molecules[13] and the
suspension of particles in Brownian movements.[14] For these
systems, the concept of an effective medium[15-16] that offers a

Darcy damping has been introduced. Likewise, Heisenberg postulated an effective viscosity in his theory of turbulence. Hence as the second method of closure, we shall assume that the relaxational cluster of many distributions represents in its ensemble a porous medium that is frictional and presents a similar Darcy damping. We shall determine this damping by a self-consistent theory of collective phenomena.

To elucidate this physical picture analytically, we take the moment of (33) to get the momentum equation in the form:

$$\left(\partial_t + u_1 \cdot \nabla\right) u^1 = - u^1 \cdot \nabla\left(\bar{u} + u^0\right) + E^1 + J^1 . \qquad (41)$$

The hydrodynamic friction is found as

$$J^1 \equiv \int dv \, v \, Z^1$$

$$\cong \int dv \, v \, C''\{f^1\}$$

$$= -\gamma^{[\nu]} u^1 , \qquad (42)$$

by (39) and (40). It is proportional to the coefficient of damping $\gamma^{[\nu]}$. Although it assumes the same role as the Darcy damping, it finds a self-consistent kinetic basis in (42).

For the role of relasation, it is more important to retain the collective behavi'or than to discriminate the phase individuality among the distributions in the cluster. Hence we can replace the cluster by an effective medium with the property (42), and write the approximation

$$\mathcal{G}^{[\nu]}\{\} \cong \mathcal{G}^{[\nu]}\{\}\Big|_{\underline{v}=0} \equiv -\gamma^{[\nu]}\{\} \ . \tag{43}$$

This simplifies (33) into the following

$$\left(\partial_t + \mathcal{L}_1\right)f^1 = -L^1(\bar{f}+f^0) - Z^0 , \tag{44}$$

with the differential operator

$$\mathcal{L}_1 \equiv L_1 + \gamma^{[\nu]} \ . \tag{45}$$

5.2 Local quasi-homogeneity

The scaling differential (25) permits the assumption of quasi-homogeneity among groups. Thus a low-order group of long correlation time can be assumed to vary more slowly than does a higher-order group of shorter correlation time. BY applying the property of quasi-homogeneity at various levels, we find that the interactions between the nearest-neighbor groups are the most pronounced. Under this circumstance, we can simplify (32) and (44) into

$$\left(\partial_t + L_0\right)f^0 = -L^0\bar{f} + Z^0 \tag{46a}$$

$$\left(\partial_t + \mathcal{L}_1\right)f^1 = -L^1 f^0 \ . \tag{46b}$$

5.3 Macro-kinetic equation

The integration of (46b) will be made by the intermediary of the propagator

1

satisfies the differential equation

$$\left(\partial_t + \mathcal{L}_1 \right) \mathcal{U}(t,t_1) = 0 .$$

(47)

The omission of $Z^{\bullet}$ in (46b) implies **9**

$$\mathcal{U}(t,t-\tau) \cong \bar{\mathcal{U}}(t,t-\tau) .$$

(48)

By assumption (48), we may use the average propagator to calculate

$$f^1 = -\int_0^t d\tau \; A^1 \bar{\mathcal{U}}(t,t-\tau) \; L^1(t-\tau) \{ f^0 \} ,$$

(49)

with $\mathcal{L}^1 = L^1$. By a substitution into (35b), we calculate the eddy collision

$$
\begin{aligned}
Z^{\bullet} &\equiv - A^{\circ} L^1 f^1 \\
&= A^{\circ} \int_0^{t \to \infty} d\tau \; L^1(t, \underset{\sim}{x}, \underset{\sim}{v}) \, \bar{\mathcal{U}}(t,t-\tau) L^1(t-\tau) \{ f^0 \} \\
&= \mathcal{C}^{[1]} \{ f^0 \} \\
&\simeq \mathcal{C}^{[\prime]} \{ f^0 \} ,
\end{aligned}
$$

(50)

with the collision coefficient

$$
\begin{aligned}
\mathcal{C}^{[\prime]} &\equiv \int_0^{t \to \infty} d\tau \left\langle L'(t, \underset{\sim}{x}, \underset{\sim}{v}) \, \bar{\mathcal{U}}(t,t-\tau) L'(t-\tau) \right\rangle \\
&= \underset{\sim}{\partial} \cdot \underset{=}{D}^{[\prime]} \cdot \underset{\sim}{\partial} ,
\end{aligned}
$$

(51)

and the eddy diffusivity

$$\underset{\approx}{D}^{[\prime]} = \int_0^{t \to \infty} d\tau \left\langle \underset{\sim}{E}'(t,\underset{\sim}{x}) \, \bar{U}(t,t-\tau) \, \underset{\sim}{E}'(t-\tau) \right\rangle. \tag{52}$$

By the scaling differential

$$t > \tau_c^{[\nu]}$$

the upper limit t has been put **to** ∞ without altering the value of the integral. The asymptoticity renders the diffusivity deterministic.

Finally upon substituting (50) into **(46a)**, we derive the kinetic equation for the macro-distribution in the form

$$\left(\partial_t + \bar{L} + L^\circ\right) f^\circ = -L^\circ \bar{f} + \mathcal{C}^{[\prime]} \{f^\circ\}, \tag{53a}$$

that can be rewritten as

$$\left(\partial_t + \bar{L} + L^\circ\right) f^\circ = -L^\circ \bar{f} + \underset{\sim}{\partial} \cdot \underset{\approx}{D}^{[\prime]} \{\cdot \underset{\sim}{\partial} f^\circ\} \tag{53b}$$

by (51). The macro-kinetic equation possesses a memory as belonging to the non-Markovian behavior through the operator

$$\underset{\approx}{D}^{[\prime]}\{\}.$$

5. Derivation of the hydrodynamic equations **of** turbulence **from** the kinetic equation

By taking the first moment of the macro-kinetic equation (53) and with the condition (22), we obtain the momentum

equation

$$\partial_t \underset{\sim}{u}^{\circ} + A^{\circ} \underset{\sim}{u}^{\circ} \cdot \nabla \underset{\sim}{u}^{\circ} = \underset{\sim}{E}^{\circ} + \underset{\sim}{J}^{\circ} \ . \tag{54}$$

The hydrodynamic friction takes the form

$$\underset{\sim}{J}^{\circ} = \int d\underset{\sim}{v} \ \underset{\sim}{v} \ \mathcal{C}^{[']} \{ f^{\circ} \} \ , \tag{55a}$$

or

$$\underset{\sim}{J}^{\circ} = \int d\underset{\sim}{v} \ \underset{\sim}{v} \ \underset{\sim}{\partial} \cdot \underset{\approx}{D}^{[']} \{ \cdot \underset{\sim}{\partial} f^{\circ} \} \ . \tag{55b}$$

by (51).

Since the macro-kinetic equation (53) is an equation espli it in f° without the involvement of f', its moments will generate $\underset{\sim}{u}^{o}$ and not $\underset{\sim}{u}'$. This is a consequence of the irrever-sibility of the macro-kinetic equation. This property has been observed in deriving the momentum equation (54).

The equation of kinetic energy is obtained upon multi-plying (54) by $\underset{\sim}{u}^{o}$ and taking an ensemble average. It has the form

$$S^{[o]} = W^{[o]} - T^{[o]} \ , \tag{56}$$

with the following transport functions:

(i) The function

$$S^{[o]} \equiv \frac{1}{2} \left\langle \left(\partial_t + \underset{\sim}{u}^{\circ} \cdot \nabla \right) \underset{\sim}{u}^{\circ 2} \right\rangle \tag{57}$$

is the rate of change of the kinetic energy.

(ii) The transfer function

$$T^{[o]} \equiv - \langle \underset{\sim}{u}^o \cdot \underset{\sim}{J}^o \rangle \tag{58}$$

governs the cascade transfer across the spectrum. The direct cascade is a transfer toward the large wavenumbers, and the reverse cascade is a transfer toward the small wavenumbers.

(iii) The coupling function

$$W^{[o]} \equiv \langle \underset{\sim}{u}^o \cdot \underset{\sim}{E}^o \rangle \tag{59}$$

governs the coupling between the kinetic energy spectrum and the density spectrum. It represents an excitation of the kinetic energy of turbulence by the density rarefaction, i.e. $W^{[o]} > 0$.

6. Transport theory of diffusion

6.1 Laarannian correlation

The diffusivity $\underset{\sim}{D}^{[\cdot]}$ that is basic to the macro-kinetic equation is defined by (52) as the time integration of the Lagrangian correlation

$$\langle \underset{\sim}{E}'(t,\underset{\sim}{x}) \, \overline{\underset{\sim}{\mathcal{U}}}(t,t-\tau) \, \underset{\sim}{E}'(t-\tau) \rangle$$

of $\underset{\sim}{E}'$-field fluctuations, where $\overline{\underset{\sim}{\mathcal{U}}}$ is the propagator for the evolution of $\underset{\sim}{E}'(t-\tau)$ along the trajectory that is perturbed by the differential operator (45).

