NASA Contractor Report 172345

ICASE REPORT NO. 84-13

ICASE

AN ITERATIVE METHOD FOR INDEFINITE SYSTEMS OF LINEAR EQUATIONS

Kazufumi Ito

Contract No. NAS1-17070
April 1984

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

AN ITERATIVE METHOD FOR INDEFINITE SYSTEMS OF LINEAR EQUATIONS

Kazufumi Ito
Institute for Computer Applications in Science and Engineering

Abstract

An iterative method for solving nonsymmetric indefinite linear systems is proposed. The method involves the successive use of a modified version of the conjugate residual method. A numerical example is given to illustrate the method.

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Introduction

In this paper we consider an iterative method for solving linear systems of the form

$$Ax = f, \tag{1.1}$$

where A is a nonsymmetric, <u>indefinite</u> matrix of order N. The main application we have in mind is the numerical approximation of solutions to elliptic partial differential equations. When A is symmetric, there exists an effective iterative method for solving indefinite problems [4],[5] (see also the references therein). We also refer to [2] and [6] for recent progress in the development of iterative methods for solving nonsymmetric systems. However, to my knowledge, no effective iterative methods for solving <u>nonsymmetric</u> and <u>indefinite</u> systems have been developed. Most of the iterative methods (especially Krylov subspace methods) are rigorously applicable only when the symmetric part of A is positive definite [1]. There are many applications in which the symmetric part is indefinite and as shown in [1], they may result from preconditionings. In this paper we introduce a method which is applicable to indefinite problems and involve the modification of Orthomin due to Vinsome [7]. It can be easily combined with preconditioning techniques.

Section 2 presents the convergence property of the minimal residual method for indefinite problems. It is the property which provides the underlying motivation in the development of the proposed method. In Section 3, the use of the method described in Section 2 in conjunction with Orthomin and a class of problems for which our method is effective are discussed. A numerical result is presented to illustrate our method in Section 4.

Throughout this paper, $\langle\cdot,\cdot\rangle$ stands for the inner product of $\mathbb{R}^N$ and $|x|$ denote the norm of the $\mathbb{R}^N$-vector x.

2. The Parallel Residual Method

In this section we introduce a method which is applicable to indefinite problems. To simplify the discussion we assume that A is nonsingular. In regard to the singular case, see the remark at the end of Section 3. The basic idea of the method is developed, using the minimal residual method (MR). As will be seen, MR is the special case of Orthomin. The sequence of approximate solutions to (1.1) is updated by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k r_k \\ r_k &= f - A x_k \end{aligned} , \tag{2.1}$$

where α_k is chosen such that $|r_{k+1}|^2$ is minimized. If we write (2.1) in terms of residuals $\{r_k\}$, then

$$r_{k+1} = r_k - \alpha_k A r_k \tag{2.2}$$

From this, it is easy to see

$$\alpha_k = \langle r_k, Ar_k\rangle \;/\; \langle Ar_k, Ar_k\rangle \tag{2.3}$$

and

$$|r_{k+1}|^2 = |r_k|^2 - \langle r_k, Ar_k\rangle^2 \;/\; |Ar_k|^2. \tag{2.4}$$

Obviously, the sequency $\{|r_k|^2\}$ is nonincreasing and hence, it converges to a positive constant. From (2.2) and (2.4), this implies

$$|r_{k+1} - r_k|^2 = \langle r_k, Ar_k\rangle^2 \;/\; |Ar_k|^2$$

converges to zero. Since $\{|r_k|^2\}$ is uniformly bounded and so is $\{|Ar_k|^2\}$, it follows that $\langle r_k, Ar_k\rangle$ converges to zero. Moreover, if $|r_{k+1}| = |r_k|$ for some k, then the same argument shows $r_{k+1} \equiv r_k$. Hence, $\{r_k\}$ converges monotonically to a vector r^1 which satisfies $\langle r^1, Ar^1\rangle = 0$.

