ON NUMERICAL INTEGRATION AND COMPUTER IMPLEMENTATION OF

VISCOPLASTIC MODELS

T.Y. Chang and J.P. Chang
Department of Civil Engineering
University of Akron
Akron, Ohio 44325

R.L. Thompson
NASA Lewis Research Center
Cleveland, Ohio 44135

Due to the stringent design requirement for aerospace or nuclear structural components, considerable research interests have been generated on the development of constitutive models for representing the inelastic behavior of metals at elevated temperatures. In particular, a class of unified theories (or viscoplastic constitutive models) have been proposed [1-10] to simulate material responses such as cyclic plasticity, rate sensitivity, creep deformations, strain hardening or softening, etc. This approach differs from the conventional creep and plasticity theory in that both the creep and plastic deformations are treated as unified time-dependent quantities. Although most of viscoplastic models give better material behavior representation, the associated constitutive differential equations have stiff regimes which present numerical difficulties in time-dependent analysis. In this connection, appropriate solution algorithm must be developed for viscoplastic analysis via finite element method.

In the past, inelastic finite element structural analyses were performed largely based on the classical concept of creep and plasticity [11-14]. Recently, some attempts have been made to incorporate a specific type of viscoplastic theories into finite element codes [15-20] for structural analysis. In this paper, three integration schemes are implemented into a nonlinear finite element program [21] to study their numerical efficiency pertaining to finite element analysis. Moreover, four viscoplastic models, namely, those due to Walker, Miller, Krieg-Swearingen-Rohde, and Robinson, were implemented into a finite element program for nonlinear analysis. A general implementation procedure is outlined in the paper.

VISCOPLASTIC THEORIES

The basic assumption embodied in viscoplastic theories is the unified treatment of inelastic strain; i.e. no distinction is given to creep and plastic deformations. In addition, both elastic and inelastic strains are considered to be present at all stages of loading and unloading processes. The unique feature of such treatment, as compared to classical theories, is that the yield condition is not explicitly involved. Consequently, the computational algorithm for complex loading history can be much simplified. In the context of small deformation, viscoplastic models may be written in the following general form

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{D}^E \dot{\underset{\sim}{\varepsilon}} - (2G\dot{\varepsilon}^I + \delta (\lambda + 3G)\gamma \dot{T}) \tag{1}$$

$$\dot{\underset{\sim}{\varepsilon}}^I = f(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}, K, T)(\underset{\sim}{\sigma} - \underset{\sim}{\alpha}) \tag{2}$$

$$\dot{\underset{\sim}{\alpha}} = h_\alpha \dot{\underset{\sim}{\varepsilon}}^2 - r_\alpha \tag{3}$$

$$\dot{k} = h_k |\dot{\varepsilon}|^1 - r_k \tag{4}$$

where ( )˙ = Time devative;   T = Temperature;

$\underset{\sim}{\varepsilon}$ = Total strain vector; σ = Stress vector;

$\underset{\sim}{\alpha}$ = Back stress vector;   K = Drag stress;

$\underset{\sim}{D}^E$ = Elasticity matrix; λ, G = Lame contant;

$\underset{\sim}{\varepsilon}^I$ = Inelastic strain vector;

γ = Linear thermal expansion coefficient;

f = Inelastic strain rate function;

h_α, h_k = Hardening functions for back and drag stresses, respectively;

r_α, r_k = Recovery functions for back and drag stresses, respectively.

Eqs. (1-4) represent a complete set of viscoplastic constitutive equations wherein the following assumptions are invoked in the extention from uniaxial case to the three-dimensional case, namely, i) isotropic material, ii) incompressible inelastic strain, and iii) linear bulk behavior. Eq. (1) defines stress rate to be proportional to elastic strain rate while Eq. (2) states the functional dependence of inelastic strain rate on applied stress, temperature, and state variables. Furthermore, Eqs. (3-4), so-called evolutional equations, are generally constructed in hardening/recovery form such that the net effect of two antagonistic mechanisms uniquely determines the growth rate of state variables α and k.

Although the mathematical expressions of viscoplastic models proposed by various researchers differ in their detailed descriptions, they do however portray several common phenomena: i) Initial linear elastic behavior wherein the inelastic effect is negligible and then nonlinear response afterwards, ii) strain-rate sensitivity, iii) time-dependent creep and relaxation, iv) cyclic hardening or softening, v) creep recovery, vi) creep plasticity interactions, and vii) Bauschinger's effect.

