Combined Trellis Coding With Asymmetric MPSK Modulation

An MSAT-X Report

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16. Abstract

Traditionally symmetric, multiple phase-shift-keyed (MPSK) signal constellations, i.e., those with uniformly spaced signal points around the circle, have been used for both uncoded and coded systems. Although symmetric MPSK signal constellations are optimum for systems with no coding, the same is not necessarily true for coded systems. This apper shows that by designing the signal constellations to be asymmetric, one can, in many instances, obtain a significant performance improvement over the traditional symmetric MPSK constellations combined with trellis coding. In particular, we consider the joint design of $n/(n + 1)$ trellis codes and asymmetric 2^{n+1}-point MPSK, which has a unity bandwidth expansion relative to uncoded 2^n-point symmetric MPSK. The asymptotic performance gains due to coding and asymmetry are evaluated in terms of the minimum free Euclidean distance d_{free} of the trellis. A comparison of the maximum value of this performance measure with the minimum distance d_{min} of the uncoded system is an indication of the maximum reduction in required E_b/N_0 that can be achieved for arbitrarily small system bit-error rates. It is to be emphasized that the introduction of asymmetry into the signal set does not affect the bandwidth or power requirements of the system; hence, the above-mentioned improvements in performance come at little or no cost. MPSK signal sets in coded systems appear in the work of Divsalar [1]. Here we expand upon these results by considering 4-, 8-, and 16-PSK asymmetric signal sets combined with the optimum (in the sense of maximum d_{min}) trellis code having 2, 4, 8, and 16 states. The numerical results obtained will clearly demonstrate the tradeoff between the additional savings in required E_b/N_0 and the additional complexity (more trellis states) needed to achieve it.

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Preface

The Mobile Satellite Experiment (MSAT-X) is managed by the Jet Propulsion Laboratory (JPL) for the National Aeronautics and Space Administration (NASA) as part of NASA's Mobile Satellite Communications Program. The thrust of MSAT-X is to develop advanced ground segment technologies and techniques for mobile communications via satellite in future-generation high-capacity systems. Areas of concentration in technology development include: vehicle antennas; mobile radios; low bit rate, near toll quality digital voice; bandwidth and power efficient modulations; and efficient network management and multiple-access schemes. NASA plans to validate these technologies by conducting experiments through the first-generation commercial mobile satellite, expected to be launched in the late 1980's.

Presently under way is an advanced MSAT-X technology development whose goal is to transmit 4800 bps, near toll quality digital speech and data over a 5-kHz Rician fading channel, the latter being characteristic of the mobile radio environment. In order to attain this goal, specific attention has been directed toward combined modulation/coding techniques which potentially achieve increased power efficiency without expansion of bandwidth. One such class of techniques is the combination of MPSK modulation and trellis coding with the possible addition of asymmetry to the modulation for further improvement in performance. It is in this context that the research presented in this report finds its motivation.

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SECTION I

INTRODUCTION

A. SYSTEM MODEL

The system under consideration is illustrated in Figure 1. Typical symmetric and asymmetric signal sets are shown in Figure 2. In particular, the asymmetric $M=2^{n+1}$-point set is created by adding together the optimum symmetrical $M/2$-point set with a rotated version of itself. The optimization problem discussed in the Abstract thus reduces itself to a determination of this angle of rotation.

Another way of looking at the M-point asymmetric construction, which is more in keeping with Ungerboeck's "set partitioning" technique, is to imagine partitioning the symmetric M-point constellation into two M/2-point constellations with maximally separated signals, and then to perform an appropriate rotation of one subset with respect to the other. Upon optimization of the rotation angle, the resulting two subsets can be used as the first level of set partitioning in Ungerboeck's method. In the next section of this report, we briefly discuss this procedure and illustrate its application.

B. ASSIGNMENT OF SIGNALS TO STATE TRANSITIONS OF TRELLIS CODES

The approach of assigning signals to transitions of the trellis code is based on a mapping rule called "mapping by set partitioning" [2]. This mapping results from successive partitioning of a signal set into subsets. Each subset (including the original set) is partitioned into two subsets with an equal number of signals and with the largest minimum distance between signals within the subset. Figure 3 demonstrates the set partitioning method as applied to asymmetric 8-PSK. What remains is to optimize the rotation angle ϕ.

As in [2], the criterion of optimization will be to maximize the free Euclidean distance (or its square) of the trellis code. In the next section, we review the relation of this performance measure and likewise the average bit error probability of the overall coded system to the transition structure of the trellis diagram.

1
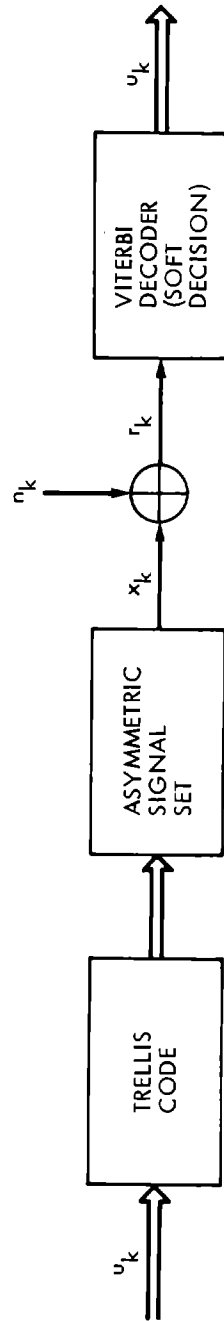
Figure 1. System Block Diagram

2
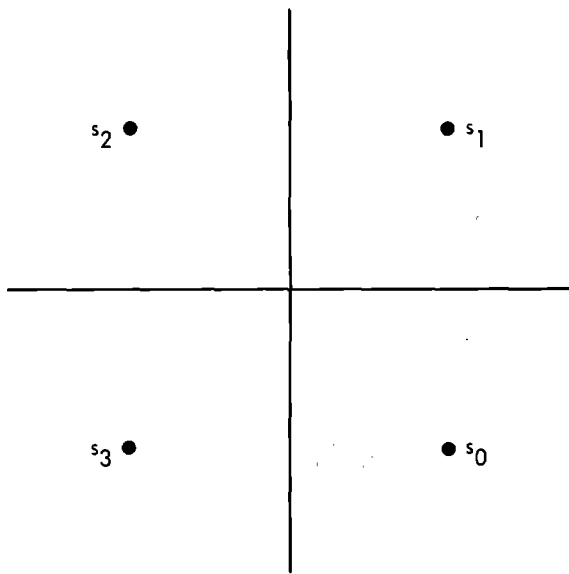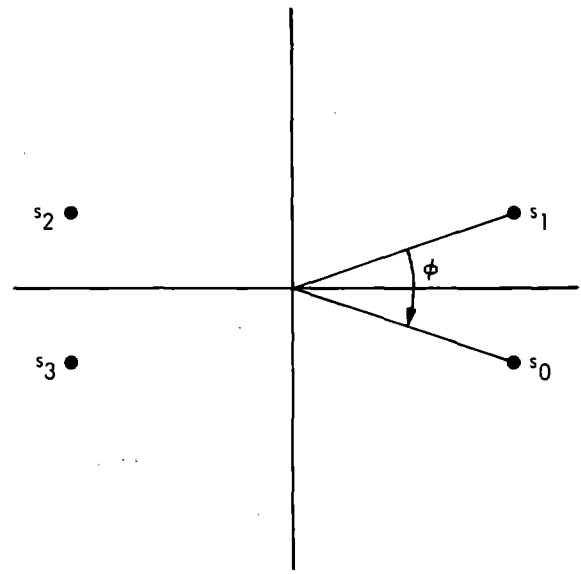
SYMMETRIC 4-PSK

ASYMMETRIC 4-PSK
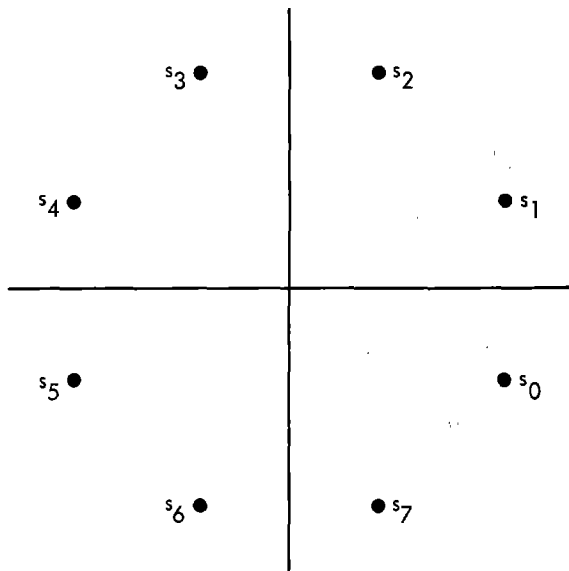
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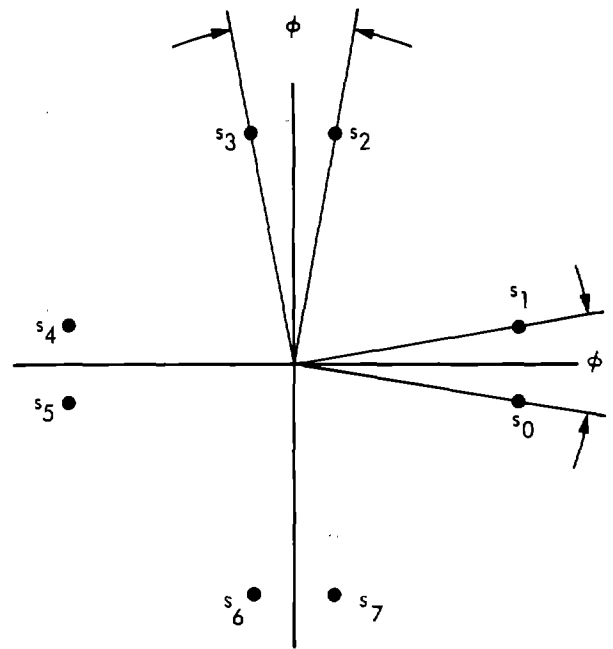
ASYMMETRIC 8-PSK

Figure 2. Symmetric and Asymmetric MPSK Signal Sets

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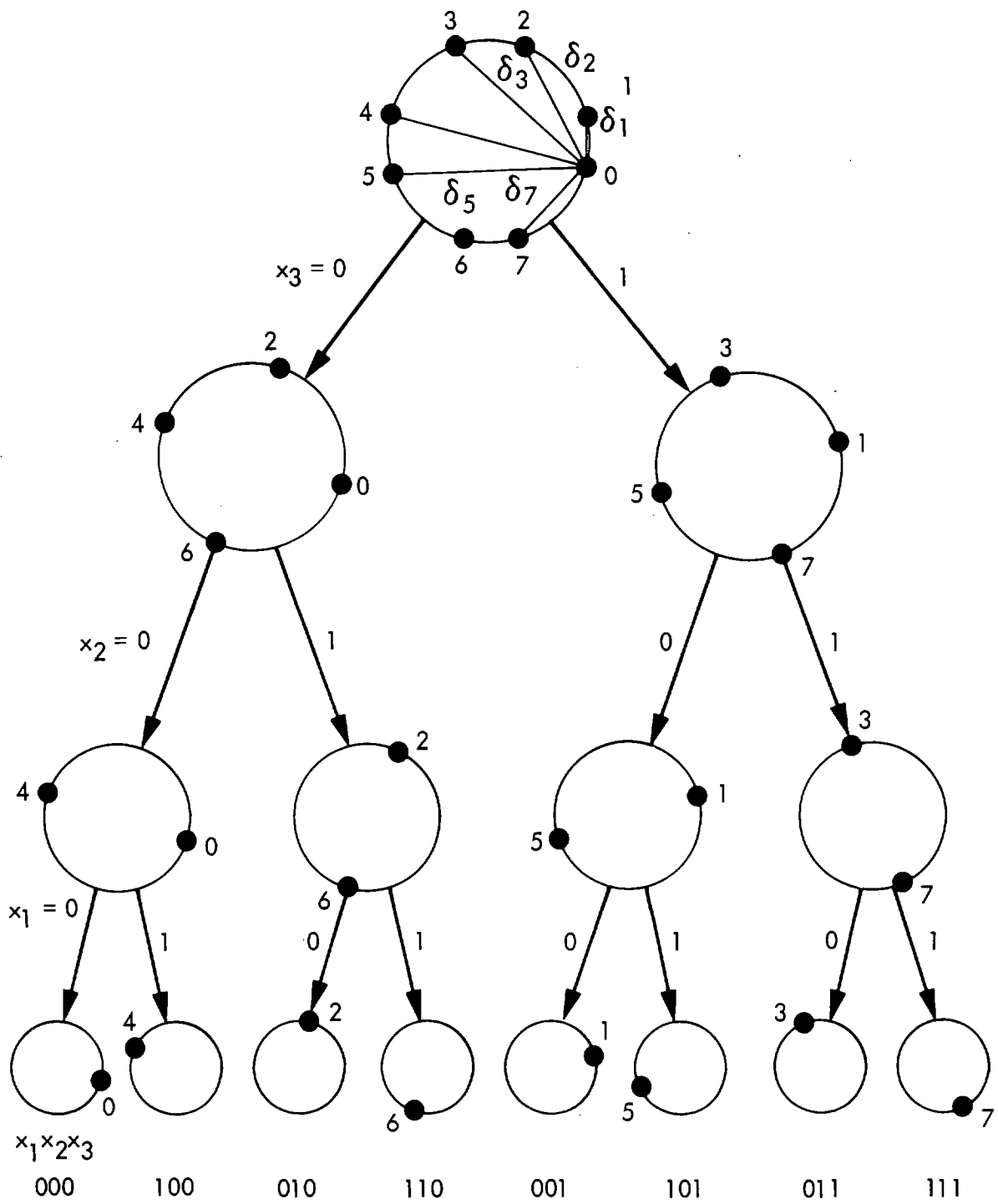
Figure 3. Set Partitioning of Asymmetric 8-PSK

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SECTION II

PERFORMANCE ANALYSIS

For every n information bits, the rate $n/(n + 1)$ trellis encoder produces $n + 1$ output coded symbols. These symbols are assigned to a unique member of the asymmetric 2^{n+1} signal set in accordance with the above mapping procedure. Thus, each transmitted signal x_k at time k is a nonlinear function of the state of the encoder s_k and the n information bits at its input denoted by u_k, i.e.,

$$x_k = f(s_k, u_k) \tag{1}$$

The next state of the encoder s_{k+1} is a nonlinear function of the present state and the input u_k. In mathematical terms,

$$s_{k+1} = g(s_k, u_k) \tag{2}$$

The received signal sample at time k is

$$r_k = x_k + n_k \tag{3}$$

where n_k is a sample of a zero mean Gaussian-noise process with variance σ^2.