By Fourier decomposition of $\underset{\sim}{E}'(t,\underset{\sim}{x})$ into the frequency ω'' and wavenumber $\underset{\sim}{k}''$, we have

$$\underset{\sim}{E}'(t,\underset{\sim}{x}) = \iint d\omega'' d\underset{\sim}{k}'' \, \underset{\sim}{E}'(\omega'',\underset{\sim}{k}'') e^{i(\omega''t - \underset{\sim}{k}'' \cdot \underset{\sim}{x})}. \tag{60}$$

The Lagrangian correlation has the form

$$\left\langle \underline{E}'(t,\underline{x})\, \bar{\underline{U}}(t,t-\tau)\, \underline{E}'(t-\tau) \right\rangle = \underline{\underline{A}}^{[\cdot]}(\omega'',\underline{k}'')\, e^{-i(\omega''-\underline{k}''\cdot\underline{v})\tau}$$
$$\times\, e^{-\gamma_k^{[\nu]}\tau}\, \left\langle e^{i\underline{k}''\cdot\hat{\underline{\lambda}}(\tau)} \right\rangle , \tag{61}$$

where $\hat{\underline{\lambda}}(\tau)$ is the path-length in the time interval τ. **The damping** $\gamma^{[\nu]}$ as a differential operator **has** its Fourier form $\gamma_k^{[\nu]}$. The first factor

$$\underline{\underline{A}}^{[\cdot]}(\omega'',\underline{k}'') \equiv \chi \left\langle \underline{E}'(\omega'',\underline{k}'')\, \underline{E}'(-\omega'',-\underline{k}'') \right\rangle \tag{62}$$

in the right hand side is the spectral function for the field fluctuations, and

$$\chi \equiv \frac{\pi}{T}\left(\frac{\pi}{X}\right)^3 \tag{63}$$

is the factor of truncation in a Fourier decomposition that is truncated within a time interval 2T and a length interval 2X in three dimensions. The spectral function is followed by a number of esponential functions, called the components of the orbit function.

The component

$$h_v(\omega'',\underline{k}'';\tau,v) = e^{-i(\omega''-\underline{k}''\cdot\underline{v})\tau} \tag{64}$$

represents the linear streaming at velocity $\underline{v}$ in an unperturbed trajectory. It defines the Eulerian correlation

$$\left\langle \underline{E}'(t,\underline{x})\, \underline{E}'(t-\tau,\underline{x}-\underline{v}\tau) \right\rangle = \iint d\omega''\, d\underline{k}''\, \underline{\underline{\mathcal{A}}}^{[']}(\omega'',\underline{k}'')\, h_{\underline{v}}(\omega'',\underline{k}'',\tau,\underline{v}), \tag{65}$$

and forms the diffusivity

$$\underline{\underline{D}}^{[']} = \iint d\omega''\, d\underline{k}''\, \underline{\underline{\mathcal{A}}}^{[']}(\omega'',\underline{k}'')\int_0^\infty d\tau\; h_{\underline{v}}(\omega'',\underline{k}'',\tau,\underline{v})$$
$$= \tfrac{1}{2}\iint d\omega''\, d\underline{k}''\, \underline{\underline{\mathcal{A}}}^{[']}(\omega'',\underline{k}'')\,\delta(\omega''-\underline{k}''\cdot\underline{v}) \tag{66}$$

in weak turbulence.

The component

$$h_{\gamma}(\underline{k}'',\tau) \equiv e^{-\gamma_k^{['']}\tau} \tag{67}$$

represents the effective medium. By

$$\hat{\underline{\ell}} = \underline{\ell}_1 + \underline{\ell}'', \tag{68}$$

the esponential function

$$\left\langle e^{\,i\underline{k}''\cdot\hat{\underline{\ell}}(\tau)} \right\rangle = h_{[1]}(\underline{k}'',\tau)\; h^{['']}(\underline{k}'',\tau) \tag{69}$$

can be written as the product of two components:

$$h_{[1]}(\underline{k}'',\tau) \equiv \left\langle e^{\,i\underline{k}''\cdot\underline{\ell}_1(\tau)} \right\rangle \tag{70}$$

$$h^{['']}(\underline{k}'',\tau) = \left\langle e^{\,i\underline{k}''\cdot\underline{\ell}''(\tau)} \right\rangle. \tag{71}$$

They represent the nonlinear streaming and the diffusive relaxation, respectively, along a trajectory that is perturbed by turbulence.

We are dealing witn strong turbulence, **so** that the linear

effect by the orbital component

$$h_v(\tau, \underset{\sim}{k}'', \underset{\sim}{v}) \cong e^{i\underset{\sim}{k}'' \cdot \underset{\sim}{v}\tau} \qquad (72)$$

for streaming is negligible, escept when calculating the collision (51).

By collecting the components into the orbit function

$$h(\underset{\sim}{k}'', \underset{\sim}{v}, \tau) \equiv h_v(\underset{\sim}{k}'', \underset{\sim}{v}, \tau) \; h_\gamma(\underset{\sim}{k}'', \tau) \; \underset{[1]}{h}(\underset{\sim}{k}'', \tau) \; h^{[v]}(\underset{\sim}{k}'', \tau), \qquad (73)$$

we can write the Lagrangian correlation in the form:

$$\langle E''(t, x) \overline{\mathcal{U}}(t, t-\tau) E'(t-\tau) \rangle \cong \int dk'' \; \underset{\approx}{s}^{[v]}(\underset{\sim}{k}'') \; h(\underset{\sim}{k}'', \tau) \qquad (74)$$

by (61), with

$$\underset{\approx}{s}^{[v]}(\underset{\sim}{k}'') \equiv \chi \langle E'(\underset{\sim}{k}'') E'(-\underset{\sim}{k}'') \rangle \qquad (75)$$

and

$$\chi \equiv (\pi/\lambda)^3 .$$

The diffusivity (52), rewritten as

$$\underset{\approx}{D}^{[1]} = \int dk'' \; \underset{\approx}{s}^{[v]}(\underset{\sim}{k}'') \; \tau_c^{[v]}(\underset{\sim}{k}'', \underset{\sim}{v}) \qquad (76)$$

has a correlation time

$$\tau_c^{[v]}(\underset{\sim}{k}'', \underset{\sim}{v}) = \int_0^\infty d\tau \; h(\underset{\sim}{k}'', \tau, \underset{\sim}{v}). \qquad (77)$$

Here and in the following the integrals without limits are

understood to estend from $-\infty$ to ∞.

6.3 Path dispersion

The evolution of the orbit is governed by the dynamical equations in the form

$$\frac{d\,\hat{\underset{\sim}{x}}(t_1)}{dt_1} = \hat{\underset{\sim}{u}}(t_1)\ , \qquad \frac{d\hat{\underset{\sim}{u}}(t_1)}{dt_1} = \hat{\underset{\sim}{E}}(t_1)\ , \tag{78}$$

with initial conditions

$$\hat{\underset{\sim}{x}}(t_1 = t) = \underset{\sim}{x}\ , \qquad \hat{\underset{\sim}{u}}(t_1 = t) = \underset{\sim}{v}\ , \qquad \hat{\underset{\sim}{E}}(t_1 = t) = \hat{\underset{\sim}{E}}(t, \underset{\sim}{x})\ , \tag{79}$$

or, in the alternative form

$$\frac{d\,\hat{\underset{\sim}{l}}(\tau)}{d\tau} = -\hat{\underset{\sim}{V}}(\tau)\ , \qquad \frac{d\,\hat{\underset{\sim}{V}}(\tau)}{d\tau} = -\hat{\underset{\sim}{E}}(t-\tau)\ , \tag{80}$$

with the conditions

$$\hat{\underset{\sim}{l}}(\tau = 0) = 0\ , \qquad \hat{\underset{\sim}{V}}(\tau = 0) = 0 \tag{81}$$

at $\tau = 0$.

The path

$$\hat{\underset{\sim}{l}}(\tau) = \bar{\underset{\sim}{l}}(\tau) + \tilde{\underset{\sim}{l}}(\tau) \tag{82}$$

can be decomposed into an average

$$\bar{\underset{\sim}{l}}(\tau) = \frac{1}{2}\,\bar{\underset{\sim}{E}}\,\tau^2 \tag{83}$$

and a fluctuation

$$\underaccent{\tilde}{\tilde{\ell}}(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \; \underaccent{\tilde}{\tilde{E}}(t-\tau'') \quad . \tag{84}$$

In stationary turbulence, the t-dependence can be neglected. By (81), the path is independent of $\underaccent{\smile}{v}$.

