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RELATIVISTIC TRANSPORT THEORY FOR COSMIC-RAYS

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ABSTRACT

We discuss various aspects of the transport of cosmic-rays in a relativistically moving magnetized plasma supporting a spectrum of hydromagnetic waves that scatter the cosmic-rays. A local Lorentz frame (the co-moving frame) moving with the waves or turbulence scattering the cosmic-rays is used to specify the individual particle momentum. Since the co-moving frame is in general a non-inertial frame in which the observer's volume element is expanding (contracting) and shearing, geometric energy change terms appear in the cosmic-ray transport equation which consist of the relativistic generalization of the adiabatic deceleration term obtained in previous analyses, and a further term involving the acceleration vector of the scatterers (this term may be thought of as a gravitational redshift effect). We also present a relativistic version of the pitch angle evolution equation, including the effects of adiabatic focussing, pitch angle scattering, and energy changes.

1. Introduction. The transport equations for cosmic-rays in a bulk plasma flow (such as the solar wind) were originally obtained by Parker (1965), and later by Dolginov and Toptygin (1966), Gleeson and Axford (1967), Skilling (1975) and Webb and Gleeson (1979). These developments assumed that background plasma flow was non-relativistic; whereas the present work is concerned with obtaining equations that also apply in relativistic flows. Our derivation relies mainly on relativistic transport theory for Liouville's equation (or the Boltzmann equation) as developed by Lindquist (1966).

2. The Equations. The development of a relativistic transport theory for the Boltzmann (or Liouville equation) for charged particle propagation in electromagnetic fields starts with the equation of motion for the individual particle:

$$m_0\, dx^a/d\tau = p^a , \tag{1}$$

$$m_0\, dp^a/d\tau = -\Gamma^a_{bc}\, p^b p^c + q\, F^a{}_b\, p^b , \tag{2}$$

where Γ^a_{bc} are the affine connection coefficients (the term involving Γ^a_{bc} represents non-inertial forces). The electromagnetic ($\underline{E}$ and $\underline{B}$) forces are contained in the last term in (2) involving the Faraday Tensor $F^a{}_b$; q denotes the particle charge; $\{p^a\}$ a = 0,1,2,3 is the momentum four vector, m_0 is the particle rest mass, and τ is the proper time. As a consequence of (1) and (2), the 1-particle phase space distribution function satisfies the relativistic Liouville equation

$$L(F) = p^a \partial F/\partial x^a - \Gamma^a_{bc} p^b p^c \frac{\partial F}{\partial p^a} + q F^a{}_b p^b \frac{\partial F}{\partial p^a} = 0 . \tag{3}$$

For the case of stochastic electromagnetic fields we set

$$F^a{}_b = \langle F^a{}_b \rangle + \delta F^a{}_b , \quad F = \langle F \rangle + \delta F , \tag{4}$$

and applying quasilinear theory we obtain the Boltzmann equation

$$\langle L \rangle (f) = (p^0/c)(\delta F/\delta t)_c = q^2 p^b \frac{\partial}{\partial p^i} \langle \delta F^i{}_b \langle L \rangle^{-1} (\delta F^j{}_c \, p^c \frac{\partial f}{\partial p^j}) \rangle \tag{5}$$

where the angular brackets denote ensemble averages $f = \langle F \rangle$, and δF and $\delta F^a{}_b$ denote the random components of F and $F^a{}_b$.
The operator $\langle L \rangle$ in (5) is the particle propagator in the average background field, and $\langle L \rangle^{-1}$ its inverse.

Taking moments of (5) with respect to directions of the particle momentum leads to moment equations for cosmic-ray transport. It is convenient to use a local Lorentz frame moving with the scatterers Σ' to specify the individual particle momentum p'. In the special case where the observer's frame is a global Lorentz frame (the more general case is considered in Webb, 1985), the cosmic-ray continuity equation, or zeroth order moment of (5) with respect to momentum directions of p' is:

$$\begin{aligned} &\partial/\partial t [\gamma (n'^0 + \underline{V} . \underline{J}'/c^2)] + \partial/\partial \underline{x} \cdot [\gamma \underline{V} n'^0 + \underline{J}' + (\gamma - 1) \underline{V} \, \underline{V} . \underline{J}'/V^2] \\ &+ \partial/\partial p' [(q \underline{E}' - m' \underline{a}') \cdot \underline{J}'/v' - \tfrac{1}{3} p' n'^0 (\partial \gamma/\partial t + \underline{\nabla} . (\gamma \underline{V}))] = 0 , \end{aligned} \tag{6}$$

where

$$\underline{J}' = - \underline{\underline{K}} \cdot [\partial n'^0/\partial \underline{x}' + p' m' (q \underline{E}' - m' \underline{a}') \, \partial (n'^0/p'^2)/\partial p'] , \tag{7}$$

is the particle current in the co-moving frame;

$$\underline{a}' = \gamma^2 \{ d\underline{V}/dt + \underline{V} (d\gamma/dt) \; / [\gamma (\gamma + 1)] \} \tag{8}$$

is the acceleration vector of the scatterers; n'^0 is the particle number density ($n'^0 = 4\pi p'^2 f_0$ where f_0 is the isotropic part of the distribution function in Σ'); $\gamma = (1 - V^2/c^2)^{-1/2}$ is the Lorentz factor corresponding to V, with $\underline{V}$ the velocity of the scattering frame relative to the observer's frame. $\underline{\underline{K}}$ is the cosmic-ray diffusion tensor, $\underline{E}'$ is the mean electric field in Σ' (which is negligible for a highly conducting plasma); m', v' denote the relativistic particle mass and speed in Σ' (p' = m'v'), and the time derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial \underline{x}} , \tag{9}$$

occurring in (8) is the Lagrangian derivative moving with the scatterers. The effects of second order Fermi acceleration have been omitted in (6), but can be included in the analysis if necessary.