To find the average bit error probability performance of the Viterbi decoder, we must first find the pair-wise error probability $p(\underline{x} \to \hat{\underline{x}})$ between the coded sequence $\{x_k\}$ and the estimated sequence $\{\hat{x}_k\}$, denoted by $\underline{x}$ and $\hat{\underline{x}}$ respectively. Assume that $|x_k|^2 = 1$. Then, using the Bhattacharyya bound [4], we have

$$p(\underline{x} \to \hat{\underline{x}}) \leq D^\Delta \; ; \; \Delta = \sum_k \delta^2(S_k, U_k) \tag{4}$$

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where

$$\delta^2(S_k, U_k) \triangleq \left| f(s_k, u_k) - f(\hat{s}_k, \hat{u}_k) \right|^2 \tag{5}$$

with $\hat{s}_k$ and $\hat{u}_k$ the estimates of the state of the decoder and the information symbol, respectively. Also, D is the Bhattacharyya distance which in this case is given by

$$D = \exp\left(-\frac{1}{8\sigma^2}\right) \tag{6}$$

The pair-state S_k and the pair-information symbol U_k [4 and 5] are defined as

$$S_k \triangleq (s_k, \hat{s}_k)$$

$$U_k \triangleq (u_k, \hat{u}_k) \tag{7}$$

We are in a correct pair-state when $\hat{s}_k = s_k$ and in an incorrect pair-state when $\hat{s}_k \neq s_k$.

In terms of the above definitions, it can be shown that

$$P_b \leq \frac{1}{n} \frac{d}{dz} T(D, z) \Big|_{z=1} \tag{8}$$

where

$$T(D, z) = \frac{1}{m} \underline{V}^t [\underline{I} - \underline{A}]^{-1} \underline{W} \tag{9}$$

and m is the number of code states. The vectors $\underline{V}$ and $\underline{W}$ have dimension $m^2 + m$ with elements taking on values 1 and 0. $\underline{A}$ is a $(m^2 + m) \times (m^2 + m)$ pair-state transition matrix with elements

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$$a(S_k, S_{k+1}) = \begin{cases} \sum_{u_k \in U_k} \frac{1}{2^n} z^{w(U_k)} D^{\delta^2(S_k, U_k)}; \\ \qquad \text{if } U_k \text{ is nonempty set} \\ \\ 0; \text{ otherwise} \end{cases} \qquad (10)$$

where

$$U_k = \left\{ (u_k, \hat{u}_k) \middle| (\hat{s}_k, \hat{u}_k) \neq (s_k, u_k), S_k \notin \mathcal{S}_d, S_{k+1} = G(S_k, U_k) \notin \mathcal{S}_t \right\} \qquad (11)$$

in which $\mathcal{S}_t$ and $\mathcal{S}_d$ are sets of all true and dummy correct pair-states respectively, and

$$G(S_k, U_k) \triangleq \left(g(s_k, u_k), g(\hat{s}_k, \hat{u}_k) \right) \qquad (12)$$

Finally, the free Euclidean distance of the code [5] is

$$d^2_{free} = \lim_{D \to 0} \log_2 \frac{T(2D, 1)}{T(D, 1)} \qquad (13)$$

Asymptotically for large signal-to-noise ratios (SNR), maximizing d_{free} is synonomous with minimizing the average bit-error probability. This relation is true provided that the distances between individual points in the signal set do not become too small. As we shall see, in some cases, optimization of the asymmetry condition produces signal sets wherein the limiting signal points tend to merge together. Thus, in these instances, the reader is cautioned that the performance advantage achieved in terms of improvement in d_{free} no longer translates directly into improvement in the required SNR; thus, one is forced to back off somewhat from this optimum condition.

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Based on the discussion of the previous section, the procedure for designing good trellis codes, combined with optimum asymmetric MPSK signal constellations, can be summarized by the following steps:

Step 1: Use the mapping by set-partitioning method to partition the signal constellation as the example in Figure 3.

Step 2: Assign signals from either of the two partitions (each containing 2^n signals) generated at the first level of partitioning in step 1 to transitions diverging from a given state. Similarly, assign signals from the other of these two partitions to transitions re-emerging to a given state. These assignments should be made such that the minimum distance between diverged and the minimum distance between re-emerged transitions are as large as possible.

Step 3: Find the free Euclidean distance of the code using Eq. (13) or the bit error probability using Eq. (8) or the pair-state transition diagram.

Step 4: Maximize the free Euclidean distance or minimize the bit error probability of step 3 with respect to the rotation angle ϕ. This value of ϕ then defines the optimum asymmetric MPSK signal constellation.

A. BEST RATE 1/2 CODES COMBINED WITH ASYMMETRIC 4-PSK (A4PSK)

The signal partitioning for trellis coded A4PSK is as in Figure 4. For a rate 1/2 code, there will be two transitions leaving (diverging from) each state. We begin by considering the signal point assignment for the simplest case of 2 states.

1. 2-State Trellis

For a 2-state trellis, one has only two choices for transition assignment. Either there exists multiple (two) transitions between like states or the two transitions leaving a given state go to different states. In the case of the former, the shortest error event path will be length one (i.e., the parallel path); hence, the maximum value of d_{free}^2 is limited to the Euclidean distance between this pair of signal points. For the set partitioning of Figure 4, this corresponds to the squared distance between points 0 and 2 (or 1 and 3) which has a value of 4.0. If on the other hand, the latter choice of assignment is made as illustrated in Figure 5, then the shortest error event path, i.e., the one yielding the minimum distance, is of length two. This path, corresponding to the error event of choosing signal 2 followed by signal 1, when, in fact, signals 0 and 0 were successively transmitted,* clearly has a larger value of d_{free}^2 than 4.0 since the squared distance of the first branch of this path is by itself 4.0. Thus, this assignment is obviously the better choice.

We shall define a <u>state transition matrix</u>, T, which describes the possible transitions between states corresponding to successive discrete time instants separated by a channel symbol. The ij^{th} entry in the matrix represents the output MPSK symbol assigned to the transition from state i to state j. The absence of an entry implies that a transition between those states is not possible. Thus, for the trellis of Figure 5, we have

$$T = \begin{bmatrix} 0 & 2 \\ & \\ 1 & 3 \end{bmatrix} \qquad (14)$$

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* In all of our discussions, we shall assume that the all zeros path, which corresponds to the all zeros input bit sequence, is the transmitted path. This implies that the signal point assignment to the trellis should satisfy the uniform error probability (UEP) criterion, i.e., the probability of error is independent of the transmitted sequence. A further discussion of this implication will follow shortly.
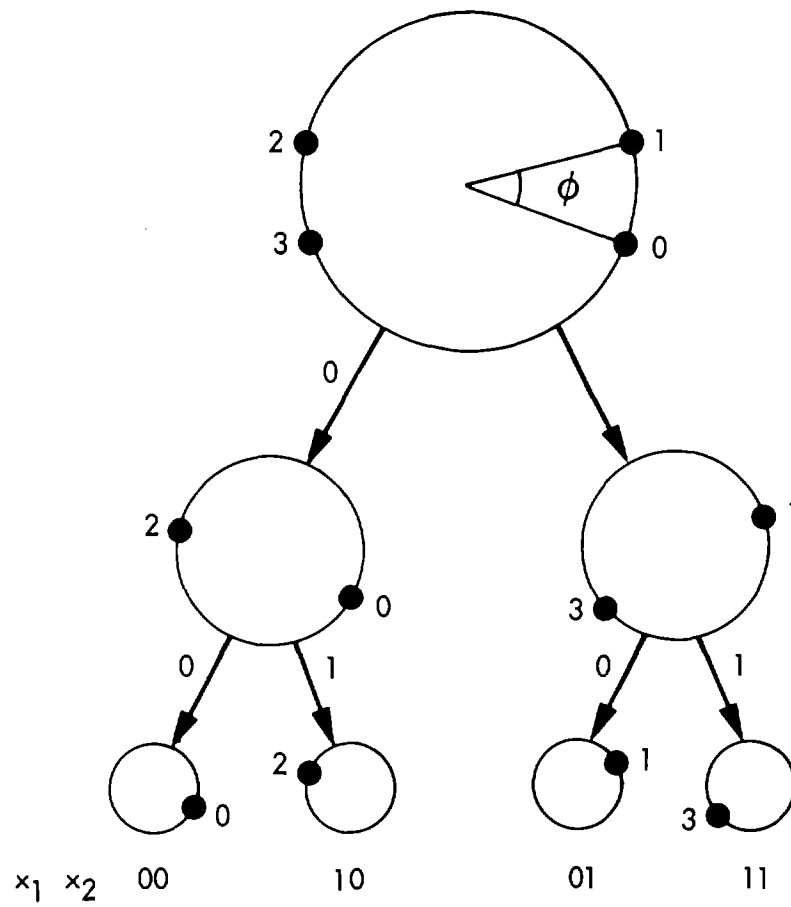
Figure 4. Set Partitioning of Asymmetric 4-PSK

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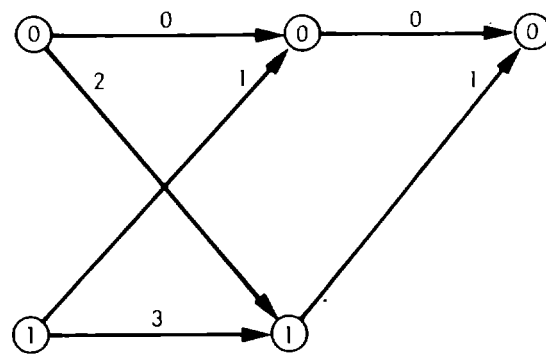
Figure 5. Trellis Diagram and MPSK Signal Assignment for 4-PSK

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We note that the signal point constellation of Figure 4 can be regarded as a special case of an unbalanced QPSK (UQPSK) where the data rates on the two channels are equal and the symbol transition times are aligned, but the powers are unbalanced. The ratio of powers between the I and Q channels can be related to the angle ϕ that defines the asymmetry. In particular, letting $\alpha = P_Q/P_I$, then

$$\alpha = \tan^2 \frac{\phi}{2} \tag{15}$$

The trellis of Figure 5 can be implemented by a constraint length 2, rate 1/2 linear convolutional code. The pair-state transition diagram for this code is illustrated in Figure 6 and has the transfer function bound

$$T(D, z) = \frac{zD^{\frac{4(1 + 2\alpha)}{1 + \alpha}}}{1 - zD^{\frac{4}{1 + \alpha}}} \tag{16}$$

where D is defined by Eq. (6). Using Eq. (16) in Eq. (8) gives the upper bound on the average bit error probability, namely,

$$P_b \leq \frac{D^{\frac{4(1 + 2\alpha)}{1 + \alpha}}}{\left(1 - D^{\frac{4}{1 + \alpha}}\right)^2} \tag{17}$$

where the unit radius circle in Figure 4 implies that $P_I + P_Q = 1$.

