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LONG-TIME ASYMPTOTICS OF A SYSTEM FOR PLASMA DIFFUSION

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16 Abstract

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LONG TIME ASYMPTOTICS OF A SYSTEM FOR PLASMA DIFFUSION

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ABSTRACT

We analyze a system of parabolic nonlinear equations that describe the diffusion of a fully collisional plasma across a strong magnetic field. We demonstrate that the solution to this system tends to a time asymptotic state which is of space-time separable form, $\phi(t)f(x)$. Furthermore, $f(x)$ is independent of the initial conditions and $\phi(t)$ depends slightly on the initial conditions. The rate of decay of the temporal part is governed by a nonlinear eigenvalue problem. Since the equations are considered in a bounded domain we are able to analyze the effect of boundary conditions on the evolution of the system. Additional effects as radiation, heating, and particle injection can also be accounted for. Essential differences between the behavior of a fully-coupled system and a scalar equation are observed.

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Derivation of Equations

We consider the problem of radial diffusion and transport of a magnetically confined fully collisional plasma. We are interested in the long term behavior of the solution subject to boundary conditions and various forcing functions. We verify numerically that the long time asymptotics are actually achieved within several transit times. Thus, the solution quickly evolves into a universal pattern independent of initial conditions.

In order to simplify the equations, we consider the case that the ions and electrons have the same temperature. In this case the mass and energy diffusion tensor terms are essentially the one-temperature Braginskii equations. Rosenbluth and Kaufman[1] present some special solutions for the equations in an infinite domain. In this study we shall, in contrast, only consider bounded domains. Indeed, the effect of boundary conditions is very important. We ignore topological changes of the plasma and assume that the plasma extends to the wall.

The resultant equations are

$$n_t + \nabla\cdot(n u_d) = S_p \tag{1a}$$

$$P_t + \frac{7}{6}\nabla\cdot(p u_d) = \frac{2}{3}\nabla\cdot(K\nabla\kappa T) + S_E. \tag{1b}$$

We consider the equations in either a Cartesian or cylindrical frame of reference. We then igonore all spatial derivatives except in the x or radial direction. The diffusion drift velocity u_d is given by

$$u_d = \frac{-\eta_\perp c^2}{B^2}\left(\nabla p - \frac{3}{4}\nabla \kappa T\right)$$

$$\eta_\perp = \eta_0 (T/T_0)^{-3/2}$$

$$K = \frac{1}{4}\left(\frac{2m_i}{m_e}^{1/2} \frac{n^2 c^2 \kappa T}{B^2}\right)\eta_\perp$$

where n is the total (ion plus electron) particle density number, κT is the temperature in ev, $\eta_\perp$ is the perpendicular resistivity and B is the magnetic field which is assumed to be constant and in the z direction.

We first consider these equations in slab symmetry. We shall see later that the changes introduced by considering cylindrical symmetry are minor. Temporarily ignoring the source terms S_p, S_E and using normalized units, (1) can be written as

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left[D_\perp p^{-1/3}\right]\frac{\partial}{\partial x}(np^{1/3}) \; , \quad D_\perp = \left(\frac{n^3}{p}\right)^{1/2} \tag{2a}$$

$$\frac{\partial p}{\partial t} = \frac{2}{3} K_1 \frac{\partial}{\partial x}\left[D_\perp n^\alpha \frac{\partial}{\partial x}(pn^{-\alpha})\right] \tag{2b}$$

where

$$K_1 = \frac{1}{3}\left(K_0 + \frac{7}{4}\right), \quad K_0 = \left(\frac{2m_i}{m_e}\right)^{1/2}, \quad \alpha = 1 - \frac{7}{3K_1} \; . \tag{2c}$$

It follows from (2c) that α and K_1 both depend on the ratio of ion mass to electron mass given by K_0. We shall consider two specific cases:

$$\text{Hydrogen:} \quad K_0 \simeq 60.6, \quad K_1 \simeq 20.78, \quad \alpha \simeq 0.888 \tag{3a}$$

$$\text{Deuterium:} \quad K_0 \simeq 86, \quad K_1 \simeq 29.25, \quad \alpha \simeq 0.920. \tag{3b}$$

