Rapid Estimation of Frequency Response Functions by Close-Range Photogrammetry

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NASA

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Contents

Summary

The accuracy is computed of a computational procedure for rapid estimation of frequency response functions from stereoscopic dynamic data. It is shown that reversal of the order of the operations of coordinate transformation and Fourier transformation, which provides a significant increase in computational speed, introduces error. A portion of the error can be made arbitrarily small by prescaling frequency data prior to coordinate transformation. The remainder of the error, which cannot be eliminated, results from components of motion normal to the camera focal planes. Spectral analysis of the dominant error term shows the presence of harmonic and cross-modulated spectral peaks caused by the nonlinearity of the perspective projection from spatial coordinates onto focal plane coordinates. Only when the camera geometry is such that all motion is parallel to both camera planes is the spatial transformation linear and no error introduced.

The full 3×3 frequency response matrix of a vibrating structure can be estimated from a single stereoscopic dynamic data set only if the driving function consists of three orthogonal mutually independent stochastic processes. Otherwise three independent sets of separate measurements are necessary. A least-squares procedure for estimation of the full frequency response matrix, column by column, for the latter case is developed.

Introduction

Close-range photogrammetric techniques have application to remote measurement of displacement in aerospace structural testing. Through the use of these techniques, the motion of multiple targets on a vibrating surface can be recorded by means of a stereoscopic digital camera system as a pair of two-dimensional (or equivalently, a four-dimensional) sampled-data time series. Originally the four-dimensional coordinates of each stereoscopic data point, recorded for each target at each sampling time, were transformed into three-dimensional spatial coordinates by means of the direct linear transformation (DLT) algorithm (ref. 1). Cross power spectral matrices were then estimated in spatial coordinates through the use of averaged discrete Fourier transformation (DFT). Kroen and Tripp (ref. 2) showed that the operations of coordinate transformation (DLT) and frequency transformation (DFT) may be commuted subject to certain restrictions, i.e., limitation of displacement normal to camera focal planes. The computation time per point required for Fourier transformation and averaging is negligible compared with that for coordinate transformation, which is required only for each averaged data point in the DFT-DLT transformational sequence. Thus, reversal of the order of operations produces a significant computational time savings (typically a factor of 125, ref. 2) at the cost, however, of error due to the nonlinearity of the direct linear transformation whenever components of motion exist normal to either camera focal plane.

This paper derives an analytical upper bound on the magnitude of the error incurred by the DFT-DLT operation sequence as a function of test geometry and perturbation amplitude. A procedure using the DFT-DLT transformational sequence is also developed for least-squares estimation of the general 3×3 frequency response matrix.

Symbols

$\mathbf{A}$	projection matrix defined in equation (20)
$\mathbf{A_j}.$	jth row of matrix $\mathbf{A}$
a	damping rate; power series coefficient; general scalar element
a_{ij}	element of matrix $\mathbf{A}$
a_λ	defined in equation (111)
$\mathbf{B}$	general matrix

b_i	ratio defined in equation (107)
$b_{nm}(t)$	power series coefficient
b_λ	defined in equation (112)
C	maximum perturbation magnitude defined in equation (74)
$\mathbf{C}(\delta\mathbf{w})$	perturbation matrix defined in equation (24)
$\mathbf{C}[F(\delta\mathbf{w})]$	matrix defined in equation (50)
c	camera focal length
c_{mk}, c'_{mk}	power series coefficients
D	mapping defined in equation (28)
D_A	mapping defined in equation (21)
$\mathbf{D}_z(t)$	diagonal matrix defined in equation (70)
d	power series coefficient
$d_j(t)$	power series defined in equation (118)
d_λ	defined in equation (113)
E	expected value
$\mathbf{E}(\omega)$	error defined in equation (35)
$\mathbf{E}_K(\omega)$	error defined in equation (63)
$\mathbf{e}_F(t)$	error defined in equation (65)
$e_j(t)$	defined in equation (116)
$\mathbf{e}_K(t)$	error defined in equation (66)
$F[f(t)]$	Fourier transformation of function $f(t)$
$f(t)$	perturbation function
$g(t)$	driving function
$\mathbf{H}(\omega)$	frequency response matrix
$\mathbf{h}$	vector in OBJ coordinates
$h_{ii}(\omega)$	ith diagonal element of $\mathbf{H}(\omega)$
$\hat{h}_{ii,j}(\omega)$	defined in equation (157)
$\mathbf{h}_{.j}(\omega)$	jth column of $\mathbf{H}(\omega)$
h_z	defined in equation (58)
$\mathbf{I}$	identity matrix
i	$\sqrt{-1}$; integer
j, m, n, r, q	integers
K	scale factor
$\mathbf{M}$	rotation matrix defined in equation (1)
$\mathbf{M}^{1,2}$	rows 1 and 2 of matrix $\mathbf{M}$

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$\mathbf{M}^3$	row 3 of matrix $\mathbf{M}$
$\mathbf{M}_\varsigma$	composite matrix defined in equation (13)
N	number of averaged records
$\mathbf{P_A}$	generalized inverse of matrix $\mathbf{A}$ defined in equation (23)
$\mathbf{P}_n$	constant matrix
$\mathbf{R_{UU}}(\tau)$	autocovariance matrix
$\mathbf{R_{WU}}(\tau)$	cross covariance matrix
$\mathbf{S_{UU}}(\omega), \mathbf{S_{uu}}(\omega)$	autospectral matrices
$\mathbf{S_{WU}}(\omega), \mathbf{S_{wu}}(\omega)$	cross spectral matrices
$s_{gg}(\omega)$	autopower spectrum of $g(t)$
$\hat{\mathbf{s}}_{\mathbf{uu}_{.j}}(\omega)$	jth column of $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$
T	record length, seconds
t	time, seconds
$\mathbf{U}_k$	constant vector
$\delta\mathbf{U}(t)$	input driving vector in object space
$\delta\mathbf{U}(\omega)$	$= F[\delta\mathbf{U}(t)]$
$\mathbf{u},\mathbf{v}$	general vectors
$\delta\mathbf{u}(t)$	input driving vector in DFP space
v_i	ith element of $\mathbf{v}$
W	point in object space
$\mathbf{W}$	coordinates of point W
W'	focal plane image of W
$W_F(t)$	normalized magnitude of $\delta\mathbf{W}(t)$
$\delta\mathbf{W}(t)$	time-varying perturbation in object space
$\delta\mathbf{W}(\omega)$	$= F[\delta\mathbf{W}(t)]$
$\mathbf{w}$	point in DFP space
$\delta\mathbf{w}(t)$	time-varying perturbation in DFP space
X_c, Y_c, Z_c	object space coordinates
X_p, Y_p, Z_p	object space coordinates of perspective center
x, y, z	focal plane coordinates
$\mathbf{Z}$	diagonal matrix defined in equation (19a)
$\delta\mathbf{Z}$	diagonal perturbation matrix defined in equation (19b)
$\delta\mathbf{Z}_F[F(\delta\mathbf{w})]$	diagonal matrix defined in equation (40)
α, β	constants
$\mathbf{\Gamma}_k$	matrix defined in equations (162), (165), and (167)

$\hat{\gamma}_k$	least value of $\gamma_k(t)$
γ_{kj}	element of $\boldsymbol{\Lambda}_k$
$\gamma_k(t)$	angle between $\delta\mathbf{W}(t)$ and ς_k-axis
δ	perturbation symbol
ϵ_h	RMS error defined in equation (173)
ϵ_M	maximum relative perturbation defined in equation (73)
ϵ_{RMS}	relative RMS error defined in equation (82)
ϵ_u	RMS error defined in equation (171)
ϵ_z	maximum negative relative perturbation defined in equation (78)
ς_M	vector defined in equation (11)
θ, ϕ, ψ	orientation angles (fig. 3)
$\boldsymbol{\Lambda}_k$	$= \mathbf{Z}^{-1}\mathbf{A}\boldsymbol{\Gamma}_k$
λ	eigenvalue of $(\mathbf{A}^T\mathbf{A})^{-1}$
$\boldsymbol{\lambda}_{.j}$	jth column of $\boldsymbol{\Lambda}_k$
ν	constant defined in equation (53)
ξ, η, ς	perspective center coordinates
$\boldsymbol{\Sigma}_{\mathbf{uu}}(\omega)$	transformed autospectral matrix
$\boldsymbol{\Sigma}_{\mathbf{wu}}(\omega)$	transformed cross spectral matrix
σ	condition number of matrix $\mathbf{A}$ defined in equation (69)
$\sigma_{\mathbf{uu}_{ij}}(\omega), \sigma_{\mathbf{wu}_{ij}}(\omega)$	elements of $\boldsymbol{\Sigma}_{\mathbf{uu}}(\omega)$ and $\boldsymbol{\Sigma}_{\mathbf{wu}}(\omega)$
$\boldsymbol{\sigma}_{\mathbf{wu}_{.j}}(\omega)$	jth column of $\boldsymbol{\Sigma}_{\mathbf{wu}}(\omega)$
τ	time
$\boldsymbol{\Upsilon}$	PC coordinates of point W
$\boldsymbol{\Upsilon}'$	focal plane image of $\boldsymbol{\Upsilon}$
ω	frequency, rad/sec
ω_0	damped frequency, rad/sec

Subscripts:

k	$k = 1$ for camera 1; $k = 2$ for camera 2
p	perspective center
1	camera 1
2	camera 2

Superscripts:

T	matrix transpose
*	complex conjugate

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Abbreviations:

DFP	dual focal plane
DFT	discrete Fourier transformation
DLT	direct linear transformation
FP	focal plane
max	maximum
OBJ	object
PC	perspective center
RMS	root mean square

Mathematical notation:

$\triangleq$	is defined to be
$\| \ \|$	vector or matrix norm

A caret ($\hat{\ }$) over a symbol denotes an estimated value.