The optimum value of α (or equivalently ϕ), i.e., the value that minimizes the bound on P_b of Eq. (17), is

$$\alpha = - \frac{4 \ln D}{\ln 3} - 1 \tag{18}$$
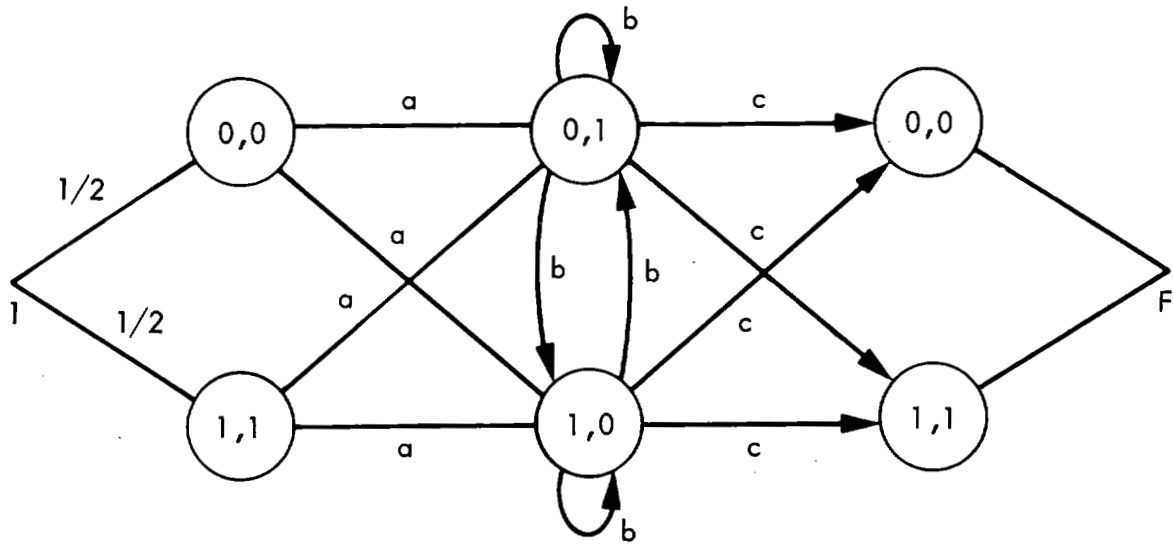
$$T(D,z) = \frac{4ac}{1-2b}$$

$$a = \frac{z}{2} D^4$$

$$b = \frac{z}{2} D^{\frac{4}{1+\alpha}}$$

$$c = \frac{1}{2} D^{\frac{4\alpha}{1+\alpha}}$$

Figure 6. Pair-State Transition Diagram for Trellis Diagram of Figure 5

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The parameter D of Eq. (6) can be related to the system bit energy-to-noise ratio E_b/N_0 by first recognizing that $\sigma^2 = (2E_s/N_0)^{-1}$ where E_s is the MPSK symbol energy. Since, for $n/(n + 1)$ trellis coding, n input bits of energy E_b, produce $n + 1$ code symbols, which in turn result in a single MPSK symbol of energy E_s, then clearly $E_s = nE_b$. Using these observations in Eq. (6) gives the desired relation for D in terms of E_b/N_0, namely,

$$D = \exp\left(-\frac{nE_b}{4N_0}\right)$$ (19)

Table 1 below gives the optimum value of α and ϕ versus E_b/N_0 in accordance with Eqs. (15) and (18) together with Eq. (19).

Substituting Eq. (18) in Eq. (17) gives the optimum (in the sense of the best asymmetric 4-PSK signal design) upper bound on the average bit error probability, namely,

$$P_b \le \frac{1}{4} 3^{3 - \frac{2E_b/N_0}{\ln 3}}$$ (20)

Table 1. Optimum Values of Power Ratio and Asymmetry Angle Versus E_b/N_0

E_b/N_0, (dB)	α	ϕ (rad)
4	1.3	1.70
5	1.9	1.89
6	2.6	2.03
7	3.5	2.16
8	5.7	2.35
9	7.2	2.43
10	9.1	2.50
11	11.5	2.57

For the symmetric signal design ($\phi = \pi/2$, $\alpha = 1$), the upper bound in Eq. (17) becomes

$$P_b \leq \frac{\exp{(-3E_b/2N_0)}}{[1 - \exp{(-E_b/2N_0)}]^2} \tag{21}$$

Finally for uncoded PSK, the corresponding upper bound would be

$$P_b \leq \exp{\left(-\frac{E_b}{N_0}\right)} \tag{22}$$

Figure 7 illustrates the three upper bounds of Eqs. (20), (21), and (22) versus E_b/N_0. For sufficiently large values of E_b/N_0, the denominator of Eq. (21) can be approximated by unity. Thus, asymptotically, the gain in E_b/N_0 of the coded symmetric 4-PSK system over the uncoded PSK system is $10 \log_{10}(3/2) = 1.76$ dB. To determine how much additional gain due to asymmetry is achievable in the same asymptotic limit, we turn to a discussion of the free distance behavior of the coded system.

Let δ_j^2 denote the squared distance from signal point 0 to signal point $j = 1,2,3$. Then, for the asymmetric constellation of Figure 4,

$$\delta_1^2 = 4 \sin^2{\frac{\phi}{2}} \; ; \; \delta_2^2 = 4 \; ; \; \delta_3^2 = 4 \cos^2{\frac{\phi}{2}} \tag{23}$$

For the minimum distance path of length 2, we have

$$d_{free}^2 = \delta_2^2 + \delta_1^2 = 4 \left(1 + \sin^2{\frac{\phi}{2}}\right) \tag{24}$$

which for the symmetric signal design ($\phi = \pi/2$) becomes

$$d_{free}^2 = 4 \left(1 + \frac{1}{2}\right) = 6 \tag{25}$$

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In the more general asymmetric case, substituting Eq. (15) into Eq. (24) gives

$$d_{free}^2 = 4 \left(1 + \frac{\alpha}{1 + \alpha} \right)$$

$$= 4 \left(\frac{1 + 2\alpha}{1 + \alpha} \right) \tag{26}$$

Thus, the improvement in d_{free}^2 due to asymmetry is from Eqs. (25) and (26)

$$\eta \stackrel{\Delta}{=} 10 \ \log_{10} \frac{d_{free}^2 \Big| \ asymm.}{d_{free}^2 \Big| \ symm.} = 10 \ \log_{10} \frac{2(1 + 2\alpha)}{3(1 + \alpha)} \tag{27}$$

For example, for E_b/N_0 = 10 dB, we have from Table 1 that α = 9.1. Thus, the performance improvement of the asymmetric constellation over the symmetric one is 1.03 dB.

If instead of minimizing the bit error probability, we select the asymmetry angle that maximizes d_{free}^2 of Eq. (24), then the value of this angle will be independent of the SNR. From Eq. (24), we see that d_{free}^2 is maximized when $\phi = \pi$, i.e., signal points 1 and 2 merge together and likewise for signal points 0 and 3. In this limiting case, $d_{free}^2 = 8$ and the <u>gain relative to the symmetric constellation</u> is 10 $\log_{10}(8/6)$ = <u>1.25 dB.</u> Note that this result represents the limiting case of Table 1 as E_b/N_0 approaches infinity. It also represents the asymptotic improvement in the E_b/N_0 performance due to asymmetry, as would be obtained by letting the symmetric and asymmetric coded curves in Figure 7 approach infinite E_b/N_0. Finally, for any finite E_b/N_0, using $\phi = \pi$ or, equivalently, $\alpha = \infty$ in Eq. (17), results in an infinite upper bound as would be expected.

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The figure plots UPPER BOUNDS ON P_b (vertical axis, from 10^{-8} to 10^{-1}) versus E_b/N_0, dB (horizontal axis, from 5.0 to 11.0).

Curves labeled:

UNCODED PSK
$$\exp\left(-E_b/N_0\right)$$

OPTIMUM RATE 1/2 TRELLIS CODED ASYMMETRIC 4-PSK
$$\frac{1}{4}\ 3^3 - \frac{2E_b/N_0}{\ln 3}$$

RATE 1/2 TRELLIS CODED SYMMETRIC 4-PSK
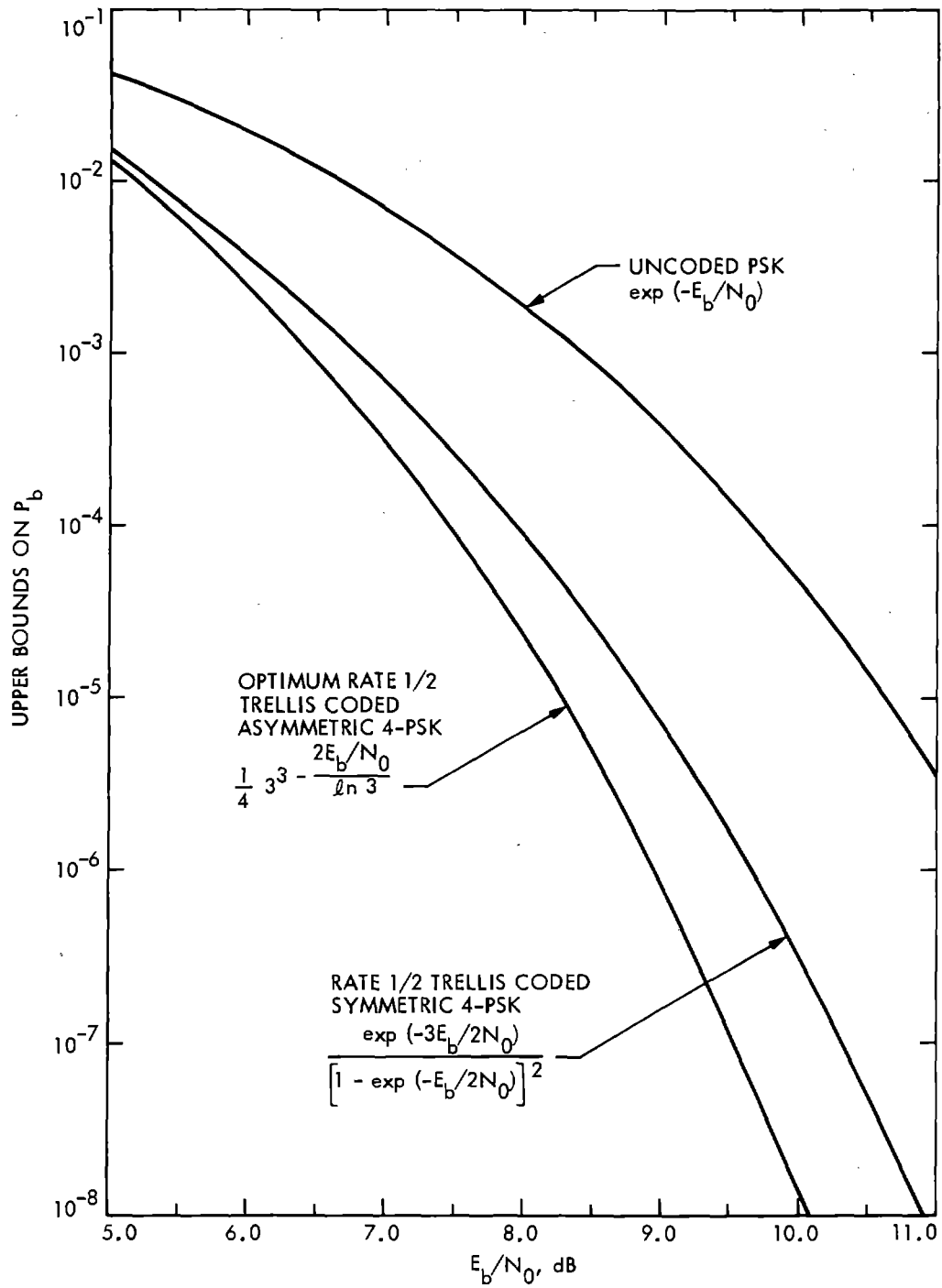$$\frac{\exp\left(-3E_b/2N_0\right)}{\left[1 - \exp\left(-E_b/2N_0\right)\right]^2}$$

Figure 7. Upper Bounds on Average Bit Error Probability Performance for Rate 1/2 Trellis Coded Symmetric and Optimum Asymmetric 4-PSK

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Since for uncoded 2-PSK (or simply PSK), the square of the minimum distance is 4 (two signal points diametrically opposed on a circle of diameter 2), then the limiting gain of the 2-state trellis coded asymmetric 4-PSK relative to this equivalent bandwidth uncoded system is $10\log_{10}(8/4) = 3.01$ dB. The relative gain of trellis coded symmetric 4-PSK to uncoded 2-PSK would, from the above discussion, be 1.25 dB less, or 1.76 dB, which agrees with the statement above.