The system (2) is augmented by initial conditions

$$n(x,0) = n_0(x), \qquad p(x,0) = p_0(x). \tag{4}$$

We consider (2) in the interval $0 \leq x \leq 1$. At $x = 0$ we impose symmetry conditions while at the edge $x = 1$ we impose convective boundary conditions. Thus, we have

$$\left.\begin{aligned} n_x + h_1 n = 0 \\ p_x + h_2 p = 0 \end{aligned}\right\} \qquad \text{at} \quad x = 1, \tag{5a}$$

$$p_x = n_x = 0 \qquad \text{at } x = 0. \tag{5b}$$

The boundary conditions at the edge (5a) simulate a plasma-wall interaction. It follows from (5) that the temperature T at the wall satisfies $T_x = (h_1 - h_2)T$. Hence, $h_1 > h_2$ implies heat deposition while $h_1 < h_2$ implies heat injection at the plasma edge.

We can also rewrite (4) as a quasilinear system of equations

$$\frac{\partial}{\partial t}\binom{n}{p} = \frac{\partial}{\partial x}\left[D_\perp A \frac{\partial}{\partial x}\binom{n}{p}\right] \tag{6}$$

where

$$D_\perp = \left(\frac{n^3}{p}\right)^{1/2}$$

and

$$A = \begin{pmatrix} 1 & n/3p \\ \frac{-2\alpha K_1}{3} \cdot \frac{p}{n} & \frac{2}{3} K_1 \end{pmatrix} .$$

We note that the eigenvalues of A are constant and strictly positive. From (3) we see that K_1 is a relatively large number between 20 and 30. Considering the case of K_1 large, the eigenvalues of A are

$$a_1 = \frac{2}{3} K_1 + O(1)$$

and

$$a_2 = \frac{4}{3} + O(1/K_1).$$

We conjecture that the long term behavior of the system (2) - (5) is given by a time separable form independent of the initial conditions (4). Thus, we conjecture that

$$\lim_{t\to\infty} n(x,t) = \phi_1(t)N(x)$$
$$\lim_{t\to\infty} p(x,t) = \phi_2(t)P(x). \tag{7}$$

Numerical experiments indicate that these asymtotic states are reached quite quickly. For single nonlinear diffusion equations, this asymptotic behavior is well known.[2] It is also valid for a 2×2 system when the diffusion tensor is diagonal. In this study we indicate that it is still valid for a 2×2 system with a nondiagonal tensor term.

2. Lypanov Functional

In this section we construct a Lypanov functional for (2) - (5) which is a strictly decreasing function of time. Hence, in the absence of forcing functions n(x,t) must approach zero. In some cases we can simplify the algebra by assuming that K_1 is large; however, the results are true for all $K_1 > 0$.

Let

$$Q = \left(n^2 p^{-1/K_1}\right)^{\sigma}, \qquad \sigma > 0; \tag{8}$$

then

$$\frac{d}{dt}\int_0^1 Q dx = -2\sigma\int_0^1 \frac{D_\perp Q}{n^2}\left\{\left[2\sigma\left(1+\frac{\alpha}{3}\right)-1\right]n_x^2 + \frac{1}{3}\left(\frac{n}{p}\right)^2 p_x^2 - \frac{1}{3}\left[1+\alpha+\frac{\sigma(3+\alpha)}{K_1}\right]\left(\frac{n}{p}\right)n_x p_x\right\}dx + 2\sigma\left(1+\frac{\alpha}{3}\right)\cdot\left[\frac{D_\perp Q n_x}{n}\right]_0^1 .$$

By the Cauchy-Schwarz inequality we have that

$$\left(\frac{n}{p}\right) n_x p_x < \frac{\varepsilon}{2} n_x^2 + \frac{1}{2\varepsilon} p_x^2 .$$

We choose

$$\varepsilon = \frac{1}{3}\left[1+\alpha+\frac{\sigma(3+\alpha)}{K_1}\right] > \frac{\alpha+1}{3} > 0.$$

It then follows that

$$\frac{d}{dt}\int_0^1 Q\,dx <$$

$$- 2\sigma \int_0^1 D_\perp Q \left\{\frac{1}{n^2}\left[2\sigma\left(1+\frac{\alpha}{3}\right) - \frac{\alpha + 7 + (3+\alpha)\sigma/K_1}{6}\right] n_x^2 + \frac{p_x^2}{6p^2}\right\} dx$$

$$+ 2\sigma\left(1+\frac{\alpha}{3}\right)\left[\frac{D_\perp Qn_x}{n}\right]_0^1 . \tag{9}$$

