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A REVIEW OF SPECTRAL METHODS

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Abstract

This paper presents a brief outline of spectral methods for partial differential equations. The basic ideas, together with simple proofs are discussed. An application to potential transonic flow is also reviewed.

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Introduction

This paper is meant as a brief survey of spectral methods, particularly as applied to partial differential equations. No attempt is made to cover any topic in depth, but rather to present a general outline to scientists unacquainted with the subject.

We begin by defining the basic spectral algorithms, emphasizing collocation and discussing the main advantage of the method, namely the infinite order of accuracy attained in problems with smooth solutions.

In the next section we present examples of theoretical numerical analysis of spectral calculations. These are elementary proofs for simple problems, but still may be taken as representative of more sophisticated results.

We conclude with an application of spectral methods to transonic flow. The full potential transonic equation is among the best understood among nonlinear equations; although there are few analytic solutions, there are many efficient finite difference codes. It is, moreover, of great engineering interest because in spite of all the simplifying assumptions introduced, it fits experimental data quite well.

It was very interesting to see what a spectral method could achieve in this problem, especially since there are several mathematical points not yet covered by any theory: nonlinearity, singularities, shock waves and entropy conditions. It turned out that the results are very satisfactory, and the algorithm as a whole is as efficient and as accurate as the best finite difference schemes, with the bonus of reduced memory requirements.

Finally, a few words about the special functions mentioned in the paper. Only the Chebyshev polynomials

$$T_n(x) = \cos\left(n \cos^{-1}(x)\right)$$

are considered in any detail. Various identities and quadrature formulas will be taken for granted; further information can be found in references [1] and [2] and in the treatise [3].

1. Spectral Methods - How and Why?

Assume one has to find an unknown function u, satisfying some differential equation.

A spectral method of solution starts by expanding u in a series of eigenfunctions of a Sturm-Liouville problem. Then, using orthogonality and various identities among such special functions one may define approximations to the derivatives of u, and employ those to compute u. In practice, the eigenfunctions will usually be trigonometric functions or orthogonal polynomials.

As a simple example, consider the wave equation with periodic initial data:

$$\text{(1.1)} \qquad \begin{cases} u_t = u_x & t > 0 \\ u(x,\ t=0) = \phi(x); & \phi(x) = \phi(x + 2\pi) \end{cases}$$

The solution will be periodic, suggesting a Fourier series rather than a polynomial expansion. Let

$$\text{(1.2)} \qquad u_N(x) = \sum_{k=0}^{N-1} a_k(t) e^{ikx};$$

then the derivative $\frac{\partial u_N}{\partial x}$ is given by

$$\frac{\partial u_N}{\partial x} = \sum_{k=0}^{N-1} ika_k(t)e^{ikx}, \tag{1.3}$$

and the original problem (1.1) reduces to

$$\frac{da_k}{dt} = ik\, a_k, \qquad 0 \leq k < N. \tag{1.4}$$

This is now a system of uncoupled ordinary differential equations for the coefficients a_k. The only input needed for solving (1.4) is the set of initial values of $a_k(0)$. These are the Fourier coefficients of the function ϕ; they are defined as integrals

$$a_k(0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx}\, \phi(x)dx, \tag{1.5}$$

or, in a more practical fashion, as sums

$$a_k(0) = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{-ik\frac{2\pi\ell}{N}}\, \phi\left(\frac{2\pi\ell}{N}\right) \tag{1.6}$$

which can be computed in $O(N \log N)$ operations by means of the fast Fourier transform (FFT).

An alternative approach resulting in (1.4), but generalizable to arbitrary orthogonal series is Galerkin's method: Substitute (1.2) into (1.1), multiply (1.1) by e^{ikx} for $k = 0,1,\ldots,N-1$ and integrate over the period. (For another system of eigenfunctions, substitute an approximating sum, multiply by the eigenfunction and the corresponding weight function, and integrate over the interval of orthogonality.)