Development of Projection Equations

The three-dimensional position of a vibrating target point W is remotely sensed as a function of time by means of a stereoscopic camera system which employs two cameras focused on W, as shown in figure 1. The image of W, denoted by W', appearing in each camera focal plane is modeled as a
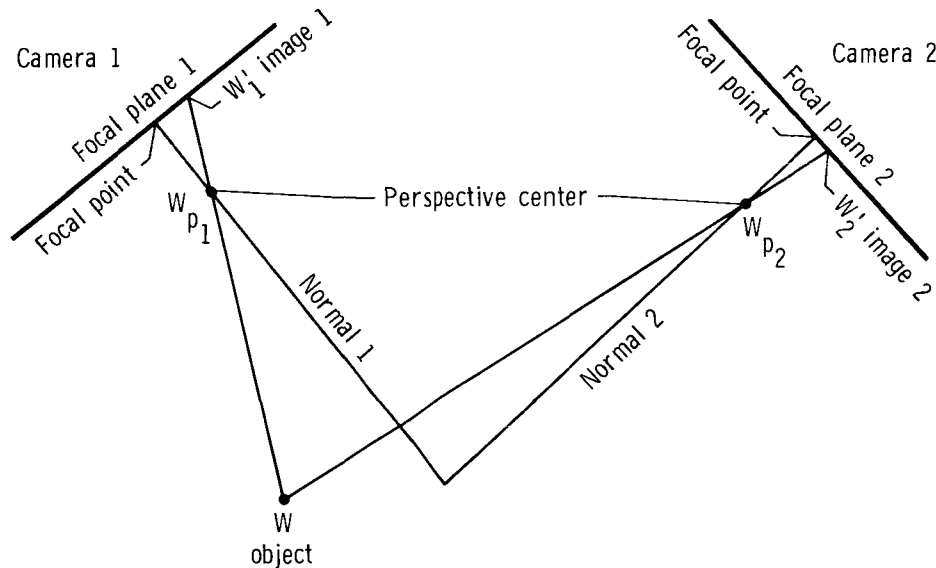

Figure 1. Stereoscopic camera geometry showing images of object W on camera focal planes.

perspective projection. That is, W projects onto the point of intersection of the camera focal plane with the line passing from W through the persective center, W_p (the lens center, ref. 3). For the perspective projection geometry, as shown in figure 2, define three sets of spatial coordinates: object (OBJ) coordinates X_c, Y_c, Z_c; perspective center (PC) coordinates ξ, η, ς whose origin is the perspective center; and focal plane (FP) coordinates x, y, z whose origin lies on the focal plane. The PC and FP coordinate systems are parallel and displaced from one another by focal length c in the z-direction, but they may be rotated
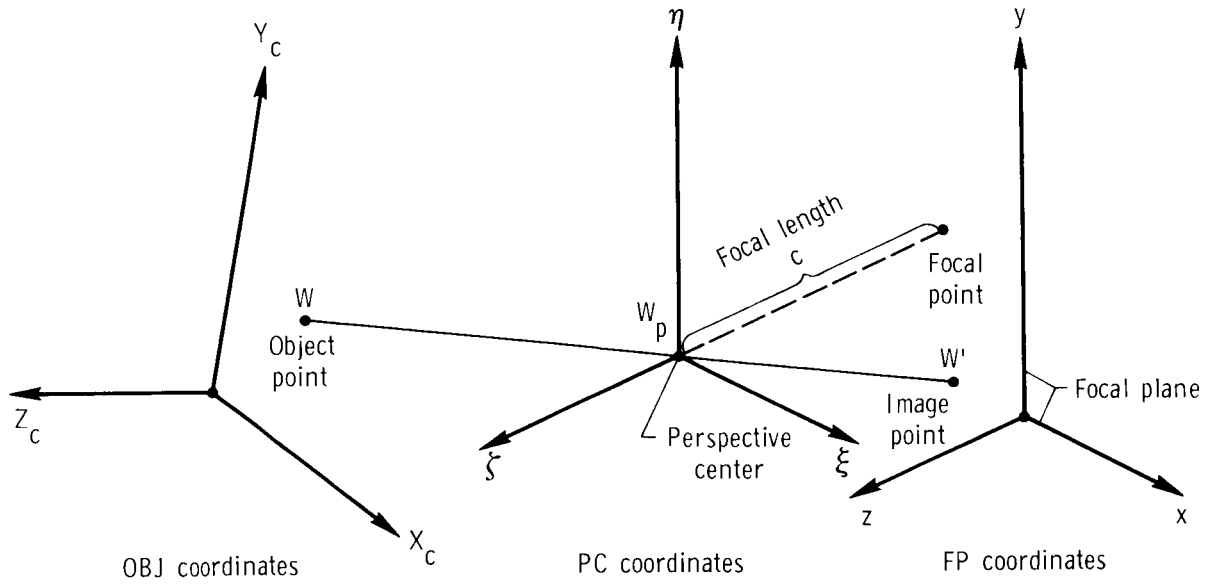
Figure 2. Perspective projection coordinate systems showing projection of object W through perspective center onto image point W' in focal plane.
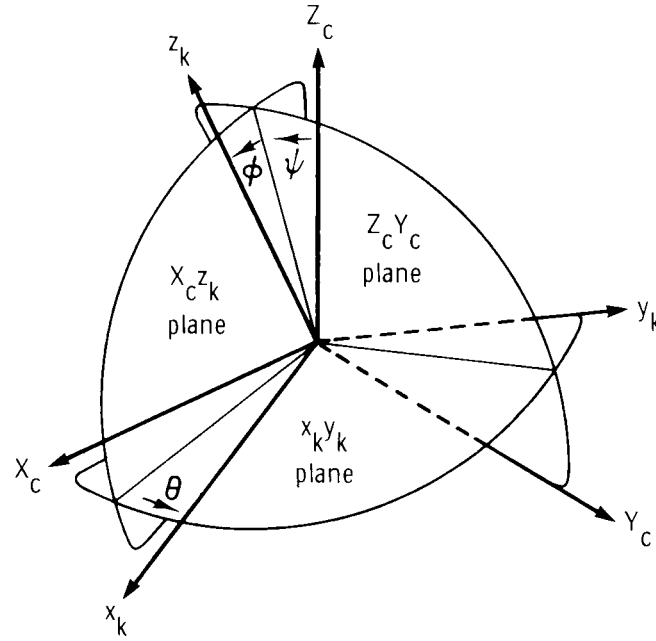

Figure 3. Angles of exterior orientation showing rotation from coordinates (X_c, Y_c, Z_c) into coordinates (x_k, y_k, z_k).

in space relative to the OBJ coordinate system. The rotational transformation from OBJ orientation to PC-FP orientation is computed via rotation matrix $\mathbf{M}$ (ref. 3) where

$$\mathbf{M} = \begin{bmatrix} \cos\phi\cos\theta & \sin\psi\sin\phi\cos\theta + \cos\psi\sin\theta & \sin\psi\sin\theta - \cos\psi\sin\phi\cos\theta \\ -\cos\phi\sin\theta & \cos\psi\cos\theta - \sin\psi\sin\phi\sin\theta & \sin\psi\cos\theta + \cos\psi\sin\phi\sin\theta \\ \sin\phi & -\sin\psi\cos\phi & \cos\psi\cos\phi \end{bmatrix} \quad (1)$$

See figure 3 for angle and coordinate definitions.

Refer to figure 2 for the following steps. Let the object (OBJ) coordinates of object point W be denoted by

$$\mathbf{W} = [X_c \quad Y_c \quad Z_c]^T_{\text{OBJ}} \tag{2}$$

and let the perspective center (PC) coordinates of W be denoted by

$$\mathbf{\Upsilon} = [\xi \quad \eta \quad \varsigma]^T_{\text{PC}} \tag{3}$$

Let $\mathbf{W}_p$ denote the coordinates of the perspective center in OBJ coordinates as

$$\mathbf{W}_p = [X_p \quad Y_p \quad Z_p]^T_{\text{OBJ}} \tag{4}$$

The transformation from OBJ coordinates to PC coordinates is then

$$\mathbf{\Upsilon} = \begin{Bmatrix} \xi \\ \eta \\ \varsigma \end{Bmatrix}_{\text{PC}} = \mathbf{M} \begin{Bmatrix} X_c - X_p \\ Y_c - Y_p \\ Z_c - Z_p \end{Bmatrix}_{\text{OBJ}} \tag{5}$$

Image point $\mathbf{\Upsilon}' = [\xi' \quad \eta' \quad \varsigma']^T$, the point of intersection of the focal plane with the line passing from $\mathbf{\Upsilon}$ through the perspective center (the origin in PC coordinates), is obtained by scaling $\mathbf{\Upsilon}$ so that its ς component equals $-c$. Thus,

$$\mathbf{\Upsilon}' = \begin{Bmatrix} \xi' \\ \eta' \\ -c \end{Bmatrix}_{\text{PC}} = -\frac{c}{\varsigma} \begin{Bmatrix} \xi \\ \eta \\ \varsigma \end{Bmatrix}_{\text{PC}} = -\frac{c}{\varsigma} \mathbf{M} \begin{Bmatrix} X_c - X_p \\ Y_c - Y_p \\ Z_c - Z_p \end{Bmatrix}_{\text{OBJ}} \tag{6}$$

is the perspective projection point of $\mathbf{W}$ in PC coordinates. It is transformed into focal plane (FP) coordinates (whose axes are parallel to the PC coordinate axes) by translation to the focal plane. Let $(x_p, y_p, c)_{\text{FP}}$ denote the FP coordinates of the perspective center and let $(x, y, 0)_{\text{FP}}$ denote the FP coordinates of the perspective projection image point. Then the PC coordinates of the perspective projection image point may be expressed as

$$\begin{Bmatrix} \xi' \\ \eta' \\ -c \end{Bmatrix}_{\text{PC}} = \begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix}_{\text{FP}} - \begin{Bmatrix} x_p \\ y_p \\ c \end{Bmatrix}_{\text{FP}} \tag{7}$$

