A SURVEY OF UNIFIED CONSTITUTIVE THEORIES*

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A literature survey has been conducted to assess the state-of-the-art of time-temperature dependent elastic-viscoplastic constitutive theories which are based on the unified approach. This class of constitutive theories is characterized by the use of kinetic equations and internal variables with appropriate evolutionary equations for treating all aspects of inelastic deformation including plasticity, creep, and stress relaxation. The review identifies more than ten such unified theories which are shown to satisfy the uniqueness and stability criteria imposed by Drucker's postulate and Ponter's inequalities. These theories are compared on the basis of the types of flow law, kinetic equation, evolutionary equation of the internal variables, and treatment of temperature dependence. The similarities and differences of these theories are first outlined in terms of mathematical formulations and then illustrated by comparisons of theoretical calculations with experimental results which include monotonic stress-strain curves, cyclic hysteresis loops, creep and stress relaxation rates, as well as thermomechanical loops. Numerical methods used for integrating these stiff time-temperature dependent constitutive equations are also briefly reviewed.

INTRODUCTION

Constitutive theories based on the classical concepts of plasticity and creep generally decompose the inelastic strain rate into a time-independent plastic strain rate and a time-dependent creep rate with independent constitutive relations describing plastic and creep behavior. While this approach can be rationalized on historical grounds and perhaps on computational convenience, experimental evidence collected on structural alloys at elevated temperature indicates inherent time-dependency and creep/plasticity interactions [1]. This suggests that inelastic deformation might be primarily controlled by a single overall mechanism and should be treated in a unified manner.

In recent years, a number of formulations of elastic-viscoplastic constitutive equations have been presented in the engineering literature. Such equations are sometimes referred to as "unified" since inelastic deformations are represented and treated by a single kinetic equation and a discrete

* This work was performed under NASA Contract No. NAS3-23925.

set of internal variables. In this context, creep, stress relaxation, and plastic flow are different manifestations of time-dependent inelastic deformations under particular loading conditions with consequently different response characteristics.

There are more than ten unified constitutive theories in the literature. These constitutive equations have some common properties and some essential differences which have been reviewed recently by Walker [2]. Since then, there have been more advances in the development of the unified theories. The purpose of this survey is to update Walker's previous work by reviewing the state-of-the-art and the numerical integration techniques for these unified theories. This survey also serves to identify areas for further model developments.

The unified theories which are reviewed in this survey include those of Walker [2], Bodner and Partom [3,4], Miller [5], Krieg, Swearengen and Rhode [6], Chaboche [7], Robinson [8], Hart and co-workers [9], Stouffer and Bodner [10], Lee and Zaverl [11], Ghosh [12], and Kagawa and Asada's modification of Miller's model [13].

GENERAL CHARACTERISTICS OF UNIFIED CONSTITUTIVE EQUATIONS FOR ELASTIC-VISCOPLASTIC MATERIALS

Constitutive equations for elastic-viscoplastic material could be formulated either with or without the use of a yield criterion. A basic assumption for this class of constitutive theories is that in the range where inelastic strains are present, the total strain rate $\dot{\varepsilon}_{ij}$ can be divided into elastic and inelastic components which are both nonzero, i.e.

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p} \tag{1}$$

This equation is applicable for the small strain case and a similar decomposition is assumed to hold for the deformation rates in the case of large strains. Those are equivalent to strain rates if the strains are small.

For the small strain case considered here, the elastic strain rate is given by the time derivative of Hooke's Law. An important question related to Equation (1) is an appropriate definition of $\dot{\varepsilon}_{ij}^{p}$. One possibility is to define $\dot{\varepsilon}_{ij}^{p}$ as the total strain rate contribution that is both thermodynamically and geometrically irreversible , i.e., non-elastic in all respects. An alternative definition of the incremental plastic strain is the residual strain upon loading and unloading from a stress increment. This seems to be the definition adopted by E. H. Lee in his treatment of large plastic strains, see e.g. [14]. Since non-elastic strains are also generated during unloading, constitutive equations based on this definition would be different than in the former case. Each approach seems possible, but the proper definition and use of the $\dot{\varepsilon}_{ij}^{p}$ term should be indicated and be consistent with the constitutive equations.

The expression "unified" applied to such theories is generally taken to mean that all aspects of inelastic behavior such as plastic flow, creep, and stress relaxation are included in the $\dot{\varepsilon}_{ij}^{p}$ function and are particular response characteristics for different loading histories. This broad definition of "unified" would admit theories with and without a yield criterion and alternative specifications of $\dot{\varepsilon}_{ij}^{p}$. Separation of the non-elastic strain rate into geometrically reversible (anelastic) and non-reversible components could be a convenient procedure and does not detract from the "unified" concept.

Constitutive theories which are formulated without the use of a yield criterion include that of Bodner and his associates [3,4], Walker [2], Miller [5], and Krieg, Swearengen and Rhode [6]. Since these models do not contain a completely elastic regime, the function that describes the inelastic strain rate should have the property that the inelastic strain rate be very small for low stress levels.

For theories with a yield criterion, $\dot{\varepsilon}_{ij}^{p}$ is identically zero until an invariant function of the stress reaches a prescribed value; the function, by definition, is independent of strain rate. For stresses at or exceeding the yield value, Equation (1) applies and $\dot{\varepsilon}_{ij}^{p}$ and the stress σ_{ij} are functionally related. The fully elastic state, i.e. $\dot{\varepsilon}_{ij}^{p} = 0$, would apply only for stress states less than the rate independent yield value, and loading and unloading paths above that are controlled by the loading conditions through the constitutive equations. Theories of this type have been developed by Perzyna [15] for the case of isotropic hardening and by Chaboche [7], Robinson [8], and Lee and Zaverl [11] for the case of both isotropic and directional hardening.

All the unified models are formulated on the basis of internal variables which depend on the loading history. The essential features of these unified theories are: (1) a flow law which functional form depends on the method of treatment of directional (kinematic) hardening, (2) a kinetic equation which is the temperature dependent functional relationship between the strain rate and stress invariants and includes internal variables, and (3) a set of evolution equations for describing the growth of the internal variables. Here, the internal variables are used to represent the current resistance to inelastic flow of the deformed solid. Two deforming solids with identical values of their internal variables would have identical inelastic responses under the same imposed stress state. Both the choice and the number of internal variables vary with the unified models. Most of the unified models use two internal variables or one variable with two components: one to represent isotropic hardening and another to represent directional (kinematic) hardening. In most models, the isotropic hardening variable is represented by a scalar quantity, either the drag stress (K) or the yield stress (Y), while directional hardening is represented by a second order tensor Ω_{ij} or a scalar function of such a tensor.

Basic Flow Laws

Four basic forms of the inelastic flow law have been identified. Plastic incompressibility is always assumed and these flow laws are:

$$(1) \quad \dot{\varepsilon}_{ij}^{p} = \lambda_1 S_{ij} \qquad \dot{\varepsilon}_{kk}^{p} = 0 \tag{2a}$$

$$(2) \quad \dot{\varepsilon}_{ij}^{p} = \lambda_2 \Sigma_{ij} = \lambda_2 (S_{ij} - \Omega_{ij}) \quad , \quad \dot{\varepsilon}_{kk}^{p} = 0 \tag{2b}$$

$$(3) \quad \dot{\varepsilon}_{ij}^{p} = \lambda_{ijkl}\, S_{kl} \quad , \quad \dot{\varepsilon}_{iikl}^{p} = \dot{\varepsilon}_{ijkk}^{p} = 0 \tag{2c}$$

$$(4) \quad \dot{\varepsilon}_{ij}^{p} = \frac{\partial f}{\partial \sigma_{ij}} \quad , \quad \dot{\varepsilon}_{kk}^{p} = 0 \tag{2d}$$

where S_{ij}, σ_{ij}, and Σ_{ij} are the deviatoric, direct and effective stresses, respectively. The tensor Ω_{ij} represents the "equilibrium stress" which has also been referred to as the "back stress" and the "rest stress." The parameter f is a yield function or a flow potential. It should be noted that the first three laws can be considered or can be derived from Equation (2d) if they are associated with a flow potential.