The term $- m' \underline{a}' \cdot \underline{J}'/v'$ in (6) involving the acceleration vector of the scatterers may be thought of as a gravitational redshift effect: it arises because the scattering frame is in general an accelerating frame and it is then necessary to introduce non-inertial forces to properly describe the particle motion in Σ'. The form of this term is similar to that of the Joule heating term $\underline{J}'\cdot(q\underline{E}')/v'$ but with $q\underline{E}'$ replaced by $-m'\underline{a}'$. The second momentum change term in (6)

$$\langle \dot{p}' \rangle^{*} = -\tfrac{1}{3} p' \left[\frac{\partial \gamma}{\partial t} + \underline{\nabla}\cdot(\gamma \underline{V}) \right] \quad , \tag{10}$$

may be thought of as the relativistic generalization of adiabatic deceleration. However, care is required in its interpretation; it is intimately connected with the fact that the co-moving frame is a non-inertial frame. More generally one can show that

$$\langle \dot{p}' \rangle^{*} = -\tfrac{1}{3} p' c \, \Gamma'^{\alpha}_{0\alpha} = -\tfrac{1}{3} p' c \, U^{\alpha}{}_{;\alpha} \quad , \tag{11}$$

where in (11) we have used the summation convention on the index α (α = 0,1,2,3) and $\underline{U}$ denotes the four velocity of the scattering frame. Thus (11) shows that $\langle \dot{p}' \rangle^{*}$ is associated with the affine connection coefficients in the co-moving frame. The streaming equation (7) follows from the first moment of the Boltzmann equation (5) with respect to directions of $\underline{p}'$. In the derivation of (6) and (7) the diffusion approximation has been used (the distribution function is assumed to be near isotropic with respect to $\underline{p}'$) and the cosmic-ray inertia in Σ' has been neglected in the derivation of (7)).

An alternative basis for deriving cosmic-ray transport equations is to use the drift approximation in which the particle gyrofrequency is assumed to be large compared to the scattering frequency ν, so that to a first approximation the distribution function is independent of gyrophase. By averaging the Liouville equation (5) over gyrophase in Σ' one obtains the pitch angle scattering equation:

$$\begin{aligned}
&\gamma\,(1+\mu' v' \underline{n}\cdot\underline{V}/c^2)\,\partial F/\partial t + \left[\underline{V} + (\mu'/\gamma)\left(v'\underline{n} + (\gamma-1)\, v'\, \underline{n}\cdot\underline{V}\,\underline{V}/V^2\right)\right]\cdot\underline{\nabla} F \\
&+ p'\,\partial F/\partial p' \left\{ \tfrac{1}{2}(1-3\mu'^2)(\gamma+1)\,\underline{n}\cdot\partial/\partial x_3'\,(\gamma\underline{V}/(\gamma+1)) \right. \\
&\qquad \left. - \tfrac{1}{2}(1-\mu'^2)\, c\, U^{a}{}_{;a} - (\mu'/v')\,\underline{n}\cdot\underline{a}' \right\} \\
&+ \tfrac{1}{2}(1-\mu'^2)\,\partial F/\partial\mu' \left\{ v'\,\underline{\nabla}'\cdot\underline{n} + \mu'\left[c\,U^{a}{}_{;a} - 3(\gamma+1)\,\underline{n}\cdot\partial/\partial x_3'\,(\gamma\underline{V}/(\gamma+1))\right] \right. \\
&\left. - \frac{2}{v'}\,\underline{n}\cdot\underline{a}' \right\} = \frac{\partial}{\partial\mu'}\left[\tfrac{1}{2}(1-\mu'^2)\,\nu\,\frac{\partial F}{\partial\mu'}\right] \quad ,
\end{aligned} \tag{12}$$

where $\mu' = \cos\theta'$ with θ' the particle pitch angle,

$$\frac{\partial}{\partial x_3'} = \gamma\,\frac{\underline{n}\cdot\underline{V}}{c^2}\,\frac{\partial}{\partial t} + \underline{n}\cdot\underline{\nabla} + (\gamma-1)\,\underline{n}\cdot\frac{\underline{V}(\underline{V}\cdot\underline{\nabla})}{V^2} \quad , \tag{13}$$

and $\underline{n} = \underline{e}'_3 = \underline{B}'/B'$. The electric field in Σ' has been assumed to be zero in the derivation of (12). Equation (12) is the special relativistic generalization of the pitch scattering equation used by Skilling (1975) to derive the cosmic-ray transport equations. It also contains for example the adiabatic focussing term used in solar cosmic-ray propagation theory.

3. Conclusions. The development presented here has concentrated on deriving kinetic equations for cosmic-ray transport in special relativistic flows; in addition, hydrodynamical forms of the equations can be obtained.

These equations are being used to study cosmic-ray acceleration in relativistic shocks.

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