We, therefore, require that

$$2\sigma\left(1+\frac{\alpha}{3}\right) > \frac{\alpha + 7 + (3+\alpha)\sigma/K_1}{6} .$$

Since by (2c), $\alpha = 1 - 7/3K_1$, we require that

$$\sigma > \frac{8 - 7/3K_1}{16 - 10/K_1 + 7/3\ K_1^2} = \frac{1}{2}\ 1 + \frac{1}{3K_1} + O(K_1^2). \tag{10}$$

We next treat the boundary terms using (5). We then conclude that

$$\frac{d}{dt}\int_0^1 Q\,dx < - 2\sigma \int_0^1 D_\perp\ Q\ \frac{A}{n^2}\ n_x^2 + \frac{1}{6p^2}\ p_x^2\ dx$$

$$- 2\sigma(1+\alpha)D_\perp(1)Q(1)h_1 \tag{11}$$

where

$$Q = \left(\frac{n^2}{p^{1/K_1}}\right)^\sigma, \qquad \sigma > 0, \qquad D_\perp = \left(\frac{n^3}{p}\right)^{1/2}$$

$$A = 2\sigma\ 1 + \frac{\alpha}{3}\ - \frac{\alpha + 7 + (3+\alpha)\sigma/K_1}{6} > 0, \tag{12}$$

and σ also satisfies the inequalities (10).

Note:

(1) It is not necessary to choose the exponent of p to be $-1/K_1$ in the definition of Q. In fact, one can choose the exponent so that the coefficient of $n_x\ p_x$ in (7) is exactly zero. For large K_1 this gives an exponent of the order of $-1/K_1 + O(1/K_1^2)$.

(2) The boundary term depends only on h_1 and not on h_2, i.e., the decay of Q is affected by mass deposition but not by pressure boundary conditions. If the exponent of p in Q is not exactly $-1/K_1$, then h_2 does enter into the energy inequality but with a coefficient of order $1/K_1$.

We further note that the right-hand side of (11) depends on spatial derivatives of n and p. Hence, as long as n and p are not constant, the average of Q must approach zero in the absence of source terms. Since σ is arbitrary, subject only to the inequality (10), and the exponent of p need not be exactly $-1/K_1$ (see Note 1), we conclude that n and p must decay to zero.

3. Decay of Solution

In the previous section we were able to get a priori estimates on the decay of the functional Q, (8), (11). In order to get further information on the decay rates, we now assume that the time-separable conjecture (6) is true. More exactly we set

$$n(x,t) = \phi_1(t)N(x)$$
$$p(x,t) = \phi_2(t)P(x) \tag{13}$$

and substitute this hypothesis into (2). As usual we introduce two separation constants λ_1, λ_2. We then find that

$$\frac{d\phi_1}{dt} = -\lambda_1 \phi_1^{2.5} \phi_2^{-1/2}$$
$$\frac{d\phi_2}{dt} = -\lambda_2 \phi_1^{1.5} \phi_2^{1/2} \tag{14}$$

and also

$$\frac{d}{dx}\left[D_1 P^{-1/3}\right]\frac{d}{dx}(NP^{1/3}) + \lambda_1 N = 0$$
$$K_1 \frac{d}{dx}\left[D_1 N^{\alpha} \frac{d}{dx}(PN^{-\alpha})\right] + \frac{3}{2}\lambda_2 P = 0, \tag{15}$$

with boundary conditions

$$\left.\begin{aligned} \frac{dN}{dx} + h_1 N = 0 \\ \frac{dP}{dx} + h_2 P = 0 \end{aligned}\right\} \qquad \text{at} \quad x = 1, \tag{16a}$$

$$\frac{dN}{dx} = \frac{dP}{dx} = 0, \qquad \text{at } x = 0, \tag{16b}$$

where $D_1 = N^{3/2} P^{-1/2}$.