Equation (2a) is the Prandtl-Reuss flow law associated with the von Mises yield criterion. However, it can be considered as a basic material equation in its own right independently of a yield condition. As such, this equation is usually taken to be applicable for proportional loading conditions for which isotropic hardening would be appropriate. The equation states that the material response (i.e., the plastic strain rate) to stress is isotropic even though λ_1 could be stress history dependent. Since stress is directional, λ_1 could have a directional character within the context of incremental isotropy and thereby account for induced directional hardening effects.

Equation (2b) is the flow law obtained by introducing the kinematic hardening variable of Prager [16] into the classical plasticity formulation to account for directional hardening (the Bauschinger effect). In this context, the term Ω_{ij} would represent the new origin of a translating von Mises yield surface in deviatoric stress space, and Equation (2b) would be the associated flow rule. As before, Equation (2b) can be taken to be a basic material equation in a formulation without a yield criterion and the "equilibrium stress" tensor Ω_{ij} is generally intended to serve the following functions: (a) to account for directional hardening (the multi-dimensional Bauschinger effect), and for the non-coaxiality of $\dot{\varepsilon}_{ij}^{p}$ and S_{ij} under nonproportional loading histories (Figure 1); (b) to account for reversed plastic straining effects, e.g. reversed creep, relaxation through zero stress, when the effective stress Σ_{ij} is negative; (c) for theories without a fully elastic range (i.e., a yield criterion), to account for low plastic straining within a given range.

Equation (2c) is the generalized anisotropic form of the Prandtl-Reuss flow law which can be rewritten in a 6D stress and strain rate space to take the form,

$$\dot{E}^p_\alpha = \Lambda_{\alpha\beta} T_\beta \qquad \begin{array}{l} \alpha = 1,\!-\!\!-6 \\ \beta = 1,\!-\!\!-6 \end{array} \tag{3}$$

where $\dot{E}_\alpha$ and T_β are related to the usual plastic strain rates and stresses in a simple manner, see [10], and $\Lambda_{\alpha\beta}$ is the 6x6 matrix of coefficients. If the material is initially isotropic and the law for plastically induced directional hardening does not lead to off diagonal terms, then $\Lambda_{\alpha\beta}$ is initially and remains diagonal. Under these conditions, Equation (2c) is equivalent to Equation (2b) since 6 material constants determine the anisotropic flow behavior. All the flow equations, Equations (2a,b,c), would be equivalent for the case of proportional loading, including cyclic conditions. The real differences in those equations would show up for nonproportional loading histories.

For constitutive theories with a flow potential, both the flow law and the growth law of the directional (kinematic) hardening variable Ω_{ij} are derivable from a single flow potential. The associated flow law of a basic form of such a flow potential is [8].

$$\dot{\varepsilon}^p_{ij} = \frac{1}{2\mu} F^{\frac{n-1}{2}} \quad (\sigma_{ij} - \Omega_{ij}) \qquad \text{for inelastic loading} \tag{4a}$$

and $\dot{\varepsilon}^p_{ij} = 0$ for elastic loading/unloading (4b)

where F is the von Mises yield function, n and μ are material parameters. The conditions for inelastic loading and elastic unloading have been identified in [8]. It can be easily seen that (4a) and (2b) are equivalent. In both cases, the direction of the inelastic strain rate vector is coaxial with the current effective stress vector (see Figure 1).

<u>Kinetic Equations</u>

The flow laws, Equations (2a) and (2b) can be squared to give respectively,

$$\lambda_1 = [D_2{}^p/J_2]^{1/2} \tag{5a}$$

$$\lambda_2 = [D_2{}^p/J_2']^{1/2} \tag{5b}$$

where $D_2{}^p$ is the second invariant of the plastic strain rate, $D_2{}^p = (1/2)\ \dot{\varepsilon}^p_{ij}\ \dot{\varepsilon}^p_{ij}$, and J_2 and J_2' are the second invariants of the deviatoric stress and effective deviatoric stress, respectively,

$$J_2 = (1/2)\ S_{ij}\ S_{ij} \tag{6a}$$

$$J_2' = (1/2)\ (S_{ij} - \Omega_{ij})(S_{ij} - \Omega_{ij}) \tag{6b}$$

Fundamental to all "unified" viscoplastic formulations based on flow laws of the forms listed in Equations (2) is that inelastic deformations are governed by a functional relation between D_2p and J_2 (or J_2') that could involve load history dependent variables. These variables are intended to represent properties of the inelastic state with respect to resistance to plastic flow, e.g. hardening, and damage. Some functions that have been suggested are the following.

$$\text{(a)} \quad D_2^p = D_o X^n \tag{7a}$$

$$\text{(b)} \quad D_2^p = D_o \exp\left[-\left(\frac{1}{X}\right)^n\right] \tag{7b}$$

$$\text{(c)} \quad D_2^p = D_o \left[\sinh(X)^m\right]^n \tag{7c}$$

where $\quad X = 3J_2/K^2 \quad , \text{ or } 3J_2'/K^2$

and n, m, and D_o are constants. The inelastic strain rate components can then be obtained as a function of the stress by the use (2a) or (2b) and one of Equations (7). Expression (7b) would seem to have some advantage over (7a) or (7c) in theories without a yield criterion in that the value of D_2^p is almost zero for some range of J_2 regardless of the value of n. In (7b), Do is the limiting value of the inelastic strain rate in shear; (7a) and (7b) do not contain such a limit. These differences between the kinetic equations are illustrated in a normalized plot of log (D_2^p/D_o) vs X in Figure 2 for the case of $n = 3$ and $m = 1.0$.

In all the preceding equations (7a,b,c) the exponent n influences the slope of the D_2, J_2 relation and therefore has the major influence on strain rate sensitivity. That parameter also affects the overall level of stress-strain curves although the level also depends on the hardening parameter K.

Temperature (T) dependence of plastic flow is a first order phenomenon comparable to strain rate sensitivity and should appear directly in the kinetic equation. In the case of Equations (7a,b), this can be achieved by taking the exponent n to be a function of T, e.g. $n = ck/T$ (k is Boltzmann's constant and c a material constant) which leads to strong temperature dependence of the stress parameter $X=3J_2/K^2$ (or $3J_2'/K^2$). Numerical results for this dependence are shown in Figure 3 for both the power law and exponential kinetic equations at different non-dimensionalized strain rates.

The method of including temperature dependence in Equations 7a,b is comparable to an activation energy formulation. Table I lists temperature-dependent kinetic equations based on four different functional expressions for the activation energy. Some of the consequences of the various relations are discussed in [17].

Another procedure for including temperature dependence in the kinetic equations is to multiply the stress function, the right hand side of equations (9) by a temperature function. The temperature factor can again be motivated by thermal activation considerations and the Arrhenius expression seems to be the reasonable function to use (Table I). This is the approach taken by Miller [5].