We now impose boundary condition on the equation:

$$(1.7)\qquad \begin{cases} u_t = u_x, & |x| < 1,\ t > 0 \\ u(x,\ t=0) = \phi(x) \\ u(x=1,\ t) = \psi(t). \end{cases}$$

A Fourier series is no longer appropriate, so we expand in terms of Chebyshev polynomials:

$$(1.8)\qquad u_N = \sum_{k=0}^{N} a_k(t) T_k(x).$$

The formula for the derivatives is more complicated

$$(1.9)\qquad \frac{\partial u_N}{\partial x} = \sum_{k=0}^{N} b_k(t) T_k(x)$$

$$c_k\, b_k = \sum_{\substack{p=k+1 \\ p+k \text{ odd}}}^{N} p a_p; \qquad c_k = \begin{cases} 2 & \text{if } k = 0 \\ 1 & \text{if } k > 0, \end{cases}$$

but the final equation for the coefficients is similar to (1.4)

$$(1.10)\qquad \frac{d}{dt}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = A \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix},$$

with a certain matrix A. The initial values for the coefficients can again be taken to be the Chebyshev coefficients of the initial data which can be efficiently computed by FFT. There is, however, an additional equation to be satisfied:

$$\sum_{k=0}^{N} a_k(t)T_k(1) = \psi(t), \tag{1.11}$$

which represents the boundary condition. Note that (1.10) and (1.11) overspecify the N unknown coefficients; this would be no problem if infinite series were used, but for a calculation with a finite number of modes some compromise has to be made, e.g., satisfying only N of the N+1 equations (1.10), together with (1.11). Reserving some of the coefficients to satisfy boundary conditions - as done here - is called a Lanczos tau-method.

Finally, consider a wave equation with variable coefficients:

$$\begin{cases} u_t = \big(c(x)u\big)_x, & |x| < 1,\ t > 0 \\ u(x,\ t=0) & \\ u(x=1,t) = \psi(t), & a(x) > \infty > 0. \end{cases} \tag{1.12}$$

One should again approximate u by an Nth degree polynomial, but the coefficients of c(x)u cannot usually be defined in terms of the coefficients of u. This forces us to adopt a different approach:

Take N+1 points $x_0, x_1, \ldots, x_N$ in $[-1,1]$. These define a unique polynomial of degree N which is identical with u at the points - the interpolant of u. We now replace c(x)u by the interpolant of c(x)u - which is readily available, since all is needed are the values of $u(x_j)$ - and

compute its derivative to advance (1.12) in time. The boundary condition is satisfied by having $x_0 = 1$ and by setting $u(x_0, t) = \psi(t)$.

This procedure - which can, obviously, be applied to nonlinear equations too, is called a collocation method or a pseudospectral method. It is more general than the Galerkin and tau methods mentioned above, and boils down to defining the values of $f'(x_j)$ given $f(x_j)$, accurately for all polynomials or trigonometric polynomials of degree $\leq N$. This can be of course done for any set of points, and the corresponding operator is represented by a matrix multiplication

$$f'(x_j) = \sum_k D_{jk} f(x_k). \tag{1.13}$$

However, this is an inefficient numerical procedure, needing $O(N^2)$ operations. For special sets of collocation points the matrix multiplication can be done by FFT in $O(N \log N)$ operations. This is, in fact, one of the reasons why trigonometric functions and Chebyshev polynomials are usually employed in spectral calculations. Note that once the algorithm (1.13) is available, the expansion coefficients are no longer needed - in contrast to formulas (1.4), (1.10), (1.11).

In conclusion, we have introduced three kinds of spectral methods: Galerkin, tau, and collocation. We have singled out Chebyshev and Fourier collocation methods as the most useful for two reasons:

a) They may be used in variable coefficient and nonlinear problems;

b) They allow the fast Fourier transform.