2.   4-State Trellis

For a rate 1/2, 4-state trellis code combined with 4-PSK, the assignment of signals to the branches according to steps 2 and 3 of the previous section, leads to the trellis illustrated in Figure 8. Depending on the value of ϕ, there are two possibilities for the shortest path with the minimum free distance. For small values of ϕ, the length-4 path corresponding to MPSK signals 2,3,3,2 is the dominant one; whereas, for values of ϕ near π, the length-3 path corresponding to MPSK signals 2,1,2 is dominant. The squared Euclidean distances for these paths are

$$d^2(2,1,2) = 4 + 4 \sin^2 \frac{\phi}{2} + 4$$

$$d^2(2,3,3,2) = 4 + 8 \cos^2 \frac{\phi}{2} + 4 \tag{28}$$

To find the optimum value of ϕ, we equate the two squared distances in Eq. (28) which results in *

$$\tan^2 \frac{\phi}{2} = 2 \quad \Rightarrow \phi = 1.91 \text{ rad} \tag{29}$$

with a corresponding value of d^2_{free},

$$d^2_{free} = 4 + 8 \left(\frac{1}{1 + 2} \right) + 4 = \frac{32}{3} = 10.67 \tag{30}$$

---

* Since the two squared distance functions in Eq. (28) are monotonic functions (one increasing and one decreasing) of ϕ over the interval $(0, \pi)$, their crossover point results in the maximum value of the smaller of the two evaluated at each ϕ.
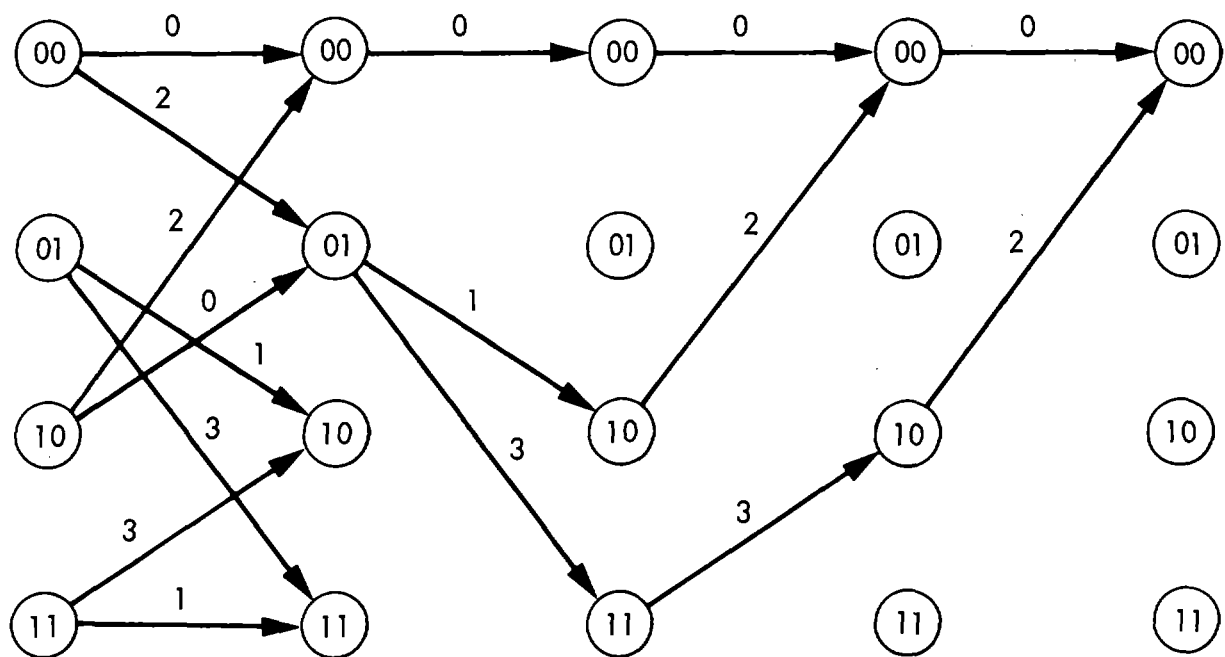
Figure 8. 4-State Trellis Diagram

For the symmetric case ($\phi = \pi/2$), the length-3 path gives the smaller minimum distance, which from Eq. (28) is

$$d^2_{free} = 4 + 4 \left(\frac{1}{2}\right) + 4 = 10 \tag{31}$$

Thus, from Eqs. (30) and (31), the gain in d^2_{free} due to asymmetry is

$$\eta = 10 \log_{10} \frac{32/3}{10} = 0.28 \text{ dB} \tag{32}$$

Again, relative to an uncoded PSK, the gains are as follows:

$$\eta \bigg|_{asymm.} \triangleq 10 \log_{10} \frac{d^2_{free} \big|_{asymm.}}{d^2_{min}} = 10 \log_{10} \frac{32/3}{4} = 4.26 \text{ dB}$$

$$\eta \bigg|_{symm.} \triangleq 10 \log_{10} \frac{d^2_{free} \big|_{symm.}}{d^2_{min}} = 10 \log_{10} \frac{10}{4} = 3.98 \text{ dB} \tag{33}$$

Although we have only discussed the minimum distance paths with respect to the all zeros path as the transmitted one, we have also checked our results against all possible transmitted paths with the conclusion that the signal assignment in Figure 8 leads to a UEP code, i.e., its average bit error probability is independent of the transmitted sequence.

In general, it would be desirable to have a necessary and sufficient set of conditions which would determine whether a particular signal assignment to a given trellis diagram has the UEP property. Indeed, one would like to have these conditions, <u>independent of the implementation of the code and independent of its linearity</u>. Thus far, finding such a set of necessary and sufficient conditions has eluded the authors of this paper and thus it is essential to check (typically by computer search) each signal assignment made for the UEP property. To make this task a bit simpler, we shall define an approximation to the UEP property, denoted by "UEP" which, for the purpose of system

comparison in terms of minimum free distance, is quite suitable. In particular, we shall say that a code is "UEP" if, independent of the input sequence, the trellis diagram produces the same minimum free distance and same number of error event paths at this distance. This approximate definition is equivalent to requiring that the leading term in the transfer function polynomial be independent of the input sequence. The stricter UEP definition would require that all terms of the polynomial be independent of the input sequence.

There is an important point to be emphasized here that is true regardless of whether the UEP or "UEP" definition is applied. When dealing with Euclidean (rather than Hamming) distance as a performance measure, the lengths and composition of the first error event paths at a given distance from the transmitted path may vary with the transmitted path itself. More specifically, the individual terms in the transfer function polynomial are characterized by a coefficient that specifies only the number (regardless of their length) of first error event paths at a given distance from the transmitted path and an exponent of D (the Bhattacharyya distance) which specifies the distance itself. Thus, even though a code is UEP, which implies a unique set of coefficients and exponents independent of the transmitted path, the makeup of the paths, i.e., their lengths and corresponding output MPSK symbols, contributing to a given term in the polynomial may well vary with the transmitted sequence.

3.   8-State Trellis

Following the steps previously discussed for the design of good codes, one arrives at the 8-state trellis diagram on Figure 9 with state transition matrix, T, given by
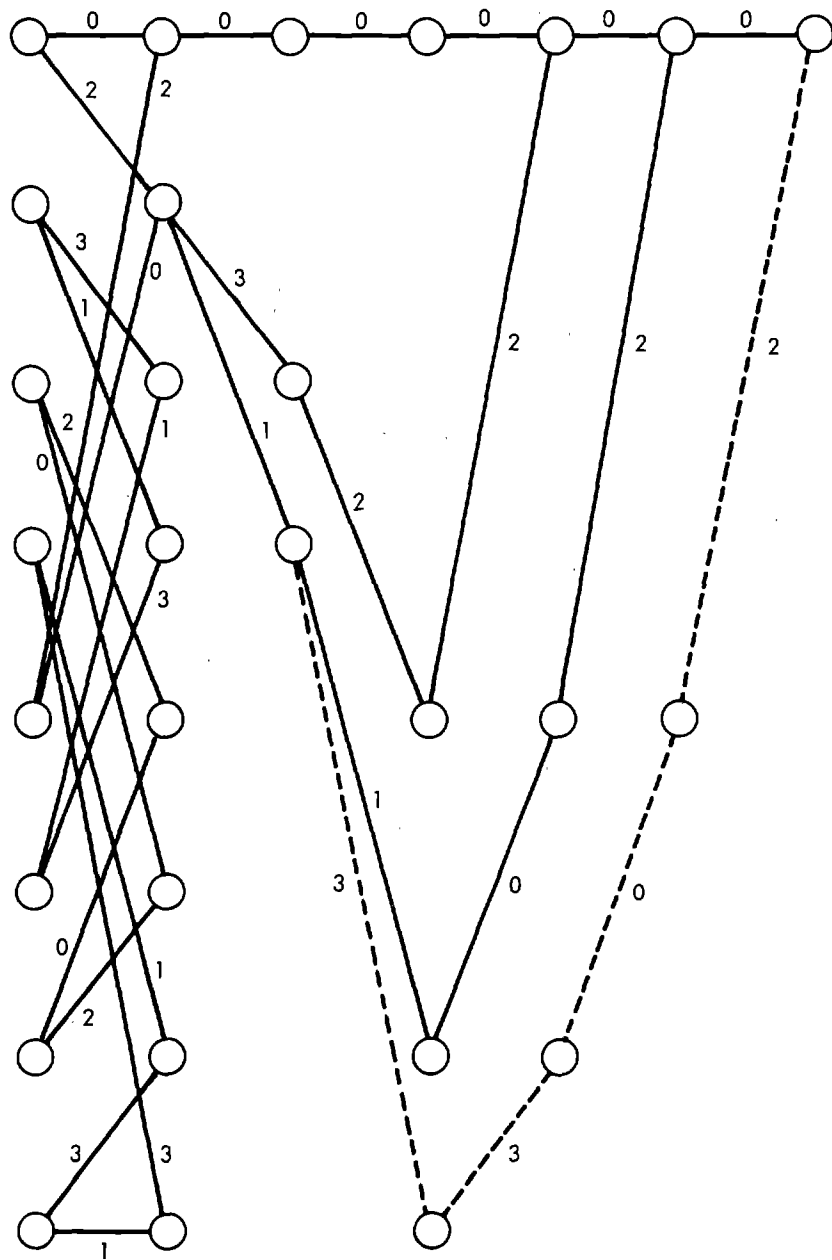
Figure 9. 8-State Trellis Diagram for Asymmetric 4-PSK

$$
T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\left[\begin{array}{cccccccc}
0 & 2 & & & & & & \\
 & & 3 & 1 & & & & \\
 & & & & 2 & 0 & & \\
 & & & & & & 1 & 3 \\
2 & 0 & & & & & & \\
 & & 1 & 3 & & & & \\
 & & & & 0 & 2 & & \\
 & & & & & & 3 & 1
\end{array}\right]
\end{array}
\qquad (34)
$$

As for the 4-state trellis, there are two shortest-length paths (solid lines) that, depending on the value of ϕ, yield the minimum free distance. The squared distance of these paths is given by

$$d^2(2,3,2,2) = 4 + 4 \cos^2 \frac{\phi}{2} + 4 + 4 = 12 + 4 \cos^2 \frac{\phi}{2}$$

$$d^2(2,1,1,0,2) = 4 + 4 \sin^2 \frac{\phi}{2} + 4 \sin^2 \frac{\phi}{2} + 0 + 4 = 8 + 8 \sin^2 \frac{\phi}{2} \qquad (35)$$

When these distances are equated, the optimum value of ϕ is found to be *

$$\sin^2 \frac{\phi}{2} = \frac{2}{3} \quad \Rightarrow \quad \phi = 1.23 \text{ rad} \qquad (36)$$

and the corresponding squared minimum free distance is

$$d^2_{\text{free}} = 8 + 8 \left(\frac{2}{3} \right) = \frac{40}{3} = 13.33 \qquad (37)$$

---

* Again, the two distances in Eq. (35) are monotonic with ϕ and thus equating them results in the maximum value of the smaller of the two over all ϕ ϵ (0, π).

For the symmetric signal design with $\phi = \pi/2$, the length-5 path provides the smaller distance with the value

$$d^2_{free} = 8 + 8 \left(\frac{1}{2} \right) = 12 \tag{38}$$

Thus, gain due to asymmetry is

$$\eta = 10 \log_{10} \frac{40/3}{12} = 0.46 \text{ dB} \tag{39}$$

and the gains of the asymmetric and symmetric 8-state trellis coded 4-PSK system over the uncoded PSK system are

$$\eta \Big|_{asymm.} = 10 \log_{10} \frac{40/3}{4} = 5.23 \text{ dB}$$

$$\eta \Big|_{symm.} = 10 \log_{10} \frac{12}{4} = 4.77 \text{ dB} \tag{40}$$

There is another path illustrated by dashed lines in Figure 9 which corresponds to the length-6 error event "2,1,3,3,0,2". The squared distance of this path from the all zeros path is identical to that of the length-4 path found above, and thus does not change the relative gains given in Eqs. (39) and (40). One might wonder then why we even mention this path at this time. We shall see later when we discuss the signal assignment for an 8-state trellis code for 16-PSK that indeed the paths found in Figure 9 still provide the minimum distance. However, because the distances between points in a 16-point MPSK constellation are obviously not the same as in the 4-point constellation being discussed here, we shall find that there the solid line length-4 path and the dotted line length-6 path <u>do not have the same distance</u>. In fact, to get the optimum asymmetric design one must equate the distance of the length-5 path with that of the length-6 path. We shall delay further detailed discussion of this interesting point until later on. Suffice it to say that one must not be complacent with finding the <u>shortest</u> minimum distance paths for a given modulation level and assume that they also control the optimum design of a system employing the same trellis code but a different number of modulation levels. Rather, in each case, one must be certain to check all possible paths <u>of all lengths</u>.

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B.   BEST RATE 2/3 CODES COMBINED WITH ASYMMETRIC 8-PSK (A8PSK)

The signal partitioning for rate 2/3 trellis coded A8PSK is as in Figure 3. Here there are four paths that diverge from each state. Thus, one now has more flexibility as to how many parallel paths, e.g., 1, 2, or 4 should be assigned per transition between states. For the 2-state trellis the choice is somewhat obvious; thus, we shall again begin our discus- sion with this simple case.