Thus the transformation from OBJ coordinates to FP coordinates, obtained from equations (6) and (7), is

$$\begin{Bmatrix} x - x_p \\ y - y_p \\ -c \end{Bmatrix}_{\text{FP}} = -\frac{c}{\varsigma} \mathbf{M} \begin{Bmatrix} X_c - X_p \\ Y_c - Y_p \\ Z_c - Z_p \end{Bmatrix}_{\text{OBJ}} \tag{8}$$

By means of the two cameras, a subset of the three-dimensional object space is mapped into the two focal plane coordinate systems, which can be viewed as a four-dimensional, dual-focal-plane (DFP) space. Let subscripts 1 and 2 in the sequel denote cameras 1 and 2. Let $\mathbf{M}^{1,2}$ denote the 2×3 matrix consisting of rows 1 and 2 of $\mathbf{M}$, and let $\mathbf{M}^3$ denote the third row of $\mathbf{M}$. Also let

$$\mathbf{w} = [x_1 \quad y_1 \quad x_2 \quad y_2]^T \tag{9}$$

$$\mathbf{w}_1 = [x_1 \quad y_1]^T \tag{10a}$$

$$\mathbf{w}_2 = [x_2 \quad y_2]^T \tag{10b}$$

and

$$\varsigma_M = [\varsigma_1 \quad \varsigma_2]^T \tag{11}$$

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Apply equation (5) to cameras 1 and 2 and combine to obtain an expression for ς_M as

$$\varsigma_M = \left\{ \begin{array}{c} \varsigma_1 \\ \varsigma_2 \end{array} \right\} = \mathbf{M}_\varsigma \left\{ \begin{array}{c} X_c \\ Y_c \\ Z_c \end{array} \right\} - \left[\begin{array}{c|c} \mathbf{M}_1^3 & 0 \\ \hline 0 & \mathbf{M}_2^3 \end{array} \right] \left\{ \begin{array}{c} X_{p_1} \\ Y_{p_1} \\ Z_{p_1} \\ -- \\ X_{p_2} \\ Y_{p_2} \\ Z_{p_1} \end{array} \right\} \tag{12}$$

where

$$\mathbf{M}_\varsigma = \left[\begin{array}{c} \mathbf{M}_1^3 \\ -- \\ \mathbf{M}_2^3 \end{array} \right] \tag{13}$$

To obtain the elements of $\mathbf{w}$, similarly apply equation (8) to cameras 1 and 2. Extracting rows 1 and 2 of equation (8) for each camera and combining gives

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} x_1 - x_{p_1} \\ y_1 - y_{p_1} \end{array} \right\} \varsigma_1 \\ \hline \left\{ \begin{array}{c} x_2 - x_{p_2} \\ y_2 - y_{p_2} \end{array} \right\} \varsigma_2 \end{array} \right\} = \left[\begin{array}{c} -c_1 \mathbf{M}_1^{1,2} \\ -- \\ -c_2 \mathbf{M}_2^{1,2} \end{array} \right] \left\{ \begin{array}{c} X_c \\ Y_c \\ Z_c \end{array} \right\} + \left[\begin{array}{c|c} c_1 \mathbf{M}_1^{1,2} & 0 \\ \hline 0 & c_2 \mathbf{M}_2^{1,2} \end{array} \right] \left\{ \begin{array}{c} X_{p_1} \\ Y_{p_1} \\ Z_{p_1} \\ -- \\ X_{p_2} \\ Y_{p_2} \\ Z_{p_2} \end{array} \right\} \tag{14}$$

Let $\delta\mathbf{W} = [\delta X_c \quad \delta Y_c \quad \delta Z_c]^T$ be a perturbation in $\mathbf{W}$. From equation (12) the corresponding perturbation $\delta\varsigma_M$ is

$$\delta\varsigma_M \triangleq \left\{ \begin{array}{c} \delta\varsigma_1 \\ \delta\varsigma_2 \end{array} \right\} = \mathbf{M}_\varsigma \left\{ \begin{array}{c} \delta X_c \\ \delta Y_c \\ \delta Z_c \end{array} \right\} = \mathbf{M}_\varsigma \, \delta\mathbf{W} \tag{15}$$

Now substitute the elements of $\mathbf{w} + \delta\mathbf{w}$, $\varsigma_M + \delta\varsigma_M$, and $\mathbf{W} + \delta\mathbf{W}$ into equation (14) for the elements of $\mathbf{w}, \varsigma$, and $\mathbf{W}$, and subtract equation (14) from the result. This yields

$$\left\{ \begin{array}{c} \delta\varsigma_1 \left\{ \begin{array}{c} x_1 - x_{p_1} \\ y_1 - y_{p_1} \end{array} \right\} + (\varsigma_1 + \delta\varsigma_1) \left\{ \begin{array}{c} \delta x_1 \\ \delta y_1 \end{array} \right\} \\ \hline \delta\varsigma_2 \left\{ \begin{array}{c} x_2 - x_{p_2} \\ y_2 - y_{p_2} \end{array} \right\} + (\varsigma_2 + \delta\varsigma_2) \left\{ \begin{array}{c} \delta x_2 \\ \delta y_2 \end{array} \right\} \end{array} \right\} = \left[\begin{array}{c} -c\mathbf{M}_1^{1,2} \\ -- \\ -c\mathbf{M}_2^{1,2} \end{array} \right] \left\{ \begin{array}{c} \delta X_c \\ \delta Y_c \\ \delta Z_c \end{array} \right\} \tag{16}$$

Equations (15) and (16) are combined to obtain

$$\left[\left[\begin{array}{c} -c_1 \mathbf{M}_1^{1,2} \\ -- \\ -c_2 \mathbf{M}_2^{1,2} \end{array} \right] - \left[\begin{array}{c|c} x_1 - x_{p_1} & 0 \\ y_1 - y_{p_1} & 0 \\ \hline 0 & x_2 - x_{p_2} \\ 0 & y_2 - y_{p_2} \end{array} \right] \left[\begin{array}{c} \mathbf{M}_1^3 \\ \mathbf{M}_2^3 \end{array} \right] \right] \left\{ \begin{array}{c} \delta X_c \\ \delta Y_c \\ \delta Z_c \end{array} \right\} = \left\{ \begin{array}{c} (\varsigma_1 + \delta\varsigma_1) \left\{ \begin{array}{c} \delta x_1 \\ \delta y_1 \end{array} \right\} \\ (\varsigma_2 + \delta\varsigma_2) \left\{ \begin{array}{c} \delta x_2 \\ \delta y_2 \end{array} \right\} \end{array} \right\} \tag{17}$$

or in matrix notation, by letting $\delta\mathbf{w} = [\delta x_1 \quad \delta y_1 \quad \delta x_2 \quad \delta y_2]^T$

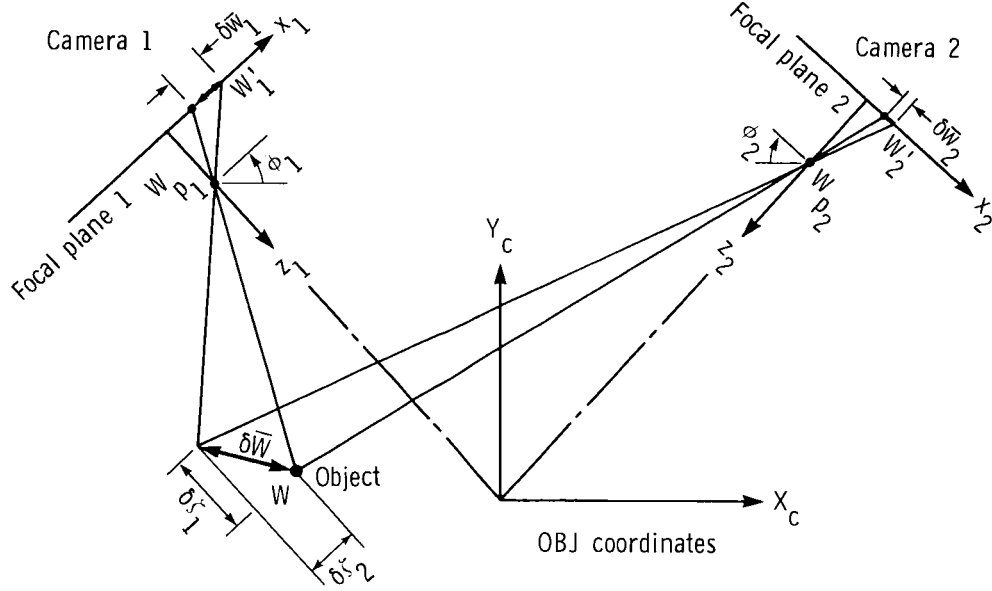$$\mathbf{A} \, \delta\mathbf{W} = (\mathbf{Z} + \delta\mathbf{Z})\delta\mathbf{w} \tag{18}$$

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Figure 4. Convergent projection geometry showing projection of object perturbation $\delta\mathbf{W}$ onto image perturbations $\delta\mathbf{w}_1$ and $\delta\mathbf{w}_2$ in focal planes 1 and 2.

where

$$\mathbf{Z} = \begin{bmatrix} \varsigma_1 & 0 & 0 & 0 \\ 0 & \varsigma_1 & 0 & 0 \\ 0 & 0 & \varsigma_2 & 0 \\ 0 & 0 & 0 & \varsigma_2 \end{bmatrix} \tag{19a}$$

$$\delta\mathbf{Z} = \begin{bmatrix} \delta\varsigma_1 & 0 & 0 & 0 \\ 0 & \delta\varsigma_1 & 0 & 0 \\ 0 & 0 & \delta\varsigma_2 & 0 \\ 0 & 0 & 0 & \delta\varsigma_2 \end{bmatrix} \tag{19b}$$

and

$$\mathbf{A} = \begin{bmatrix} -c_1\mathbf{M}_1^{1,2} \\ \hline -c_2\mathbf{M}_2^{1,2} \end{bmatrix} - \begin{bmatrix} x_1 - x_{p_1} & 0 \\ y_1 - y_{p_1} & 0 \\ \hline 0 & x_2 - x_{p_2} \\ 0 & y_2 - y_{p_2} \end{bmatrix} \begin{bmatrix} \mathbf{M}_1^3 \\ \hline \mathbf{M}_2^3 \end{bmatrix} \tag{20}$$

An illustration of perturbations $\delta\mathbf{W}$ and $\delta\mathbf{w}$ appears in figure 4.