From (2.3) and (2.4) we have

$$|\alpha_k r_k|^2 = (\,|r_k|^2 - |r_{k+1}|^2)\;|r_k|^2/|Ar_k|^2.$$

If A is nonsingular, then there exists a positive constant K such that

$$|r|^2/|Ar|^2 \le K \qquad \text{for all} \quad r \in \mathbb{R}^N.$$

Hence $\sum_k |\alpha_k r_k|^2 \le K(\,|r_0|^2 - |r^1|^2)$. It now follows from (2.1) that $\{x_k\}$ converges to x^1 such that

$$r^1 = f - Ax^1 \qquad \text{and} \qquad \langle r^1, Ar^1\rangle = 0. \tag{2.5}$$

If the symmetric part of A, say M; i.e.,

$$M = (A + A^T)/2$$

or

$$\langle x, Ax \rangle = \langle x, Mx \rangle \qquad \text{for all} \quad x \in \mathbb{R}^N;$$

is positve definite, then $r^1 \equiv 0$ which follows from the fact that $\langle r^1, Ar^1 \rangle = 0$. However, when M is indefinite, this may not be true, i.e., $|r^1| \neq 0$. In other words, MR may fail to converge for indefinite problems. The method described below makes use of the fact that the sequence $\{x_k, r_k\}$ converges to the pair (x^1, r^1) that satisfies (2.5).

We consider the following algorithm.

Algorithm $PR^{(1)}$

1. Choose x_0. Compute $r_0 = f - Ax_0$.
   Set $\tilde{r}_0 = r_0 - \langle r_0, g \rangle g$.
2. Iterate: For $k = 0,1,2,\cdots$ until convergence DO:

$$x_{k+1} = x_k + \alpha_k \tilde{r}_k \tag{2.6a}$$

$$q_k = A\tilde{r}_k - \langle A\tilde{r}_k, g \rangle g \tag{2.6b}$$

$$\tilde{r}_{k+1} = \tilde{r}_k - \alpha_k q_k \tag{2.6c}$$

$$\alpha_k = \langle \tilde{r}_k, q_k \rangle / |q_k|^2, \tag{2.6d}$$

where $g = r^1/|r^1|$ and r^1 is the residual vector obtained from MR. It is easily shown by induction that

$$\langle \tilde{r}_k, g \rangle = 0 \qquad \text{for all} \quad k \geqslant 0$$

and

$$\tilde{r}_k = r_k - \langle r_k, g\rangle\, g.$$

The idea here is that in the second step (2.2) of MR, the residual vector perpendicular to g; i.e., $\tilde{r}_{k+1}$ is minimized in the least-squares sense. Using the same arguments used in the proof of convergence of MR, one can show that $\{\tilde{r}_k\}$ converges monotonically to a vector $\tilde{r}^2$ which satisfies

$$\langle \tilde{r}^2, A\tilde{r}^2\rangle = 0 \quad \text{and} \quad \langle \tilde{r}^2, g\rangle = 0. \tag{2.7}$$

If $\tilde{r}^2 \equiv 0$ (where a sufficient condition for this is that (2.7) implies $\tilde{r}^2 \equiv 0$ and also see the discussions in Section 3), then the algorithm (2.6) yields the pair (x^2, r^2) such that

$$r^2 = f - Ax^2 \tag{2.8}$$

and r^2 is parallel to r^1 since $r^2 = \langle r^2, g\rangle g$.

If $s_i = \langle r^i, g\rangle$ for $i = 1,2$, then $s_2 r^1 = s_1 r^2$. From (2.5) and (2.8)

$$A(s_2\, x^1 - s_1\, x^2) = (s_2 - s_1) f,$$

so that one can find a solution x to (1.1):

$$x = (s_2\, x^1 - s_1\, x^2)/(s_2 - s_1) \tag{2.9}$$

A choice of the startup vector x_0 in the algorithm (2.6) is given by

$$x_0 = x^1 + cg, \qquad c \neq 0. \tag{2.10}$$

In this way, it can be shown that $s_1 \neq s_2$ in (2.9) if A is nonsingular. Indeed, since $\langle\tilde{r}_k,g\rangle = 0$ in the algorithm $PR^{(1)}$,

$$\langle x^2,g\rangle = \langle x_0,g\rangle = \langle x^1,g\rangle + c.$$

If $s_1 = s_2$, then $A(x^1 - x^2) = 0$, so that $x^1 = x^2$, which contradicts the above statement.

The algorithm (2.6) can be viewed as a method which finds the pair (x^2,r^2) whose second element r^2 is <u>parallel</u> to r^1 where r^1 is the residual vector obtained from MR. It also leads to the steepest descent algorithm for minimizing

$$|A\,x - \langle Ax,g\rangle g - f|^2$$

over the vector x satisfying $\langle x - x_0,g\rangle = 0$.