From (84), we calculate the variance

$$\left\langle \underaccent{\tilde}{\tilde{\ell}}(\tau) \underaccent{\tilde}{\tilde{\ell}}(\tau) \right\rangle = 2 \int_0^\tau d\tau' \left(\tfrac{1}{3}\tau^3 - \tfrac{1}{2}\tau^2\tau' + \tfrac{1}{6}\tau'^3 \right) \left\langle \underaccent{\tilde}{\tilde{E}}(\tau) \underaccent{\tilde}{\tilde{E}}(\tau-\tau') \right\rangle, \tag{85}$$

the diffusivity by path dispersion

$$\underaccent{\approx}{K}_\ell(\tau) \equiv \tfrac{1}{2} \frac{d}{d\tau} \left\langle \underaccent{\tilde}{\tilde{\ell}}(\tau) \, \underaccent{\tilde}{\tilde{\ell}}(\tau) \right\rangle$$

$$= \tau \int_0^\tau d\tau' (\tau - \tau') \left\langle \underaccent{\tilde}{\tilde{E}}(\tau) \, \underaccent{\tilde}{\tilde{E}}(\tau-\tau') \right\rangle . \tag{86}$$

and the integral

$$\int_0^\tau d\tau'' \underaccent{\approx}{K}_\ell(\tau'') \equiv \tfrac{1}{2} \left\langle \underaccent{\tilde}{\tilde{\ell}}(\tau) \, \underaccent{\tilde}{\tilde{\ell}}(\tau) \right\rangle$$

$$= \int_0^\tau d\tau' \left(\tfrac{1}{3}\tau^3 - \tfrac{1}{2}\tau^2\tau' + \tfrac{1}{6}\tau'^3 \right) \left\langle \underaccent{\tilde}{\tilde{E}}(\tau) \underaccent{\tilde}{\tilde{E}}(\tau-\tau') \right\rangle . \tag{87}$$

The asymptotic values are:

(i) for large τ

$$\left\langle \underaccent{\tilde}{\tilde{\ell}}(\tau) \, \underaccent{\tilde}{\tilde{\ell}}(\tau) \right\rangle \simeq \tfrac{2}{3} \tau^3 \underaccent{\approx}{D}(\tau \to \infty, \tag{88}$$

$$\underaccent{\approx}{K}_\ell(\tau) \simeq \underaccent{\approx}{D}(\tau \to \infty) \tau^2 \tag{89}$$

$$k''k'' : \int_0^\tau d\tau'' \underaccent{\approx}{K}_\ell(\tau'') \simeq \omega_D^3 \quad , \tag{90}$$

with

$$\omega_D^3 \equiv \tfrac{1}{3} \underaccent{\smile}{k''}\,\underaccent{\smile}{k''} : \underaccent{\approx}{D} \quad ; \tag{91}$$

(ii) for small τ

$$\left\langle \tilde{\underaccent{\tilde}{l}}(\tau) \, \tilde{\underaccent{\tilde}{l}}(\tau) \right\rangle \cong \frac{1}{4} \tau^4 \left\langle \tilde{\underaccent{\sim}{E}}(\tau) \, \tilde{\underaccent{\sim}{E}}(\tau) \right\rangle \tag{92}$$

$$\underaccent{\approx}{K}_l(\tau) \cong \frac{1}{2} \tau^3 \left\langle \tilde{\underaccent{\sim}{E}}(\tau) \, \acute{\underaccent{\sim}{E}}(\tau) \right\rangle \tag{93}$$

$$\underaccent{\sim}{k}'' \underaccent{\sim}{k}'' : \int_0^\tau d\tau'' \, \underaccent{\approx}{K}_l(\tau'') \cong m^4 \tag{94}$$

with

$$m^4 \equiv \frac{1}{8} \underaccent{\sim}{k}'' \underaccent{\sim}{k}'' : \left\langle \tilde{\underaccent{\sim}{E}}(\tau) \, \tilde{\underaccent{\sim}{E}}(\tau) \right\rangle . \tag{95}$$

6.3 <u>Orbital component</u> $\quad h^{[\text{o}]}(\underaccent{\sim}{k}'', \tau)$

Introduce the probability of retrograde transition

$$p(\tau, \underaccent{\sim}{l}) , \tag{96}$$

with the condition of normalization

$$\int d\underaccent{\sim}{l} \; p(\tau, \underaccent{\approx}{l}) = I_o \tag{97}$$

to calculate

$$h^{[\text{o}]}(\underaccent{\sim}{k}', \tau) \equiv \left\langle e^{i \underaccent{\sim}{k}'' \cdot \tilde{\underaccent{\sim}{l}}(\tau)} \right\rangle = \int d\underaccent{\sim}{l} \; e^{i \underaccent{\sim}{k}'' \cdot \underaccent{\sim}{l}} \; p(\tau, \underaccent{\sim}{l}) . \tag{98}$$

The dispersion of path fluctuations is governed by the following equation of transition

$$\partial_\tau \, p(\tau, \underaccent{\sim}{l}) = \underaccent{\approx}{K}_l^{[\text{o}]}(\tau) : \nabla_{\underaccent{\sim}{l}} \nabla_{\underaccent{\sim}{l}} \, p(\tau, \underaccent{\sim}{l}) , \tag{99}$$

where $\underaccent{\approx}{K}_l^{[\text{o}]}$ is the diffusivity for the dispersion of path,

rewritten in the form

$$K_{\underset{\approx}{\ell}}^{[\prime\prime]}(\tau) = \tau \int_0^\tau d\tau' (\tau-\tau') \left\langle \underset{\sim}{E}''(\tau) \underset{\sim}{E}''(\tau-\tau') \right\rangle , \tag{100}$$

from (86).

By ·Fourier decomposition with the formula

$$\hat{p}(\tau,\underset{\sim}{\ell}) = \frac{1}{(2\pi)^3} \int d\underset{\sim}{\ell}\, e^{-i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}}\, p(\tau,\underset{\sim}{\ell}) , \tag{101}$$

we transform (99) into

$$\partial_\tau \hat{p}(\tau,\underset{\sim}{k}'') = - \underset{\sim}{k}''\underset{\sim}{k}'' : K_{\underset{\approx}{\ell}}^{[\prime\prime]}(\tau)\, \hat{p}(\tau,\underset{\sim}{k}'') , \tag{102}$$

and obtain the solution

$$\hat{p}(\tau,\underset{\sim}{k}'') = \frac{1}{(2\pi)^3} \exp\left[- \underset{\sim}{k}''\underset{\sim}{k}'' : \int_0^\tau d\tau'\, K_{\underset{\approx}{\ell}}^{[\prime\prime]}(\tau') \right] . \tag{103}$$

The coefficient $1/(2\pi)^3$ is found by the condition (97), when we put $\underset{\sim}{k}''=0$ in (101).

By comparing (98) with (101), and using (103), we find

$$\hat{h}_{\underset{\sim}{\omega}}^{[\prime\prime]}(\underset{\sim}{k}'',\tau) \equiv \left\langle e^{\,i\underset{\sim}{k}''\cdot\underset{\sim}{\ell}''(\tau)} \right\rangle$$
$$= \exp\left[- \underset{\sim}{k}''\underset{\sim}{k}'' : \int_0^\tau d\tau''\, K_{\underset{\approx}{\ell}}^{[\prime\prime]}(\tau) \right]$$
$$= \exp\left(- \omega_D^{[\prime\prime]3} \tau^3 \right) , \tag{104}$$

with

$$\omega_D^{[\prime\prime]3} \equiv \frac{1}{3} \underset{\sim}{k}''\underset{\sim}{k}'' : D_{\underset{\approx}{}}^{[\prime\prime]} \simeq \frac{1}{3} k''^2 D^{[\prime\prime]} . \tag{105}$$

The asymptotic value (90) at large τ is taken, in view of

the scaling differential

$$\tau > \tau_c^{[\nu]} \quad .$$

The diffusivity $D^{[\nu]}$, as belonging to the highest group, regulates the relasation, and will be called the relasational diffusivity.

6.4 Orbital component $h_{[1]}(k'', \tau)$

BY the same procedures as for (104), we calculate

$$h_{[1]}(k'', \tau) = \left\langle e^{i k'' \cdot l_1(\tau)} \right\rangle = \exp\left\{ -k''k'' : \int_0^\tau d\tau'' \left[K_1(\tau'') \right]_{[1]} \right\}$$
$$= \exp\left(- m_{[1]}^4 \tau^4 \right), \tag{106}$$

with

$$m_{[1]}^4 \equiv \frac{1}{8} k'' k'' : \left\langle E_1(\tau) E_1(\tau) \right\rangle$$
$$\cong \frac{1}{8} k''^2 \, trace \left\langle E_1(\tau) E_1(\tau) \right\rangle . \tag{107}$$

The asymptotic value (94) at small τ is taken in view of the scaling differential

$$\tau < \tau_c^{\circ} \quad .$$

6.5 Collision diffusivity $D^{[\cdot]}$

BY collecting all the components

$$h_v(\tau, k'', v) \equiv e^{i k'' \cdot v \tau} \tag{108a}$$

$$h_\gamma(\tau, k'') \equiv e^{- \gamma_b^{[\nu]} \tau} \tag{108b}$$

$$h^{[\nu]}(\tau, k'') = e^{- \omega_b^{*3} \tau^3} \tag{108c}$$

$$h_{[1]}(\tau, k'') \equiv e^{-m_{[1]}^4 \tau^4} \tag{108d}$$

of the orbit function $h(\tau, k'', v)$ from (73), (72), (104), and (106),
we determine the correlation time

$$\tau_c^{[\prime]}(k', v) \equiv \int_0^\infty d\tau \ h(k''; \tau, v)$$
$$= \int_0^\infty d\tau \ h_v(k''; \tau, v) h_\gamma(k''; \tau) h_{[1]}(k'; \tau) h^{[\prime\prime]}(k'; \tau) \tag{109}$$

for the collisional diffusivity, which can now be written in
the form

$$\underset{\approx}{D}^{[\prime]} = \int dk' \ \underset{\approx}{s}^{[\prime]}(k'') \ \tau_c^{[\prime]}(k'', v) \tag{110a}$$

and

$$D^{[\prime]} = \frac{2}{3} \int_k^\infty dk' \ F_E(k'') \ \tau_c^{[\prime]}(k''; v) \ , \tag{110b}$$

a tensor and a trace, respectively.