The projection equation for $\delta\mathbf{w}$ as a function of $\delta\mathbf{W}$ can be obtained from equation (18) as a mapping D_A from OBJ coordinates into DFP coordinates as

$$\delta\mathbf{w} = (\mathbf{Z} + \delta\mathbf{Z})^{-1}\,\mathbf{A}\,\delta\mathbf{W} \stackrel{\triangle}{=} D_A(\delta\mathbf{W}) \tag{21}$$

Matrix $(\mathbf{Z} + \delta\mathbf{Z})^{-1}$ exists whenever no perturbation in the ς-direction intersects the (ξ, η) plane (which intersects the perspective center and is parallel to the focal plane, as shown in fig. 2). Equation (21) defines $\delta\mathbf{w}$ as a function of $\delta\mathbf{W}$, since the elements of $\delta\mathbf{Z}$ are obtained from equation (15), which is dependent only on $\delta\mathbf{W}$. If matrix $\mathbf{A}$ has full rank (rank 2), then from equation (18) one can obtain

$$\delta\mathbf{W} = \mathbf{P_A}(\mathbf{Z} + \delta\mathbf{Z})\,\delta\mathbf{w} \tag{22}$$

where

$$\mathbf{P_A} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T \tag{23}$$

Matrix $\mathbf{P_A}$, a generalized inverse (ref. 4), is the two-sided inverse of $\mathbf{A}$ over the range (column space) of $\mathbf{A}$.

It is now shown that $\delta \mathbf{Z}$ is dependent only on $\delta \mathbf{w}$ in equation (22) and that equation (22) therefore defines a mapping D from DFP coordinates into OBJ coordinates which is the inverse of mapping D_A. Define a matrix containing the elements of $\delta \mathbf{w}$ (with the help of eq. (9)) as

$$\mathbf{C}(\delta \mathbf{w}) = \begin{bmatrix} \delta x_1 & 0 \\ \delta y_1 & 0 \\ 0 & \delta x_2 \\ 0 & \delta y_2 \end{bmatrix} \tag{24}$$

which yields the identity

$$\delta \mathbf{Z} \, \delta \mathbf{w} = \mathbf{C}(\delta \mathbf{w}) \, \delta \varsigma_M \tag{25}$$

From equations (15), (22), and (25) one obtains an expression for $\delta \varsigma_M$ as

$$\delta \varsigma_M = \mathbf{M}_\varsigma \mathbf{P_A} \mathbf{Z} \, \delta \mathbf{w} + \mathbf{M}_\varsigma \mathbf{P_A} \mathbf{C}(\delta \mathbf{w}) \, \delta \varsigma_M \tag{26}$$

which is solved for $\delta \varsigma_M$ to obtain

$$\delta \varsigma_M = \left[\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A} \, \mathbf{C}(\delta \mathbf{w}) \right]^{-1} \mathbf{M}_\varsigma \mathbf{P_A} \mathbf{Z} \, \delta \mathbf{w} \tag{27}$$

Thus, whenever $\left[\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A} \, \mathbf{C}(\delta \mathbf{w}) \right]$ is nonsingular, equations (22), (25), and (27) can be combined to obtain

$$\delta \mathbf{W} = \mathbf{P_A} \left\{ \mathbf{Z} + \mathbf{C}(\delta \mathbf{w}) \, \left[\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A} \, \mathbf{C}(\delta \mathbf{w}) \right]^{-1} \mathbf{M}_\varsigma \mathbf{P_A} \mathbf{Z} \right\} \delta \mathbf{w} \stackrel{\triangle}{=} D(\delta \mathbf{w}) \tag{28}$$

From equations (21), (22), and (28) it is seen that

$$\delta \mathbf{W} = D \left[D_A(\delta \mathbf{W}) \right] \tag{29}$$

and D is a left inverse of D_A. In the applications for which this study is intended, every value of $\delta \mathbf{w}$ will be contained in the range of mapping D_A; that is, $\delta \mathbf{w}$ will be the image under D_A of some perturbation $\delta \mathbf{W}$ in OBJ coordinates. Thus, if $\delta \mathbf{w} = D_A(\delta \mathbf{W})$, then

$$D_A[D(\delta \mathbf{w})] = D_A \left\{ D \left[D_A(\delta \mathbf{W}) \right] \right\} = D_A(\delta \mathbf{W}) = \delta \mathbf{w} \tag{30}$$

Hence, D is the right inverse of D_A over the range of D_A. Equation (28) shows that, in general, D is nonlinear in $\delta \mathbf{w}$.

Error Caused by Commutation of Coordinate Transformation and Fourier Transformation

Let $\delta \mathbf{W}(t)$ and $\delta \mathbf{w}(t)$ denote time-varying perturbations of $\delta \mathbf{W}$ and $\delta \mathbf{w}$. (The (t) notation will often be suppressed in the sequel except where needed for emphasis or clarity.) Fourier transformation of vector functions $\delta \mathbf{W}(t)$ and $\delta \mathbf{w}(t)$ is obtained by componentwise scalar Fourier transformation as

$$F\left[\delta \mathbf{W}(t) \right] = \left\{ \begin{array}{c} \delta X_c(\omega) \\ \delta Y_c(\omega) \\ \delta Z_c(\omega) \end{array} \right\} \tag{31}$$

and

$$F\left[\delta \mathbf{w}(t) \right] = \left\{ \begin{array}{c} \delta x_1(\omega) \\ \delta y_1(\omega) \\ \delta x_2(\omega) \\ \delta y_2(\omega) \end{array} \right\} \tag{32}$$

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which are complex-valued vector functions of ω. Mapping D is extended to complex-valued vectors by the relation

$$D(\mathbf{u} + i\mathbf{v}) \overset{\triangle}{=} D(\mathbf{u}) + i\, D(\mathbf{v}) \tag{33}$$

If D were linear, then (since Fourier transformation is a linear operation) composition of mapping under D and Fourier transformation would be commutative. However, because of the nonlinearity of D, it follows that

$$F\left[\delta \mathbf{W}(t)\right] = F\left\{D\left[\delta \mathbf{w}(t)\right]\right\} \neq D\left\{F\left[\delta \mathbf{w}(t)\right]\right\} \tag{34}$$

If the perturbation magnitudes are small, the error

$$\mathbf{E}(\omega) \overset{\triangle}{=} D\left\{F\left[\delta \mathbf{w}(t)\right]\right\} - F\left[\delta \mathbf{W}(t)\right] \tag{35}$$

may be acceptably small. An expression for $\mathbf{E}(\omega)$ is now obtained.

It is easily shown that for nonsingular matrices $\mathbf{Z}$ and $\mathbf{Z} + \delta \mathbf{Z}$

$$(\mathbf{Z} + \delta \mathbf{Z})^{-1} = \mathbf{Z}^{-1}\left[\mathbf{I} - \delta \mathbf{Z}(\mathbf{Z} + \delta \mathbf{Z})^{-1}\right] \tag{36}$$

Substitute equation (36) into equation (21) to obtain

$$\delta \mathbf{w}(t) = \mathbf{Z}^{-1}\left\{\mathbf{A}\,\delta \mathbf{W}(t) - \delta \mathbf{Z}(t)[\mathbf{Z} + \delta \mathbf{Z}(t)]^{-1}\mathbf{A}\,\delta \mathbf{W}(t)\right\} \tag{37}$$

The Fourier transform of equation (37) is then

$$F(\delta \mathbf{w}) = \mathbf{Z}^{-1}\left\{\mathbf{A}F(\delta W) - F[\delta Z(\mathbf{Z} + \delta \mathbf{Z})^{-1}\,\mathbf{A}\,\delta W]\right\} \tag{38}$$

To obtain $D[F(\delta \mathbf{w})]$, substitute $F(\delta \mathbf{w})$ into equation (28) for $\delta \mathbf{w}$ to yield

$$D[F(\delta \mathbf{w})] = \mathbf{P_A}\left\{\mathbf{Z} + \delta \mathbf{Z}_F[F(\delta \mathbf{w})]\mathbf{Z}\right\}F(\delta \mathbf{w}) \tag{39}$$

where

$$\delta \mathbf{Z}_F[F(\delta \mathbf{w})] = \mathbf{C}[F(\delta \mathbf{w})]\left\{\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A}\,\mathbf{C}[F(\delta \mathbf{w})]\right\}^{-1}\mathbf{M}_\varsigma \mathbf{P_A} \tag{40}$$

and $\mathbf{C}[F(\delta \mathbf{w})]$ is obtained by substitution of $F(\delta \mathbf{w})$ for $\delta \mathbf{w}$ into equation (24). Combine equations (38) and (39) to obtain

$$
\begin{aligned}
D[F(\delta \mathbf{w})] = {}&F(\delta \mathbf{W}) - \mathbf{P_A}\,F\left[\delta \mathbf{Z}(\mathbf{Z} + \delta \mathbf{Z})^{-1}\,\mathbf{A}\,\delta \mathbf{W}\right] \\
&+ \mathbf{P_A}\delta \mathbf{Z}_F[F(\delta \mathbf{w})]\left\{\mathbf{A}\,F(\delta \mathbf{W}) - F\left[\delta \mathbf{Z}(\mathbf{Z} + \delta \mathbf{Z})^{-1}\mathbf{A}\,\delta \mathbf{W}\right]\right\}
\end{aligned} \tag{41}
$$