We now address the second question in the title of this section: Why use spectral methods? To answer this, notice that the spectral approximation obtained in (1.2) - (1.4) differs from the exact solution of problem (1.1) by the quantity

$$\sum_{k=N}^{\infty} a_k e^{ik(x+t)}, \tag{1.14}$$

where a_k are the Fourier coefficients of ϕ. The error is certainly majorized by the sum of the absolute values of a_k. Now, if one assumes a smooth ϕ, i.e., one possessing continuous derivatives of all orders, it turns out that the coefficients decay faster than any power of k:

(1.15)

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} \phi(x)dx = \frac{1}{2\pi ik} \int_0^{2\pi} e^{-ikx} \phi'(x)dx$$

$$= \frac{1}{2\pi(ik)^2} \int_0^{2\pi} e^{-ikx} \phi''(x)dx = \cdots = \frac{1}{2\pi(ik)^M} \int_0^{2\pi} e^{-ikx} \phi^{(M)}(x)dx,$$

(simple integration by parts; M is arbitrary). Therefore, a spectral method using N modes and applied to smooth functions will admit an error estimate

$$|\text{error}| \lesssim \frac{C(M)}{N^M}\ , \tag{1.16}$$

for any M, with some constant $C(M)$. (In contrast, a finite difference method with N grid points will usually have an error of the form $\frac{C}{N^p}$ where p is some small integer such as $p = 2$ for a second order method.) Formula (1.16) is usually referred to as the "infinite order of accuracy of a spectral method," or, for short, "spectral accuracy." It also holds for general eigenfunction expansions, provided one deals with smooth functions. Computationally, (1.16) means one can obtain very accurate numerical solutions, using relatively few data points.

We should also say a few words about nonsmooth functions. This is not an academic question, since most nonlinear hyperbolic systems admit discontinuous solutions, and even allow discontinuities to evolve in time from smooth initial data. For example, consider the Euler equations of gasdynamics, which produce solutions with shock waves, contact discontinuities, and rarefaction waves. In this case one cannot expect high accuracy - rather, the Gibbs phenomenon which occurs at discontinuities will produce an oscillating error component which does not vanish as $N \to \infty$. However, proper treatment of this problem may filter out the noise and produce good approximations "away from shocks" [4]. In fact, the Gibbs phenomenon itself may serve as a shock locator, accurately pinpointing sharp transitions in the solution.

Although there exist several satisfactory spectral calculations of nonsmooth solutions [5, 6, 7], this area is very much in need of a firm theoretical basis.

2. Proofs

In this section we present two examples of analysis of spectral methods. The problems we treat are of the form

$$\frac{\partial u}{\partial t} = Lu$$

and are replaced by numerical approximations

$$\frac{\partial u_N}{\partial t} = L_N u_N.$$

We present elementary convergence proofs, showing that the numerical results actually approach the unknown functions sought.

We use the Lax equivalence theorem, which states that a scheme which is consistent and stable is convergent. Consistency is the following limit property

$$\lim_{N\to\infty} L_N u = Lu.$$

It expresses the fact that the spectral operator approaches the differential operator as the number of modes increases, and will be taken for granted, under the assumption of spectral accuracy. What we prove is stability, i.e., estimates of the form

$$\|u_N(x,t)\| \leq C \|u_N(x, t=0)\|, \qquad 0 < t < T,$$

which should hold in some norm, with constants C which may depend on T, but not on N, the number of modes employed.

2.1 Fourier Collocation

We solve the wave equation with periodic boundary conditions

$$(2.1) \qquad \begin{cases} u_t = u_x, & t > 0 \\ u(x, t=0) = \phi(x); & \phi(x + 2\pi) = \phi(x), \end{cases}$$

by Fourier collocation, using the points $x_j = \frac{2\pi j}{N}$, $0 \leq j < N$; we assume that N is even, $N = 2M$. The approximate solution u is a trigonometric polynomial of degree N, representable as a linear combination of the functions:

$$(2.2) \qquad \begin{cases} 1, & \cos(x), \cos(2x), \ldots, \cos(Mx) \\ & \sin(x), \sin(2x), \cdots, \sin\big((M-1)x\big) \end{cases}$$

which satisfies:

$$\frac{\partial u_N}{\partial t}(x_j) = \frac{\partial u_N}{\partial x}(x_j) = \sum_k D_{jk}\, u_N(x_j). \tag{2.3}$$