NUMERICAL INTEGRATION SCHEMES

For finite element applications, it is useful to choose an appropriate integration scheme for handling the nonlinear viscoplastic equations concerned. Krieg (22) pointed out the existance of numerical stiff regions in viscoplastic formulation together with a discussion of potential difficulties. The stiffness of the equations originates from the nonlinear relationship assumed in Eq. (1) and the hardening/recovery form in evolutional equations. Formal definition of the stiffness of a set of differential equations can be found in Ref.(23) where the measure of "stiffness" is given in terms of the spectra of eigenvalues obtained from the Jacobian matrix of associated equation system.

Numerical approaches intended for integrating stiff differential equations have been developed by a number of researchers. Among them

Gear's method is the most famous one. Although Gear's package has been used quite effectively in solving one-dimensional constitutive equations, (1) it is not suitable for large scale finite element analysis simply because its solution procedure is of a multistep nature. When employed in finite element analysis, this method usually requires a large amount of storage in order to follow the deformation history of the material. For this reason, one-step method is much preferable.

For the purpose of discussion, the constitutive equations are re-written as follows

$$\dot{\underset{\sim}{y}} = \underset{\sim}{f}(\underset{\sim}{y}, t) \tag{5}$$

where y represents stress, inelastic strain and state variables while f denotes nonlinear functions. One-step method for solving inelastic rate problems in the field of finite element has been investigated by several researchers (24-26). In a broad sense, it can be written in terms of one-parameter (θ) family of implicit algorithm (the - method) as follows.

$$\underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \Delta t_n [(1-\theta) \underset{\sim}{f}_n + \theta \underset{\sim}{f}_{n+1}] \tag{6}$$

where $\Delta t_n = t_{n+1} - t_n$ is the n-th time step size and θ is an integration parameter which has the range of (0,1). In Eq. (6) it is assumed that a numerical solution at the beginning of time step n is known, the solution at the end of the step is to be sought.

The simplest integration scheme is the explicit forward Euler scheme corresponding to $\theta = 0$. It is an explicit scheme since the solution at time t_{n+1} is completely determined from conditions existing at time t_n. Therefore, in the forward Euler method, the solution at time t_{n+1} is approximated by

$$\underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \Delta t_n \underset{\sim}{f}_n \tag{7}$$

When this method is employed in solving stiff equations, very small step size must be used in order to obtain stable and accurate solutions.

On the other hand, the case $\theta = 1/2$ results in the so-called implicit trapezoidal scheme which is also widely known as Crank-Nicholson rule in the context of linear differential equations. Then

$$\underset{\sim}{y}_{n+1} = \underset{\sim}{y}_n + \frac{\Delta t_n}{\gamma} (\underset{\sim}{f}_n + \underset{\sim}{f}_{n+1}) \tag{8}$$

Note that $\underset{\sim}{f}_{n+1} = \underset{\sim}{f}(y_{n+1}, t_{n+1})$ is unknown. Nonlinear implicit equation

is best solved by the Newton-Raphson iteration. To this end, Eq. (8) is rewritten in the form

$$\underset{\sim}{F}_{n+1}^i = \underset{\sim}{y}_{n+1}^i - \underset{\sim}{y}_n - \Delta t_n(\underset{\sim}{f}_{n+1}^i + \underset{\sim}{f}_n)/2 = 0 \tag{9}$$

The right superscript "i" denotes iteration number. Since y_n and f_n are known, Newton-Raphson iteration gives

$$\underset{\sim}{y}_{n+1}^{i+1} = \underset{\sim}{y}_{n+1}^i - \frac{\underset{\sim}{F}_{n+1}^i}{\partial \underset{\sim}{F}_{n+1}^i / \partial \underset{\sim}{y}_{n+1}^i} \tag{10}$$

Rearranging Eq. (10) yields

$$\frac{\partial \underset{\sim}{F}_{n+1}^i}{\partial \underset{\sim}{y}_{n+1}^i} = \underset{\approx}{I} - \frac{\Delta t_n \partial \underset{\sim}{f}_{n+1}^i}{2 \partial \underset{\sim}{y}_{n+1}^i} \tag{11}$$

Defining

$$\Delta \underset{\sim n+1}{y} = \underset{\sim n+1}{y}^{(i+1)} - \underset{\sim n+1}{y}^i \tag{12}$$

and performing differentiation, one obtains

$$[\underset{\approx}{I} - \frac{\Delta t_n}{2} \frac{\partial \underset{\sim}{f}_{n+1}^i}{\partial \underset{\sim}{y}_{n+1}^i}] \Delta \underset{\sim}{y}_{n+1}^{i+1} = \underset{\sim}{y}_n - \underset{\sim}{y}_{n+1}^i + \frac{\Delta t}{2} (\underset{\sim}{f}_n + \underset{\sim}{f}_{n+1}^i) \tag{13}$$

where the initial value of $\underset{n+1}{y}^1$ may be obtained by an explicit scheme.