The expression for $\mathbf{E}(\omega)$ is obtained from equations (35) and (41) as

$$\mathbf{E}(\omega) = -\mathbf{P_A}\left\{\mathbf{I} + \delta \mathbf{Z}_F[F(\delta \mathbf{w})]\right\}F\left[\delta \mathbf{Z}(\mathbf{Z} + \delta \mathbf{Z})^{-1}\mathbf{A}\,\delta \mathbf{W}\right] + \mathbf{P_A}\,\delta \mathbf{Z}_F[F(\delta \mathbf{w})]\,\mathbf{A}\,F(\delta \mathbf{W}) \tag{42}$$

Determination of Error Bounds

Let $\|\mathbf{v}\|$ denote the Euclidean norm of an n-element vector $\mathbf{v}$, where

$$\|\mathbf{v}\| \overset{\triangle}{=} \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} \tag{43}$$

and let $\|\mathbf{A}\|$ denote the norm of a rectangular matrix $\mathbf{A}$, where

$$\|\mathbf{A}\| \triangleq \max_{\mathbf{v} \neq 0} \frac{\|\mathbf{A}\mathbf{v}\|}{\|\mathbf{v}\|} \tag{44}$$

It can be shown that the value of $\|\mathbf{A}\|$ is equal to the largest singular value of $\mathbf{A}$ (ref. 5). The value $\|\mathbf{A}^{-1}\|$, when $\mathbf{A}^{-1}$ exists, equals the inverse of the smallest singular value of $\mathbf{A}$. It follows from equation (44) that

$$\|\mathbf{A}\mathbf{v}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{v}\| \tag{45}$$

and that there always exists a $\mathbf{v}$ for which equality holds in equation (45). It can also be shown (ref. 5) that

$$\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\| \tag{46}$$

where $\mathbf{A}$ and $\mathbf{B}$ are conformable matrices. The triangle inequality for vector norms given by

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \tag{47}$$

is true also for matrix norms (ref. 5) as

$$\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\| \tag{48}$$

In the sequel, it is assumed, realistically, that the norms $\|\delta\mathbf{W}(t)\|$ and $\|F(\delta\mathbf{W})\|$ are bounded, and that the integral $\int_{-\infty}^{\infty} \|F(\delta\mathbf{W})\|^2 d\omega$ exists.

Take the norm of equation (42) and apply inequalities (45) to (47) to obtain

$$\|\mathbf{E}(\omega)\| \leq \|\mathbf{P_A}\| \cdot \|\mathbf{I} + \delta\mathbf{Z}_F[F(\delta\mathbf{w})]\| \cdot \left\|F\left[\delta\mathbf{Z}(\mathbf{Z} + \delta\mathbf{Z})^{-1}\mathbf{A}\,\delta\mathbf{W}\right]\right\| \\ + \|\mathbf{P_A}\| \cdot \|\mathbf{A}\| \cdot \|\delta\mathbf{Z}_F[F(\delta\mathbf{w})]\| \cdot \|F(\delta\mathbf{W})\| \tag{49}$$

Inspection of equation (49) shows that when $\|\delta\mathbf{Z}_F[F(\delta\mathbf{w})]\|$ is sufficiently small, $\mathbf{E}(\omega)$ is dominated by $\mathbf{P_A}\,F\left[\delta\mathbf{Z}(\mathbf{Z} + \delta\mathbf{Z})^{-1}\mathbf{A}\,\delta\mathbf{W}\right]$. It is now shown through the use of equation (40) that $\|\delta\mathbf{Z}_F[F(\delta\mathbf{w})]\|$ can be made arbitrarily small by appropriate scaling of $F(\delta\mathbf{w})$.

From equation (24) it is seen that

$$\mathbf{C}[F(\delta\mathbf{w})] = \begin{bmatrix} \delta x_1(\omega) & 0 \\ \delta y_1(\omega) & 0 \\ 0 & \delta x_2(\omega) \\ 0 & \delta y_2(\omega) \end{bmatrix} \tag{50}$$

It is readily shown that the norm of $\mathbf{C}[F(\delta\mathbf{w})]$ is bounded by

$$\|\mathbf{C}[F(\delta\mathbf{w})]\| = \max\left[\|\delta\mathbf{w}_1(\omega)\|, \|\delta\mathbf{w}_2(\omega)\|\right] \leq \|F(\delta\mathbf{w})\| \tag{51}$$

It can be shown (ref. 4) that for any square matrix $\mathbf{A}$, if $\|\mathbf{A}\| < 1$, then $(\mathbf{I} - \mathbf{A})^{-1}$ exists, and its norm is bounded by

$$\|(\mathbf{I} - \mathbf{A})^{-1}\| \leq \frac{1}{1 - \|\mathbf{A}\|} \tag{52}$$

Let

$$\nu = \|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \cdot \|F(\delta\mathbf{w})\| \tag{53}$$

It follows from equations (46), (51), and (53) that

$$\|\mathbf{M}_\varsigma\mathbf{P_A}\,\mathbf{C}[F(\delta\mathbf{w})]\| < \nu \tag{54}$$

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By the result cited above (inequality (52)), if $\nu < 1$, then $\{\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A}\, \mathbf{C}[F(\delta\mathbf{w})]\}^{-1}$ exists, and thus $\delta\mathbf{Z}_F[F(\delta\mathbf{w})]$ (eq. (40)) exists. It follows from equations (52) and (54) that

$$\| \{\mathbf{I} - \mathbf{M}_\varsigma \mathbf{P_A}\, \mathbf{C}[F(\delta\mathbf{w})]\}^{-1} \| \leq \frac{1}{1 - \|\mathbf{M}_\varsigma \mathbf{P_A} \mathbf{C}[F(\delta\mathbf{w})]\|} \leq \frac{1}{1 - \nu} \tag{55}$$

Take the norm of equation (40) and use inequalities (46), (51), and (55) to give for $\nu < 1$

$$\|\delta\mathbf{Z}_F[F(\delta w)]\| \leq \|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \cdot \left(\frac{1}{1-\nu}\right) \|F(\delta\mathbf{w})\| \tag{56}$$

Inequality (56) may be rewritten as

$$\|\delta\mathbf{Z}_F[F(\delta\mathbf{w})]\| \leq h_z \tag{57}$$

where

$$h_z = \|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \left(\frac{1}{1-\nu}\right) \|F(\delta\mathbf{w})\|_{\max} \tag{58}$$

Let $F(\delta\mathbf{w})$ be scaled by factor $\frac{1}{K}$ so that equations (53) and (57) become, respectively,

$$\nu = \frac{1}{K}\|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \cdot \|F(\delta\mathbf{w})\|_{\max} \tag{59}$$

and

$$\left\|\delta\mathbf{Z}_F\left[\frac{1}{K}F(\delta\mathbf{w})\right]\right\| \leq \frac{h_z}{K} \tag{60}$$

Thus K can be chosen sufficiently large to ensure that $\nu < 1$ is satisfied and to make $\left\|\delta\mathbf{Z}_F\left[\frac{1}{K}F(\delta\mathbf{w})\right]\right\|$ arbitrarily small, as was to be shown.

The effect of scaling $F(\delta\mathbf{w})$ in equation (49) can now be determined. Replacement of $D[F(\delta\mathbf{w})]$ in equation (39) by $KD\left[\frac{1}{K}F(\delta\mathbf{w})\right]$ yields

$$KD\left[\frac{1}{K}F(\delta\mathbf{w})\right] = \mathbf{P_A}\left\{\mathbf{Z} + \delta\mathbf{Z}_F\left[\frac{1}{K}F(\delta\mathbf{w})\right]\mathbf{Z}\right\}F(\delta\mathbf{w}) \tag{61}$$

Define scaled error $\mathbf{E}_K(\omega)$, corresponding to $\mathbf{E}(\omega)$ in equation (35), as

$$\mathbf{E}_K(\omega) \overset{\triangle}{=} KD\left[\frac{1}{K}F(\delta\mathbf{w})\right] - F(\delta\mathbf{W}) \tag{62}$$

The expression for $\mathbf{E}_K(\omega)$ analogous to equation (42) is written as

$$\begin{aligned}
\mathbf{E}_K(\omega) = &-\mathbf{P_A}\left\{I + \delta\mathbf{Z}_F\left[\frac{1}{K}F(\delta\mathbf{w})\right]\right\}\, F\left[\delta\mathbf{Z}(\mathbf{Z} + \delta\mathbf{Z})^{-1}\mathbf{A}\,\delta\mathbf{W}\right]\\
&+ \mathbf{P_A}\delta\mathbf{Z}_F\left[\frac{1}{K}F(\delta\mathbf{w})\right]\mathbf{A}\,F(\delta\mathbf{W})
\end{aligned} \tag{63}$$

Through the use of equations (60) and (63) it follows that the norm $\|\mathbf{E}_K(\omega)\|$ is bounded by

$$\begin{aligned}
\|\mathbf{E}_K(\omega)\| \leq &\left(1 + \frac{1}{K}h_z\right)\|\mathbf{P_A}\| \cdot \left\|F\left\{\delta\mathbf{Z}(t)[\mathbf{Z} + \delta\mathbf{Z}(t)]^{-1}\mathbf{A}\,\delta\mathbf{W}(t)\right\}\right\|\\
&+ \frac{1}{K}h_z\|\mathbf{P_A}\| \cdot \|\mathbf{A}\| \cdot \|F[\delta\mathbf{W}(t)]\|
\end{aligned} \tag{64}$$