From (2.3) one can deduce the stability of this numerical scheme, namely:

$$\sum_{0\le j<N} u_N^2(x_j,t) = \sum_{0\le j<N} u_N^2(x_j,\ t=0). \tag{2.4}$$

One way of proving this is by explicitly computing the collocation derivative matrix D_{jk}, which turns out to be antisymmetric. Therefore, when (2.3) is multiplied by $u_N(x_j,t)$ and summed, one obtains

$$\frac{\partial}{\partial t}\frac{1}{2}\sum_{0\le j<N} u_N^2(x_j,\ t) = \sum_{0\le j,k<N} D_{jk}\, u_N(x_j,\ t)u_N(x_k,t) = 0 \tag{2.5}$$

which implies (2.4).

The second method to establish stability is based on a different interpretation of the sums in (2.5):

$$\frac{\partial}{\partial t}\frac{1}{2}\sum_{0\le j<N} u_N^2(x_j,t) = \sum_{0\le j<N} u_N(x_j,t)\frac{\partial u_N}{\partial x}(x_j,t). \tag{2.6}$$

An examination of the functions in (2.2) shows that one may replace the second sum by an integral, because for any combination appearing in $u_N \frac{\partial u_N}{\partial x}$ the trapezoidal rule with N points is exact:

$$\int_0^{2\pi} f(x)dx = \frac{2\pi}{N}\sum_{0\le j<N} f(x_j). \tag{2.7}$$

Thus,

$$(2.8)\qquad \frac{\partial}{\partial t}\frac{1}{2}\sum_{0\le j<N} u_N^2(x_j,t) = \frac{N}{2\pi}\int_0^{2\pi} u_N(x)\,\frac{\partial u_N}{\partial x}\,dx = 0 \quad \text{(by periodicity)},$$

leading again to (2.4).

2.2 Chebyshev Collocation for an Initial-Boundary Value Problem

The differential equation to be solved is:

$$(2.9)\qquad \begin{cases} u_t = u_x & |x| < 1, \quad t > 0 \\ u(x,\ t=0) = \phi(x) \\ u(x=1,\ t) = 0. \end{cases}$$

Consider the points:

$$(2.10)\qquad x_j = \cos\frac{j\pi}{N}, \qquad 0 \le j \le N.$$

The identity

$$T_m(x_j) = \cos\left(m\,\cos^{-1}(x_j)\right) = \cos\frac{jm\pi}{N}$$

shows clearly how one can use FFT (a cosine transform) to fit interpolating polynomials to data at these points. To solve (2.9) we begin by satisfying the boundary condition by taking:

$$(2.10)\qquad u_N(x_0) = 0,$$

while the collocation procedure is:

$$\frac{\partial u_N}{\partial t}(x_j) = \frac{\partial u_N}{\partial x}(x_j), \qquad 0 < j < N. \tag{2.11}$$

(Note that the point $x_N = -1$ is not a collocation point.) For the initial condition, one should set

$$u_N(x_j, t=0) = \phi(x_j), \qquad 0 < j < N. \tag{2.12}$$

Conditions (2.10) - (2.12) uniquely define a polynomial $u_N(x,t)$, of degree $N-1$ in x, with coefficients depending on t.

A little thought shows that (2.11) may be replaced by

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} + \tau(t) p_N(x) \tag{2.13}$$

where $p_N(x)$ is the unique polynomial of degree $N-1$ which vanishes at x_j, $0<j<N$ and takes the value 1 at x_0. In fact, since $T_N(x)$ attains its extrema in $[-1,1]$ at the points x_j, one can explicitly identify p_N

$$p_N(x) = \frac{T_N'(x)}{N^2} \cdot \tag{2.14}$$

Thus one may extend formula (2.11) - equality at certain points - to the formula (2.13) which holds everywhere. This is done at the expense of an unknown function $\tau(t)$; we stress again that $p_N(x)$ is an explicitly known function.