Evolutionary Equations for Internal Variables

The general framework of the evolutionary equations of internal variables is based on the now well-accepted Bailey-Orowan theory [18,19] which theorizes inelastic deformation to occur under the actions of two simultaneously competing mechanisms, a hardening process proceeding with deformation and a recovery or softening process proceeding with time. The evolution rate of an internal variable is then the difference between the hardening rate and the recovery rate as given by

$$\dot{X}_i = h_1(X_i)\,\dot{M}_1 - r_1(X_i, T) \qquad (8)$$

where $\dot{X}_i$ is the evolution rate of the internal variable X_i, and h_1 and r_1 are the hardening and the thermal recovery functions, respectively. h_1 and r_1 are functions of X_i, temperature, T, and the hardening measure, $\dot{M}_1$ is either $\dot{\varepsilon}_{ij}^p$ or $\dot{W}_p$ depending on the model.

(1.) Isotropic Hardening

The quantity K in Equation (7) is usually interpreted as the isotropic hardening internal variable and is often referred to as the drag stress. Evolutionary equations for the isotropic hardening parameter generally follow the hardening/recovery format shown in Equation (8). A comparison of these hardening and recovery functions in various unified theories is shown in Table II. The rate of isotropic hardening is usually given by a function of the hardening variable K, which may saturate to a limiting value, shown as K_1 in Table II, multiplied by a measure of the hardening rate. Both the inelastic work rate and the effective inelastic strain rate have been proposed as the scalar hardening measure. On the other hand, the rate of softening or recovery is often taken to be a power function of K and a temperature-dependent constant K_0 which value represents the reference state for that particular temperature. This recovery model, sometimes credited to Friedel [20], theorizes that recovery occurs only when the current internal state exceeds the reference state.

(2.) Directional or Kinematic Hardening

Probably the main difference in the various unified theories is the treatment of directional or kinematic hardening. Differences exist not only in the choice of the flow law but also in the evolutionary equations. The general framework of these evolutionary equations follows the hardening/recovery formulation represented in Equation (8) with indexes to indicate the directions of hardening and recovery.

$$\dot{\Omega}_{ij} = h_2\,(\Omega_{ij})\dot{M}_{ij} - d(\Omega_{ij}, T)\dot{N}_{ij} - r_2(\Omega_{ij}, T)V_{ij} + \theta(\Omega_{ij}, T)\dot{T}\,W_{ij} \qquad (9)$$

where h_2, d, and r_2 are the hardening, dynamic recovery , and static thermal recovery functions, respectively. θ represents hardening and/or recovery associated with the rate of temperature change. $\dot{M}_{ij}$, $\dot{N}_{ij}$, V_{ij} and W_{ij} are the directional indexes of h_2, d, r_2, and θ, respectively. The main differences among the various theories, as summarized in Table III, are in the choices of the directional index and the hardening and recovery functions.

As indicated in Table III, unified models based on the equilibrium stress utilize the inelastic strain rate as the directional index for hardening and contain a "dynamic recovery" term in the hardening function. The proposed hardening rule is thus similar to the Prager rule [16] in conventional plasticity which requires the translation of a yield surface to occur in the direction of the plastic strain increment. On the other hand, the evolutionary equation proposed in conjunction with Equation (2a) is based on the direct stress as the index for directional hardening [3,4]. This formulation avoids the cross-softening effect associated with inelastic strain rate as the index and the theory is more compatible with Ziegler's modification [21] of the Prager hardening rule. The directional index for "dynamic recovery" is generally in the opposite direction of the directional hardening variable Ω_{ij}. The "dynamic recovery" term is treated in [3,4] as a saturation term in the direction of the direct stress but the index has recently been modified to be in the direction of $-\Omega_{ij}$ also [22]. The unit vector which represents the direction cosines of the directional hardening variable is usually taken to be the directional index for static thermal recovery. Recovery always occurs in the opposite direction of the unit vector and tends to reduce the magnitude of the directional (kinematic) hardening variable. Most unified theories utilize Friedel's recovery model and take zero magnitude of Ω_{ij} as the reference state. Table III shows that a temperature rate term is also included in the theories of Walker and Chaboche. In principle, similar terms could be added to the other theories.

The temperature dependence of the internal variables is also important. The experience with the unified models to date indicates that all the material constants in the formulations would depend on temperature and must be evaluated at a number of base temperatures.

Uniqueness and Stability Criteria

For stability, unified theories with internal variables must, according to Ponter [23], obey the following inequality:

$$d\sigma_{ij}\, d\dot{\varepsilon}_{ij}^{p} - dXi\, d\dot{X}i > 0 \tag{10}$$

where $d\sigma_{ij}$, $d\dot{\varepsilon}_{ij}$, dXi, and $d\dot{X}i$ represent incremental changes in stress, inelastic strain rate, the current value and the evolution rate of the internal variables. The inequality admits classical plastic flow, creep, and stress relaxation behavior. It also admits recovery phenomena involving negative inelastic work provided that the corresponding changes in the internal variables are sufficiently large to make the inequality in Equation (10) remain valid. The basic requirement of Equation (10) is that the dissipation rate must be nonnegative.

For a constant internal state, a small change in σ_{ij} results in a corresponding change in $\dot{\varepsilon}_{ij}^{p}$ so that [23]

$$d\sigma_{ij}\ d\dot{\varepsilon}_{ij}^{p} > 0 \quad , \quad \dot{X}_i = 0 \tag{11}$$

The inelastic work inequality is identical to Drucker's postulate [24] in classical plasticity that for a stable material flow the plastic work done must be nonnegative. For proportional loading the kinetic equations represented in Equation (7a) to Equation (7c) all yield convex "flow potentials" to which the inelastic strain rate vectors are normal. The consequence is that the inelastic work is always positive, and unified theories based on Equation (7a) to (7c) obey the inelastic work inequality.

For uniqueness, it appears that the inelastic strain rate must be a single-valued function of stress and internal variables. To satisfy the requirement for stable flow, Equation (10) dictates that stress-strain curves at constant strain rate must have positive slope but must decrease with increasing strain. On the other hand, stress-strain curves at constant plastic strain or plastic work must have positive slope, but the slope may either increase or decrease with increasing strain rate [23].

Most, if not all, of the unified theories listed in Table II satisfy the Ponter inequalities and met the uniqueness and stability requirements. The stability requirement is, however, not essential for constitutive theory developments. Unified theories admit unstable inelastic flow and are generally modeled by including softening mechanisms such as thermal softening and continuum damage in the evolution and/or kinetic equations.

NUMERICAL METHODS FOR INTEGRATING UNIFIED CONSTITUTIVE EQUATIONS

The unified constitutive equations can be characterized as mathematically "stiff." That is, in these equations, dependent variables are susceptible to large changes from small increments of the independent variables or from small time steps. This "stiff" behavior occurs usually with the onset of a significant amount of inelastic strains in the loading cycle and is due to the generally nonlinear nature of the functional forms that are employed in the kinetic equations of these theories.

A systematic comparison of a variety of approaches for integrating unified constitutive equations has been reported by Kumar, Morjaria, and Mukherjee [25]. This study concluded that for the constitutive theory of Hart, a relatively simple Euler integration method, together with a time step control strategy, was optimal when compared with the more sophisticated methods. The Walker constitutive theory has been integrated using the Euler single step approach usually without automatic time step control, but rather by determining an optimum step size for each problem. Efficiency obtained by using this approach has been acceptable and has shown considerable improvement over more sophisticated approaches such as higher order Runge-Kutta methods.