The first Fourier transform term in equation (64) cannot be analytically evaluated even for simple time functions because of the inverse factor $[\mathbf{Z} + \delta\mathbf{Z}(t)]^{-1}$. This fact prevents determination of a useful bound. However, the integral of $\|\mathbf{E}_K(\omega)\|^2$ over ω can be bounded analytically, as will now be shown. Let

$$\mathbf{e}_F(t) = \|\mathbf{P_A}\|\delta\mathbf{Z}(t)[\mathbf{Z} + \delta\mathbf{Z}(t)]^{-1}\mathbf{A}\,\delta\mathbf{W}(t) \tag{65}$$

and

$$\mathbf{e}_K(t) = \left(1 + \frac{1}{K}h_z\right)\mathbf{e}_F(t) + \frac{1}{K}h_z\|\mathbf{P_A}\|\cdot\|\mathbf{A}\|\delta\mathbf{W}(t) \tag{66}$$

From equations (64), (65), and (66), it is seen that

$$\|\mathbf{E}_K(\omega)\| \le \|F\left[\mathbf{e}_K(t)\right]\| \tag{67}$$

Take the norm of equation (65) and apply inequalities (45) and (46) to give

$$\|\mathbf{e}_F(t)\| \le \sigma\|\mathbf{D}_z(t)\|\cdot\|\delta\mathbf{W}(t)\| \tag{68}$$

where

$$\sigma = \|\mathbf{P_A}\|\cdot\|\mathbf{A}\| \tag{69}$$

and

$$\mathbf{D}_z(t) = \delta\mathbf{Z}(t)[\mathbf{Z} + \delta\mathbf{Z}(t)]^{-1} \tag{70}$$

Each nonzero element of $\mathbf{D}_z(t)$, which is diagonal, is of the form

$$\frac{\delta\varsigma_k(t)/\varsigma_k}{1 + [\delta\varsigma_k(t)/\varsigma_k]}$$

where $k = 1$ or 2. Factor $\delta\varsigma_k(t)$ is obtained from equation (15) as

$$\delta\varsigma_k(t) = \mathbf{M}_k^3\delta\mathbf{W}(t) = \cos\gamma_k(t)\|\delta\mathbf{W}(t)\| \tag{71}$$

where $\gamma_k(t)$ is the angle between $\delta\mathbf{W}(t)$ and the ς_k-axis. If $\hat{\gamma}_k$ is the smallest angle attained for any t between $\delta\mathbf{W}(t)$ and the ς_k-axis, then it follows from equation (71) that

$$\left|\frac{\delta\varsigma_k(t)}{\varsigma_k}\right| \le \left|\frac{\cos\hat{\gamma}_k}{\varsigma_k}\right|\cdot\|\delta\mathbf{W}(t)\| \tag{72}$$

and that

$$\epsilon_M \overset{\triangle}{=} \max_{k,t}\left[\frac{\delta\varsigma_k(t)}{\varsigma_k}\right] = C\max_k\left|\frac{\cos\hat{\gamma}_k}{\varsigma_k}\right| \tag{73}$$

where

$$C = \max_t\|\delta\mathbf{W}(t)\| \tag{74}$$

Define

$$W_F(t) = \frac{1}{C}\|\delta\mathbf{W}(t)\| \tag{75}$$

Combine equations (72) through (75) to obtain

$$\left|\frac{\delta\varsigma_k(t)}{\varsigma_k}\right| \le \epsilon_M\,W_F(t) \tag{76}$$

Then it follows from equations (70) and (76) that

$$\|\mathbf{D}_z(t)\| \le \frac{\epsilon_M}{1 - \epsilon_z}W_F(t) \tag{77}$$

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where

$$\epsilon_z = \left| \min_{k,t} \left[0, \frac{\delta \varsigma_k(t)}{\varsigma_k} \right] \right| \tag{78}$$

Therefore from equations (68), (75), and (77), $\|\mathbf{e}_F(t)\|$ is bounded by

$$\|\mathbf{e}_F(t)\| \leq \sigma \frac{\epsilon_M}{1 - \epsilon_z} C \, W_F^2(t) \tag{79}$$

To bound $\|\mathbf{e}_K(t)\|$ apply equations (69), (75), and (79) to the norm of equation (66) and simplify to obtain

$$\|\mathbf{e}_K(t)\| \leq \left(1 + \frac{1}{K} h_z\right) \sigma \frac{\epsilon_M}{1 - \epsilon_z} C \, W_F^2(t) + \frac{1}{K} h_z \sigma C \, W_F(t) \tag{80}$$

Square equation (80) and use the fact that $|W_F(t)| \leq 1$ to obtain

$$\|\mathbf{e}_K(t)\|^2 \leq (\sigma C)^2 \left\{ \left(1 + \frac{h_z}{K}\right)^2 \left(\frac{\epsilon_M}{1 - \epsilon_z}\right)^2 W_F^4(t) \right.$$
$$\left. + \frac{h_z}{K} \left[2 \left(1 + \frac{h_z}{K}\right) \left(\frac{\epsilon_M}{1 - \epsilon_z}\right) + \frac{h_z}{K} \right] W_F^2(t) \right\} \tag{81}$$

The relative RMS error is defined as

$$\epsilon_{\text{RMS}} = \frac{\left[\int_{-\infty}^{\infty} \|\mathbf{E}_K(\omega)\|^2 \, d\omega\right]^{1/2}}{\left[\int_{-\infty}^{\infty} \|F[\delta \mathbf{W}]\|^2 \, d\omega\right]^{1/2}} \leq \frac{\left[\int_{-\infty}^{\infty} \|\mathbf{e}_K(t)\|^2 \, dt\right]^{1/2}}{\left[\int_{-\infty}^{\infty} \|\delta \mathbf{W}(t)\|^2 \, dt\right]^{1/2}} \tag{82}$$

The inequality in equation (82) follows from Parseval's theorem (ref. 6) and inequality (67). The integral in the denominator of equation (82) is evaluated by means of equation (75) to obtain

$$\int_{-\infty}^{\infty} \|\delta \mathbf{W}(t)\|^2 \, d\omega = C^2 \int_{-\infty}^{\infty} W_F^2(t) \, dt \tag{83}$$

Integration of equation (81) over t and division of the result by equation (83) yields the bound to ϵ_{RMS}^2 as

$$\epsilon_{\text{RMS}}^2 \leq \sigma^2 \left\{ \left(1 + \frac{h_z}{K}\right)^2 \left(\frac{\epsilon_M}{1 - \epsilon_z}\right)^2 \frac{\left[\int_{-\infty}^{\infty} W_F^4(t) \, dt\right]}{\left[\int_{-\infty}^{\infty} W_F^2(t) \, dt\right]} + \frac{h_z}{K} \left[2 \left(1 + \frac{h_z}{K}\right) \left(\frac{\epsilon_M}{1 - \epsilon_z}\right) + \frac{h_z}{K} \right] \right\} \tag{84}$$

where

$$\sigma = \|\mathbf{P_A}\| \cdot \|\mathbf{A}\| \tag{69}$$

$$h_z = \|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \left(\frac{1}{1 - \nu}\right) \|F(\delta \mathbf{w})\|_{\text{max}} \tag{58}$$

$$\nu = \frac{1}{K} \|\mathbf{M}_\varsigma\| \cdot \|\mathbf{P_A}\| \cdot \|F(\delta \mathbf{w})\|_{\text{max}} \tag{59}$$

$$\epsilon_M = \max_{k,t} \left[\frac{\delta \varsigma_k(t)}{\varsigma_k} \right] \tag{73}$$

$$\epsilon_z = \left| \min_{k,t} \left[0, \frac{\delta \varsigma_k(t)}{\varsigma_k} \right] \right| \tag{78}$$

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and from equations (74) and (75)

$$W_F(t) = \frac{\|\delta \mathbf{W}(t)\|}{\max_t \|\delta \mathbf{W}(t)\|}$$

The terms involving h_z/K in equation (84) can be made negligible by scaling $F(\delta \mathbf{w})$ through suitable choice of factor K. Scaling $F(\delta \mathbf{w})$ in effect limits the distortion caused by its projection from DFP coordinates to OBJ coordinates via mapping D. On the other hand, the error due to distortion caused by projection of $\delta \mathbf{W}(t)$ from OBJ coordinates to DFP coordinates, represented by the remaining term in equation (84), is independent of scale factor K. This portion of the error is roughly proportional to the ς component of $\delta \mathbf{W}(t)$, represented by the ratio $\epsilon_M/(1 - \epsilon_z)$.

If K is chosen sufficiently large so that the terms involving h_z/K are negligible in equation (84), then ϵ_{RMS} is bounded by

$$\epsilon_{\text{RMS}} \leq \sigma \left(\frac{\epsilon_M}{1 - \epsilon_z} \right) \frac{\left[\int_{-\infty}^{\infty} W_F^4(t)\, dt \right]^{1/2}}{\left[\int_{-\infty}^{\infty} W_F^2(t)\, dt \right]^{1/2}} \leq \sigma \left(\frac{\epsilon_M}{1 - \epsilon_z} \right) \tag{85}$$

The second bound in equation (85) follows from the fact that since $0 \leq W_F(t) \leq 1$, then

$$0 \leq W_F^4(t) \leq W_F^2(t) \tag{86}$$

and therefore the ratio of integrals in equation (85) is bounded by unity.