6.6 Relaxational diffusivity $D^{[\prime\prime]}$

From our basic kinetic'system (46) for fo and f$\perp$,
that was used to derive the kinetic equation (53) and the
diffusivity $\underset{\approx}{D}^{[\prime]}$, it is seen that the submicro-group f" has no
esplicit governing equation like (38), but is represented by an
effective medium without identification of individual velocity
distributions. This means that the collisional diffusivity $\underset{\sim}{D}^{[\prime]}$
and the relasational diffusivity of groups

$$D^{[\prime]} \ . \ D^{[\prime\prime]} \ , \ldots \tag{111}$$

have to be treated differently. While the collisional

diffusivity $\underset{\approx}{D}^{[l]}$ is calculated by including all orbital components, as being based upon the propagator, the relasational diffusivity of various groups do not have their individual kinetics and propagators, because they all form part of the same effective medium. For this reason, the diffusivities (111) are self-generating in the symbolic form

$$D^{[\prime\prime]} = D^{[\prime\prime]}\left\{D^{[\prime\prime\prime]}\right\} \, .$$ (112)

by selecting the diffusive component $h^{[\prime\prime]}(\underset{\sim}{k}'',\tau)$ of the orbit function.

More esplicitly, (112) is

$$D^{[\prime\prime]} \cong \int_0^\infty d\tau \int d\underset{\sim}{k}''' \, \mathcal{J}^{[\prime\prime]}(\underset{\sim}{k}''') \, h^{[\prime\prime]}(\underset{\sim}{k}''',\tau)$$

$$= \frac{2}{3} \int_{k''}^\infty dk''' \, F_E(k''') \, \tau_c^{[\prime\prime]}(k''') \, ,$$ (113)

with

$$\tau_c^{[\prime\prime]}(k''') \equiv \int_0^\infty d\tau \, exp\left(-\omega_D^{[\prime\prime]3}\tau^3\right) = \Gamma\left(\tfrac{4}{3}\right) \omega_D^{[\prime\prime]-1} \, .$$ (114')

Upon substitution, we obtain

$$D^{[\prime\prime]} = \frac{2}{3}\Gamma\left(\tfrac{4}{3}\right) \int_{k''}^\infty dk''' F_E(k''') \, \omega_D^{[\prime\prime]-1}$$

$$= \frac{2}{3}\Gamma\left(\tfrac{4}{3}\right) \int_{k''}^\infty dk''' F_E(k''') \left(\tfrac{1}{3}k''^2 D^{[\prime\prime\prime]}\right)^{-1} \, .$$ (115)

The use of (110), (77) and (108c) has been made.

The diffusivities $D^{[\prime\prime]}$ and $D^{[\prime\prime\prime]}$ have an identical integral structure, but with contingent limits of integration

(k'', ∞) and (k''', ∞)

respectively, *so* that we can identifv $\overset{[\prime\prime\prime]}{D}$ as a diffusivity $D^{[\prime\prime\prime]}$ in the integrand making (115) an integral equation. For its solution, we differentiate (115) with respect to the lower limit of integration k", as denoted by an upper dot, and obtain the differential equation

$$\overset{\cdot\,[\prime\prime]}{D} = -\frac{2}{3}\Gamma\left(\frac{4}{3}\right)E_E(k'') \Big/ \left(\frac{1}{3}k''^2 D^{[\prime\prime]}\right)^{\frac{1}{3}}. \tag{116}$$

This is solved to give the diffusivity for relaxation in the form

$$D^{[\prime\prime]} = \left[\frac{8}{9}\times 3^{\frac{1}{3}}\,\Gamma\left(\frac{4}{3}\right)\int_{k''}^{\infty}dk''\,k'''^{-2/3}F_E(k''')\right]^{\frac{3}{4}}. \tag{117}$$

The relasational diffusivity $\overset{[\prime\prime]}{D}$ forms the last link of the chain of many diffusitivities. The approximate formula provides with the necessarv relaxation process for the approach of $\overset{[\prime]}{D}$ to equilibrium and for the closure of the sequence of diffusivities.

7. Enhancement of turbulence by density waves

The momentum equation (54) can be written in the following alternative form:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u^{\,\circ}\right)u^{\circ} = \underset{\sim}{E}^{\circ}, \tag{118}$$

with the differential operator

$$\mathcal{L}_u^{\,\circ} \equiv \underset{\sim}{u}^{\circ}\cdot\underset{\sim}{\nabla} + \gamma^{[\prime]} \tag{119}$$

The nonlinear term $\underline{u}^0 \cdot \nabla \underline{u}^0$ represents the steepening of sawtooth waves. The steepening cannot increase indefinitely to give rise to a discontinuity in view of the damping by $\gamma^{[\prime]}$. This damping has a kinetic origin as due to the interaction between the waves and the distribution function.

By considering the left hand side of (118) as the total time derivative, we can integrate to obtain the macro-velocity

$$\underline{u}^0 = \int_0^t d\tau \; \bar{\mathcal{U}}_u(t, t-\tau) \, \underline{E}^0(t-\tau) \, . \tag{120}$$

and the diffusivity

$$\langle \underline{E}^0 \underline{u}^0 \rangle = \int_0^t d\tau \langle E^0(t,x) \, \bar{\mathcal{U}}_u(t, t-\tau) E^0(t-\tau) \rangle \equiv \underline{\underline{D}}^{[0]} \, . \tag{121}$$

The asymptoticity is obtained on the same basis as for $\underline{D}^{[\prime]}$. The evolution operator $\bar{\mathcal{U}}_u$ determines the Lagrangian evolution of $\underline{u}^0$ under the differential operator (119). The same earlier argument has been used for approximating $\mathcal{U}_u$ by $\bar{\mathcal{U}}_u$. The operator $\bar{\mathcal{U}}_u$, being of hydrodynamic origin, does not involve $\underline{v}$ in the perturbed trajectory, so that $\underline{D}^{[0]}$ has the same structure as $\underline{D}^{[\prime]}$ from (110a), but $\underline{D}^{[0]}$ is independent of $\underline{v}$ and has different limits of integration. Thus we can write

$$D^{[0]} = \frac{2}{3} \int_0^k dk' \; F_E(k') \, \tau_c^{[0]}(k') \, , \tag{122}$$

with the correlation time

$$\tau_c^{[0]}(k') \equiv \int_0^\infty d\tau \; h(k', \tau, \underset{\sim}{v}=0) \; . \tag{123}$$

The orbit function

$$h(k', \tau, \underset{\sim}{v}=0) = h_\gamma(k', \tau) \; h_{[0]}(k', \tau) \; h^{[\prime]}(k', \tau) \tag{124}$$

and its components

$$h_\gamma(k', \tau) \equiv e^{-\gamma^{[\prime]}\tau} \tag{125a}$$

$$h_{[0]}(k', \tau) \equiv e^{-m_{[0]}^4 \tau^4} \tag{125b}$$

$$h^{[\prime]}(k', \tau) \equiv e^{-\omega_D^{[\prime]3}\tau^3} \tag{125c}$$

are modified forms of (73) and (108). Here the diffusion time is $\omega_D^{[\prime]-1}$, such that

$$\omega_D^{[\prime]3} \equiv \frac{1}{3} k'^2 D^{[\prime]}\Big|_{\underline{v}=0} \tau^3 \; , \tag{126}$$

and the diffusivity

$$D^{[\prime]}\Big|_{\underset{\sim}{v}=0} = \int dk'' \; \mathcal{I}^{[\prime]}(k'') \; \tau_c^{[\prime]}(k'')\Big|_{\underset{\sim}{v}=0} \tag{127}$$

has a correlation time

$$\tau_c^{[\prime]}(k'')\Big|_{\underset{\sim}{v}=0} = \int_0^\infty d\tau \; h(k'', \tau)\Big|_{\underset{\sim}{v}=0} \; , \tag{128}$$

with the orbit function

$$h\Big|_{v=0} = h_\gamma \; h_{[1]} \; h^{[\prime\prime]} \; . \tag{129}$$

The use of (105), (110a), (109) and (73) has been made.

By comparing the trace of the diffusivity tensor (121) with the coupling function (59) we find

$$W^{[o]} = 3 \, D^{[o]} \, .$$

(130)

with

$$D^{[o]} \equiv \text{trace} \, \underset{\approx}{D}^{[o]} \, .$$

(131)

This diffusivity governs the enhancement of turbulence from the coupling between $\underset{\sim}{E}^o$ and $\underset{\sim}{u}^o$. In the spectral sense there is an enhancement of the velocity spectrum at the expense of the density spectrum by an amout $W^{[o]}$.