Tanaka and Miller recently developed a noniterative, self-correcting solution (NONSS) method for integrating stiff time-dependent constitutive equations [26]. In this approach, implicit quantities are removed by Taylor expansions of σ, $\dot{\epsilon}$, and $\dot{X}$ through the incorporation of the integration operator α. The method which reduces to the explicit Euler method when $\alpha = 0$ and to the implicit Euler method when $\alpha > 0$ is unconditionally stable for $\alpha \geq 1/2$ and is noniterative. Accuracy is maintained through self-adaptive time control and by correcting previous errors at the current step.

A summary of these various numerical techniques and their applications to several unified theories as well as to Norton's law for integrating a uniaxial stress-strain curve to a total strain of 1-2% is shown in Table IV. As illustrated in Table IV, the explicit Euler is stable when the size of the strain increment is kept below 10^{-4}. The size of the strain increment can be increased by using an implicit method such as the NONSS or α-method with $\alpha = 1$ (implicit Euler). By restricting the comparison to the explicit methods only, it appears that there is no substantial difference between the integrability of Walker and Miller theories nor between these unified theories and the classical Norton law. The size of the strain increment is, however, somewhat sensitive to the values of model constants which describe material strain rate sensitivity.

PREDICTIVE AND SIMULATIVE CAPABILITIES OF UNIFIED CONSTITUTIVE THEORIES

Four of the unified models which have been successfully applied for simulating and/or predicting monotonic, cyclic, creep, and stress relaxation behavior are those of Robinson [8], Walker [2], Bodner-Partom [3,4], and Miller [5]. Robinson's model is based on a yield condition and utilizes loading and unloading criteria, while the latter three do not. The kinetic equations commonly used in unified theories without a yield surface or flow potential are based on the power-law, exponential, and hyperbolic sine functions; these kinetic equations are represented in Walker, Bodner-Partom and Miller theories, respectively. These four theories will be used to illustrate the simulative and predictive capabilities of the unified theories.

(1) Monotonic Stress-Strain Behavior

All unified theories are capable of reproducing the monotonic stress-strain curve. Figure 4 shows an experimental uniaxial tensile stress-strain curve of Hastelloy-X deformed at a strain rate of 1.3×10^{-4} sec^{-1} at 922 K and model simulation using Bodner-Partom theory. The computed curve includes contributions from both work hardening and thermal recovery.

(2) Cyclic Stress-Strain (Hysteresis) Behavior

Bauschinger Effect is represented in most unified theories by a kinematic or directional hardening internal variable. Cyclic hardening, however, can be represented by increases in the isotropic hardening variable, the directional hardening variable, or both. These different types of cyclic hardening behavior are illustrated in Figure 5 for Bodner-Partom theory which does not use an equilibrium stress. The use of different evolutionary equations for the equilibrium stress in different regions of stress space allows Robinson's model to reproduce rounded hysteresis loops. Examples of cyclic saturated hysteresis loops calculated using Robinson's model are compared with experimental results of 2-1/2Cr-1Mo steel in Figure 6 [27].

(3) Creep Responses

Most of the unified models can predict primary and secondary creep responses of material subjected to constant load. Steady state creep rates are predicted by these unified models to occur when the evolutionary rates of the isotropic and/or directional hardening variable vanish as the hardening terms are balanced by the thermal recovery. Examples of calculated steady state creep rate under constant stress and comparison with experimental data are shown in Figure 7 for Bodner-Partom's model. According to the unified theories, the steady state creep rate is a function of stress and temperature only; it should not depend on the loading histories. This is demonstrated in both experimental data and predictions by Miller's model in Figure 8 [28].

(4) Stress Relaxation Response

The behavior of unified constitutive models under stress relaxation is analogous to the creep behavior. Under a constant strain condition, the relaxation rate would, again, depend on the current values of the internal variables and on the growth laws which describe their changes with time and inelastic deformation. Stress relaxation calculations based on Walker theory is compared with experimental data of Hastelloy-X [2] in Figure 9.

(5) Thermomechanical Response

The behavior of unified constitutive theories under thermomechanical cycling depends critically on the change of material constants with temperature. In particular, the shape of the predicted thermomechanical loop is sensitive to the growth of the kinematic hardening variable (the equilibrium stress) with temperature. Walker's model prediction of thermomechanical loop of Hastelloy-X is shown in Figure 10.

(6) Multiaxial Behavior

All the unified theories utilize single-valued kinetic equations formulated in terms of either $3J_2/K^2$ or $3J_2'/K^2$. For a constant value of the internal variable K and under proportional paths, these kinetic equations predict a locus of constant inelastic strain rate invariant in stress space; the shape of the predicted "yield surface" or "flow potential" is identical to von Mises yield function. For unified models formulated based on the equilibrium stress, the size of the "yield surface" is proportional to K, while the center of the "yield surface" is at Ω_{ij} and translates according to the evolution rate of Ω_{ij}.

Recent studies [22,29], indicate that materials exhibit considerably more cyclic hardening when tested under nonproportional paths of combined tension and torsion than under proportional paths of tension or torsion only. As a result, most if not all,of the constitutive models need to be modified to take into account the hardening behavior due to out-of-phase loading.

SUMMARY AND CONCLUSIONS

1. A review of more than ten time-temperature dependent elastic-viscoplastic constitutive theories indicates that these theories differ in the choice of flow law, kinetic equation, and evolutionary equation of the internal variables.

2. The unified approach treats all aspects of inelastic deformation including plasticity, creep, and stress relaxation using the same set of flow law, kinetic equation, and internal variables.

3. The unified constitutive theories satisfy the uniqueness and stability criteria imposed by Drucker's postulate for rate independent stable plastic flow and Ponter's inequalities for constitutive theories based on internal variables.

4. The unified theories can be formulated either with or without the use of a yield criterion. Three basic flow laws are identified in theories without a yield criterion. For theories with a yield criterion, the associated flow law is derived from the yield function or the flow potential.

5. Three different formulations of the kinetic equations are identified, and they include the exponential, power law, and hyperbolic sine functions. The exponential formulation gives a limiting inelastic strain rate and appears to give better results for high strain rate applications.

6. All three forms of kinetic equations are functions of $3J_2/K^2$ (or $3J_2'/K^2$) and result in "yield surfaces" and equi-creep rate surfaces which are described by the J_2-based von Mises criterion.

7. The number of internal variables varies among the unified theories. Most unified theories use two internal variables, one to represent isotropic hardening and one to present kinematic or directional hardening. The measure of hardening is either the inelaslastic strain rate or the inelastic work rate.

8. Directional (kinematic) hardening can be modeled with or without the use of an equilibrium stress. The directional index of kinematic hardening can be based on either the inelastic strain rate or the direct stress.

9. Material constants in the unified models are necessarily temperature-dependent and required to be evaluated at the temperatures of interest. There are indications that a temperature rate term is also required in the unified theories.

10. All of the unified theories which are reviewed do not automatically predict additional cyclic hardening under nonproportional loading paths. Additional terms are needed in the unified theories to include such hardening behavior.

11. The equilibrium-stress-based unified theories can describe reverse creep and/or reverse stress relaxation behavior without further modifications. Unified models which are not based on the equilibrium stress would require modification by adding an anelastic term in order to take into account these types of behavior.

12. The unified constitutive equations are stiff but can be integrated using either explicit or implicit methods.

ACKNOWLEDGEMENTS

The financial support of NASA through Contract No. NAS3-23925 is gratefully acknowledged.