Recall that $\|\mathbf{A}\|$ equals the largest singular value of $\mathbf{A}$. It is easily shown that $\|\mathbf{P_A}\|$ equals the square root of the largest eigenvalue of $\left(\mathbf{A}^T \mathbf{A} \right)^{-1}$, which equals the inverse of the smallest nonzero singular value of $\mathbf{A}$. Thus σ equals the ratio of the largest singular value of $\mathbf{A}$ to the smallest nonzero singular value of $\mathbf{A}$, defined as the condition number of $\mathbf{A}$ (ref. 5). Matrix $\mathbf{A}$ tends to be poorly conditioned (σ large) if cameras 1 and 2 are closely spaced with nearly parallel focal planes (angles ϕ_1 and ϕ_2 small in fig. 4) or if they are nearly opposed (ϕ_1 and ϕ_2 close to $\pi/2$ in fig. 4). Matrix $\mathbf{A}$ is well conditioned (σ close to 1) if the principal axes are coplanar and at right angles to each other (ϕ_1 and ϕ_2 equal to $\pi/4$ in fig. 4).

Evaluation of Bound for Exponentially Damped Sinusoidal Perturbation Functions

Inequality (85) is evaluated analytically for the special case where $\delta \mathbf{W}(t)$ is a vector of fixed direction of varying magnitude. Let

$$\delta \mathbf{W}(t) = \mathbf{h}\, f(t) \tag{87}$$

where $\mathbf{h}$ is a vector in OBJ coordinates.

A typical perturbation function encountered in structural vibration testing is the exponentially damped sinusoid

$$f(t) = \left\{ \begin{array}{ll} e^{-at} \sin \omega_0 t & (t \geq 0) \\ 0 & (t < 0) \end{array} \right\} \tag{88}$$

The minimum value of $e^{-at} \sin \omega_0 t$ occurs for $\omega_0 t$ approximately equal to $3\pi/2$. Therefore

$$\epsilon_z = \epsilon_M e^{-3\pi a/2\omega_0} \tag{89}$$

Analytic evaluation of inequality (85) for this case yields the bound

$$\epsilon_{\text{RMS}} \leq \sigma \left(\frac{\epsilon_M}{1 - \epsilon_z} \right) \left[\frac{3\omega_0^2}{8(4a^2 + \omega_0^2)} \right]^{1/2} \tag{90}$$

A similar computation can be performed for the exponentially damped cosine perturbation function

$$f(t) = \left\{ \begin{array}{ll} e^{-at} \cos \omega_0 t & (t \geq 0) \\ 0 & (t < 0) \end{array} \right\} \tag{91}$$

to give

$$\epsilon_{\text{RMS}} \leq \sigma \left(\frac{\epsilon_M}{1 - \epsilon_z} \right) \left(\frac{16a^4 + 28a^2\omega_0^2 + 3\omega_0^4}{64a^4 + 48a^2\omega_0^2 + 8\omega_0^4} \right) \tag{92}$$

For metal structures, typical values of a/ω_0 seldom exceed 0.02, in which case inequalities (90) and (92) reduce to

$$\epsilon_{\text{RMS}} \leq 0.613\sigma \left(\frac{\epsilon_M}{1 - \epsilon_z} \right) \tag{93}$$

Numerical studies showing computed errors for damped sinusoidal perturbations compared with maximum errors predicted by equation (93) are presented in reference 2.

Geometric Examples

Figure 4 illustrates an example of typical geometry in which the y_1- and y_2-axes are parallel to the Z_c-axis, and the coordinates (ξ_1, ς_1) and (ξ_2, ς_2) are rotated from the (X_c, Y_c) coordinates by angles ϕ_1 and ϕ_2, respectively. Let the object W be located at the intersection of the normals to each focal plane.

Rotation matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are then

$$\mathbf{M}_k = \begin{bmatrix} \cos\phi_k & \sin\phi_k & 0 \\ 0 & 0 & 1 \\ \sin\phi_k & -\cos\phi_k & 0 \end{bmatrix} \tag{94}$$

for $k = 1, 2$. Matrices $\mathbf{A}$ and $\mathbf{M}_\varsigma$, defined in equations (20) and (13), are constructed from $\mathbf{M}_1$ and $\mathbf{M}_2$ as

$$\mathbf{A} = \begin{bmatrix} c_1 \cos\phi_1 & c_1 \sin\phi_1 & 0 \\ 0 & 0 & c_1 \\ c_2 \cos\phi_2 & -c_2 \sin\phi_2 & 0 \\ 0 & 0 & c_2 \end{bmatrix} \tag{95}$$

and

$$\mathbf{M}_\varsigma = \begin{bmatrix} \sin\phi_1 & -\cos\phi_1 & 0 \\ \sin\phi_2 & -\cos\phi_2 & 0 \end{bmatrix} \tag{96}$$

If constants are chosen such that $\phi_2 = -\phi_1$, $\varsigma_1 = \varsigma_2$, $c_1 = c_2$, $\psi_1 = \psi_2 = \pi/2$, and $\theta_1 = \theta_2 = 0$, then the matrix product $\mathbf{A}^T\mathbf{A}$ is

$$\mathbf{A}^T\mathbf{A} = 2c^2 \begin{bmatrix} \cos^2\phi & 0 & 0 \\ 0 & \sin^2\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{97}$$

The norm $\|\mathbf{A}\|$, equal to the square root of the largest eigenvalue of $\mathbf{A}^T\mathbf{A}$ (ref. 5), is then

$$\|\mathbf{A}\| = \sqrt{2}c \tag{98}$$

The norm $\|\mathbf{P_A}\|$, equal to the square root of the largest eigenvalue of $(\mathbf{A}^T\mathbf{A})^{-1}$, is

$$\|\mathbf{P_A}\| = \frac{1}{\sqrt{2}c} \max(\sec\phi, \csc\phi) \tag{99}$$

Similarly, it is determined that

$$\|\mathbf{M}_\varsigma\| = \sqrt{2} \max(\sin\phi, \cos\phi) \tag{100}$$

For this case

$$\sigma = \|\mathbf{P_A}\| \cdot \|\mathbf{A}\| = \max(\sec\phi, \csc\phi) \tag{101}$$

Note that if $\delta\mathbf{W}(t)$ is confined to the Z_c-direction then $\hat{\gamma}_1 = \hat{\gamma}_2 = \pi/2$, and hence by equation (73) $\epsilon_M = 0$.
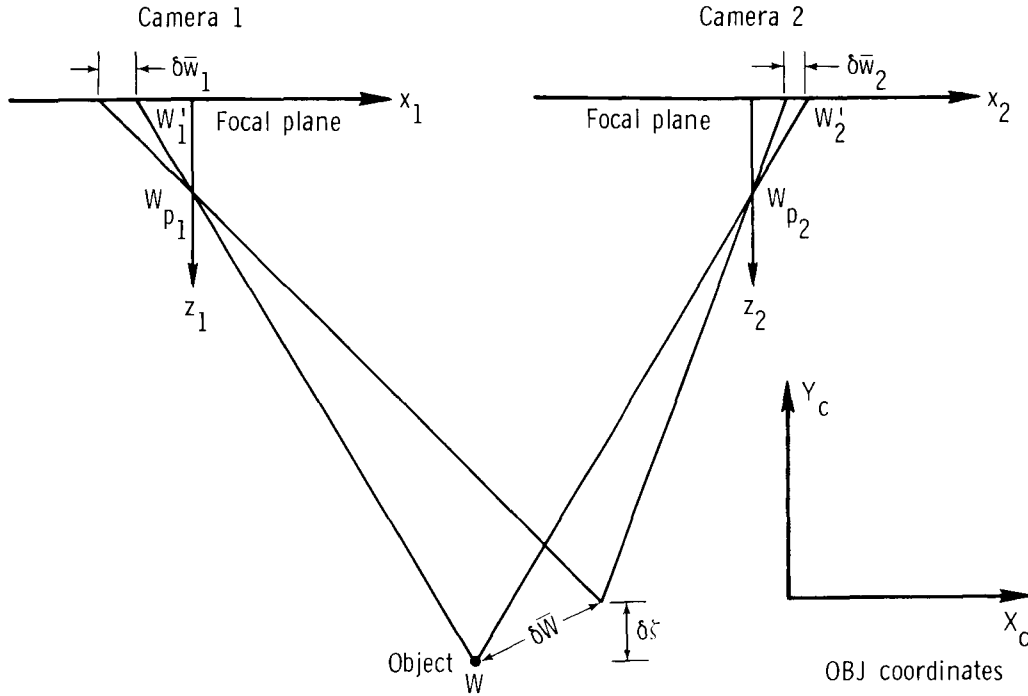
Figure 5. Parallel projection geometry showing projection of object perturbation $\delta \mathbf{W}$ onto parallel focal planes.