If $\tilde{r}^2$, the limit of $\{\tilde{r}_k\}$ is nonzero, then inductively one can consider the further algorithm $PR^{(2)}$ in which (2.6b) is replaced as

$$q_k = A\tilde{r}_k - \langle A\tilde{r}_k,g_1\rangle g_1 - \langle A\tilde{r}_k,g_2\rangle g_2.$$

These now lead to the successive algorithms $\{PR^{(i)}\}$.

Algorithm $PR^{(i)}$

1. Choose x_0. Computer $r_0 = f - Ax_0$.

   Set $\tilde{r}_0 = r_0 - \sum_{j=1}^{i} \langle r_0, g_j \rangle g_j$.

2. Iterate: For $k = 0,1,2,\cdots$ until convergence DO;

$$x_{k+1} = x_k + \alpha_k \tilde{r}_k \tag{2.11a}$$

$$q_k = A\tilde{r}_k - \sum_{j=1}^{i} \langle A\tilde{r}_k, g_j \rangle g_j \tag{2.11b}$$

$$\tilde{r}_{k+1} = \tilde{r}_k - \alpha_k q_k \tag{2.11c}$$

$$\alpha_k = \langle \tilde{r}_k, q_k \rangle / |q_k|^2 \tag{2.11d}$$

where $g_j = \tilde{r}^j / |\tilde{r}^j|$, $1 \leq j \leq i$. $\{\tilde{r}^j\}$ is the sequence of orthogonal vectors where $\tilde{r}^{j+1}$ is the limiting residual vector in (2.11c) of $PR^{(j)}$, and $PR^{(0)}$ is identified with MR. For the first integer i such that $\tilde{r}^{i+1} \equiv 0$, the sequence of algorithms $\{PR^{(k)}\}_{k>0}$ is terminated. As will be shown in Section 3, if (x^{k+1}, r^{k+1}) is the limiting pair of $PR^{(k)}$ for $0 \leq k \leq i$, then a solution to (1.1) is given by

$$x = \left(x^{i+1} - \sum_{k=1}^{i} \xi_k x_k\right) / \left(1 - \sum_{k=1}^{i} \xi_k\right) \tag{2.12}$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_i)^T$ is the solution of the following system of linear equation

$$C\xi = b. \tag{2.13}$$

Here, C is an upper triangular matrix of order i with (k,ℓ) - element $c_{k,\ell} = \langle g_k, r^\ell \rangle$, $k \leq \ell$, and $b_k = \langle g_k, r^{i+1} \rangle$, $1 \leq k \leq i$. Since $|c_{k,k}| = |\tilde{r}^k| \neq 0$ for $1 \leq k \leq i$, $\det(C) \neq 0$ and hence, (2.13) has a unique solution.

3. Orthomin and Parallel Residual Methods

In order to accelerate the convergence of MR one can consider the following algorithm, so-called Orthomin [7].

Algorithm Orthomin (i)

1. Choose x_0. Compute $r_0 = f - Ax_0$.

   Set $p_0 = r_0$.

2. Iterate: For $k = 0,1,2,\cdots$ until convergence DO;

$$x_{k+1} = x_k + \alpha_k p_k \tag{3.1a}$$

$$r_{k+1} = r_k - \alpha_k Ap_k \tag{3.1b}$$

$$\alpha_k = \langle r_k, Ap_k \rangle / |Ap_k|^2 \tag{3.1c}$$

$$p_{k+1} = r_{k+1} - \sum_{j=1}^{i} \beta_k^j p_{1-j+1} \tag{3.1d}$$

$$\beta_k^j = \langle Ar_{k+1}, A p_{k-j+1} \rangle / |Ap_{k-j+1}|^2 \tag{3.1e}$$

for $1 \leqslant j \leqslant i$

In (3.1d) and (3.1e), the direction vectors $\{P_k\}$ are chosen so that the A^TA - orthogonality holds among vectors $\{P_j\}_{k-i+1 \leq j \leq k+1}$. For the special case $i = 0$, Orthomin is identical to MR (the minimal residual method). The case $i = 1$ is most attractive as far as the work and storage costs of performing one loop are concerned and it is exactly the same as the conjugate residual method when A is symmetric and positive definite. So, in the following discussion we only consider Orthomin (1). However, all results given in this section remain valid for the cases, $i \geq 1$.