8. Transport theory of cascade

8.1 Two memories of the transfer function

In sections 4 and 5, we have derived the macro-kinetic equation (53) and the hydrodynamic equation of momentum (54). The eddy collision is

$$\partial_j \, D_{j\lambda}^{[\prime]} \left\{ \partial_\lambda f^o (t-\tau) \right\} \, .$$

(132)

by (51) and the hydrodynamic friction is

$$J_i^o = \int dv \, v_i \, \partial_j \, D_{j\lambda}^{[\prime]} \left\{ \partial_\lambda f^o (t-\tau) \right\} \, ,$$

(133)

by (55b).

Upon multiplying by $\underset{\sim}{u}^{o}$ and averaging, we have the transfer function (58) which we rewrite as follows:

$$T^{[o]} \equiv -\left\langle \underset{\sim}{u}^{\cdot} \cdot \underset{\sim}{J}^{o} \right\rangle = -\int d\underset{\sim}{v}\, v_i\, \partial_j \cdot D^{[\prime]}_{jn} \left\{ \partial_n \varphi^{[o]}_i (\tau, \underset{\sim}{x}, \underset{\sim}{v}) \right\}, \tag{134}$$

where

$$\varphi^{[o]}_i (\tau, \underset{\sim}{x}, v) \equiv \left\langle u^{o}_i (t, \underset{\sim}{x})\, f^{o}(t-\tau) \right\rangle \tag{135}$$

is a Lagrangian correlation.

Like in (61) we use the Fourier method to transform (135) and write

$$\varphi^{[o]}_i (\tau, \underset{\sim}{x}, v) = \int d\underset{\sim}{k}'\, h(\underset{\sim}{k}')\, \varphi^{[o]}_i (\underset{\sim}{k}'), \tag{136a}$$

with the Fourier component

$$\varphi^{[o]}_i (\underset{\sim}{k}') \equiv \chi \left\langle u^{o}_i (\underset{\sim}{k}')\, f^{o}(-\underset{\sim}{k}') \right\rangle . \tag{136b}$$

By taking the moment, we have the spectrum

$$\int d\underset{\sim}{v}\, v_i\, \varphi^{[o]}_i (\underset{\sim}{k}') = \mathcal{E}^{[o]} (\underset{\sim}{k}') \tag{137a}$$

and the energy

$$\int d\underset{\sim}{k}' \int d\underset{\sim}{v}\, v_i\, \varphi^{[o]}_i (\underset{\sim}{k}') = \int d\underset{\sim}{k}'\, \mathcal{E}^{[o]} (\underset{\sim}{k}') \equiv \mathcal{E}^{[o]} (t, \underset{\sim}{x}), \tag{137b}$$

with the definitions

$$\mathcal{E}_6^{[o]}(k') = \chi \left\langle u^\circ(k') \cdot u^\circ(-k') \right\rangle \tag{138a}$$

$$\mathcal{E}_6^{[o]}(t,x) = \int dk' \, \mathcal{E}_6^{[o]}(k') \equiv \left\langle u^{\circ 2} \right\rangle . \tag{138b}$$

By substituting for the Fourier form of $D_{j\iota}^{[\prime]}$ and $\varphi_i^{[o]}$ from (110a) and (136a), we transform (134) into

$$T^{[o]} = -\iint dk' \, dk'' \, \mathcal{S}_{j\iota}^{[\prime]}(k'') \int_0^\infty d\tau \, N_{j\iota}^{[o]}(k; k'') . \tag{139}$$

For the sake of abbreviation, we introduce the partition function

$$N_{j\iota}^{[o]}(k, k'') \equiv \int dv \, v_i \, \partial_f \left\{ h(k'') \, \partial_\iota \left[h(k') \, \varphi_i^{[o]}(k') \right] \right\} . \tag{140}$$

The dependence of

$$h(k'), \quad h(k''), \quad \varphi_i^{[o]}(k') \tag{141a}$$

on

$$\tau, \quad v, \quad \gamma^{[\prime]}, \quad \gamma^{[\prime\prime]}, \quad \omega_D^{[\prime]}, \quad \omega_D^{[\prime\prime]}, \quad m_{[1]}^{[\prime]}, \quad m^{[o]} \tag{141b}$$

is understood by their definitions.

In order to simplify the expression (140), we consider the following properties:

(i) In strong turbulence, the streaming effect by v in h_v is negligible, escept with the derivatives

a

36

$$\partial_j \partial_\ell h_\nu(\underset{\sim}{k'}) = -k'_j k'_\ell h_\nu(\underset{\sim}{k'})\Big|_{\underset{\sim}{\nu}=0}$$
$$= -\tfrac{1}{3} k'^2 \delta_{j\ell}$$

(142a)

and

$$\partial_j \partial_\ell h(\underset{\sim}{k'}) = -\tfrac{1}{3}\delta_{j\ell} k'^2 h(\underset{\sim}{k'}, \underset{\sim}{\nu}=0),$$

(142b)

and with similar derivatives of h(k").

(ii) Since $T^{[o]}$ is finite and real, and so is $A_{j\ell}^{[\nu 3]}$, the integrand in (139) must be even. This rules out the contribution by $\partial_{\underset{\sim}{h}}$.

(iii) With the partial integration with respect to $\underset{\sim}{\nu}$, we have

$$\varphi_i^{[o]}(\underset{\sim}{\nu}=\infty, \underset{\sim}{k'}) = 0, \quad h_\nu\Big|_{\underset{\sim}{\nu}=0} = 0.$$

(143)

(iv) The transfer function controls the transfer of energy across the spectrum. Consequently, the integrand of (139) must retain ν_i.

By means of these properties, we can simplify the partition function (140). We write

$$N_{j\ell}^{[o]} = \left(N_{j\ell}^{[o]}\right)_a + \left(N_{j\ell}^{[o]}\right)_b,$$

(144)

and calculate the two components

$$\left(N_{j\ell}^{[o]}\right)_a \cong \int d\underset{\sim}{\nu}\, \nu_i\, \varphi_i^{[o]}(\underset{\sim}{k'}) \left[M_{j\ell}(k,k'')\right]_a$$
$$= \mathcal{E}^{[o]}(k') \left[M_{j\ell}(k',k'')\right]_a$$

(145a)

and

$$\left(N_{j\ell}^{[o]}\right)_b \cong \int d\nu\, \nu_i\, \varphi_i^{[o]}(k') \left[M_{j\ell}(k',k'')\right]_b$$
$$= \mathcal{E}^o(k') \left[M_{j\ell}(k',k'')\right]_b.$$

145b)

The two memory-loss functions are:

$$\left[M_{jr}(k',k'')\right]_a \equiv h(k'')\,\partial_j\,\partial_r\,h(k')\Big|_{v=0} = \tfrac{1}{3}\,h(k'')\,\partial^2 h(k')\Big|_{v=0}\,\delta_{jr} \qquad (146a)$$

and

$$\left[M_{jr}(k',k'')\right]_b \equiv \partial_j\,\partial_r\,h(k')\,h(k')\Big|_{v=0} = \tfrac{1}{3}\,\partial^2 h(k'')\,h(k')\Big|_{v=0}\,\delta_{jr}\,. \qquad (146b)$$

The loss of v in the orbit function makes the memory-loss functions only dependent on the wavenumbers k', k'' as scalars. It is seen that (144) describes the partition of the macro-energy of turbulence $\mathcal{E}^{[o]}(k')$ in the k'-space, while accounting for the memory losses along the two orbits. of orbit functions

$$h(k') \quad \text{and} \quad h(k'')\,.$$

By the two memory losses, the transfer function (139) can also be divided into two parts, as follows:

$$T^{[o]} = (T^{[o]})_a + (T^{[o]})_b, \qquad (147)$$

with

$$\left(T^{[o]}\right)_a = -\iint dk'\,dk''\,s^{[\prime]}_{jr}(k'')\,\mathcal{E}^{[o]}(k')\int_0^\infty d\tau\,\left[M_{jr}(k',k'')\right]_a \qquad (148a)$$

$$\left(T^{[o]}\right)_b = -\iint dk'\,dk''\,s^{[\prime]}_{jr}(k'')\,\mathcal{E}^{[o]}(k')\int_0^\infty d\tau\,\left[M_{jr}(k',k'')\right]_b\,. \qquad (148b)$$

by (145).

The transfer function **is** energized by the macro-energy $\mathcal{E}^{[o]}(k')$ and is diffused by the micro-field fluctuations of intensity $s^{[\prime]}(k'')$. Since the process takes place in the Lagrangian space, the two orbit functions cause two memory losses.