1.   2-State Trellis

The 2-state trellis used here is exactly of the form given in Figure 5 except that now each branch represents two parallel paths (see Figure 10). The minimum free distance path is once again of length two and corresponds to the error event "2,1". Since from Figure 3 the set of squared distances from signal point 0 to signal point $j = 1,2,3,\ldots,7$ is now

$$\delta_1^2 = 4 \sin^2 \frac{\phi}{2} = 2(1 - \cos \phi); \qquad \delta_5^2 = 4 \sin^2 \left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 2(1 + \cos \phi)$$

$$\delta_2^2 = 2; \qquad\qquad\qquad\qquad \delta_6^2 = 2$$

$$\delta_3^2 = 4 \sin^2 \left(\frac{\pi}{4} + \frac{\phi}{2}\right) = 2(1 + \sin \phi); \quad \delta_7^2 = 4 \sin^2 \left(\frac{\pi}{4} - \frac{\phi}{2}\right) = 2(1 - \sin \phi)$$

$$\delta_4^2 = 4 \tag{41}$$

then the squared minimum free distance is given by

$$d_{free}^2 = \delta_2^2 + \delta_1^2 = 4 - 2 \cos \phi \tag{42}$$

which is maximized when $\phi = \pi/2$, i.e., the signal points 1, 3, 5, and 7 merge respectively with points 2, 4, 6, and 0. In this limiting case, the maximum value of Eq. (42) becomes

$$d_{free}^2 = 4 \tag{43}$$

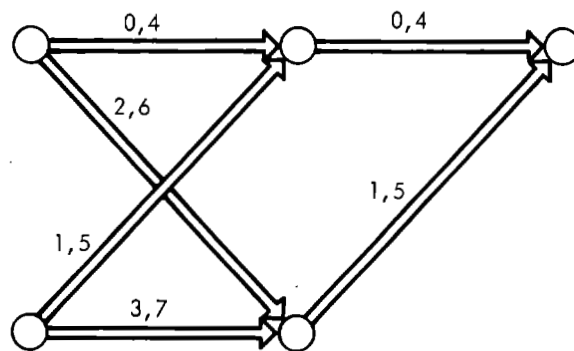Figure 10. 2-State Trellis Diagram and Signal Assignment for 8-PSK

For the symmetric 8-PSK constellation ($\phi = \pi/4$), Eq. (42) becomes

$$d^2_{free} = 4 - \sqrt{2} \tag{44}$$

Thus, the gain due to asymmetry is

$$\eta = 10 \log_{10} \frac{4}{4 - \sqrt{2}} = 1.895 \text{ dB} \tag{45}$$

Since rate 2/3 trellis coded A8PSK is equivalent in bandwidth to uncoded 4-PSK, and since the latter has $d^2_{min} = 2$, then the relative gains for the asymmetric and symmetric coded signal designs are, respectively,

$$\eta \Big|_{asymm.} = 10 \log_{10} \frac{4}{2} = 3.01 \text{ dB}$$

$$\eta \Big|_{symm.} = 10 \log_{10} \frac{4 - \sqrt{2}}{2} = 1.116 \text{ dB} \tag{46}$$

As was true for the 2-state rate 1/2 trellis coded A4PSK case, the optimum asymmetric signal design corresponds to a merger of alternate signal points in the original symmetric set. This implies that the gain due to asymmetry as dictated by Eq. (45) only translates into an equivalent E_b/N_0 gain, in the limit of infinite E_b/N_0 (zero average bit-error rate). Thus, it behooves us to investigate the practical gain achievable with asymmetry. This is done once again by finding the pair state transition diagram for the trellis, evaluating its transfer function $T(D,z)$, and differentiating this result in accordance with Eq. (8) to find an upper bound on the average bit error rate. Minimization of this bit error rate bound with respect to the asymmetry angle ϕ then results in an optimum asymmetric signal point design as a function of E_b/N_0. The details of this procedure are as follows [3].

Figure 11 illustrates the pair-state transition diagram for the rate 2/3 trellis code. The transfer function of this diagram is, by inspection, given by

$$T(D, z) = d + \frac{(a_1 + a_2) c}{1 - 2b} \tag{47}$$

Applying Eq. (8) and simplifying the algebra results in

$$P_b \leq \frac{1}{2} \frac{d}{dz} T(D, z) \Big|_{z=1} = \frac{1}{4} D^{\delta_4^2} + \frac{1}{2} \frac{D^{\delta_2^2}\left(D^{\delta_1^2} + D^{\delta_5^2}\right)\left(2 - D^{\delta_7^2}\right)}{\left(1 - D^{\delta_7^2} - D^{\delta_3^2}\right)^2} \tag{48}$$

The upper bound in Eq. (48) is implicitly a function of the asymmetry angle ϕ through the distances between signal points defined in Eq. (41). Minimizing Eq. (48) with respect to ϕ does not lead to an exact closed form expression for the optimum asymmetry angle as was possible in Eq. (18). Thus, we have elected to perform the minimization by numerical analysis with the resulting values tabulated below.

Table 2. Optimum Values of Asymmetry Angles Versus E_b/N_0

E_b/N_0, dB	ϕ (rad)
5	0.7854
6	0.9189
7	1.037
8	1.139
9	1.217
10	1.280
11	1.327

$$a_1 = \frac{1}{2}\left[D^{\delta_1^2}z^2 + D^{\delta_5^2}z\right]$$

$$a_2 = \frac{1}{2}\left[D^{\delta_1^2}z + D^{\delta_5^2}z^2\right]$$

$$b = \frac{1}{2}\left[D^{\delta_7^2}z + D^{\delta_3^2}z^2\right]$$

$$c = \frac{1}{2}\left[D^{\delta_6^2}(z+1)\right]$$

$$d = \frac{1}{2}\left[D^{\delta_4^2}z\right]$$

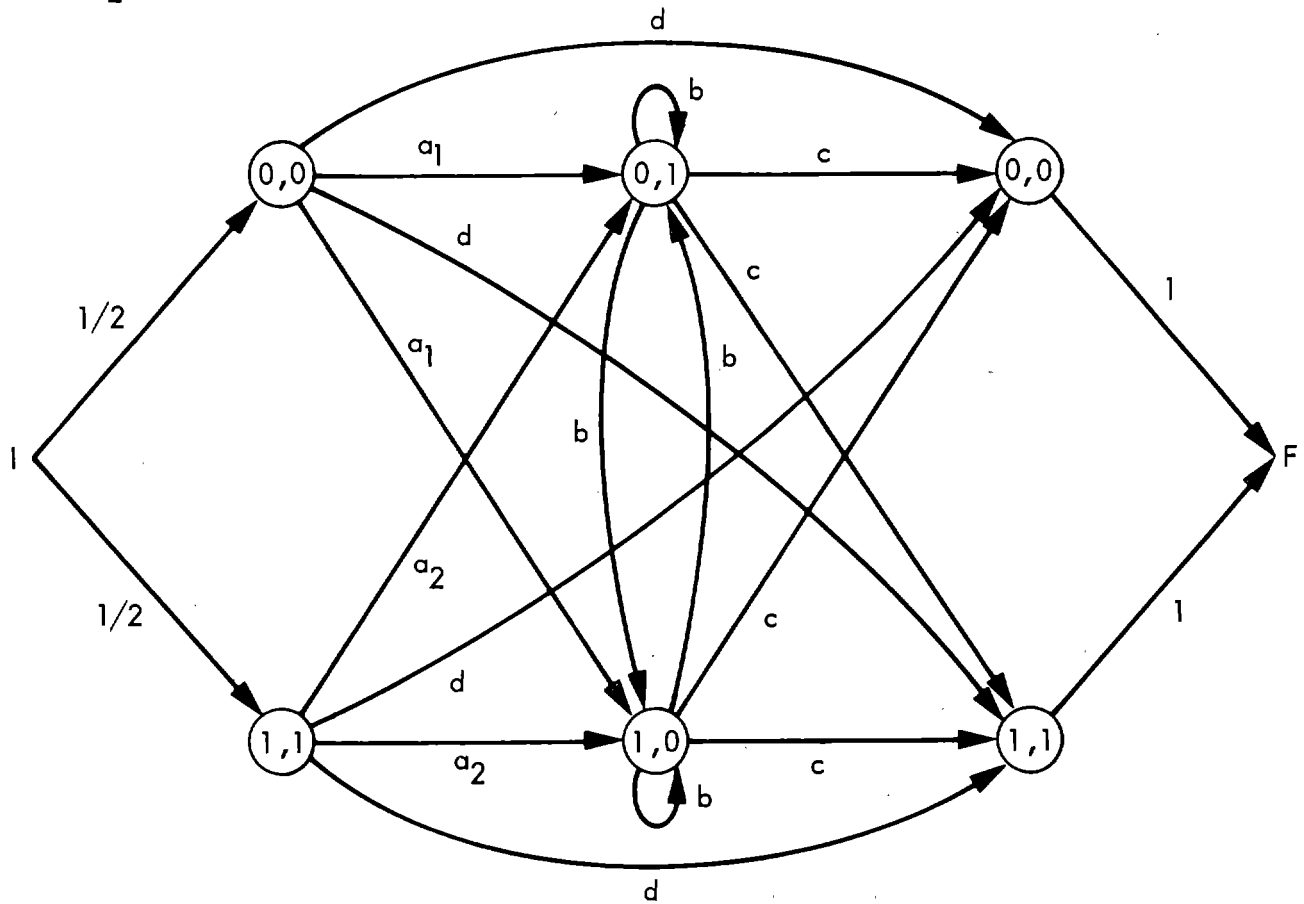Figure 11. Pair-State Transition Diagram for Rate 2/3 Trellis Code

As a check on the above, we note that, for large values of E_b/N_0, Eq. (48) can be approximated by

$$P_b \leq \frac{1}{2} \frac{D^{\delta_2^2 + \delta_1^2} \left(2 - D^{\delta_7^2}\right)}{\left(1 - D^{\delta_7^2}\right)^2}$$

(49)

Differentiating Eq. (49) with respect to ϕ and equating the result to zero leads to the transcendental equation

$$D^{2(1-\sin \phi)} = \frac{3 - \sqrt{\dfrac{\sin \phi + 9 \cos \phi}{\sin \phi + \cos \phi}}}{2} \quad ; \quad D = \exp\left(-\frac{E_b}{2N_0}\right)$$

(50)

Solutions of Eq. (50) agree extremely well with Table 2 even for moderate values of E_b/N_0.

Substituting the values of ϕ from Table 2 into Eq. (48) results in the optimum upper bound on the average bit error rate and is illustrated in Figure 12. Also illustrated in that figure is the result for the symmetric case, i.e., Eq. (48) evaluated for $\phi = \pi/4$, and the corresponding upper bound for uncoded 4-PSK (one half the result in Eq. (22)).

Before going on, we should point out that the selection of a trellis with parallel paths, as in Figure 10, limits the achievable free distance. The minimum distance path is of length one and corresponds to the transition between like states in the trellis. Thus, for 8-PSK, if one is to achieve a larger d_{free}^2 than four, i.e., the squared distance between signal points 0 and 4, then one must choose a trellis with no parallel paths. For four states we shall demonstrate that this is not possible, i.e., for any amount of asymmetry, the selection of a trellis with no parallel paths achieves a smaller d_{free}^2 than the trellis with parallel paths.
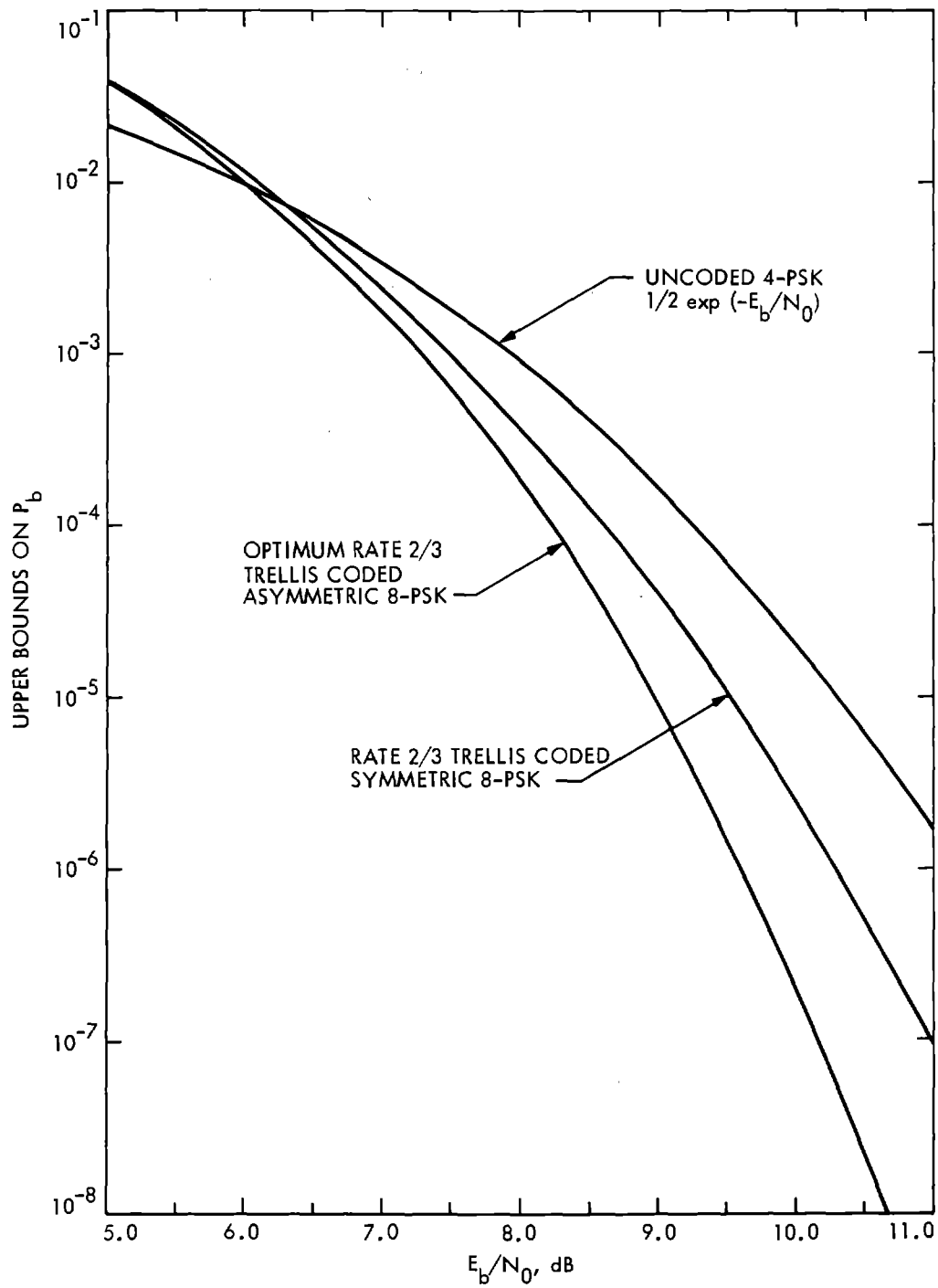
30