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TABLE I

FIVE FORMS OF TEMPERATURE-DEPENDENT KINETIC EQUATIONS WITH THE CORRESPONDING ACTIVATION ENERGY FUNCTION

Activation Energy	Temperature-Dependent Kinetic Equations
$\Delta H = C\ell n\left(\frac{K^2}{3J_2}\right)$	$D_2^p = D_0\left[\frac{3J_2}{K^2}\right]^{C/kT}$
$\Delta H = H_0 - Vg(J_2)$	$D_2^p = D_0 \exp\left[-\frac{H_0 - Vg(J_2)}{kT}\right]$
$\Delta H = \frac{H^*K^2}{3J_2}$	$D_2^p = D_0 \exp\left[-\frac{H^*}{kT}\left(\frac{K^2}{3J_2}\right)\right]$
$\Delta H = kT\left[\frac{K^2}{3J_2}\right]^{C/kT}$	$D_2^p = D_0 \exp\left[-\left(\frac{K^2}{3J_2}\right)^{C/kT}\right]$
$\Delta H = Q$	$D_2^p = D_0 \exp\left[-\frac{Q}{kT}\right]\left[\sinh\left(\frac{3J_2}{K^2}\right)^m\right]^n$

where C, D_0, H^*, H_0, Q, m, and n are constants; V is the activation volume; and k is the Boltzmann's constant.

TABLE II

THE SPECIFIC FORMS OF ISOTROPIC HARDENING AND STATIC THERMAL RECOVERY FUNCTIONS USED IN THE SELECTED UNIFIED CONSTITUTIVE THEORIES

$$\dot{K} = h_1(K)\dot{M}_1 - r_1(T,K)$$

$$\text{where} \quad \dot{M}_1 = \dot{\bar{\varepsilon}}; \; \dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}\,\dot{\varepsilon}_{ij}^{p}\,\dot{\varepsilon}_{ij}^{p}}$$

$$\text{or} \quad \dot{M}_1 = \dot{W}_p \text{ (Bodner-Partom's Theory)}$$

Model	Hardening Function, $h_1(K)$	Static Thermal Recovery Function, $r_1(T,K)$
Bodner-Partom	$C_1(K_1 - K)$	$C_2(K - K_o)^p$
Walker	$C_1(K_1 - K)$	-
Krieg et al	C_1	$C_2(K - K_o)^p$
Robinson	C_1	-
Chaboche	$C_1(K_1 - K) + f_1(\bar{\varepsilon}, \dot{\bar{\varepsilon}}, \Omega_{ij})$	-
Lee and Zaverl	$C_1(K_1^* - K)/\sqrt{J_2}$	-
Hart	C_1	-
Ghosh	C_1K^{-q}	$C_2(K - K_o)^p$
Miller	$C_1[K_1 - C_4(\sinh^{-1}C_3\lvert\dot{\bar{\varepsilon}}\rvert)^q K^m]$	$C_2[\sinh C_3 K^m]^p$

where C_1, C_2, C_3, C_4, C_5, m, p, q, K_o, and K_1 are material constants; K_1^* is the saturated value of K; K_1^* is governed by an evolutionary equation which is function of $\dot{\bar{\varepsilon}}$ and J_2.

TABLE III

THE SPECIFIC FUNCTIONS OF ANISOTROPIC HARDENING, DYNAMIC RECOVERY, STATIC THERMAL RECOVERY, AND THE TEMPERATURE RATE TERM IN SELECTED UNIFIED CONSTITUTIVE THEORIES

$$\dot{\Omega}_{ij} = h_2(\Omega_{ij})\dot{M}_{ij} - d(\Omega_{ij}, T)\dot{N}_{ij} - r_2(\Omega_{ij}, T)v_{ij} + \theta(\Omega_{ij}, T)\dot{T}W_{ij}$$

where: $v_{ij} = \Omega_{ij}/[I_2]^{1/2}$, $I_2 = \beta/2\ \Omega_{kl}\Omega_{kl}$ (except Walker, $v_{ij} = g_{ij}$)

$$g_{ij} = \Omega_{ij} - \mathring{\Omega}_{ij} - n_1\epsilon_{ij}$$

Model	Hardening Function, h_2	$\dot{M}_{ij}$	Dynamic Recovery Function, d	$\dot{N}_{ij}$	Static Thermal Recovery Function, r_2	Temperature Rate Function, θ	W_{ij}
Bodner-Partom	$n_1 n_2$	$\dot{W}_p u_{ij}$	$n_1[I_2]^{1/2}$	$\dot{W}_p v_{ij}$	$n_3 I_2^{s}$	-	-
Walker	$n_1 + n_2$	$\dot{\epsilon}_{ij}$	f_1	$\dot{\bar{\epsilon}} g_{ij}$	$n_6 I_2^{s}$	$\frac{\partial n_1}{\partial T}, \frac{1}{n_2}\frac{\partial n_2}{\partial T}$	ϵ_{ij}, g_{ij}
Krieg et al	$n_1 + n_2\exp[-f_0/n_2]$	$\dot{\epsilon}_{ij}$	-	-	$n_3 I_2^{s}$	-	-
Robinson	$n_1 I_2^{-s}$	$\dot{\epsilon}_{ij}$	-	-	$n_2 I_2^{b}$	-	-
Chaboche	$n_1 n_2 f_2$	$\dot{\epsilon}_{ij}$	$n_2 f_2$	$\dot{\bar{\epsilon}}\Omega_{ij}$	$n_3 n_2 I_2^{s}$	f_3	Ω_{ij}
Lee and Zaverl	$1.5n_1[J_2']^{-1/2}$	$\dot{\bar{\epsilon}}\Omega^*_{ij}$	$1.5n_1[J_2']^{-1/2}$	$\dot{\bar{\epsilon}}\Omega_{ij}$	-	-	-
Hart	n_1	$\dot{\epsilon}_{ij}$	-	-	$n_2 n_3\left(\frac{\sigma^*}{\mu}\right)^{b}\left[\log\left(\frac{\sigma^*}{\sqrt{J_2}}\right)\right]^{-s}$	-	-
Ghosh	-	-	-	-	-	-	-
Miller	n_1	$\dot{\epsilon}_{ij}$	n_2/f_4	$\dot{\epsilon}\Omega_{ij}$	$n_3[\sinh n_4\sqrt{I_2}]^{s}$	-	-

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where: $f_0 = (\sigma_{ij} - \Omega_{ij})\Omega_{ij}$

$f_1 = n_3 + n_4\exp(-n_5\bar{\epsilon})$

$f_2 = n_4 + (1 - n_4)\exp(-n_5\bar{\epsilon})$

$f_3 = \frac{1}{n_2}\frac{\partial n_2}{\partial T} + \frac{1}{n_1}\frac{\partial n_1}{\partial T} + \frac{1}{f_2}\frac{\partial f_2}{\partial T}$

$f_4 = \sinh^{-1}(n_6\dot{\bar{\epsilon}})^{b}$

$u_{ij} = \sigma_{ij}/(\sigma_{kl}\sigma_{kl})^{1/2}$

$J_2' = 1/2(s_{ij} - \Omega_{ij})(s_{ij} - \Omega_{ij})$

β = 1, 2, or 4/3, depending on the model

n_1, n_2, n_3, n_4, n_5, n_6, b, and s are material constants.

Ω^*_{ij} are the saturated values of Ω_{ij}; Ω^*_{ij} are governed by evolutionary equations which are functions of $\dot{\bar{\epsilon}}$ and J_2'.