8.2 Direct and reverse cascades

By noting the scaling differential (25) and by making use
of the properties of the partition function mentioned above, **we**
reduce (146a) and (146b) into

$$\left[M_{j\imath}(k',k'')\right]_a \cong -\tfrac{1}{3}k'^2\tau^2\, h(k'',\tau,\underline{v}=0)\,\delta_{j\imath} \tag{149a}$$

$$\left[M_{j\imath}(k',k')\right]_\ell \cong -\tfrac{1}{3}k''^2\tau^2\, h(k'',\tau,\underline{v}=0)\,\delta_{j\imath}. \tag{149b}$$

We can see that the loss of memory is effectively caused
by the orbit function

$$h(k'',\tau,\underline{v}=0), \tag{150a}$$

as the result of the perturbations of the trajectory carrying
the field

$$\overline{\mathcal{U}}(t,t-\tau)\,\underline{E}'(t-\tau)$$

for shaping the diffusivity $D^{c.3}$. The other orbit function

$$h(k',\tau,\underline{v}=0) \tag{150b}$$

merely serves to provide a macro-gradient of a spectrum

$$k'^2\,\xi^{[o]}(k'),$$

by the transformation

$$\left.\partial_{\underline{v}}^2 h(k')\right|_{\underline{v}=0} = -k'^2\tau^2\, h(k',\tau,\underline{v}=0). \tag{151}$$

The substitution of (149) into (148) leads to the transfer
function in two components:

$$\left(T^{[0]}\right)_a = \iint dk'\,dk''\,s^{[']}(k'')\,k'^2\,\mathcal{E}^{[0]}(k')\int_0^\infty d\tau\,\tau^2\,h(k'',\tau,\underline{v}=0)$$ (152a)

$$\left(T^{[0]}\right)_b = \iint dk'\,dk''\,k''^2\,s^{[']}(k'')\,\mathcal{E}^{[0]}(k')\int_0^\infty d\tau\,\tau^2\,h(k'',\tau,\underline{v}=0).$$ (152b)

with

$$s^{[']} \equiv \text{trace}\ \underset{=}{s}^{[']}\ .$$

We write (152) in the following concise form

$$\left(T^{[0]}\right)_a = K^{[']}\,R^{[0]}$$ (153a)

$$\left(T^{[0]}\right)_b = \Omega^{[']}\langle u^{o2}\rangle\ ,$$ (153b)

after a rearrangement by separating the variables k', k". The governing functions are: the vorticity function

$$
\begin{aligned}
R^{[0]} &= \langle(\nabla u^o)^2\rangle \\
&= \int dk'\,k'^2\,\chi\langle u^o(k')\,u^o(-k')\rangle \\
&= \int dk'\,k'^2\,\mathcal{E}^{[0]}(k') \\
&= 2\int_0^k dk'\,k'^2\,F_u(k')\ ,
\end{aligned}
$$ (154)

the eddy viscosity as a scalar trace

$$
\begin{aligned}
K^{[']} &= \int dk''\,s^{[']}(k'')\,G^{[']}(k'') \\
&= \frac{2}{3}\int_k^\infty dk''\,F_E(k'')\,G^{[']}(k'')\ ,
\end{aligned}
$$ (155)

and the eddy damping

$$
\begin{aligned}
\Omega^{[']} &= \int dk''\,k''^2\,s^{[']}(k'')\,G^{[']}(k'') \\
&= \frac{2}{3}\int_k^\infty dk''\,k''^2\,F_E(k'')\,G^{[']}(k'')\ .
\end{aligned}
$$ (156)

The modulation function

$$G^{[\prime]}(k'') = \int_0^\infty d\tau \; \tau^2 \; \lambda(k'', \tau, \underline{v}=0) \equiv \left[\tau_M^{[\prime]}(k'') \right]^3$$

(157)

finds a modulation time $\tau_M^{[\prime]}(k'')$.

The orbit function $h(k'', \tau, \underline{v}=0)$ represents a damping as the result of the perturbations of the trajectory. It determines the relaxation for the approach of the diffusivity $D^{[\prime]}$ to equilibrium. Likewise it determines. by (157), the time $\tau_M^{[\prime]}$ of modulation for the approach of the eddy viscosity (155) and the damping coefficient (156) to equilibrium.

The two parts (153a) and (153b) of the transfer function contain the second moments (154) and (156) of the spectral distributions with limits of integrations

$$(o,k) \quad \text{and} \quad (k, \infty),$$

respectively. The two moments gain their importance for large and small values of k. respectively. Hence we can conclude that the two parts (153a) and (153b) of the transfer function control the direct cascade near a sink at large k and the reverse cascade near a source of small k. respectively.

In his theory of incompressible turbulence, Heisenberg has proposed a transfer function in the form of a product of the vorticity function with the eddy viscosity5 His formalism was based upon the empirical concept of mixing-length. In addition, he proposed an empirical formula of eddy viscosity on the basis of dimensional consideration. Here we have developed a kinetic theory of transport for the transfer process. Our analytical formula of transfer(153a) by direct cascade supports his phenomenological formalism. Our theoretical result of eddy

viscositv (155) is based upon the macro-kinetic equation and has undergone the process of relaxation, modulation and memory losses. It is obviously more complicated than Heisenberg's formula.

En analogy with the derivation of the kinematic viscosity by the kinetic theory of gases, the derivation of the eddy viscosity needs the kinetic equation as an essential basis. This explains why a purely hydrodynamical method, e.g. the direct interaction approximation,[17] cannot produce an eddy viscosity.

8.3 Modulation function

The modulation function $G^{[\prime]}(k")$ in (157) controls the eddy viscosity (155) and the eddy damping (156), and is calculated by the second moment of the orbit function of three components

$$h(k",\tau)\Big|_{v=0} \equiv h_\gamma(k",\tau)\, h^{[\prime\prime]}(k",\tau)\, h_{[1]}(k",\tau)$$
$$= exp\left[-\left(\gamma_k^{[\prime]}\tau + \omega_D^{[\prime\prime]3}\tau^3 + m_{[1]}^4\tau^4\right)\right] \qquad (158)$$

of time scales as given bv the inverse of the frequency scales

$$\gamma_k^{[\prime]}, \qquad \omega_D^{[\ast]}, \qquad m_{[1]}. \qquad (159)$$

respectively. Each component could form its own individual modulation. An approximate evaluation of the second moment of (158) gives the composite modulation, as defined bv (157) with a time scale $\tau_M^{[\prime]}$ such that

$$\tau_M^{[\prime]-1} \cong \left\{ \left(\tfrac{1}{2}\right)^{2/3} \gamma_k^{[\prime]2} + 3^{2/3} \omega_D^{[\prime]2} + \left[\tfrac{1}{4}\Gamma\left(\tfrac{3}{4}\right)\right]^{2/3} m_{[1]}^2 \right\}^{\frac{1}{2}} \tag{160}$$

is the resultant of the frequency scales (159). The numerical coefficients are adjusted in such a way that the formula (160) gives the esact values of the three individual modulation functions at the three-scales (159). Thus the formula (160) for the composite modulation is an interpolation of three esact components.

8.4 Identification of the effective medium

From (153a), we see that the eddy viscosity $K^{[\prime]}$ describes the turbulent property of the medium of small eddies that carry the $\underset{\sim}{E}'$-field fluctuations.

On a phenomenological basis, we can conceive that this medium offers a friction

$$\underset{\sim}{I}^\circ = K^{[\prime]} \nabla^2 \underset{\sim}{u}^\circ \tag{161}$$

to the evolution of u°, and causes an energy dissipation

$$K^{[\prime]}\langle \underset{\sim}{u}^\circ \cdot \nabla^2 \underset{\sim}{u}^\circ \rangle \cong -K^{[\prime]}\langle (\nabla \underset{\sim}{u}^\circ)^2 \rangle$$
$$= -K^{[\prime]} R^{[\sigma]} \tag{162}$$

to the energy balance in homogenous turbulence. Since the energy dissipation (162) is the result of interaction between the micro-group and the macro-group, we see here that the physical picture of cascade emerges. By pursuing the comparison of (161) with (42), we can identity

$$\gamma^{[\prime]} = K^{[\prime]} \nabla^2, \quad \gamma_k^{[\prime]} = -k^2 K^{[\prime]}. \tag{163a}$$

for the micro-group, and similarly for the submicro-group

$$\gamma^{[\prime\prime]} = K^{[\prime\prime]} \nabla^2, \qquad \gamma_k^{[\prime\prime]} = -k^2 K^{[\prime\prime]}. \tag{163b}$$

Recall that the realization of a direct cascade requires a transfer of large wavenumbers in a transport of the gradient tspe. Both conditions are met by the effective medium that represents the cluster of high-order distributions.

The identification of the effective friction eddy viscositv in (163) connects the concept of effective medium to the kinetic basis of cascade process in a self-consistent way.

9. Equation of state of turbulence

9.1 Local nonlinearity

The density-induced turbulence is governed by the following system of two differential equations:

$$\left(\partial_t + \hat{\underline{u}} \cdot \underline{\nabla} \right) \hat{\underline{u}} = \hat{\underline{E}} \tag{164}$$

$$\left(\partial_t^2 - \underline{\nabla} \cdot \hat{c}^2 \underline{\nabla} \right) \hat{N} = \hat{n} \tag{165}$$

with the driving forces

$$\hat{\underline{E}} = -\hat{c}^2 \underline{\nabla} \ln \hat{N} \tag{166}$$

and

$$\hat{n} = \underline{\nabla} \underline{\nabla} : \hat{N} \hat{\underline{u}} \hat{\underline{u}}, \tag{167}$$

from (8), (9), (7) and (10), respectively.