Figure 12. Upper Bounds on Average Bit Error Probability Performance for Rate 2/3 Trellis Coded Symmetric and Optimum Asymmetric 8-PSK

Before concluding our discussion of the 2-state case, we note that had we selected a trellis with four parallel paths between like states and no cross transitions, then d_{free}^2 would have been limited to two, i.e., the squared distance between signal points 0 and 2 (or 6). Thus, the selection made in Figure 10, which achieves a d_{free}^2 larger than two, is optimum.

2.   4-State Trellis

For four states, we can either have a trellis with two parallel paths between states or one with no parallel paths. These two possibilities and their corresponding signal point assignments are illustrated in Figure 13. The state transition matrix for the latter trellis is

$$T = \begin{bmatrix} 0 & 4 & 2 & 6 \\ 1 & 5 & 3 & 7 \\ 4 & 0 & 6 & 2 \\ 5 & 1 & 7 & 3 \end{bmatrix} \tag{51}$$

and the shortest minimum distance path is of length 3 corresponding to the MPSK output symbols "2,0,1". The squared distance of this path from the all zeros path is

$$d^2(2,0,1) = 4 - 2 \cos \phi \tag{52}$$

which for every value of ϕ between 0 and $\pi/2$ is smaller than that corresponding to any other path of any length. In the limit, Eq. (52) achieves its maximum value, i.e., $d_{free}^2 = 4$ when $\phi = \pi/2$. For the symmetric case where $\phi = \pi/4$, Eq. (52) evaluates to $d_{free}^2 = 4 - \sqrt{2}$ which is the same result as for the 2-state trellis, thus implying no gain by going to the additional complexity.

One might wonder at this point whether the selection of another signal point assignment for the trellis of Figure 13(b), still satisfying the "UEP" condition, would lead to improved results. An example of such would be the state transition matrix

32
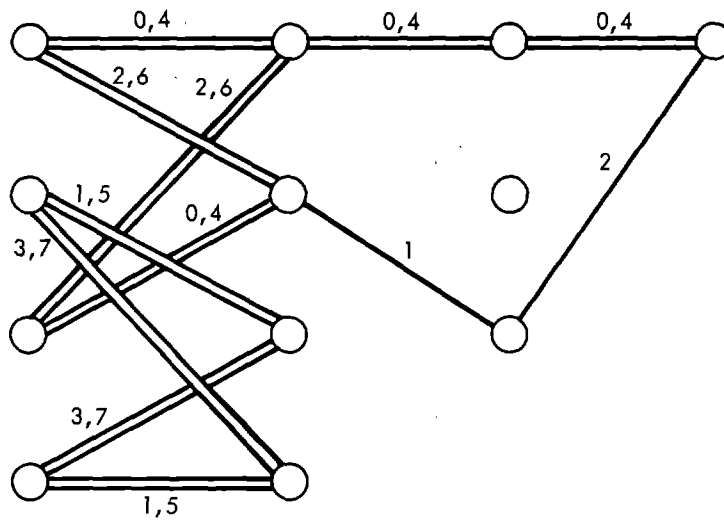
(a) With 2 Parallel Paths per Transition
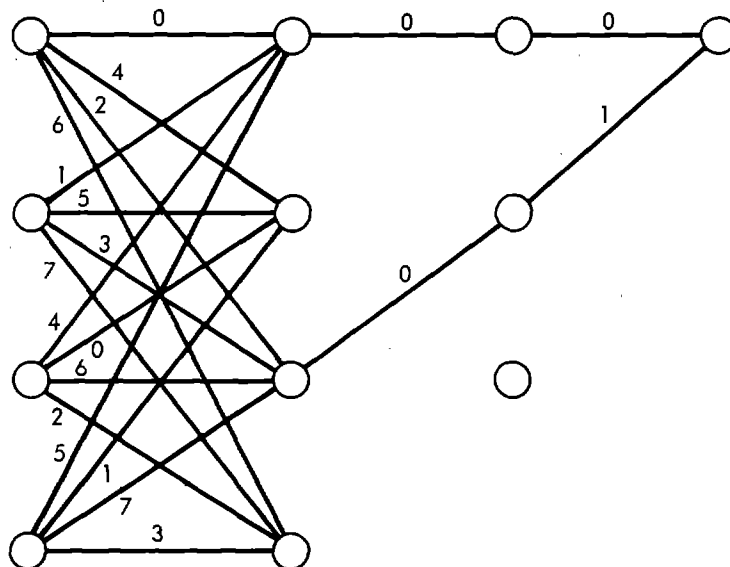
(b) With No Parallel Paths per Transition

Figure 13. 4-State Trellis Diagram

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$$T = \begin{bmatrix} 0 & 4 & 2 & 6 \\ 5 & 1 & 7 & 3 \\ 4 & 0 & 6 & 2 \\ 1 & 5 & 3 & 7 \end{bmatrix} \qquad (53)$$

For this case, the shortest minimum distance error event path is of length 2, namely, "6,1", which achieves the identical squared distance as Eq. (52). The authors have exhaustively tried many other combinations with the result that with the fully connected trellis structure of Figure 13(b) no further improvement is possible.

To show that Figure 13(a) is the preferred approach, we observe, as did Ungerboeck [2], that <u>all paths of length greater than one have a squared distance larger than four</u>. In fact, the closest to this value would be achieved by the error event path "2,1,2" with squared distance $6 - 2\cos\phi$, which is greater than four for all values of ϕ (other than $\pi/2$). In conclusion, the maximum d_{free}^2 is achieved by the 4-state trellis of Figure 13(a) and has the value of 4, <u>independent of the asymmetry angle</u>. Stated another way, for rate 2/3, 4-state trellis coded 8-PSK, there exists no gain due to asymmetry, and the gain relative to the uncoded 4-PSK case is 3.01 dB.

3. 8-State Trellis

For eight states, we again have several options of signal assignment according to whether or not there should exist parallel paths. We remind the reader that if parallel paths are assigned to the transitions, then d_{free}^2 is limited to have a value of 4, regardless of asymmetry. Thus, we should first investigate a fully connected trellis with no parallel paths and see if indeed one can achieve a larger value of free distance. In that regard, consider the 8-state trellis of Ungerboeck [2] reproduced here in Figure 14 with a state transition matrix
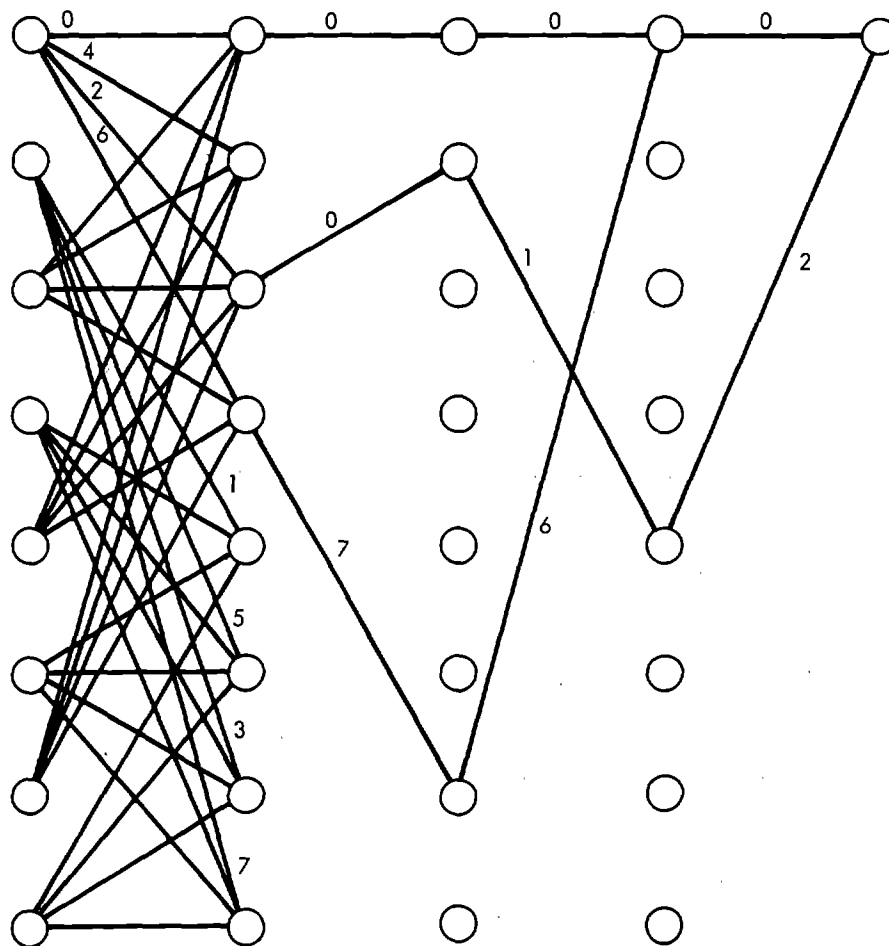
Figure 14. 8-State Trellis Code for 8-PSK

$$T = \begin{bmatrix} 0 & 4 & 2 & 6 & & & & \\ & & & & 1 & 5 & 3 & 7 \\ 4 & 0 & 6 & 2 & & & & \\ & & & & 5 & 1 & 7 & 3 \\ 2 & 6 & 0 & 4 & & & & \\ & & & & 3 & 7 & 1 & 5 \\ 6 & 2 & 4 & 0 & & & & \\ & & & & 7 & 3 & 5 & 1 \end{bmatrix} \qquad (54)$$

For this assignment the two shortest paths that, depending on the amount of asymmetry, yield the minimum distance from the all zeros path are "6,7,6" and "2,0,1,2". The squared distances for these paths are, respectively,

$$d^2(6,7,6) = 4 + 4 \sin^2 \left(\frac{\pi}{4} - \frac{\phi}{2} \right) = 6 - 2 \sin \phi$$

$$d^2(2,0,1,2) = 4 + 4 \sin^2 \frac{\phi}{2} = 6 - 2 \cos \phi \qquad (55)$$

Equating these distances and solving for ϕ, we again find that the optimum value corresponds to the symmetric constellation, namely $\phi = \pi/4$. Thus, once again there is no gain due to asymmetry.

Substituting $\phi = \pi/4$ into Eq. (55) gives

$$d^2_{free} = 6 - \sqrt{2} = 4.586 \qquad (56)$$

and a gain relative to an uncoded 4-PSK of

$$\eta = 10 \log_{10} \frac{6 - \sqrt{2}}{2} = 3.60 \text{ dB} \qquad (57)$$

36

Since d_{free}^2 of Eq. (56) is indeed larger than 4, the trellis of Figure 14 is preferred over any configuration with parallel paths assigned to the transitions.