TABLE IV

COMPARISONS OF THE INTEGRATABILITY OF VARIOUS CONSTITUTIVE MODELS

Comparison is based on the size of the average strain increment per step for integrating a uniaxial tensile stress-strain curve to a total strain of 1-2%

Constitutive Model	Integration Method	Strain Increment Per Step	Automatic Time or Strain Increment Control	Stability	Reference
Walker	Explicit Euler	10^{-5} to 10^{-4}	No	Stable	Walker [2]; Cassenti [30]
Bodner-Partom	NONSS (α-Method)		No		Present Investigation
	α = .1	1×10^{-4}		Stable	
	α = .5	1×10^{-4}		Stable	
	α = 1	1×10^{-3}		Stable	
Miller	Explicit Euler	4×10^{-4}	No	Unstable	Lee et al [31]
	Tayler Series Modified Euler	4×10^{-4}	No	Stable	Lee et al
	NONSS (α-Method, α = 1)	3×10^{-3}*	Yes	Stable	Tanaka [26]
Hart	Explicit Euler	2×10^{-5}*	Yes	Stable	Kumar et al [25]
	Predictor-Corrector	3×10^{-5}*	Yes	Stable	Kumar et al
	High Order Predictor-Corrector	3×10^{-5}*	Yes	Stable	Kumar et al
	Two-Step Adam	2×10^{-5}*	Yes	Stable	Kumar et al
Norton	Explicit Euler	1×10^{-3}	No	Unstable	Lee et al [31]
		5×10^{-4}	No	Stable	Lee et al
	Taylor Series Modified Euler	1×10^{-3}	No	Stable	Lee et al

* Average strain increment per step = total strain/number of time steps.

DEVIATORIC STRESS SPACE

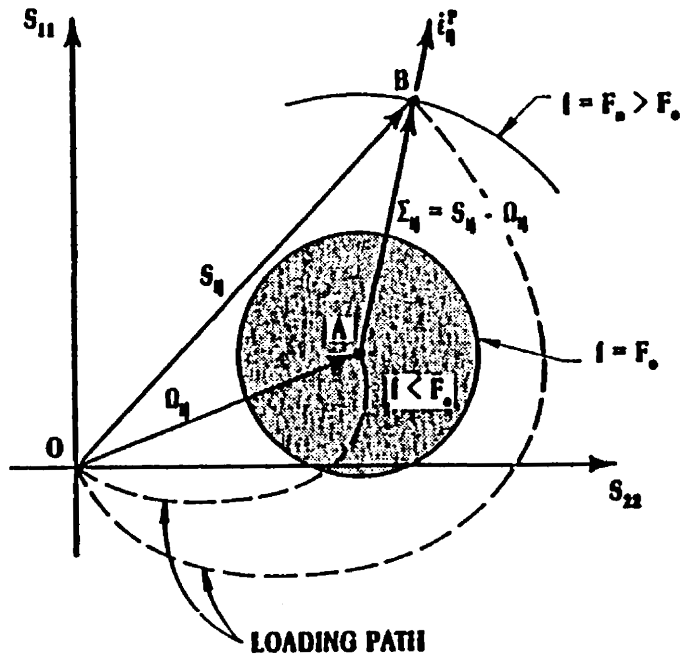

Figure 1. Graphical representation of the basic flow laws used in the unified constitutive theories. For theories based on an equilibrium stress, the inelastic strain rate vector $\dot{\varepsilon}_{ij}^{p}$ is coaxial with the effective stress Σ_{ij} and normal to the flow potential f if such a concept is used. For theories which do not include an equilibrium stress, $\dot{\varepsilon}_{ij}^{p}$ is coaxial with the deviatoric stress S_{ij} for both isotropic and incrementally isotropic cases but is noncoaxial with S_{ij} for generalized anisotropic cases.

KINETIC RELATIONS

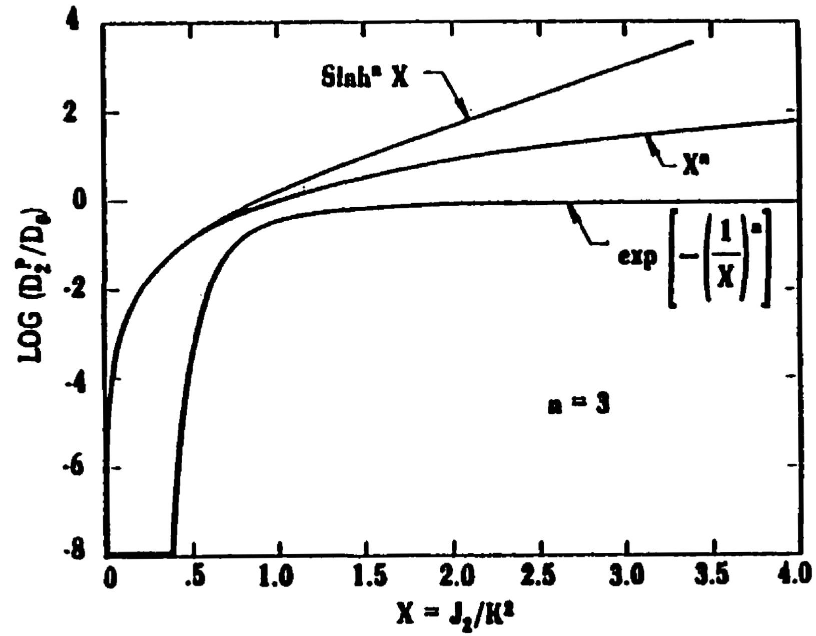

Figure 2. Functional behavior of the kinetic equations used in the unified constitutive theories. The exponential formulation in Bodner-Partom's theory is seen to give a limiting inelastic strain rate of D_0.

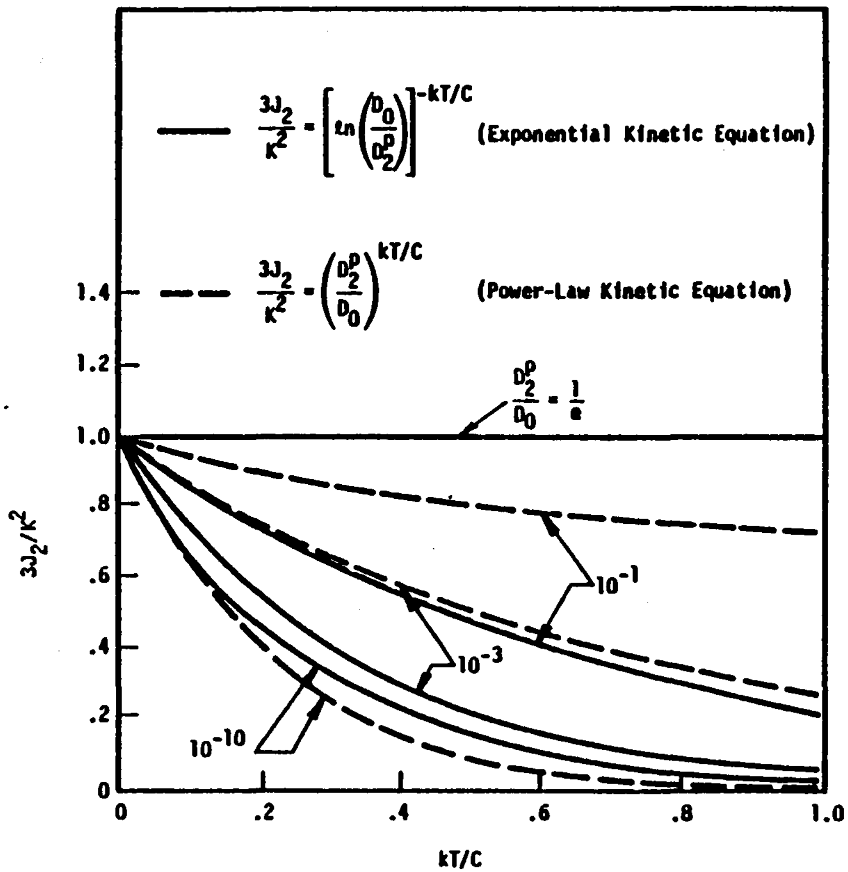

Figure 3. Functional behavior of temperature-dependent kinetic equations utilized in Bodner-Partom and Walker theories.