The system describes the microdynamical state of the

interaction between turbulence and finite amplitude density waves. The Navier-Stokes equation (164) is obviously nonlinear by its advection $\hat{\underline{u}} \cdot \nabla \hat{\underline{u}}$, and is also nonlinear by the driving force $\hat{\underline{E}}$. The latter nonlinearity arises from the **wave** propagation by the definitions (166) and (167). Such a form of nonlinearity also appears in soliton dynamics and is called the modulational nonlinearity.[18] It is non-local (in time) as in (165), or is local in the form

$$\hat{\underline{E}} \equiv -\hat{c}^2 \nabla \ln \hat{N} = \frac{1}{\hat{N}} \nabla : \hat{N} \hat{\underline{u}} \hat{\underline{u}} \cdot \tag{168}$$

as obtained by assuming that the temporal derivative is small as compared to the spatial derivative. This is the case with the strong turbulence in cascade process.

By taking the divergence, we transform (168) into the following Poisson equation

$$\begin{aligned}
\nabla \cdot \hat{\underline{E}} &= \nabla \cdot \frac{1}{\hat{N}} \nabla \cdot \hat{N} \hat{\underline{u}} \hat{\underline{u}} \\
&\cong \nabla_j \hat{u}_i \, \nabla_i \hat{u}_j = \hat{\lambda}
\end{aligned} \tag{169}$$

The approximation is made by assuming that the transverse modes of turbulence are dominant in wave scattering.[19] The Poisson equation clearly indicates the locally nonlinear character of the source in driving the turbulence.

An equation of state is obtained, first in the barotropic form (7) for relating $\hat{\underline{E}}$ to $\hat{N}$ through the speed of propagation (2), and then in the Poisson form (169) for relating $\hat{\underline{E}}$ to $\hat{\underline{u}}$ by assuming a transverse scattering of waves.

9.2 Scattering function r^o

The scattering function of the macro-group is

$$r^o \equiv A^o \hat{r}$$

$$= \nabla_j u_i^o \, \nabla_i u_j^o - \langle \nabla_j u_i^o \, \nabla_i u_j^o \rangle. \tag{170}$$

by (169). The intensity is obtained by the mean square value

$$\langle r^{o2} \rangle = \bar{A} \left[\nabla_j u_i^o \, \nabla_i u_j^o - \langle \nabla_j u_i^o \, \nabla_i u_j^o \rangle \right] \left[\nabla_l u_m^o \, \nabla_m u_l^o - \langle \nabla_l u_m^o \, \nabla_m u_l^o \rangle \right]. \tag{171}$$

The differential ∇ does not extend beyong the immediate function. For

$$\langle \nabla_j u_i^o \, \nabla_i u_j^o \, \nabla_l u_m^o \, \nabla_m u_l^o \rangle. \tag{172}$$

we shall decouple into double products by approximation. The approximation is legitimate when it does not deal with a process, as is the case with (170). We find

$$\langle r^{o2} \rangle \cong 2 \langle \nabla_j u_i^o \, \nabla_m u_l^o \rangle \langle \nabla_i u_j^o \, \nabla_l u_m^o \rangle. \tag{173}$$

or

$$\langle r^{o2} \rangle \cong 2 \iiiint dk' \, dk'' \, dk''' \, dk'''' \, i k_j' \, i k_m'' \, i k_i''' \, i k_l'''' $$
$$\times \langle u_i^o(k') u_l^o(k'') \rangle \langle u_j^o(k''') \, u_m^o(k'''') \rangle. \tag{174}$$

by Fourier transformation, where X is the factor of truncation of the Fourier decomposition.

A homogenous turbulence has the following property

$$\langle u_i^o(k') \, u_l^o(k'') \rangle = \frac{1}{3} X \langle u^o(k') \cdot u^o(-k') \rangle \delta_{il} \, \delta(k' + k''). \tag{175a}$$

and similarly, we have

$$\langle u_j^o(\underline{k}''') \, u_m^o(\underline{k}'''') \rangle = \tfrac{1}{3} \mathcal{X} \langle \underline{u}^o(\underline{k}''') \cdot \underline{u}^o(-\underline{k}''') \rangle \, \delta_{jm} \, \delta(\underline{k}'' + \underline{k}'''). \qquad (175b)$$

By the properties (17'), we can simplify (173) into

$$\langle \dot{n}^2 \rangle = \frac{2}{4} R^{[o]}. \qquad (176)$$

where $R^{[o]}$ is the vorticity function as defined by (154).

By a Fourier transformation of (168) and with the aid of (176), we obtain

$$R_E^{[o]} = \frac{2}{3} R^{[o] \, 2}, \qquad (177)$$

where $R^{[o]}$ is the vorticity function of $\underline{u}^o$-fluctuations, as defined by (154) and

$$R_E^{[o]} \equiv 2 \int_0^k dk' \, k'^2 \, F_E(k') \qquad (178)$$

is the vorticity function of $\underline{E}^o$-fluctuations.

We convert (177) into the spectral form by writing

$$2 \int_0^k dk'' \, k''^2 \, F_E(k'') = \frac{2}{3} \times 4 \int_0^k dk'' \, k''^2 F_u(k'') \int_0^{k''} dk' \, k'^2 F_u(k')$$

$$= \frac{2}{3} \times 8 \int_0^k dk'' \, k''^2 F_u(k'') \int_0^{k''} dk' \, k'^2 F_u(k'). \qquad (179)$$

The inner integral accentuates large values of k' near k'', and as a first approximation can be written as

$$k''^3 \, F_u(k''). \qquad (180)$$

yielding the relation

$$2 \int_0^k dk'' \, k''^2 \, F_E(k'') \simeq \frac{16}{3} \int_0^k dk'' \, k''^5 \big[F_u(k'') \big]^2. \qquad (181)$$

or, after a differentiation,

$$F_E(k) \cong \frac{8}{3} k^3 \left[F_u(k)\right]^2 . \tag{182}$$

The approximation is valid for small k and even smaller k" as in the case with the coupling subrange.

The relations (177) and (182) are the equation of state in the s-space and k-space. respectively.

10. Spectrum of the enhanced turbulence.

10.1 Energy spectrum in the coupling Subranne

In the coupling subrange, the turbulent energy is enhanced at the espense of the potential energy. The governing equation of energy balance is (130), rewritten as

$$3 D^{[0]} = K^{[\cdot]} R^{[0]} \tag{183a}$$

or

$$2 \int_0^k dk'\, F_E(k')\, \tau_c^{[0]}(k') = \frac{2}{3} R^{[0]} \int_k^\infty dk''\, F_E(k'')\left[\tau_M^{[\cdot]}(k'')\right]^3 , \tag{183b}$$

by (130), (122) and (155), where $R^{[0]}$ has the dimension of $(time)^{-2}$.

The equation (183a) describes the balance between the coupling function as a spectral source and the cascade transfer as a spectral sink at the larger end of the spectrum. Therefore. the formula (153a) of the transfer **by** direct cascade is taken.

An inspection of (183b) reveals that any function having the dimension of time is independent of k.

For describing the energy flow, we differentiate (183a) with respect to k, as denoted by an upper dot, to get

$$3 \dot{D}^{[o]} = K^{[']} \dot{R}^{[o]} + \dot{K}^{[']} R^{[o]}$$

$$= K^{[']} \dot{R}^{[o]} \left(1 + \frac{R^{[o]}}{\dot{R}^{[o]}} \frac{\dot{K}^{[']}}{K^{[']}} \right) \tag{184}$$

$$\simeq K^{[']} \dot{R}^{[o]}$$

in view of small $R^{[o]}$. In terms of the spectral distributions, (184) is, by (183b),

$$2 F_E(k) \tau_c^{[o]} = 2 k^2 F_u(k) \times \frac{2}{3} \int_k^\infty dk'' F_E(k'') \left[\tau_M^{[']}(k'') \right]^3.$$

or, after a rearrangement,

$$\frac{2}{3} \int_k^\infty dk'' F_E(k'') \left[\tau_M^{[']}(k'') \right]^3 = \left[F_E(k) \tau_c^{[o]} / F_u(k) \right] k^{-2}. \tag{185}$$

The factor

$$F_E(k) \tau_c^{[o]} / F_u(k)$$

having the dimensions of $(\text{time})^{-1}$, is independent of k. Another differentiation of (185) leads to

$$F_u(k) = t_u^{-2} k^{-3}. \tag{186}$$

The factor

$$t_u^{-2} = 3 \tau_c^{[o]} / \left(\tau_M^{[']} \right)^3 \tag{187}$$

is again independent of k.

Since $\tau_c^{[o]}$ is the time of steepening of sawtooth, and $\tau_M^{[']}$ is the modulation time for the cascade of energy, the balance between the two processes gives the duration t_u for the enhancement of turbulence.

By the definitions (123) and **(157)**, we can write

$$t_u^2 = \frac{1}{3} \int_0^\infty d\tau \, \tau^2 \, h(\tau) \Big|_{\underset{\sim}{v}=0} \Big/ \int_0^\infty d\tau \, h(\tau) \Big|_{\underset{\sim}{v}=0}, \tag{188}$$

as related to the normalized second moment of the orbit function $h(\tau)\Big|_{\underset{\sim}{v}=0}$.