4.   16-State Trellis

Since we have already demonstrated that an 8-state trellis with no parallel paths has a d_{free} that exceeds the maximum distance between parallel paths, it is not necessary to consider a 16-state trellis with parallel paths. Instead, we go directly to the fully connected trellis of Figure 15 as considered by Ungerboeck [2], with a state transition matrix

$$
T = \begin{bmatrix}
0\,4\,2\,6 & & & \\
 & 1\,5\,3\,7 & & \\
 & & 4\,0\,6\,2 & \\
 & & & 5\,1\,7\,3 \\
2\,6\,0\,4 & & & \\
 & 3\,7\,1\,5 & & \\
 & & 6\,2\,4\,0 & \\
 & & & 7\,3\,5\,1 \\
4\,0\,6\,2 & & & \\
 & 5\,1\,7\,3 & & \\
 & & 0\,4\,2\,6 & \\
 & & & 1\,5\,3\,7 \\
6\,2\,4\,0 & & & \\
 & 7\,3\,5\,1 & & \\
 & & 2\,6\,0\,4 & \\
 & & & 3\,7\,1\,5
\end{bmatrix}
\tag{58}
$$

For this assignment, the two shortest paths that, depending on the amount of asymmetry, yield the minimum distance from the all zeros path are "6,1,7,2" and "2,0,1,1,0,1,6". The first of these paths (the one of length 4), discovered by Ungerboeck, is concerned only with symmetric MPSK constellations. The second one, which indeed. allows a slight gain to be achieved with asymmetry, does not show up until one investigates paths of length 7. This once again emphasizes the point that paths of all lengths (up to some reasonable limit) must be looked at before deciding whether or not there can exist a gain due to asymmetry.
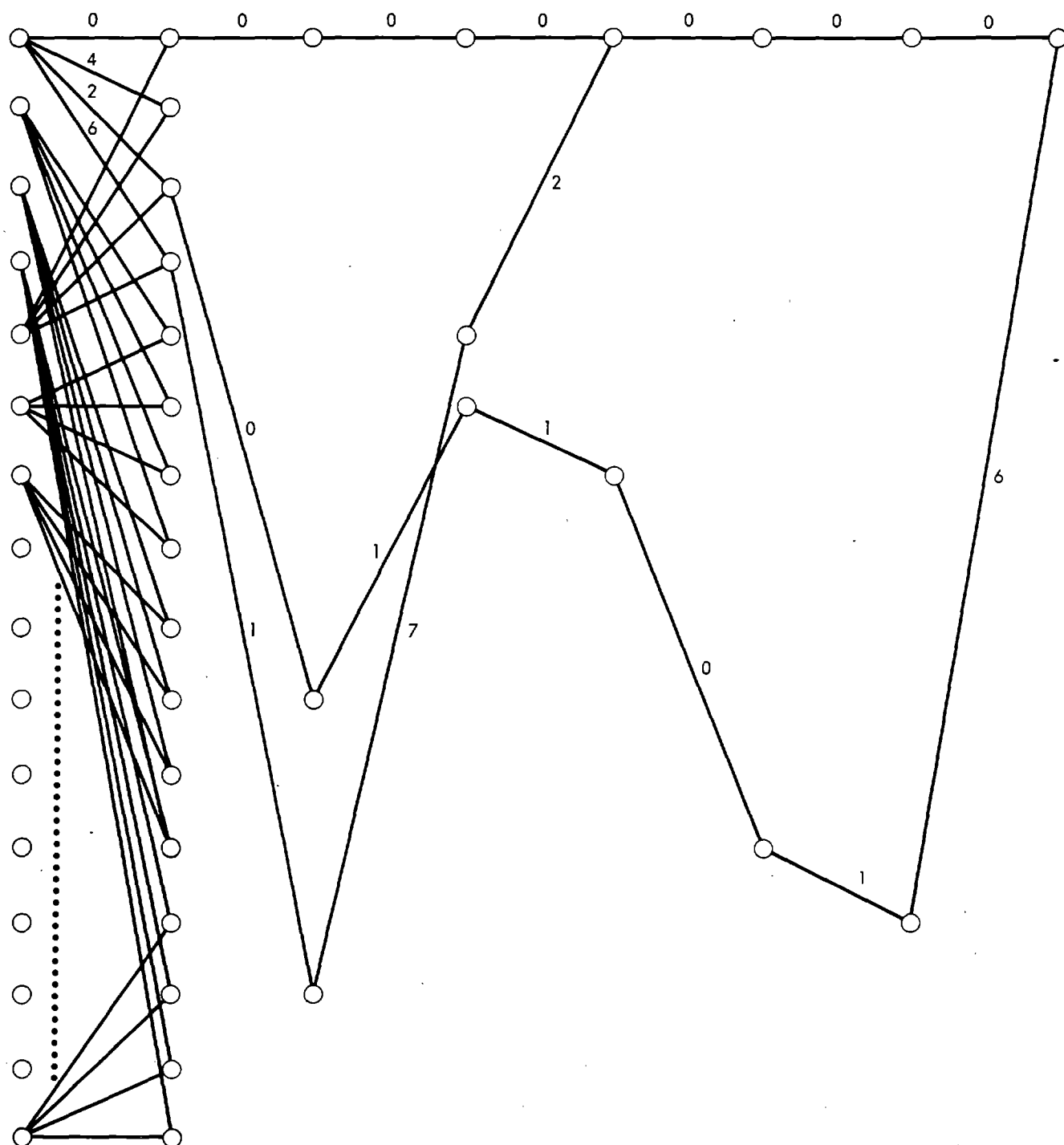
Figure 15. 16-State Trellis Code for 8-PSK

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The squared distances for the above two paths are, respectively,

$$d^2(6,1,7,2) = 8 - 2 (\sin \phi + \cos \phi)$$

$$d^2(2,0,1,1,0,1,6) = 10 - 6 \cos \phi \tag{59}$$

Equating these two distances gives the optimum asymmetric 16-PSK design corresponding to

$$\cos \phi = \frac{4}{5} \; ; \; \phi = 0.6435 \text{ rad}$$

$$d^2_{free} = \frac{26}{5} = 5.20 \tag{60}$$

It should also be pointed out that the length-8 path "6,7,0,0,0,7,7,6", which has the squared distance

$$d^2(6,7,0,0,0,7,7,6) = 10 - 6 \sin \phi \tag{61}$$

can be used to determine an alternate optimum asymmetric 16-PSK constellation with $\phi = \pi/2 - 0.6435$ rad and the same value of d_{free}.

The gain due to asymmetry is

$$\eta = 10 \log_{10} \frac{26/5}{8 - 2\sqrt{2}} = 0.024 \text{ dB} \tag{62}$$

and the gains relative to uncoded 4-PSK are

$$\eta \Big|_{asymm.} = 10 \log_{10} \frac{26/5}{2} = 4.15 \text{ dB}$$

$$\eta \Big|_{symm.} = 10 \log_{10} \frac{8 - 2\sqrt{2}}{2} = 4.126 \text{ dB} \tag{63}$$

While the gain due to asymmetry is so small as to be only of academic interest, it nevertheless points out the curiosity that, while asymmetry provided

no advantage with 4- and 8-state trellises, a theoretical gain is once again achievable when the complexity is increased to 16 states. Again, it should be emphasized that, despite its slight positive impact on system performance, the gain due to asymmetry comes free of charge.

C.   BEST RATE 3/4 CODES COMBINED WITH ASYMMETRIC 16-PSK (A16PSK)

The signal partitioning for trellis coded A16PSK follows the same steps as those leading to the partitionings in Figures 3 and 4. For a rate 3/4 code, there will be four transitions leaving (diverging from) each state. As before, we begin with the simple 2-state case. Our discussion herein will be brief since by now the reader should be thoroughly familiar with the procedure for picking a good signal assignment and when to have or not have parallel paths along the transitions.

   1.   2-State Trellis

The 2-state trellis for A16PSK is identical in form to that in Figure 5 except that now each branch represents four parallel paths. In particular, the transitions between like states correspond to signals 0,4,8,12 and 3,7,11,15, respectively, while the cross transitions correspond to 2,6,10,14 and 1,5,9,13. The minimum distance path is of length 2 and corresponds to the error event "2,1". The set of squared distances from signal point 0 to signal point $j = 1,2,3,\ldots,15$ is now

$$\delta_1^2 = 4 \sin^2 \frac{\phi}{2} = 2 (1 - \cos \phi); \quad \delta_9^2 = 4 \sin^2 \left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 2 (1 + \cos \phi)$$

$$\delta_2^2 = 4 \sin^2 \frac{\pi}{8} = 2 - \sqrt{2} = \delta_{14}^2; \quad \delta_{11}^2 = 4 \sin^2 \left(\frac{3\pi}{8} - \frac{\phi}{2}\right) = 2 \left[1 - \cos \left(\frac{3\pi}{4} - \phi\right)\right]$$

$$\delta_3^2 = 4 \sin^2 \left(\frac{\pi}{8} + \frac{\phi}{2}\right) = 2 \left[1 - \cos \left(\frac{\pi}{4} + \phi\right)\right]; \quad \delta_{13}^2 = 4 \sin^2 \left(\frac{\pi}{4} - \frac{\phi}{2}\right) = 2 (1 - \sin \phi)$$

$$\delta_4^2 = 4 \sin^2 \frac{\pi}{4} = 2 = \delta_{12}^2; \quad \delta_{15}^2 = 4 \sin^2 \left(\frac{\pi}{8} - \frac{\phi}{2}\right) = 2 \left[1 - \cos \left(\frac{\pi}{4} - \phi\right)\right]$$

$$\delta_5^2 = 4 \sin^2 \left(\frac{\pi}{4} + \frac{\phi}{2} \right) = 2 (1 + \sin \phi)$$

$$\delta_6^2 = 4 \sin^2 \frac{3\pi}{8} = 2 + \sqrt{2} = \delta_{10}^2$$

$$\delta_7^2 = 4 \sin^2 \left(\frac{3\pi}{8} + \frac{\phi}{2} \right) = 2 \left[1 - \cos \left(\frac{3\pi}{4} + \phi \right) \right]$$

$$\delta_8^2 = 4 \sin^2 \frac{\pi}{2} = 4 \tag{64}$$

Thus, the squared free distance is given by

$$d_{free}^2 = \delta_2^2 + \delta_1^2 = 2 - \sqrt{2} + 2 (1 - \cos \phi) \tag{65}$$

which is maximized when $\phi = \pi/4$, i.e., signal points 1,3,5,7,9,11,13,15 merge with points 0,2,4,6,8,10,12,14, respectively. In this limiting case, the maximum value of Eq. (65) becomes

$$d_{free}^2 = 4 - 2\sqrt{2} = 1.172 \tag{66}$$

while for the symmetric case ($\phi = \pi/8$), Eq. (65) evaluates to

$$d_{free}^2 = 4 - \sqrt{2} - 2 \cos \pi/8 = 0.738 \tag{67}$$

Thus, the gain due to asymmetry is

$$\eta = 10 \log_{10} \frac{1.172}{0.738} = 2.01 \text{ dB} \tag{68}$$

and the gains relative to the equivalent bandwidth, uncoded 8-PSK system are

$$\eta \bigg|_{asymm.} = 10 \log_{10} \frac{1.172}{2 - \sqrt{2}} = 3.01 \text{ dB}$$

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$$\eta \Big|_{symm.} = 10 \ log_{10} \ \frac{0.738}{2 - \sqrt{2}} = 1.00 \ dB \tag{69}$$

where we have made use of the fact that the latter has $d^2_{min} = 2 - \sqrt{2}$.

We note that for all 2-state cases considered, the total gain of the trellis coded asymmetric MPSK constellation over the uncoded M/2-point one is 3.01 dB. Indeed, this can be shown to be always true independent of M.

2.   4-State Trellis

The 4-state trellis for A16PSK has the structure of Figure 13(a) and is illustrated in Figure 16. Unlike the A8PSK case, the minimum distance is not determined by the length-one path between like states, i.e., there exist paths with length greater than one whose distance from the all zeros path is less than the minimum distance among the parallel transitions. In particular, the squared minimum distance among parallel paths is determined by signal points 4 or 12 and has a value of 2 (see Eq. (64)). The paths "2,1,2" and "2,15,15,2", depending on the value of ϕ, yield the optimum asymmetric design, which, as we shall see shortly, has a value of d^2_{free} which is less than two but still larger than that corresponding to a symmetric constellation.