In Orthomin (1), (3.1d) and (3.1e) are simply written as

$$P_{k+1} = r_{k+1} - \beta_k \, p_k$$

$$\beta_k = \langle Ar_{k+1}, Ap_k \rangle \, / \, |Ap_k|^2.$$

Note that for $k \geq 1$

$$\langle r_k, Ap_k \rangle = \langle r_k, Ar_k - \beta_{k-1} \, Ap_{k-1} \rangle$$

$$= \langle r_k, Ar_k \rangle,$$

Since $\langle r_k, Ap_{k-1} \rangle = 0$, and hence

$$\alpha_k = \langle r_k, Ar_k \rangle \, / \, |Ap_k|^2.$$

Thus, the same arguments given in the proof of convergence of MR enable us to show that the sequence $\{x_k, r_k\}$ converges to a pair (x^1, r^1) which satisfies

$$\begin{aligned} r^1 &= f - Ax^1 \\ \langle r^1, Ar^1\rangle &= 0. \end{aligned} \tag{3.2}$$

Adaption of the method described in Section 2 to Orthomin (1) for the case, $r^1 \neq 0$ is as follows.

<u>Algorithm $PR^{(1)}$</u>

1. Choose x_0. Compute $r_0 = f - Ax_0$.

   Set $\tilde{r}_0 = r_0 - \langle r_0, g\rangle g$ (3.3a)

   $p_0 = \tilde{r}_0$

2. Iterate: For $k = 0,1,2,\cdots$ until convergence DO;

$$x_{k+1} = x_k + \alpha_k p_k \tag{3.3b}$$

$$\tilde{r}_{k+1} = \tilde{r}_k - \alpha_k q_k \tag{3.3c}$$

$$\alpha_k = \langle \tilde{r}_k, q_k\rangle \,/\, |q_k|^2 \tag{3.3d}$$

$$p_{k+1} = \tilde{r}_{k+1} - \beta_k p_k \tag{3.3e}$$

$$q_{k+1} = (A\tilde{r}_{k+1} - \langle A\tilde{r}_{k+1}, g\rangle g) - \beta_k q_k \tag{3.3f}$$

$$\beta_k = \langle A\tilde{r}_{k+1}, q_k\rangle \,/\, |q_k|^2, \tag{3.3g}$$

where $g = r^1/|r^1|$. It is easy to show by induction that for $k \geq 0$

$$\tilde{r}_k = r_k - \langle r_k, g\rangle\, g$$

and

$$q_k = Ap_k - \langle Ap_k, g\rangle\, g.$$

Successively, one can consider the sequence of algorithms $\{PR^{(i)}\}_{i>0}$ in which (3.3a) and (3.3f) are replaced by

$$\tilde{r}_0 = r_0 - \sum_{j=1}^{i} \langle r_0, g_j\rangle\, g_j \tag{3.3a$'$}$$

$$q_{k+1} = \Big(A\tilde{r}_{k+1} - \sum_{j=1}^{i} \langle A\tilde{r}_{k+1}, g_j\rangle\, g_j \Big) - \beta_k\, q_k, \tag{3.3f$'$}$$

respectively, where $g_j = \tilde{r}^j/|\tilde{r}^j|$. For $j \geq 1$, $\tilde{r}^j$ is the limiting residual vector in (3.3c) of Algorithm $PR^{(j-1)}$. The sequence of vector $\{\tilde{r}^j\}$ is orthogonal and satisfies

$$\langle \tilde{r}^j, A\tilde{r}^j\rangle = 0 \qquad \text{for all } j. \tag{3.4}$$