The solution to (14) is given by

$$\phi_2 = \phi_0 \, \phi_1^{\Omega}, \qquad \phi_0 = \text{constant} \tag{17a}$$

and

$$\phi_1 = (t_0 + \lambda_* \, \omega t)^{-1/\omega}, \qquad t_0 = \text{constant} \tag{17b}$$

where

$$\Omega = \lambda_2/\lambda_1, \; \omega = (3 - \Omega)/2, \qquad \lambda_* = \lambda_1/\sqrt{\phi_0} \; . \tag{17c}$$

It follows that the temporal behavior depends on the ratio of the two separation constants. It is easily seen that if one changes scales then the values of λ_1 and λ_2 change but the ratio Ω is invariant. Hence, Ω is essentially a nonlinear eigenvalue. The value of Ω is determined by the global existence of a solution. In practice Ω is found by numerically integrating the system (2) and then calculating the rate of decay.

It is shown in reference 3 there are four critical values for Ω

$$\Omega = 3, \; 7/6, \; 1, 0.$$

For $\Omega > 3$ it follows that $\omega < 0$ and the diffusion process terminates in a finite time. For $\Omega = 3$ we have exponential decay while for $\Omega < 3$ we have algebraic decay. The value $\Omega = 7/6$ is a bifurcation between processes for which heat is added to the system through the boundary. Whenever $h_2 > h_1$ then heat enters the system and $\Omega > 7/6$; conversely, when $h_2 < h_1$ then heat leaves the system and $\Omega < 7/6$. Furthermore, if $1 < \Omega < 7/6$ then all the state variables decay and the temperature has an inverted profile. When $0 < \Omega < 1$ then there is a thermal instability, and the temperature grows in time even though the pressure and density decay. Finally, when $\Omega < 0$, then both the temperature and the pressure grow while the density decays.

4. Computational Results

In this section we present some numerical results for the system (2). All the computations were performed using an explicit four-stage Runge-Kutta type scheme described in reference 3. For the all the graphs used we have defined K_0 as appropriate for hydrogen (3a). Results for deuterium (3b) have been similar. Having fixed K_0 the main free parameters are h_1 and h_2 that appear in the boundary conditions at the edge (5a).

In Figure 1 we present a case in cylindrical coordinates with $h_1 = 10$, $h_2 = 20$, and so $h_1 < h_2$. A sequence of five times are plotted for density, pressure, and temperature. The plots are at time steps 1, 100, 250, 500, 900. The initial profile is linear for the density and exponential for the pressure. Other runs indicate that the asymptotic state is independent of the initial profile. We see that the solution changes very fast near the plasma edge. After 1000 time steps which correspond to a time of about .05 the asymptotic state is almost reached. In these plots the quantities are normalized to be 1 at the center, $x = 0$. Hence, we only see the spatial portion of the solution and have normalized out the decaying temporal portion of the solution. We see that the approach of the pressure to the steady state is not monotonic.

In Figure 2 we have increased h_1 to 21 without changing h_2 so that now $h_1 > h_2$. Comparing Figures 1 and 2 we see that the pressure now approaches its asymptotic form more slowly than in the previous case. The final profiles for the density and pressure look similar to Figure 1, but now the temperature has an inverted profile. It is interesting to note that the temperature approaches its asymptotic form more rapidly than before. We have also run this case in Cartesian coordinates and have included radiation effects and the graphs look similar to Figure 2.

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3. P. Rosenau and E. Turkel, to appear Phys. Fluids, 1985.