Eq. (13) stands for a linear system of equations for implicit trapezoidal method. The system is readily solved by Gaussian elimination and backward substitution. If this method is employed in an analysis, the immediate question is: how one can determine whether the solution has

converged or not? In fact, several convergence criteria could be used for this purpose. One convenient way is to check the iterative value of y such that

$$e = ||\frac{\Delta y^{(i)}}{y}|| < Tol \tag{14}$$

where $||\ ||$ = Euclidean norm

Tol = A tolerance ratio

Presently, the above criterion is employed to determine the convergence of a solution.

Comparing Eqs. (7) & (13), it is apparent that the implicit trapezoidal method requires not only much more functional evaluations but also solving a system of linear equations. As an alternative, the implicitness of f_{n+1} in Eq. (8) may be removed by using Talor series expansion, namely,

$$\tilde{f}_{n+1} = \tilde{f}_n + J_n \Delta y_{n+1} \tag{15}$$

where

$$J_n = \partial \tilde{f}_n / \partial y_n \tag{16}$$

Thus, Eq. (13) becomes

$$[I - J_n \Delta t/2] \Delta y_n = \Delta t\ \tilde{f}_n \tag{17}$$

The above equation is referred as the explicit trapezoidal scheme since the solution is completely determined from the initial conditions.

At this point, it is instructive to make some qualitative comparisons among the aforementioned numerical schemes. Comparing explicit trapezoidal scheme with forward Euler scheme reveals that they differ only in the expression $J_n \Delta t/2$, i.e. the product of Jacobian matrix and half of step size. The addition of such matrix necessitates the solution be obtained by solving a system of simultaneous equations. Like implicit trapezoidal scheme, it also requires the evaluations of Jacobian matrix. Apparently, by including the extra term, the numerical behavior of the constitutive equations have become stabilized. In this context, $J_n \Delta t/2$ essentially plays the role of a correcting factor. On the other hand, since no itera-

tion is involved in the explicit trapezoidal scheme, it can be viewed as a starter of implicit trapezoidal scheme.

We consider another extreme case, i.e. $\theta = 1$, which is called implicit Euler scheme,

$$y_{n+1} = y_n + \Delta t_n f_{n+1} \tag{18}$$

Hughes et al (24) demonstrated that for viscoplastic finite element analysis one-parameter family of implicit algorithm is unconditionally stable when $\theta > 1/2$ while only conditionally stable otherwise. In recent years, various numerical schemes have been applied to viscoplastic problems (15-20,24-26). Some of the authors have discarded the explicit Euler method due to its numerical instability. However, the validity of this conclusion needs to be further explored. In Ref. (27), present investigators evaluated three numerical techniques for integrating the viscoplastic constitutive equations for a uniaxial state of stress. The schemes evaluated were: i) forward Euler method, ii) explicit trapezoidal method, and iii) implicit trapezoidal method with Newton-Raphson iteration method. Although implicit trapezoidal method with iteration appears to be more stable and accurate than the other methods even when the step size is considerable large, its suitability for finite element analysis must be re-assessed.

In principle, inelastic analysis using finite element method consists of a sequence of incremental process. Two most widely used approaches are the initial strain and tangent stiffness methods. In consideration of the formulation presented in Eqs. (1-4), one finds that the initial strain method is the most natural way to handle viscoplastic models. The reason behind this will be elaborated below.

In Eq. (1), we invoke an assumption that the strain increment is decomposed into elastic and inelastic components. Then, the inelastic part, which is governed entirely by Eqs. (1-4) at constitutive level, is converted into an equivalent load in the finite element formulation. Thus, we have

$$K^E \Delta u = (\Delta p_0) + (\Delta p_\varepsilon) \tag{19}$$

where

K^E = Elastic structural stiffness matrix, which may vary with the temperature

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Δu = Incremental nodal displacement vector

Δp_0 = Incremental vector of applied load

Δp_ϵ = Incremental vector due to inelastic and thermal strains.