By construction, the sequence of successive algorithms $\{PR^{(k)}\}$ yields the pairs (x^k, r^k), $k \geq 1$ such that

$$r^k = f - Ax^k$$

and

$$r^k = \tilde{r}^k + \sum_{j=1}^{k-1} \langle r^k, g_j\rangle\, g_j = \sum_{j=1}^{k} c_{j,k}\, g_j$$

where $c_{j,k} = \langle r^k, g_j\rangle$, $1 \leq j \leq k$. Let i be the first integer such that $\tilde{r}^{i+1} \equiv 0$. Then

$$r^{i+1} = \sum_{j=1}^{i} b_j g_j$$

$$b_j = \langle r^{i+1}, g_j \rangle, \qquad 1 \leq j \leq i.$$

If $\xi = (\xi_1, \cdots, \xi_i)^T$ is the solution to the linear equation $C\,\xi = b$, then

$$r^{i+1} = \sum_{k=1}^{i} \xi_k r^k$$

or

$$A\Big(x^{i+1} - \sum_{k=1}^{i} \xi_k x^k\Big) = \Big(1 - \sum_{k=1}^{i} \xi_k\Big) f.$$

Hence a solution to (1.1) is given by (2.12) and (2.13).

A choice of the startup vector x_0 in $PR^{(j)}$ would be

$$x_0 = x^j + c_j g_j, \qquad c_j \neq 0. \tag{3.5}$$

With this choice, one can show that $1 - \sum_{k=1}^{i} \xi_k \neq 0$ in (2.12), using similar agrument given in Section 2.

In the above method, the sequence of vectors $\{x^k\}$ and $\{g_k\}$ need to be stored and the algorithm $PR^{(k)}$ requires $2kN$ multiplications and additions besides the basic operations of Orthomin (1) to be performed in each loop. Let us define the index of indefiniteness of the matrix A; say $INDF(A)$. If S is the set of vectors x satisfying $\langle x, Ax \rangle = 0$, then $INDF(A)$ is defined as the maximum number of orthonormal vectors contained in S. Note that S is not a linear subspace of $\mathbb{R}^N$ in general. If the symmetric part of A is positive deifnite, then $S = \{0\}$ and $INDF(A) = 0$. If A is skewsymmetric, then $S = \mathbb{R}^N$ and INDF(A) = N. In general, $N \geq$ INDF(A) $\geq$ the number of nonpositive eigenvalues of M.

It is easy to see from (3.4) that the number of successive algorithms $\{PR^{(k)}\}_{k>0}$ required to obtain a solution of (1.1) is less than or equal to INDF(A) + 1. For example, when the symmetric part of A is positive qdefinite, $PR^{(0)}$, which is identical to Orthomin (1) can yield a solution. A result of these discussions is that the method described above is effective only when INDF(A) is relatively small. The other possible applications of our method are as follows. For the stiff problem, MR or Orthomin (1) may slow down in the process of iterations; i.e., the convergence ratio λ;

$$\lambda_k = |r_{k+1}|^2 \;/\; |r_k|^2 = 1 - \langle r_k, Ar_k\rangle^2 \;/\; |r_k|^2 \; |Ap_k|^2 \qquad (3.6)$$

becomes nearly 1. In such a case, one can terminate Orthomin (1) with the directional vector r^1 which gives the convergence ratio of nearly 1 and then employ the algorithm $PR^{(1)}$ with $g = r^1/|r^1|$. This may reduce the number of computations required for convergence.

Remark. In principle, the method described above can be used for the case when A is singular. In $PR^{(j)}$, $j \geq 0$, if the sequence $\{|q_k|\}$ remains uniformly bounded below, then the same convergence property as for A nonsingular holds. However, the choice (3.5) of the startup vector x_0 may not ensure the validity of (2.12) (i.e., $1 - \sum_{k=1}^{i} \xi_k \neq 0$). If $\{q_k\}$ converges to the zero vector, then one must terminate $PR^{(j)}$, $j \geq 0$ according to a stopping criterion: $|q_k| < \varepsilon$ with small positive number ε.

4. A Numerical Example

To illustrate the method described in Sections 2 and 3, we consider the equation

$$\varepsilon\, u_{xx} + (xu)_x = g \qquad \text{in } (-1,1) \tag{4.1}$$

with boundary condition $u(\pm 1) = 0$, where $\varepsilon > 0$ and the function g is chosen so that $\cos(\frac{\pi}{2}x)$, $-1 \leq x \leq 1$ is the solution to (4.1). It is easy to show that if $\mathscr{A}$ denotes a linear operator on $L_2[-1,1]$:

$$\mathscr{A}u = \varepsilon\, u_{xx} + (xu)_x$$

with

$$\mathscr{D}(\mathscr{A}) = \{u \in L_2 \mid u_x,\ u_{xx} \in L_2 \text{ and } u(\pm 1) = 0\},$$