Multiply now (2.13) by $\frac{1+x}{\sqrt{1-x^2}} u_N$ and integrate:

(2.15)

$$\frac{d}{dt}\int_{-1}^{1}\frac{1}{2}u_N^2\frac{1+x}{\sqrt{1-x^2}}\,dx=\int_{-1}^{1}u_N\frac{\partial u_N}{\partial x}\frac{1+x}{\sqrt{1-x}}\,dx+\tau(t)\int_{-1}^{1}u_N\,p_N\frac{1+x}{\sqrt{1-x^2}}\,.$$

The first integral on the right is negative, as integration by parts shows:

$$(2.16)\qquad \int_{-1}^{1}u_N\frac{\partial u_N}{\partial x}\frac{1+x}{\sqrt{1-x^2}}=\frac{u_N^2}{2}\frac{1+x}{\sqrt{1-x^2}}\Bigg|_{x=-1}^{1}-\int_{-1}^{1}\frac{u_N^2}{2}\frac{d}{dx}\sqrt{\frac{1+x}{1-x}}\,dx$$

$$=-\int_{-1}^{1}\frac{u_N^2}{2}(1+x)^{-1/2}(1-x)^{-3/2}\,dx.$$

The other two integrals may be replaced by sums over x_j. Indeed, the following formula holds whenever g is a polynomial of degree $< 2N$

$$(2.17)\qquad \int_{-1}^{1}\frac{g(x)}{\sqrt{1-x^2}}\,dx=\sum_{j=0}^{N}c_j\,g(x_j),\qquad \left(c_0=c_N=\frac{1}{2}\,;\ c_j=1,\ 0<j<N\right).$$

Then the multiplier of τ

$$(2.18)\qquad \int_{-1}^{1}u_N\,p_N\frac{1+x}{\sqrt{1-x^2}}=\sum_{j=0}^{N}c_j\,u_N(x_j)p_N(x_j)(1+x_j)$$

is seen to vanish, since

$$1 + x = 0 \quad \text{at} \quad x = x_N = -1$$

$$u_N = 0 \quad \text{at} \quad x = x_0 = 1$$

$$p_N = 0 \quad \text{at the interior points} \quad x_j,\ 0 < j < N.$$

In conclusion, we find

$$\frac{d}{dt}\int_{-1}^{1} u_N^2 \sqrt{\frac{1+x}{1-x}} = -\int_{-1}^{1} u_N^2 (1+x)^{-1/2}\,(1-x)^{-3/2}\,dx < 0 \tag{2.19}$$

and have proved stability in the form

$$\|u_N(x,t)\| \leq \|u_N(x,\ t=0)\|$$

where the norm is defined by

$$\|u\|^2 = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, u^2(x)dx. \tag{2.20}$$

A few remarks are in order about the above results. We note that special properties of the expansion functions are extensively used throughout. Orthogonality is needed, especially in the subtler form of Gauss quadrature formulas; for instance, formula (2.17) uses N+1 points, but is accurate for polynomials of degree 2N-1. None of the methods used in this section would work for arbitrary collocation points; in particular, polynomial collocation at evenly spaced points has very different (and numerically undesirable) properties. This is, in fact, the basic reason why eigenfunctions of Sturm-Liouville problems are used in spectral approximation; again, the Chebyshev

polynomials stand out as readily available functions, allowing general boundary conditions, and admitting the Fast Fourier Transform.