In addition to the incremental procedure used for solving the global stiffness equations, a subincrementing technique is employed to calculate the constitutive material matrix. That is, let Δt be the time increment for solving the global stiffness equations. Then Δt is sub-divided into smaller increments with a constant step size, $\Delta \tau = \Delta t/m$. Moreover, the number of subincrements can be determined by an automatic stepping procedure for which an error measure is compared with a specified tolerance. Further discussion of this is given in [27].

COMPUTER IMPLEMENTATION

With the constitutive relations and numerical integration schemes outlined in the previous sections, the next step is to implement these relationships into a typical (general purpose) finite element program for intended analysis. For this purpose, the related computer subroutines are written in the form of an independent material module so that it can be easily interfaced with a finite element code.

The calculation steps for a viscoplastic model can be summarized as follows:

Step 1. Preset the strains, stresses, back stresses, inelastic strains, nodal temperatures, etc. transferred from the main program.

Step 2. For non-isothermal condition, interpolate temperature at Gauss points from nodal temperatures.

Step 3. Compute strain rate and temperature rate, and select step size of subincrements.

Step 4. Interpolate temperature dependent material constants based on the average temperature at the mid-point of a time step.

Step 5. Solve for the state variables from the constitutive equations using a subincrementing method with a selected integration technique.

Step 6. Check for solution convergence and determine whether cut-back of step size is necessary.

Step 7. Update the stresses, strains, inelastic strains and other state variables, then return to the main program.

EXAMPLE

To demonstrate the utility of the finite element procedures, Robinson's unified theory was applied to the analysis of a pressurized thick-walled cylinder which is restrained in its axial direction. Finite element mesh, its dimension and boundary condition are shown in Fig.1. The loading history consists of a 0.0028 hour ramp up to an internal pressure of 3.65 ksi followed by a hold period at that temperature for 200 hours. Explicit trapezoidal method was employed for this example.

Figs. 2 to 3 show the predicted redistribution of hoop and axial stress at several selected time following rapid pressurization, wherein zero time denotes the end of the loading ramp. As can be seen, while the internal pressure is held constant, these stresses undergo variation exhibiting rapid redistribution followed by a steady-steady response. The tendency of approaching to a saturated state is apparent.

Figs. 4 and 5 show the creep displacement at the outside wall of thick-walled cylinder using both Euler and explicit trapezoidal scheme with different time step sizes as well as number of subincrements. Solid line indicates the base-line solution using the explicit trapezoidal scheme. From the numerical experiments, it was found that the forward Euler integration scheme with an automatic stepping and error control is far more efficient in computation as compared to the explicit and implicit trapezoidal schemes.