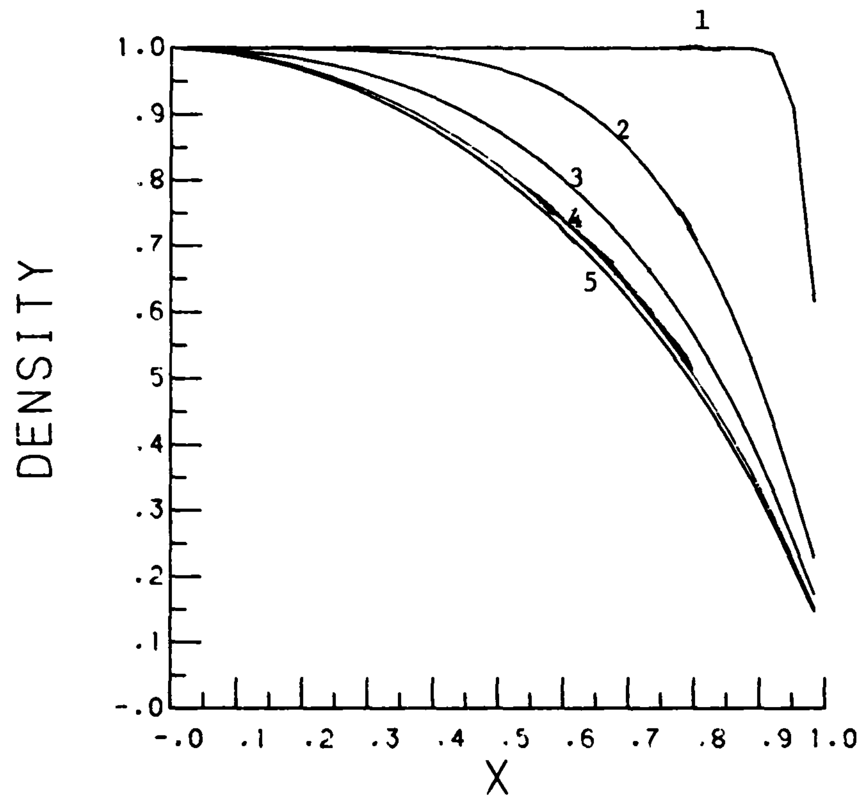

Figure 1a. Normalized plots of density, pressure, and temperature with $h_1 = 10$, $h_2 = 20$ for hydrogen in axisymmetric cylindrical coordinates. The plots are shown at time steps:

1 - 1	4 - 500
2 - 100	5 - 900
3 - 250	

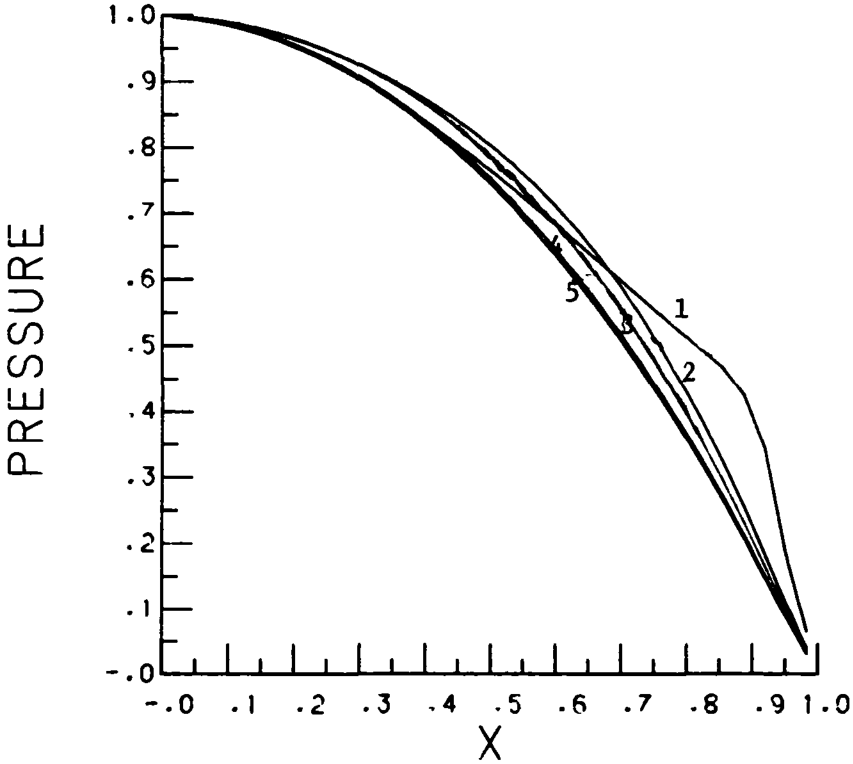

Figure 1b.