10.2 Spectrum of field and density fluctuations in the coupling

.ubrange

By the use of the equation of state (182), the relation
(165b) and the spectral result (186), we derive the spectral
distributions of field and density fluctuations as follows:

$$F_E(k) = \frac{8}{9} t_u^{-4} k^{-3} \tag{189a}$$

$$F_N(k) = \frac{8}{9} (c t_u)^{-4} k^{-5}. \tag{189b}$$

10.3 Spectral intensities

By introducing the spectral intensities

$$u_k^{[\prime]2} \equiv 2 \int_k^\infty dk' \, F_u(k') \tag{190a}$$

$$E_k^{[\prime]2} \equiv 2 \int_k^\infty dk' \, F_E(k') \tag{190b}$$

$$N_k^{[\prime]2} \equiv 2 \int_k^\infty dk' \, F_N(k'), \tag{190c}$$

the results (186), (189a) and (189b) for the spectral densities
can be converted into the following :

$$u_k^{[\prime]2} = t_u^{-2} k^{-2} \tag{192a}$$

$$E_k^{[\prime]2} = \frac{8}{3} t_u^{-2} k^{-2} \tag{192b}$$

$$N_k^{[\prime]2} = \frac{4}{3} (c t_u)^{-4} k^{-4}. \tag{192c}$$

It is to be noted that the spectral results for velocity and field fluctuations are independent of c, while the spectral results for density fluctuations will depend on c. For the density spectrum, c enters as a constant parameter, implying an isothermal propagation.

The coupling subrange is located between the production subrange and the inertia subrange, with the mean density gradient and the rate of energy dissipation as the respective external parameters. It possesses the two parameters: the constant speed of propagation in isothermal gas that enters into the equation of propagation as en external parameter, and the constant time scale of enhancement of turbulence as an internal parameter.

11. Discussions and conclusions

In incompressible turbulence, the agent of production of turbulence is usually the mean gradient of the velocity itself. However, if the density may fluctuate. such as in convective turbulence, acoustic turbulence, density -induced turbulence, and other types of compressible turbulence, the turbulent motion may interact with the large- amplitude density waves. If the turbulence motion is intensified by the density waves, we call it density-induced turbulence. The processes are as follows: often a mean density gradient is more likely to be found than a mean velocity gradient. It excites all the nonlinear mechanisms that controls the density waves, i.e. the density-velocity coupling among large eddies. and the cascade transfer from

large toward small eddies.

If the coupling by large eddies is more intense than the transfer, the density fluctuations will mainly go to feed the kinetic energy for its continuing cascade toward smaller eddies, and consequently the density spectrum will fall rapidly. At increasing wavenumbers, the tail portion of the velocity spectrum is governed by the balance between the cascade transfer and the viscous dissipation.

We need a mathematical model to describe these processes in density -induced turbulence. In Section 2 , we have named several mathematically equivalent models. The Navier-Stokes model (a) and its kinetic correspondent (d) are too formal for the purpose. The Riemann system (c) has the merit of illustrating the build-up of the kinetic energy of turbulence at the expense of the wave energy. By the symmetry, the system does not distinguish the different physical roles from the two equations. The acoustic turbulence model (b) that has clearly put the propagation into evidence is more suitable to treat the density -induced turbulence. In order to include those transport properties which take their origin from the velocity distribution of eddies, we choose a kinetic approach in the wave-kinetic model (e), upon replacing the Navier-Stokes equation of motion by a master equation. By group-scaling, we decompose the master equation into scaled equations for the distribution functions of many scales. The closure is found by considering those groups which form a relaxational cluster as an effective medium having self-consistently the same transport property as

the cluster itself. As a result, the kinetic equation for the macro-distribution can be used to derive the transport functions !coupling function and transfer function), the transport Properties (eddy diffusivity and eddy viscosity). and finally to find the spectral functions. The spectral intensities in the coupling subrange are found to have the power law k^{-2} for the velocity and field fluctuations and the power law k^{-4} for the density fluctuations. This is the first time that a kinetic equation for the singlet distribution (macro-distribution) can determine the spectral structure of turbulence without the need of the pair-distribution.

Acknowledgement

REFERENCES

1) B.B. Kadomtsev and **V.I.** Petviashvili,Dokl, Akad, Nauk. SSSR **208** (1973) 794 [Sov. Phys. Doklady **18 (1973)** 115] .

2) V.E. Zakharov and R.Z. Sagdees. Dokl. Akad. Nauk SSSR 192 (1970) 297 [Sov. Phys. Doklady 15 (1970) $39] .

3) J.M. Burgers. Adv. Appl. Mech. I (1948) 171.

4) H. Tokunaga, J. Phys. Soc. Japan 41 (1976) 328.

5) A.N. Kolmogoroff. Dokl. Akad. Nauk SSSR 30 (1941) **299.** Also Compt. Rendus Acad. Sci. U.R.S.S. 30 (1941) 301.

6) W. Heisenberg. Z. Physik 124 (1948) 628.

7) B. Riemann. Chap. **VIII** in:Collected Works **of** Bernard Riemann, 2nd ed., **H.** Weber, ed. (Dover Publications. Inc., New York, 1953).

8) C.M. Tchen. "Kinetic Approach to Compressible Turbulence." In Proc. Int'l Symposium of Rarefied Gas Dynamics, Cannes. Publ. by Commissatiat **a** l'Energie Atomique, Paris **(1979)** 105.

9) C.M. when. "Group-Kinetic Theory and Modelling of Turbulence ", NASA Contractor Report, vol. I & II **(1984)**.

10) A.S. Monin, Dokl. Akad. Nauk SSSR **177** (1967) 1036 $\left[\right.$Sov. Phys. Doklady **12** (1968) **1113** $\left]\right.$.

11) T.S. Lundgren. Phys. Fluids 10 (1967) **969.**

12) N.N. Bogoliubov, in: 'Studies in Statistical Mechanics"' transl. by E.K. Gorä, J. de Boer and G.E. Uhlenbeck, eds. (North-Holland, Amsterdam, 1962), vol. I, **p.1.**

13) P. Debye and A.M. Bueche, J. Chem. Phys. **16** (1948) 573.

14) **H.C.** Brinkman. Appl. Sci. Res. **A1** (1947) 27, **51;** Proc. Koninkl. Ned. Akad. Wetenschap (Amsterdam) 50 (1947) 618, 821; Physica 13 (1947) 447.

15) K.F. Freed and M. Yuthukumar, J. Chem. Phys. 68 (1978) 2083

16) M. Muthukumar and K.F. Freed, J. Chem. Phys. **70** (1979) **5975.**

17) R.H. Kraichnan, J. Fluid Mech. **5** (1959) 497.

18) V.E. Zakharov. Zh. Eksp. Theor. Fiz. **56** (1967) 1735 $\left[\right.$Sov. Phys.-JETP **26** (1967) 994$\left]\right.$.

19) M.J. Lighthill, Proc. Roy. Soc. London **A211** (1952) 564.

1. REPORT NO. NASA CR-3905	2. GOVERNMENT ACCESSION NO.	3. RECIPIENT'S CATALOG NO.
4. TITLE AND SUBTITLE Excitation. of Turbulence by Density Waves		5. REPORT DATE June 1985
		6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) C. M. Tchen		8. PERFORMING ORGANIZATION REPORT #
9. PERFORMING ORGANIZATION NAME AND ADDRESS Universities Space Research Association American City Building, Suite 311 Columbia, MD 21044		10. WORK UNIT NO. M-492
		11. CONTRACT OR GRANT NO. NAS8-36153
12. SPONSORING AGENCY NAME AND ADDRESS National Aeronautics and Space Administration Washington, D.C. 20546		13. TYPE OF REPORT & PERIOD COVERED Contractor Report Nov. 30, 1984–Feb. 28, 1985 Quarterly Report
		14. SPONSORING AGENCY CODE

15. SUPPLEMENTARY NOTES

Technical Monitors: Margaret B. Alexander and George H. Fichtl. Prepared for the Atmospheric Sciences Division, Systems Dynamics Laboratory, Marshall Space Flight Center, Alabama, 35812. First Quarterly Report.

16. ABSTRACT

A nonlinear system describes the microdynamical state of turbulence that is excited by density waves. It consists of an equation of propagation and a master equation. A group-scaling generates the scaled equations of many interacting groups of distribution functions. The two leading groups govern the transport processes of evolution and eddy diffusivity. The remaining sub-groups represent the relaxation for the approach of diffusivity to equilibrium. In strong turbulence, the sub-groups disperse themselves and the ensemble acts like a medium that offers an effective damping to close the hierarchy. The kinetic equation of turbulence is derived. It calculates the eddy viscosity and identifies the effective damping of the assumed medium self-consistently. It formulates the coupling mechanism for the intensification of the turbulent energy at the expense of the wave energy, and the transfer mechanism for the cascade. The spectra of velocity and density fluctuations find the power laws k^{-2} and k^{-4}, respectively.

17. KEY WORDS Turbulence Atmospheric Boundary Layer Storm and Mesoscale Processes Global Scale Processes Atmospheric and Terrestrial Environmental Processes	18. DISTRIBUTION STATEMENT Unclassified – Unlimited STAR Category: 47

19. SECURITY CLASSIF. (of this report) Unclassified	20. SECURITY CLASSIF. (of this page) Unclassified	21. NO. OF PAGES 58	22. PRICE A04