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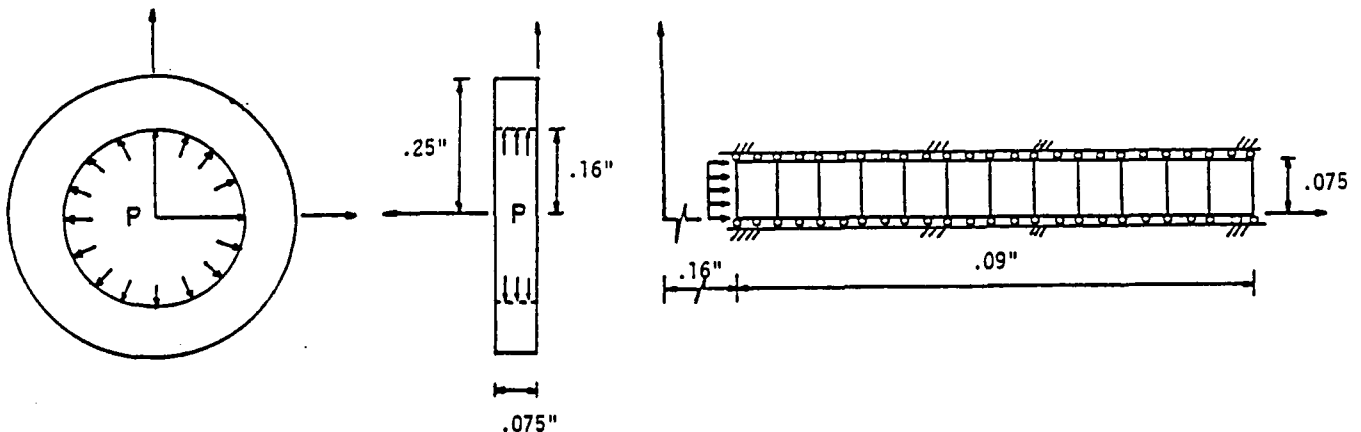
Fig. 1 - A Thick Wall Cylinder Under Internal Pressure
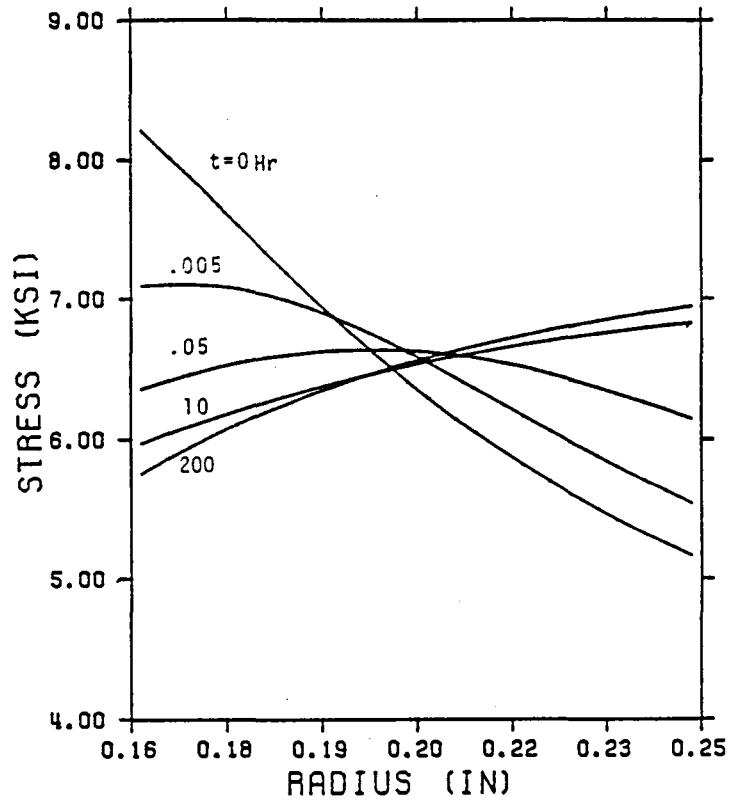
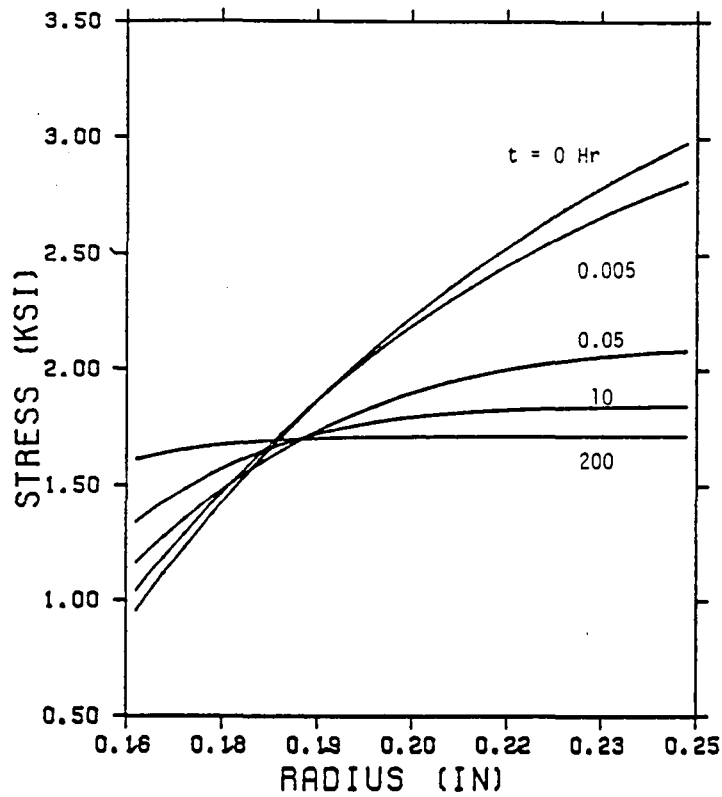Fig. 2 - Hoop Stress Distribution
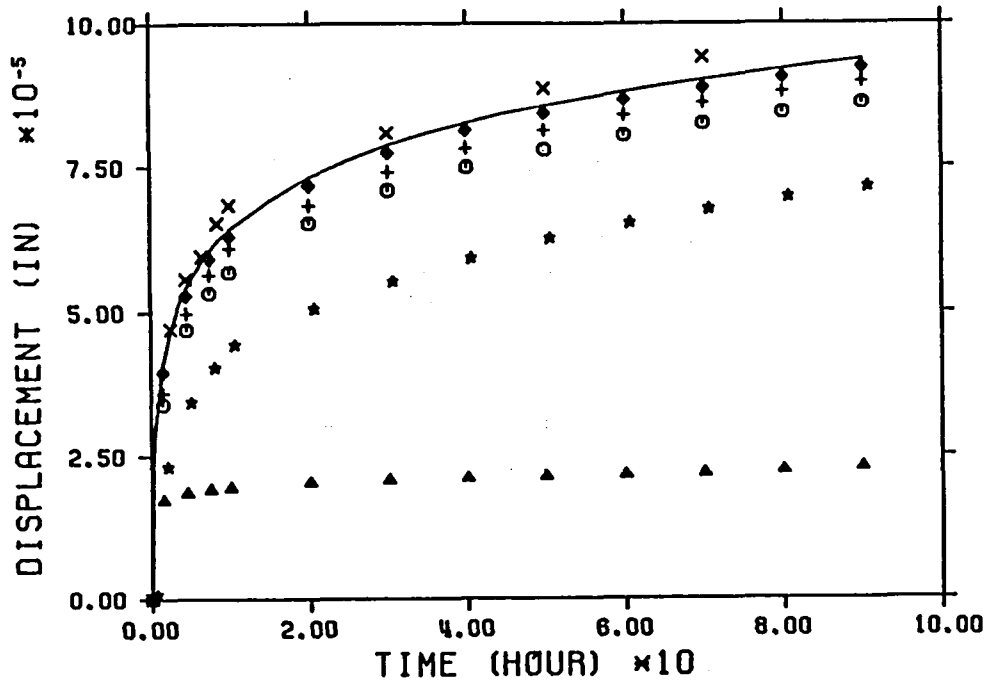