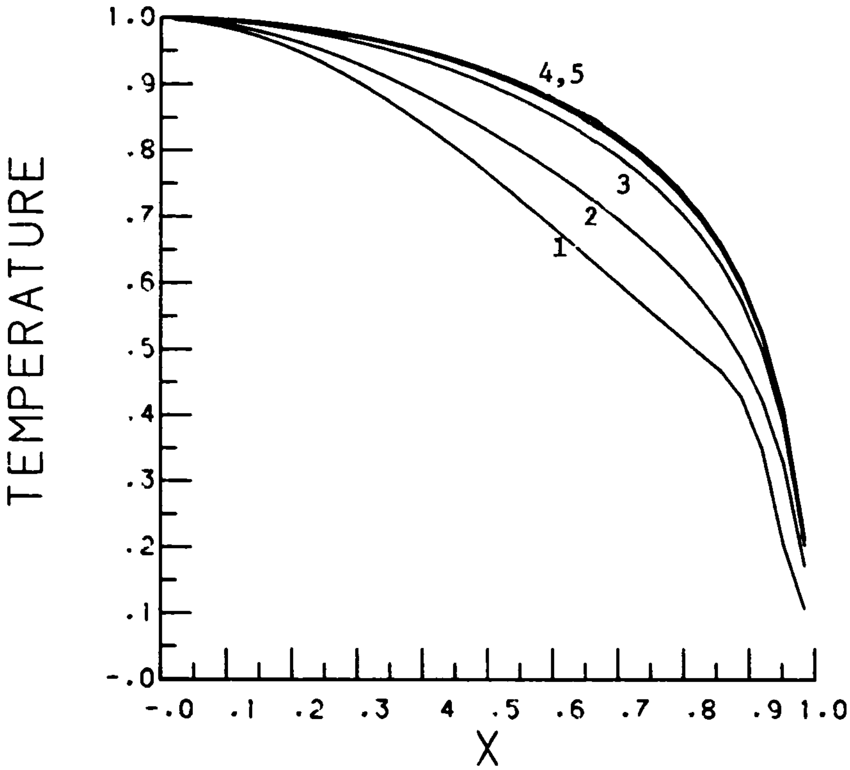

Figure 1c.

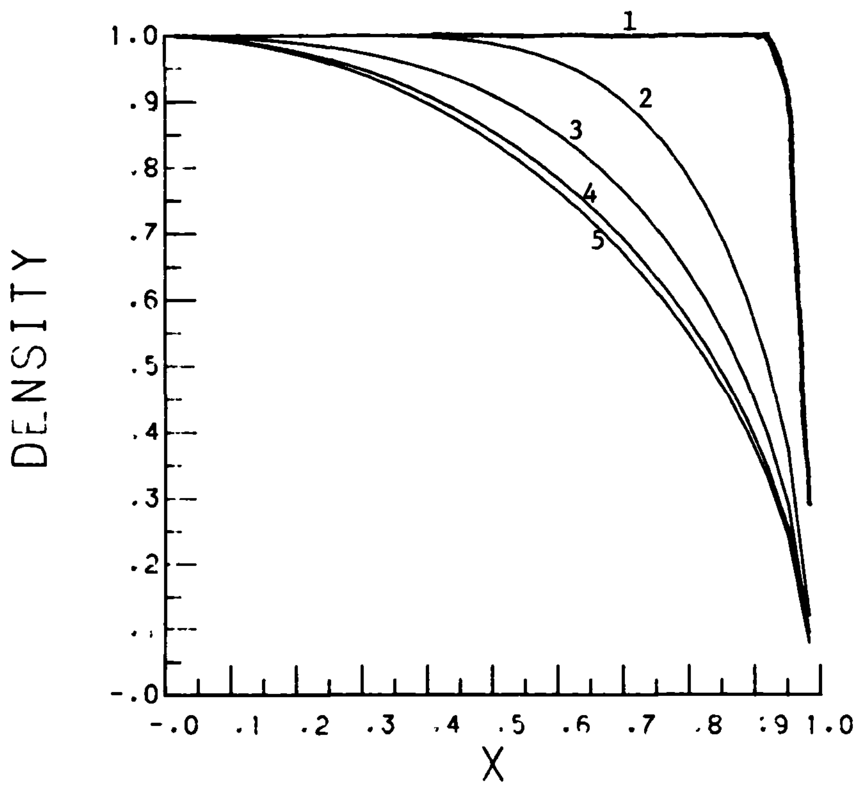

Figure 2a. Normalized plots of density, pressure, and temperature with $h_1 = 21$, $h_2 = 20$ in axisymmetric cylindrical coordinates. Same time steps as in Figure 1.