3. Applications to Transonic Flow

3.1 The Problem

Consider the steady two-dimensional flow of a compressible, inviscid gas past an airfoil, with uniform conditions at infinity. Under these assumptions, one may take the velocity vector to be the gradient of a velocity potential Φ:

$$u = \Phi_x, \qquad v = \Phi_y. \tag{3.1}$$

The density ρ is determined by Bernoulli's law

$$\rho = \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(u^2 + v^2 - 1\right)\right]^{\frac{1}{\gamma-1}} \tag{3.2}$$

where γ is the specific heat ratio and M_∞ the free stream Mach number. The equation one must solve, for the scalar unknown Φ, expresses mass conservation:

$$(\rho\Phi_x)_x + (\rho\Phi_y)_y = 0. \tag{3.3}$$

The boundary conditions for (3.3) are as follows:

a) At infinity, the potential must satisfy

(3.4) $$\Phi \sim x \quad \longleftrightarrow \quad u \sim 1$$

representing uniform flow with normalized horizontal speed 1.

b) Let the wing be located on the x axis, with a shape given by

(3.5) $$y = \pm\tau(1 - x^2), \qquad |x| < 1,$$

- a parabolic wing of (small) thickness ratio $= \tau$. We impose the boundary condition

(3.6) $$\frac{\partial \Phi}{\partial y} = \frac{dy}{dx}\frac{\partial \Phi}{\partial x} \qquad \text{at} \quad y = 0, \quad |x| < 1,$$

which approximates, to the first order in τ, the exact condition

(3.7) $$\frac{\partial \Phi}{\partial n} = 0 \qquad \text{on the body} \quad y = \pm\tau(1 - x^2).$$

Because of symmetry,

(3.8) $$\frac{\partial \Phi}{\partial y} = 0$$

on the rest of the x axis. (The symmetry about the x axis means, of course, that there is no lift force on the wing; this assumption is made only for mathematical convenience, i.e., solving for $y > 0$ only.)

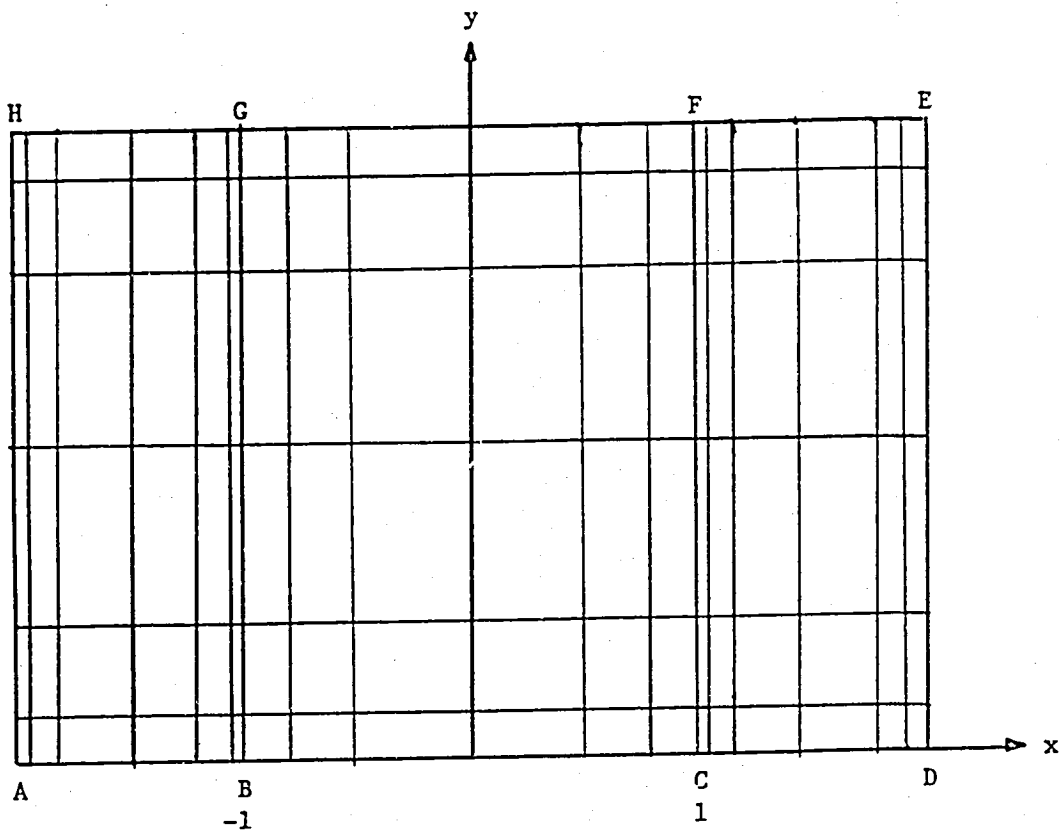
The outstanding property of equation (3.3) is that it changes type from elliptic to hyperbolic as the local speed goes from subsonic to supersonic. We shall describe briefly the qualitative behavior of the flows it represents.