then for small ε, $\mathscr{A}$ is indefinite; i.e., there exists a function $u \in \mathscr{D}(\mathscr{A})$ such that

$$\langle \mathscr{A}u, u\rangle_{L_2} = 0, \qquad u \neq 0.$$

Thus, for such an ε, the discretization of the equation (4.1) may lead to an indefinite system of linear equations. In this discussion, we are going to use the Legendre-tau method [3] to approximate the solution to (4.1) (the details of this method will be discussed in a forthcoming paper). The approximate solution $u^N(x)$ to (4.1) is represented as

$$u^N(x) = \sum_{k=0}^{N} a_k\, P^k(x),$$

where $\{P^k\}_{k\geq 0}$ are the Legendre polynomials on $[-1,1]$. These polynomials are orthogonal on $L_2[-1,1]$. The (two) underlying ideas of the tau method for solving (4.1) are (i) equating (4.1) in the sense that

$$\langle \mathscr{A}u^N - g, \chi \rangle_{L_2} = 0 \tag{4.2}$$

for all polynomials χ of degree at most $N-2$, and (ii) imposing the boundary condition on the approximate solution u^N; i.e.,

$$u^N(\pm 1) = 0. \tag{4.3}$$

Now (4.2) and (4.3) yield a system of linear equations of order $N+1$ whose coefficient matrix A is almost full. However, the matrix-vector product Av can be performed effectively in $O(N)$ operations.

In our computations, the following preconditioning technique is used. For example, the preconditioned Orthomin (1) is as follows.

<u>Algorithm (4.4)</u>

1. Choose x_0. Compute $r_0 = f - Ax_0$.

   Compute $z_0 = Q^{-1} r_0$

   Set $p_0 = z_0$.

2. Iterate for $k = 0,1,\cdots$ until convergence DO:

$$x_{k+1} = x_k + \alpha_k p_k$$

$$z_{k+1} = z_k - \alpha_k Q^{-1} Ap_k$$

$$\alpha_k = \langle z_k, Q^{-1} Ap_k\rangle / |Q^{-1} Ap_k|^2$$

$$p_{k+1} = z_{k+1} - \beta_k p_k$$

$$\beta_k = \langle Q^{-1} Az_{k+1}, Q^{-1} Ap_k\rangle / |Q^{-1} Ap_k|^2,$$

where Q is the matrix corresponding to the discretization of the equation $u_{xx} = g$ with $u(\pm 1) = 0$, also using the Legendre-tau method. The operation $Q^{-1} v$ can be performed in N multiplications and N additions. The inner product in this algorithm is weighted so that for $u = col(u_0,\ldots u_N)$ and $v = col(v_0,\ldots v_N)$,

$$\langle u,v\rangle = \int_{-1}^{1} u_x(x)\, v_x(x)\, dx$$

where

$$u(x) = \sum_{k=0}^{N} u_k P^k(x) \quad \text{and} \quad v(x) = \sum_{k=0}^{N} v_k P^k(x) \quad \text{in } [-1,1].$$

Consider the case $\varepsilon = .1$ and $N = 32$. The table below shows the convergence history of our algorithms. Here λ_k is the convergence ratio defined by (3.6). We chose $x_0 = 0$ as the startup vector for the algorithm (4.4). It was terminated after six iterations. The criterion for the termination is that $1 - \lambda_k \leq 5. \times 10^{-3}$. The preconditioned version of the algorithm (3.3) was then applied with the startup vector chosen as described in (3.5). It was also terminated after 13 iterations under the same criterion as above. Finally, the algorithm $PR^{(2)}$ with the preconditioning did converge, where the stopping criterion is that $|\tilde{z}_k|^2 \leq 1. \times 10^{-24}$. The approximate solution $u^{32}(x)$ to (4.1) is obtained by (2.12) and (2.13) in which

$$\xi_1 = -1.99$$

$$\xi_2 = .047$$

$$1 - \xi_1 - \xi_2 = 2.943$$

and

$$u^{32}(0) - 1 = .66 \times 10^{-12}.$$

It should be noted that in the algorithm $PR^{(2)}$ the convergence ratios $\{\lambda_k\}$ were distributed on the interval [.008, .412]. All computations were performed on a Control Data Corporation Cyber 170 Model 730 at NASA Langley Research Center. The total CPU time was 1.907 sec., which includes the time spent computing the Gauss quadratures on [-1,1].