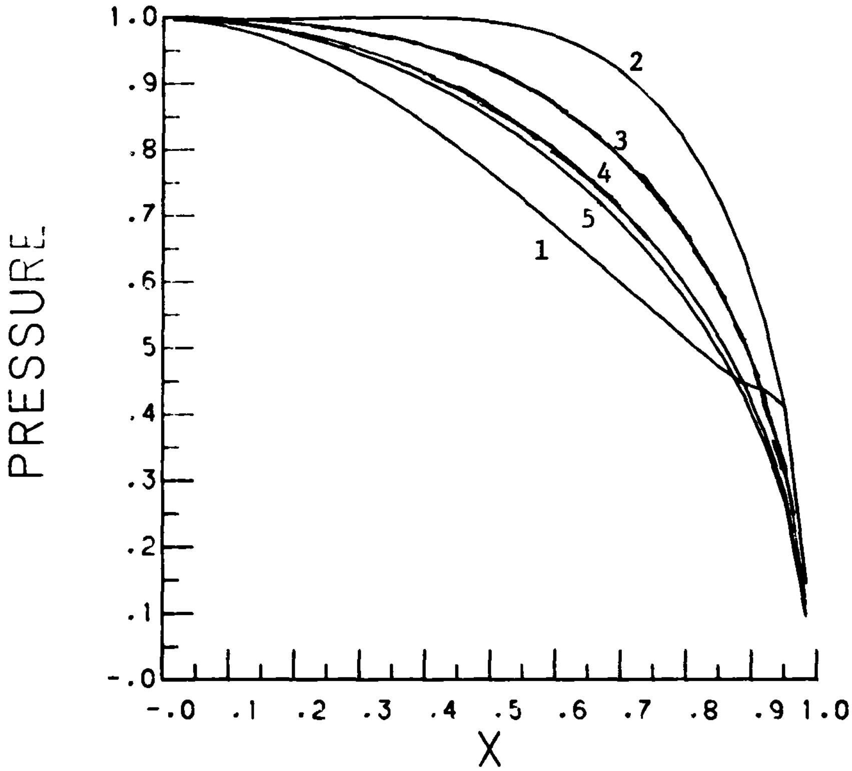

Figure 2b.

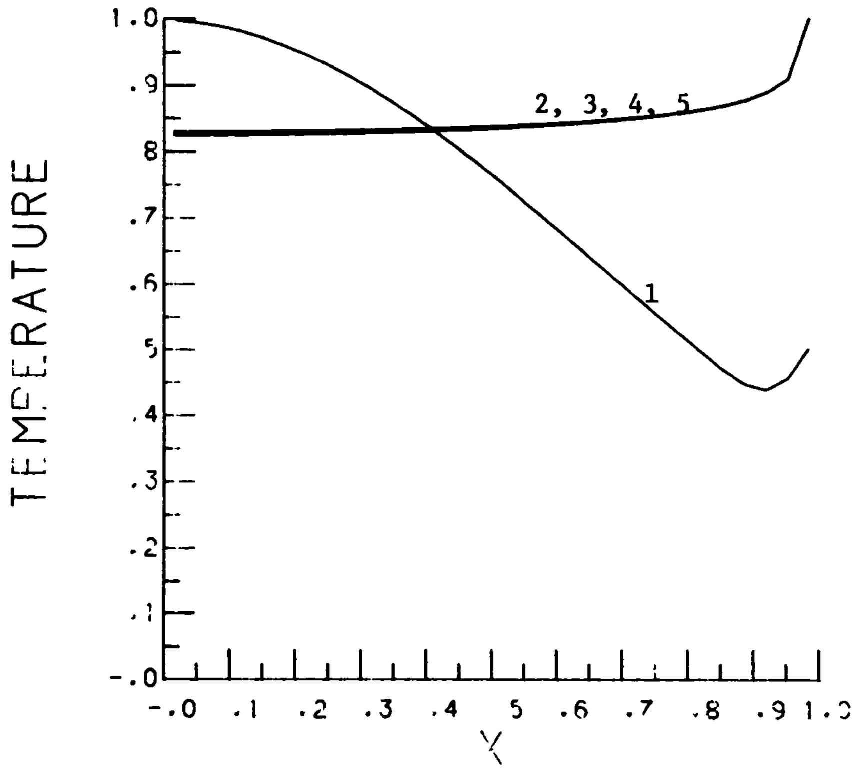

Figure 2c.

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