As long as the Mach number M_∞ is small enough, the flow is smooth and symmetrical under the transformation $x \longrightarrow -x$; the singularities at $x = \pm 1$ do not propagate into the field. The speed is either subsonic everywhere, or a small supersonic pocket may develop over the wing. As M_∞ increases over a critical value, a shock wave develops near the trailing edge, across which the speed reduces abruptly from supersonic to subsonic. Thus the solution is no longer symmetric, although there is symmetry in the differential equation and in the boundary conditions. A symmetrical flow would contain an unphysical rarefaction shock near the leading edge.

This is clearly a case of nonuniqueness of weak solutions, to be resolved by an entropy condition. It means, from the computational point of view, that some entropy inequality - or equivalently, some desymmetrizing procedure - must be enforced, in addition to equations (3.3), (3.4), (3.6), (3.8).

3.2 The Numerical Algorithm

As a first step towards the solution, we notice that the discontinuity of $\frac{\partial \Phi}{\partial y}$ at $x = \pm 1$ requires a mesh refinement there. Since we also need buffer zones in front of the airfoil and behind it, we discretize the problem by using three Chebyshev domains, as shown:



Three Chebyshev Meshes; BC Represents the Airfoil.

We impose continuity of Φ and $\frac{\partial \Phi}{\partial x}$ on the interfaces BG and CF. The various derivatives appearing in (3.3) are computed by collocation. After these obvious steps are taken, one is left with two problems:

a) to devise a desymmetrizing algorithm;

b) to find an effective iteration scheme for the solution of the (nonlinear) discrete version of (3.3).

Streett [8] has developed an algorithm to deal with these questions, which is moreover applicable to lifting flow, including the exact flow tangency condition (3.7). We summarize now Streett's method, as applied to the present problem.

In finite difference codes, desymmetrizing is done by computing derivatives by upwind differencing in supersonic regions [9] or by biasing the density towards upwind values [10]. As spectral derivative calculation is nonlocal, it allows no procedure similar to upwind differencing; it is much more convenient to use the modified density approach. A modified density $\tilde{\rho}$ is computed by the formulas:

$$\tilde{\rho} = \rho - \mu \overleftarrow{\delta}\, \rho \qquad \text{(first order)}$$

$$\tilde{\rho} = \rho - \mu\left(\overleftarrow{\delta}\rho - \varepsilon\overleftarrow{\delta E^{-1}}\rho\right) \qquad \text{(second order),}$$

with switches μ and ε defined by:

$$\mu = \max\left(0,\ 1 - \frac{1}{M^2}\right)$$

$$\varepsilon = \max\left(0,\ 1 - \frac{|\delta^0\rho|}{K}\right)$$

The difference operators E, δ are as follows:

$$(\overleftarrow{\delta} f)(x_i) = f(x_i) - f(x_{i-1})$$

$$(\delta^0 f)(x_i) = \frac{1}{2}\left(f(x_{i+1}) - f(x_{i-1})\right)$$

$$(E^{-1} f)(x_i) = f(x_{i-1})$$

where $x_{i-1} < x_i < x_{i+1}$ are three neighboring mesh points. M is the local Mach number and K of the order of the density jump across the shock. After $\tilde{\rho}$ is computed, the equation (3.3) is replaced by