Table I.

Algorithm	Iteration (k)	$\lvert\tilde{z}_k\rvert^2$	$1 - \lambda_k$
	1	$.546 \times 10^{-1}$	$.144 \times 10^{-1}$
	2	$.727 \times 10^{-2}$	.867
$PR^{(0)}$	3	$.102 \times 10^{-2}$	.860
	4	$.258 \times 10^{-3}$	.747
	5	$.251 \times 10^{-3}$	$.277 \times 10^{-1}$
	6	$.251 \times 10^{-3}$	$\underline{.968 \times 10^{-4}}$
	1	$.317 \times 10^{-1}$	$.158 \times 10^{-1}$
	2	$.170 \times 10^{-1}$	.463
	3	$.963 \times 10^{-2}$	.435
	⋮	⋮	⋮
$PR^{(1)}$	9	$.138 \times 10^{-3}$	.780
	10	$.132 \times 10^{-3}$	$.455 \times 10^{-1}$
	11	$.128 \times 10^{-3}$	$.289 \times 10^{-1}$
	12	$.127 \times 10^{-3}$	$.628 \times 10^{-2}$
	13	$.127 \times 10^{-3}$	$\underline{.321 \times 10^{-2}}$
	1	$.105 \times 10^{-2}$	.582
	2	$.124 \times 10^{-4}$	.988
	3	$.208 \times 10^{-5}$	.832
	4	$.958 \times 10^{-7}$	.954
	5	$.263 \times 10^{-7}$	.725
	⋮	⋮	⋮
$PR^{(2)}$	11	$.511 \times 10^{-12}$	.743
	12	$.153 \times 10^{-12}$	.701
	13	$.147 \times 10^{-13}$	.904
	⋮	⋮	⋮
	28	$.645 \times 10^{-23}$	.756
	29	$.208 \times 10^{-23}$	.678
	30	$\underline{.540 \times 10^{-24}}$	.740

References

[1] H. C. Elman, Iterative Methods for Large, Sparse, Nonsymmetric Systems of Linear Equations, Research Report #229, Yale University, Department of Computer Science, April 1982.

[2] H. C. Elman, Iterative Methods for Non-Selfadjoint Elliptic Problems, Technical Report #266, Yale University, Department of Computer Science, April 1983.

[3] D. Gottlieb and S. A. Orszag, Numerical Analysis of Spectral Methods: Theory and Applications, Regional Conference Series in Applied Mathematics, SIAM, 1977.

[4] C. C. Paige and M. A. Saunders, Solution of Sparse Indefinite Systems of Linear Equations, SIAM J. Numer. Anal., 12 (1975), pp. 617-629.

[5] Y. Saad, Iterative Solution of Indefinite Symmetric Linear Systems by Methods Using Orthogonal Polynomials over Two Disjoint Intervals, SIAM J. Numer. Anal., 20 (1983), pp. 784-811.

[6] Y. Saad and M. H. Schultz, Conjugate Gradient-like Algorithms for Solving Nonsymmetric Linear Systems, Research Report #283, Yale University, Department of Computer Science, August 1983.

[7] P. K. W. Vinsome, Orthomin, An Iterative Method for Solving Sparse Sets of Simultaneous Linear Equations, In Society of Petroleum Engineers of AIME, Proc. of the Fourth Symposium on Reservoir Simulation, 1976, pp. 149-159.

1. Report No. NASA CR-172345 ICASE Rept. No. 84-13	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle An Iterative Method for Indefinite Systems of Linear Equations		5. Report Date April 1984
		6. Performing Organization Code
7. Author(s) Kazufumi Ito		8. Performing Organization Report No. 84-13
		10. Work Unit No.
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665		11. Contract or Grant No. NAS1-17070
		13. Type of Report and Period Covered contractor report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Robert H. Tolson
Final Report

16. Abstract

An iterative method for solving nonsymmetric indefinite linear systems is proposed. The method involves the successive use of a modified version of the conjugate residual method. A numerical example is given to illustrate the method.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
indefinite linear systems orthomin parallel residual method	64 Numerical Analysis Unlimited-Unclassified

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	21	A02

NASA-Langley, 1984