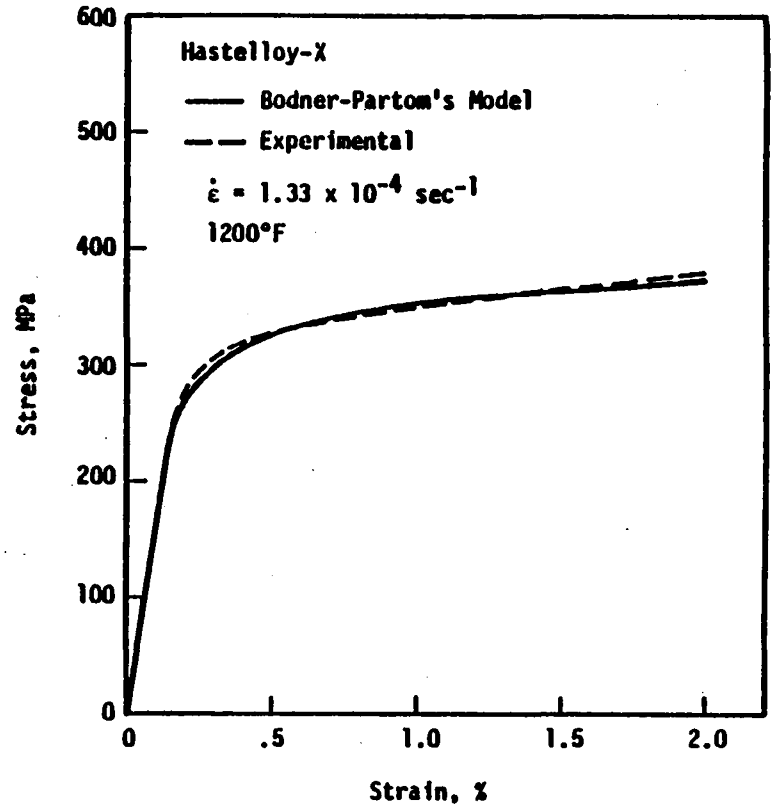

Figure 4. A monotonic stress-strain curve simulated by Bodner-Partom's Model for Hastelloy-X at 1200°F.

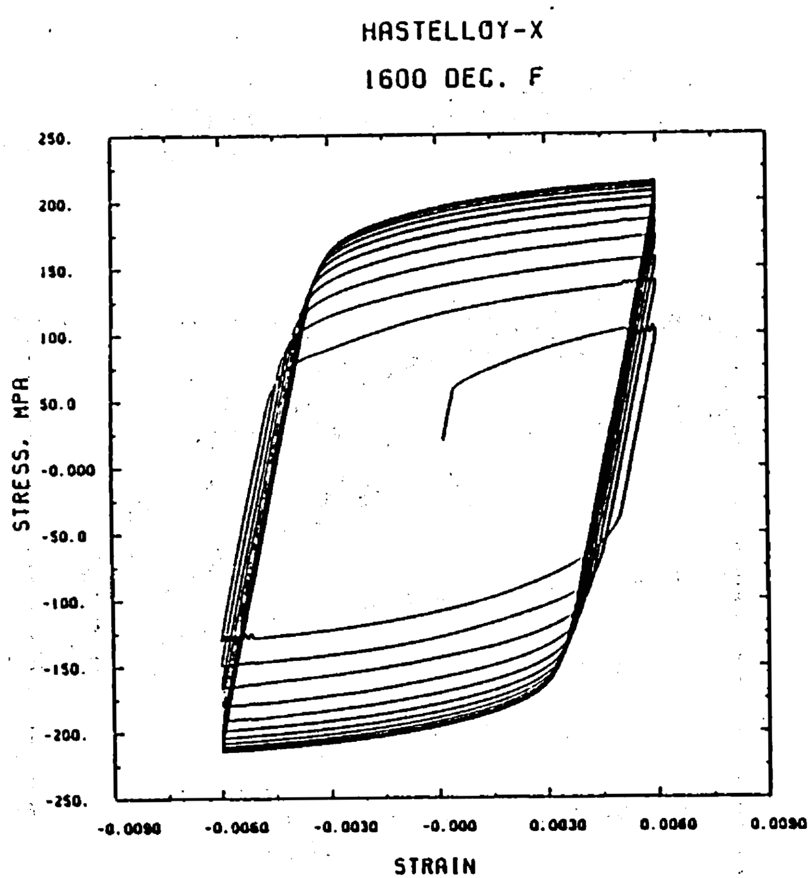

Figure 5. Cyclic stress-strain hysteresis loop simulated by Bodner-Partom's Model for Hastelloy-X at 1600°F.

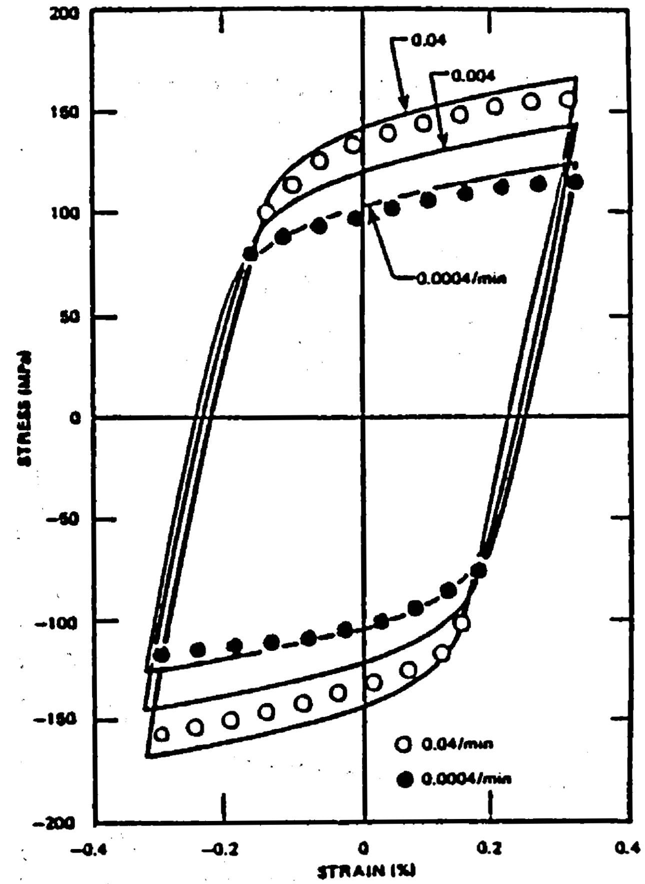

Figure 6. Stable hysteresis loops of 2-1/4Cr-1Mo steel for $\Delta\varepsilon \approx \pm.32\%$ at various strain rates at 538°C. The calculated curves (solid lines) are generated with Robinson's Model, from [27].