Fig. 3 - Axial Stress Distribution

Fig. 4 - Creep Displacement at Outside Wall of the Cylinder
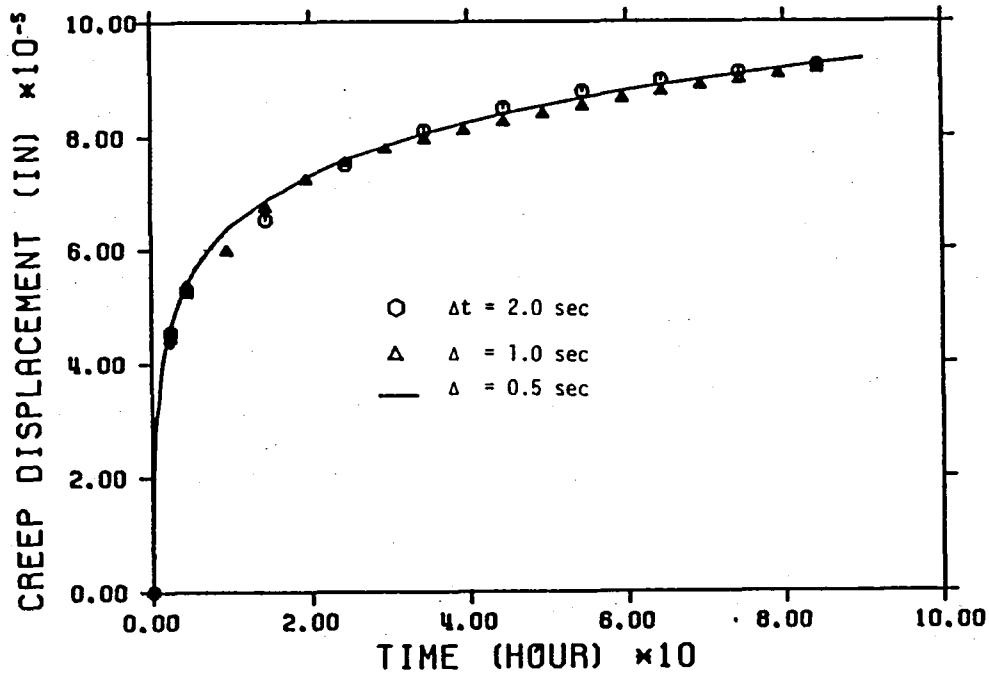by Various Integration Steps.

Fig. 5 - Creep Displacement Predicted by Euler Method with Automatic Stepping