From Eq. (65), we can determine the squared distances of the above two paths as

$$d^2(2,1,2) = 6 - 2\sqrt{2} - 2 \ cos \ \phi$$

$$d^2(2,15,15,2) = 8 - 2\sqrt{2} - 4 \ cos \ \left(\frac{\pi}{4} - \phi\right) \tag{70}$$

which when equated give the optimum A16PSK design with

$$tan \ \frac{\phi}{2} = 1 - \sqrt{2 - \sqrt{2}} \ \Rightarrow \ \phi = 0.46 \ rad$$

$$d^2_{free} = 1.38 \tag{71}$$

42
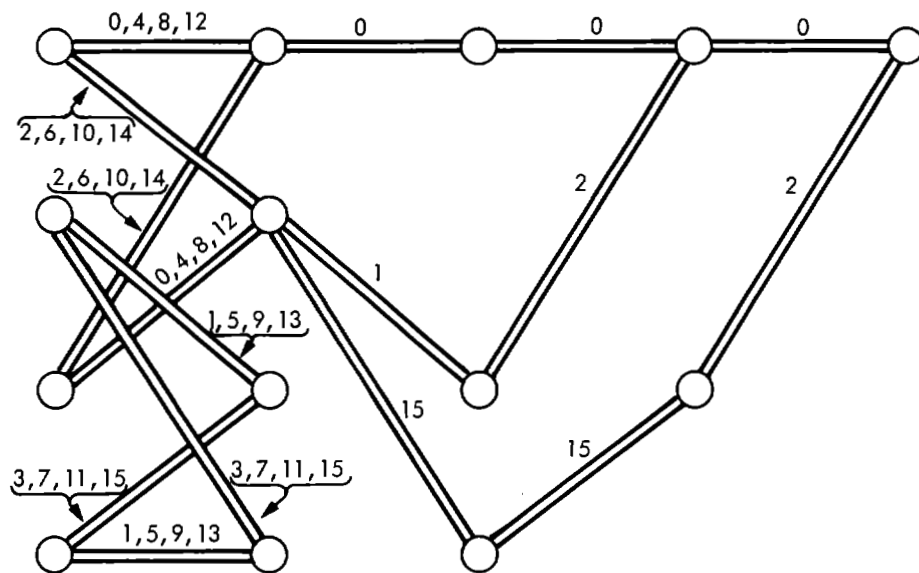
Figure 16. 4-State Trellis Diagram With 4 Parallel Paths per Transition

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For the symmetric case, the path "2,1,2" has the shorter distance which from Eq. (70) becomes

$$d^2_{free} = 1.324 \qquad (72)$$

Thus, the gain due to asymmetry is

$$\eta = 10 \log_{10} \frac{1.38}{1.324} = 0.18 \text{ dB} \qquad (73)$$

and the gains relative to the uncoded 8-PSK system are

$$\eta \Big|_{asymm.} = 10 \log_{10} \frac{1.38}{2 - \sqrt{2}} = 3.72 \text{ dB}$$

$$\eta \Big|_{symm.} = 10 \log_{10} \frac{1.324}{2 - \sqrt{2}} = 3.54 \text{ dB} \qquad (74)$$

3.   8-State Trellis

The 8-state trellis for A16PSK is as illustrated in Figure 9, except that the signal assignments are now defined by the state transition matrix

$$T = \begin{bmatrix} C0 & C2 & & & & & & \\ & & C1 & C3 & & & & \\ & & & & C2 & C0 & & \\ & & & & & & C3 & C1 \\ C2 & C0 & & & & & & \\ & & C3 & C1 & & & & \\ & & & & C0 & C2 & & \\ & & & & & & C1 & C3 \end{bmatrix} \qquad (75)$$

$$C0 = 0,4,8,12 \quad ; \quad C2 = 2,6,10,14$$
$$C1 = 1,5,9,13 \quad ; \quad C3 = 3,7,11,15$$

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Since we are only interested in determining the minimum distance paths through the trellis, we can simplify Eq. (75) by considering only the signal points which are the minimum distance from signal point 0. As such, the "reduced" state transition matrix becomes

$$
T = \begin{bmatrix}
0 & 2 & & & & & & \\
& & 1 & 15 & & & & \\
& & & & 2 & 0 & & \\
& & & & & & 15 & 1 \\
2 & 0 & & & & & & \\
& & 15 & 1 & & & & \\
& & & & 0 & 2 & & \\
& & & & & & 1 & 15
\end{bmatrix}
\tag{76}
$$

The minimum distance paths are still the three paths illustrated in Figure 9 which, using Eq. (65), now have the distance

$$
d^2(2,1,2,2) = 8 - 3\sqrt{2} - 2 \cos \phi
$$

$$
d^2(2,15,15,0,2) = 8 - 2\sqrt{2} - 4 \cos \left(\frac{\pi}{4} - \phi \right)
$$

$$
d^2(2,1,0,1,15,0,2) = 10 - 2\sqrt{2} - 4 \cos \phi - 2 \cos \left(\frac{\pi}{4} - \phi \right)
\tag{77}
$$

We note that, unlike the 8-PSK case, the length-4 (solid) and the length-7 (dashed) paths do not have the same distance. (This point was made during our discussion of trellis coded A8PSK and is now obvious from Eq. (77).) In fact, the length-7 path is, for all values of ϕ, closer in distance to the all zeros path. Thus, to find the optimum asymmetric design, we equate the distance of the lengths 5 and 7 paths which results in

$$
\tan \frac{\phi}{2} = 0.1637 \quad \Rightarrow \quad \phi = 0.3244 \text{ rad}
$$

$$
d^2_{free} = 1.589
\tag{78}
$$

and a gain due to asymmetry of

$$\eta = 10 \, \log_{10} \frac{1.589}{1.476} = 0.319 \, dB \tag{79}$$

Finally, the gains relative to an uncoded 8-PSK are

$$\eta \Big|_{asymm.} = 10 \, \log_{10} \frac{1.589}{2 - \sqrt{2}} = 4.333 \, dB$$

$$\eta \Big|_{symm.} = 10 \, \log_{10} \frac{1.476}{2 - \sqrt{2}} = 4.014 \, dB \tag{80}$$

One could obviously conceive of many different signal assignments for the trellis of Figure 16. For example, another good choice would be the state transition matrix.

$$T = \begin{bmatrix} C0 & C2 & & & & & & \\ & & C2 & C0 & & & & \\ & & & & C1 & C3 & & \\ & & & & & & C3 & C1 \\ C2 & C0 & & & & & & \\ & & C0 & C2 & & & & \\ & & & & C3 & C1 & & \\ & & & & & & C1 & C3 \end{bmatrix} \tag{81}$$

It can be easily shown that here the minimum distance paths are "2,0,15,15,2", "2,2,1,2", and "2,0,1,1,15,2" which lead to an asymmetric design with the identical T matrix as in Eq. (76).

4.   16-State Trellis

For 16 states, Wilson, Schottler, and Sleeper [6] have found a trellis code that leads to an optimum coding gain when combined with a symmetric 16-PSK constellation. In particular, there are two parallel paths per transition between states (thus each state diverges to two other states) and the signal assignment is characterized by the state transition matrix

$$
T = \begin{bmatrix}
C0 & C2 & & & & & & & & & & & & & & \\
& C2 & C0 & & & & & & & & & & & & & \\
& & C1 & C3 & & & & & & & & & & & & \\
& & & C3 & C1 & & & & & & & & & & & \\
& & & & C0 & C2 & & & & & & & & & & \\
& & & & & C2 & C0 & & & & & & & & & \\
& & & & & & C1 & C3 & & & & & & & & \\
& & & & & & & C3 & C1 & & & & & & & \\
C2 & C0 & & & & & & & & & & & & & & \\
& & C0 & C2 & & & & & & & & & & & & \\
& & & & C3 & C1 & & & & & & & & & & \\
& & & & & C1 & C3 & & & & & & & & & \\
& & & & & & & C2 & C0 & & & & & & & \\
& & & & & & & & C0 & C2 & & & & & & \\
& & & & & & & & & & C3 & C1 & & & & \\
& & & & & & & & & & & C1 & C3 & & &
\end{bmatrix}
\qquad (82)
$$

or its "reduced" version (keeping only the minimum distance parallel path)

$$
T = \begin{bmatrix}
0 & 14 & & & & & & & & & & & & & & \\
& 14 & 0 & & & & & & & & & & & & & \\
& & 1 & 15 & & & & & & & & & & & & \\
& & & 15 & 1 & & & & & & & & & & & \\
& & & & 0 & 14 & & & & & & & & & & \\
& & & & & 14 & 0 & & & & & & & & & \\
& & & & & & 1 & 15 & & & & & & & & \\
& & & & & & & 15 & 1 & & & & & & & \\
14 & 0 & & & & & & & & & & & & & & \\
& & 0 & 14 & & & & & & & & & & & & \\
& & & & 15 & 1 & & & & & & & & & & \\
& & & & & 1 & 15 & & & & & & & & & \\
& & & & & & & 14 & 0 & & & & & & & \\
& & & & & & & & 0 & 14 & & & & & & \\
& & & & & & & & & & 15 & 1 & & & & \\
& & & & & & & & & & & 1 & 15 & & &
\end{bmatrix}
\qquad (83)
$$

For this assignment, the shortest (depending on the amount of asymmetry) minimum distance error event paths are "14,0,15,15,0,15,0,14" (length 8) and "14,0,15,1,0,0,1,0,14" (length 9) with distances from the all zeros path of

$$
d^2(14,0,15,15,0,15,0,14) = 10 - 2\sqrt{2} - 6 \cos\left(\frac{\pi}{4} - \phi\right)
$$

$$
d^2(14,0,15,1,0,0,1,0,14) = 10 - 2\sqrt{2} - 2 \cos\left(\frac{\pi}{4} - \phi\right) - 4 \cos\phi \qquad (84)
$$

At $\phi = \pi/8$ (the symmetric 16-PSK constellation), the two paths have equal values, namely,

$$d^2 = 10 - 2\sqrt{2} - 6 \cos \frac{\pi}{8} = 1.628 \tag{85}$$

We note that as ϕ approaches zero, the length-9 path approaches the value

$$d^2 = 6 - 3\sqrt{2} = 1.757 \tag{86}$$

which is smaller than that of the length-8 path, but larger than the symmetric value of Eq. (85). Thus, one might jump (erroneously) to the conclusion that there exists a gain due to asymmetry of an amount determined from Eq. (86) relative to Eq. (85).

The reason why this conclusion is fallacious stems from the fact that the squared distance of the length-9 path, as given by the second relation in Eq. (84), is not a _monotonic_ function of ϕ. As a result, the crossover point ($\phi = \pi/8$) of the two functions in Eq. (84) does not necessarily yield the maximum of the smaller of the two distances over all values of ϕ. In fact, we have just observed that a larger value exists in accordance with Eq. (86). Thus, to properly determine whether or not asymmetry increases d_{free}, one must see if there exist other (longer) paths whose distance function may cross that of the length-9 path at a point where the distance from the all zeros path is smaller than Eq. (85).

The length-16 path "14,0,1,1,1,1,0,1,0,1,1,0,0,1,0,14" has a squared distance function given by

$$d^2 = 20 - 2\sqrt{2} - 16 \cos \phi \tag{87}$$

which is monotonically increasing with increasing values of ϕ. Since Eq. (87) evaluated at $\phi = 0$ has the value $4 - 2\sqrt{2}$, which is less than Eq. (86), we have the potential of a path as described above. Indeed, equating Eq. (87) with the second relation in Eq. (84) results in a crossover at $\phi = 0.226$ rad with a squared distance of 1.578. Since, this value is indeed smaller than that corresponding to the symmetric design as given by Eq. (85), then we may now make the correct conclusion that the optimum design is the symmetric one. Stated another way, the smallest of the squared distances of all three paths evaluated at each ϕ, never exceeds Eq. (85).

SECTION III

CONCLUSION

Introducing an appropriate amount of asymmetry into the constellation design of a combined modulation/trellis coding system is, under most circumstances, a cost-effective means of improving its performance. For MPSK modulation, we have shown that for low coding complexity, quite a bit of performance improvement is achievable relative to the equivalent symmetric design. As the coding complexity increases (as measured by the number of states in the trellis diagram), the amount to be gained by asymmetry typically diminishes; however, the overall improvement of the asymmetric coded system, relative to the equivalent bandwidth uncoded M/2-level system, continues to increase.

The specific numerical results obtained within the body of the paper are summarized for quick reference in Table 3. Finally, we point out that all of the numerical results derived within and summarized in Table 3 have been verified by direct numerical evaluation of Eq. (13) together with Eq. (9), with perfect agreement in all cases.

Table 3. Performance of Rate n/(n + 1) Trellis Coded MPSK Versus Uncoded M/2-PSK

No. of Mod. Levels $M=2^{n+1}$	No. of Parallel Transitions Between States	No. of States in Trellis	d^2_{min} for Uncoded Mod. With M/2 Levels	d^2_{free} for Coded Symmetric	d^2_{free} for Coded Asymmetric	ϕ_{opt} (rad)	Gain (dB) of Symm. Coded MPSK Over Uncoded M/2-PSK	Gain (dB) of Asymm. Coded MPSK Over Symm. Coded MPSK
4	1	2	4	6	8	π	1.76	1.25
4	1	4	4	10	$32/3 = 10.67$	1.91	3.98	0.28
4	1	8	4	12	$40/3 = 13.33$	1.23	4.77	0.46
8	2	2	2	$4 - \sqrt{2} = 2.586$	4	$\pi/2$	1.116	1.895
8	2	4	2	4	4	$\pi/4$	3.01	0.00
8	1	8	2	$6 - \sqrt{2} = 4.586$	$6 - \sqrt{2} = 4.586$	$\pi/4$	3.60	0.00
8	1	16	2	$8 - 2\sqrt{2} = 5.172$	$26/5 = 5.20$	0.6435	4.13	0.024
16	4	2	$2 - \sqrt{2}$	0.738	1.172	$\pi/4$	1.00	2.01
16	4	4	$2 - \sqrt{2}$	1.324	1.380	0.46	3.54	0.18
16	4	8	$2 - \sqrt{2}$	1.476	1.589	0.3244	4.014	0.319
16	4	16	$2 - \sqrt{2}$	1.628	1.628	$\pi/8$	4.440	0.00

SECTION IV

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