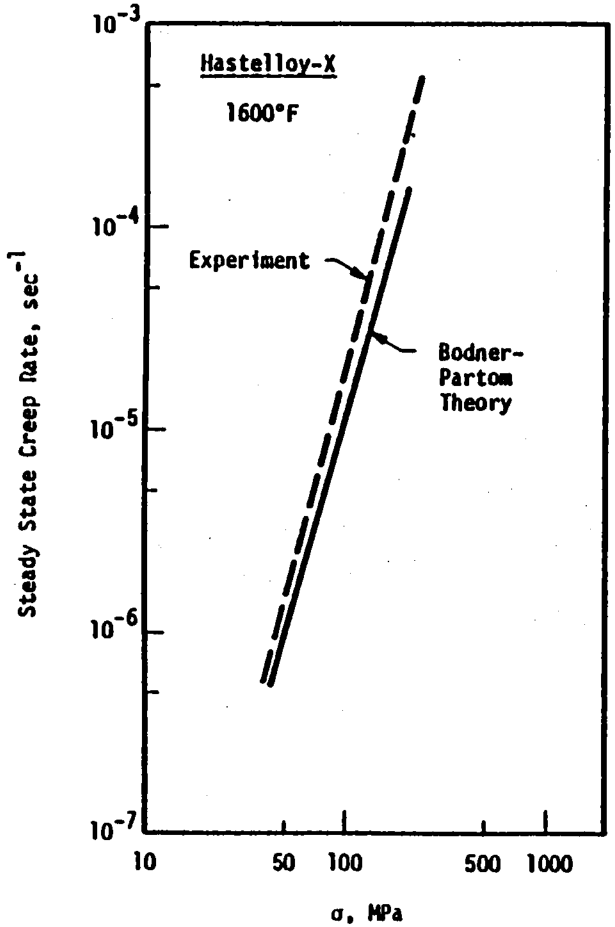

Figure 7. Steady creep rates as a function of stress simulated by Bodner-Partom's Model.

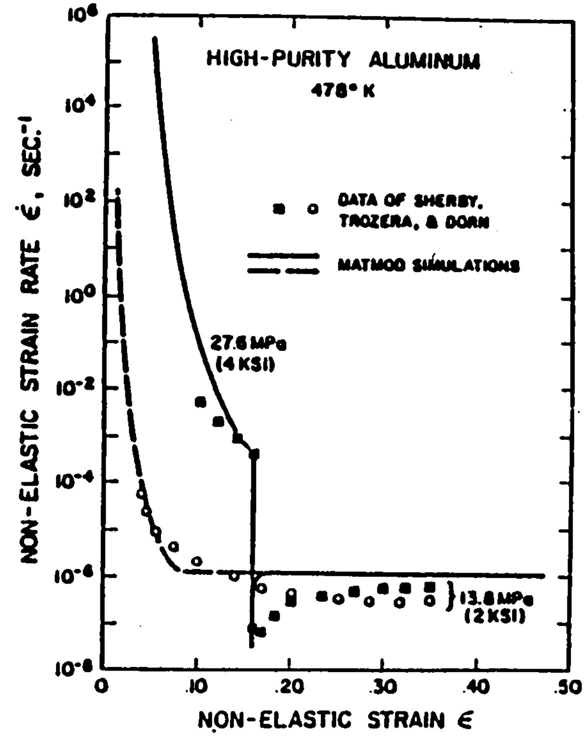

Figure 8. Miller's Model prediction compared with experimental data for a creep test with a sudden decrease in applied stress, from [28].

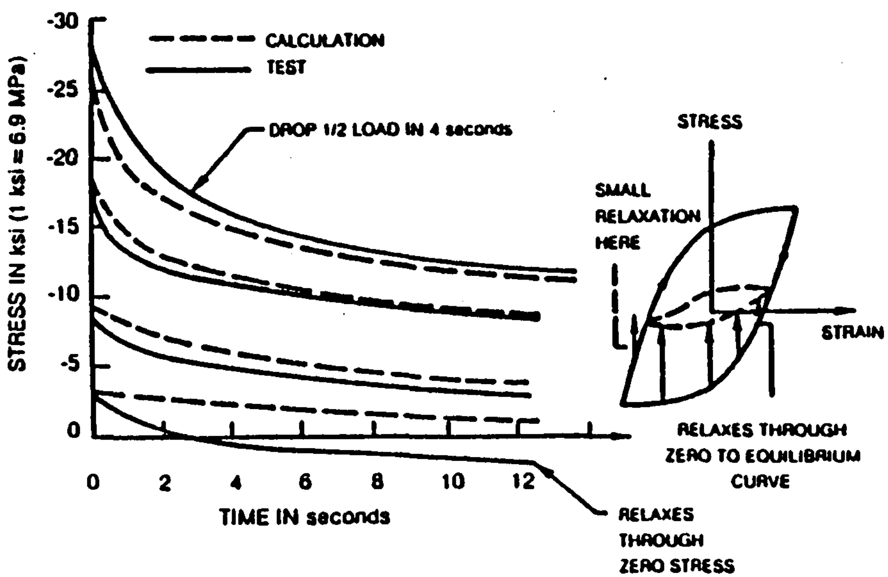

Figure 9. Negative stress relaxation response of Hastelloy-X at 871°C (1600°F) initiated from a steady state hysteresis loop executed at a constant strain rate of $\pm$ 1.35 x 10^{-3} sec^{-1} with a strain amplitude of $\pm$ .4%. The calculated curves are based on Walker's theory, from [2].

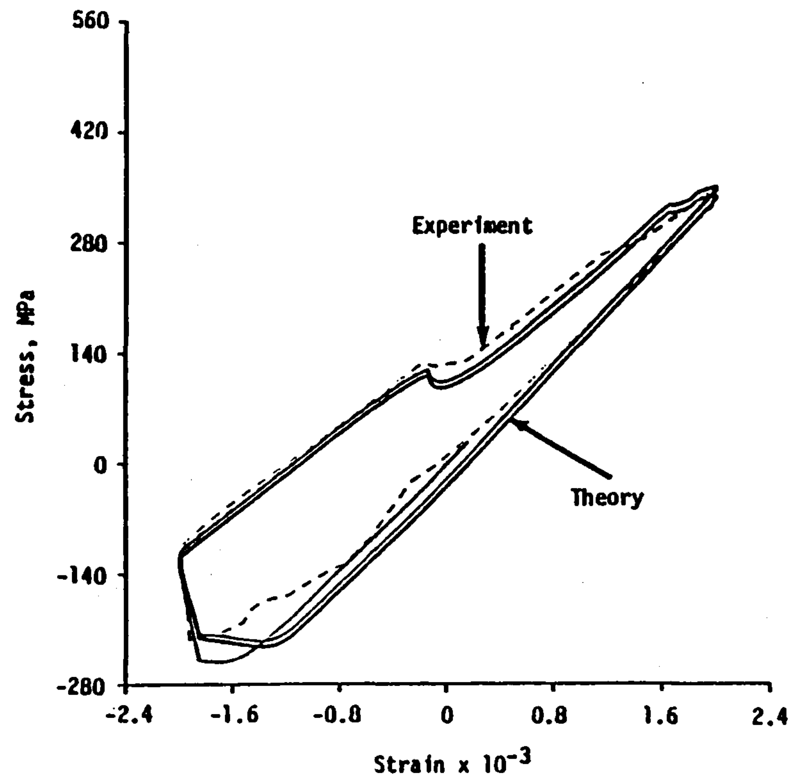

Figure 10. Thermomechanical stress-strain prediction by Walker's theory.