$$(\tilde{\rho}\Phi_x)_x + (\tilde{\rho}\Phi_y)_y = 0. \tag{3.9}$$

Finally, we address the iteration method. One may regard $\Phi(x,y)$ as the time independent solution of an equation

$$A\frac{\partial \Phi}{\partial t} = (\tilde{\rho}\Phi_x)_x + (\tilde{\rho}\Phi_y)_y \overset{\text{def}}{=} N_{sp}(\Phi) \tag{3.10}$$

where A is a linear operator and N_{sp} stands for the nonlinear part, computed spectrally. Indeed, if (3.10) possesses steady state solutions, these will satisfy (3.9). The operator A, which seems arbitrary at this point, must, in fact, be chosen very carefully to ensure convergence. The usual choice for the transonic equations is

$$\alpha\Phi_t + \beta\Phi_{xt} = N_{sp}(\Phi) \tag{3.11}$$

with variable coefficients α and β: α is nonzero at subsonic points, and β is nonzero at supersonic points.

The time introduced in equations (3.10) or (3.11) is purely artificial; we may replace the variable t with an iteration index n. Motivated by (11), the following iteration scheme is produced:

$$\alpha(\Phi^{n+1} - \Phi^n) + \beta(\Phi_x^{n+1} - \Phi_x^n) = N_{sp}(\Phi^{n+1} - \Phi^n) + N_{sp}(\Phi^n). \tag{3.12}$$

The two formulas (3.11) and (3.12) are not exactly equivalent, but it is seen that if $\Phi^{n+1} - \Phi^n \to 0$ then $N_{sp}(\Phi^n) \to 0$, so we are still solving (3.9).

As it stands, (3.12) is still hard to solve, as it involves nonlinear operators and the full matrices of spectral derivation. We overcome this difficulty by the method of approximate inverses, namely by replacing N_{sp} by a linear operator L, which is easy to invert and is near - in some sense - to N_{sp}.

We shall take L to be the finite difference representation of

$$\frac{\partial}{\partial x} \tilde{\rho} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \tilde{\rho} \frac{\partial}{\partial y}$$

on the spectral Chebyshev grid, with $\tilde{\rho}$ considered as a given coefficient. This is a five-point operator, since five points are sufficient to define the derivatives involved; it is also clear that $L\Phi \simeq N_{sp}(\Phi)$.

From the resulting iteration scheme

$$\alpha(\Phi^{n+1} - \Phi^n) + \beta(\Phi_x^{n+1} - \Phi_x^n) = L(\Phi^{n+1} - \Phi^n) + N_{sp}(\Phi^n), \tag{3.13}$$

one can readily compute $\Phi^{n+1} - \Phi^n$ by inverting the operator $\alpha + \beta \frac{\partial}{\partial x} - L$. This is efficiently done by a dimensional split - two tridiagonal matrices for separate x and y relaxation - or by an approximate LU-factorization of the five-point operator. (For additional material on iteration schemes for spectral operators, see [11].)

Using the apparatus mentioned above, the transonic flow was computed for the simple case of the symmetric parabolic airfoil, as well as for more meaningful lifting airfoil shapes [8]. The results were compared with state of the art finite difference schemes for the same problems, and found to produce the same resolution with significantly smaller number of grid

points. Shocks were accurately captured. Although a spectral iteration is more complex than the corresponding finite difference one, and usually one needs more iterations to converge in a spectral code, the running times for the two methods were comparable, and in certain cases the spectral code ran faster then the finite difference code. This clearly shows the advantage of smaller computational meshes, which are admissible because of spectral accuracy.

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Final Report

16. Abstract

This paper presents a brief outline of spectral methods for partial differential equations. The basic ideas, together with simple proofs are discussed. An application to potential transonic flow is also reviewed.

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