In the second geometric example, shown in figure 5, the OBJ and DFP coordinate systems are parallel; the camera focal planes are coplanar; focal lengths c_1 and c_2 are equal; and points W_{p_1}, W_{p_2}, and W lie on the plane $Y_c = 0$. Since $\delta\varsigma_1/\varsigma_1$ equals $\delta\varsigma_2/\varsigma_2$ for this case, it follows that

$$\delta\mathbf{Z}(\mathbf{Z} + \delta\mathbf{Z})^{-1} = \left(\frac{\delta\varsigma}{\varsigma + \delta\varsigma}\right)\mathbf{I} \tag{102}$$

Substitute equation (102) into equation (63), note that matrices $\mathbf{P_A}$ and $\mathbf{A}$ cancel, and take the norm to obtain

$$\|\mathbf{E}_K(\omega)\| \le \left(1 + \frac{1}{K}h_z\right)\left\|F\left(\frac{\delta\varsigma}{\varsigma + \delta\varsigma}\delta\mathbf{W}\right)\right\| + \frac{1}{K}h_z\|F(\delta\mathbf{W})\| \tag{103}$$

Equation (85) simplifies to

$$\epsilon_{\text{RMS}} \le \frac{\left(\frac{\epsilon_M}{1 - \epsilon_z}\right)\left[\int_{-\infty}^{\infty}W_F^4(t)\,dt\right]^{1/2}}{\left[\int_{-\infty}^{\infty}W_F^2(t)\,dt\right]^{1/2}} \tag{104}$$

(equivalent to $\sigma = 1$). Matrices $\mathbf{M}_\varsigma$ and $\mathbf{P_A}$, necessary for determination of h_z and ν via equations (58) and (59) in the computation of scale factor K, are now obtained. Rotation matrix $\mathbf{M}$ is the identity matrix $\mathbf{I}$, since the OBJ and FP coordinates are parallel. Matrix $\mathbf{M}_\varsigma$ is then

$$\mathbf{M}_\varsigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{105}$$

and $\|\mathbf{M}_\varsigma\| = \sqrt{2}$. Matrix $\mathbf{A}$ is given by

$$\mathbf{A} = c\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & 0 \\ 1 & 0 & b_2 \\ 0 & 1 & 0 \end{bmatrix} \tag{106}$$

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where

$$b_i = \frac{x_i - x_{p_i}}{c} \qquad (i = 1, 2) \qquad (107)$$

Matrix $\mathbf{A}^T\mathbf{A}$ is

$$\mathbf{A}^T\mathbf{A} = 2c^2 \begin{bmatrix} 1 & 0 & \frac{b_1+b_2}{2} \\ 0 & 1 & 0 \\ \frac{b_1+b_2}{2} & 0 & \frac{b_1^2+b_2^2}{2} \end{bmatrix} \qquad (108)$$

Its inverse is

$$(\mathbf{A}^T\mathbf{A})^{-1} = \frac{2}{(b_1 - b_2)^2 c^2} \begin{bmatrix} \frac{b_1^2+b_2^2}{2} & 0 & -\frac{b_1+b_2}{2} \\ 0 & \frac{(b_1-b_2)^2}{4} & 0 \\ -\frac{b_1+b_2}{2} & 0 & 1 \end{bmatrix} \qquad (109)$$

The eigenvalues of $(\mathbf{A}^T\mathbf{A})^{-1}$ are obtained analytically as

$$\lambda = \frac{1}{2c^2}, \frac{1}{2}\left[(a_\lambda + b_\lambda) \pm \sqrt{(a_\lambda + b_\lambda)^2 - 4\left(\frac{1}{2}a_\lambda - d_\lambda^2\right)}\right] \qquad (110)$$

where

$$a_\lambda = \frac{b_1^2 + b_2^2}{(b_1 - b_2)^2 c^2} \qquad (111)$$

$$b_\lambda = \frac{2}{(b_1 - b_2)^2 c^2} \qquad (112)$$

$$d_\lambda = -\frac{b_1 + b_2}{(b_1 - b_2)^2 c^2} \qquad (113)$$

The norm of $\|\mathbf{P_A}\|$ is the square root of the largest eigenvalue of $(\mathbf{A}^T\mathbf{A})^{-1}$. If $b_1 = -b_2$, then

$$\|\mathbf{P_A}\| = \frac{1}{\sqrt{2}c} \max\left(1, |b_1|\right) \qquad (114)$$

Effects of Error on Estimated Frequency Response Functions

The effects on estimated frequency response functions caused by the computation of $F[\delta\mathbf{W}(t)]$ by the approximation $KD\left\{\frac{1}{K} F[\delta\mathbf{w}(t)]\right\}$ can be deduced by examination of the error expression in equation (42), whose dominant part has been shown (see eqs. (63) and (70)) to be

$$\mathbf{E}_K(\omega) = \mathbf{P_A}\, F\left[\mathbf{D}_z(t)\,\mathbf{A}\,\delta\mathbf{W}(t)\right] \qquad (115)$$

for suitably large K. Consider the jth element of $\mathbf{D}_z(t)\,\mathbf{A}\,\delta\mathbf{W}(t)$, which is

$$e_j(t) \stackrel{\triangle}{=} \frac{\delta\varsigma_k(t)/\varsigma_k}{1 + [\delta\varsigma_k(t)/\varsigma_k]}\mathbf{A}_{j\cdot}\,\delta\mathbf{W}(t) \qquad (116)$$

where $\mathbf{A}_{j\cdot}$ is the jth row of $\mathbf{A}$ and $k = 1$ when $j = 1$ or 2 and $k = 2$ when $j = 3$ or 4. Recall from equation (15) that

$$\delta\varsigma_k(t) = \mathbf{M}_k^3\,\delta\mathbf{W}(t) \qquad (117)$$

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Expand $(\delta\varsigma_k/\varsigma_k)/[1+(\delta\varsigma_k/\varsigma_k)]$ into a power series

$$d_j(t) = -\sum_{n=1}^{\infty}\left[-\frac{\delta\varsigma_k(t)}{\varsigma_k}\right]^n \tag{118}$$

which is convergent for $|\delta\varsigma_k(t)/\varsigma_\kappa| < 1$. For simplicity, suppose that $\delta\mathbf{W}(t)$ contains only two damped sinusoidal components at frequencies ω_1 and ω_2. Then $\delta\varsigma(t)/\varsigma$ will be of the form

$$\frac{\delta\varsigma(t)}{\varsigma} = \alpha_1 e^{-a_1 t}\cos\omega_1 t + \alpha_2 e^{-a_2 t}\cos\omega_2 t \tag{119}$$

The jth component of $\mathbf{A}\,\delta\mathbf{W}(t)$ will be of the form

$$\mathbf{A}_j.\delta\mathbf{W}(t) = \beta_1 e^{-a_1 t}\cos\omega_1 t + \beta_2 e^{-a_2 t}\cos\omega_2 t \tag{120}$$

From equations (116), (118), (119), and (120), a power series expansion for $e_j(t)$ is obtained as

$$e_j(t) = \sum_{n=2}^{\infty}\sum_{m=0}^{n} b_{nm}(t)\cos^{n-m}\omega_1 t\cos^m\omega_2 t \tag{121}$$

where the coefficients $b_{nm}(t)$ are weighted sums of products of powers of $e^{-a_1 t}$ and $e^{-a_2 t}$. By means of trigonometric identities one obtains

$$\cos^m\omega t = \sum_{q=0}^{m/2} c_{mq}\cos q\omega t \qquad (m\text{ even}) \tag{122}$$

where coefficients c_{mq} satisfy $\sum_{q=0}^{m} c_{mq} = 1$, or

$$\cos^m\omega t = \sum_{q=0}^{(m-1)/2} c'_{mq}\cos q\omega t \qquad (m\text{ odd}) \tag{123}$$

where coefficients c'_{mq} satisfy $\sum_{q=1}^{m} c'_{mq} = 1$ and $c'_{m0} = 0$, and finally

$$\cos q\omega_1 t\cos r\omega_2 t = \frac{1}{2}\left[\cos\left(q\omega_1 + r\omega_2\right)t - \cos\left(q\omega_1 - r\omega_2\right)t\right] \tag{124}$$

Substitute formulas (122) through (124) into equation (121), and simplify to obtain

$$e_j(t) = \frac{1}{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left\{b_{nm}(t)\sum_{q=0}^{(n-m)/2}\sum_{r=0}^{m/2} d_{n-m,q}d_{m,r}\left[\cos\left(q\omega_1 + r\omega_2\right)t\right.\right.$$
$$\left.\left. + \cos\left(q\omega_1 - r\omega_2\right)t\right]\right\} \tag{125}$$

where $d_{ij} = c_{ij}$ if i is even and $d_{ij} = c'_{ij}$ if i is odd. Equation (125) shows that the error frequency spectrum contains peaks at harmonic frequencies (all integer multiples of ω_1 and ω_2) and at cross-modulated frequencies (sums and differences of all fundamental and harmonic frequencies). The amplitudes of the spectral peaks decrease rapidly with increasing frequency because the nth harmonic is weighted by $(1/\varsigma_k)^n$ in the power series.

If the experimenter can identify true modes in $D\{F[\mathbf{w}(t)]\}$ and discard harmonic and cross-modulated error peaks, then error effects are relatively unimportant to modal analysis, since the fundamental frequencies are not shifted. However, estimated damping factors may be somewhat in error because of distortion in the shapes of fundamental peaks.

Estimation of Frequency Response Matrix

The dynamics of a three-dimensional structure, whose frequency response functions are to be identified, are modeled by a three-dimensional coupled system of Nth-order linear time-invariant differential equations in OBJ coordinates. Let $\delta \mathbf{U}(t)$ be a three-element-vector driving function and let $\delta \mathbf{W}(t)$ be a three-element-vector response function at point $\delta \mathbf{W}$. Then $\delta \mathbf{W}(t)$ and $\delta \mathbf{U}(t)$ are related by

$$\sum_{n=1}^{N} \frac{d^n}{dt^n} \mathbf{P}_n \, \delta \mathbf{W}(t) + \mathbf{P}_0 \, \delta \mathbf{W}(t) = \delta \mathbf{U}(t) \tag{126}$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ are 3×3 matrices of constants. Take the Fourier transform of equation (126) to obtain

$$\left[\sum_{n=0}^{N} (i\omega)^n \mathbf{P}_n \right] \delta \mathbf{W}(\omega) = \delta \mathbf{U}(\omega) \tag{127}$$

By means of techniques of linear systems analysis (ref. 7), equation (127) may be solved for $\delta \mathbf{W}(\omega)$ as

$$\delta \mathbf{W}(\omega) = \mathbf{H}(\omega) \, \delta \mathbf{U}(\omega) \tag{128}$$

where $\mathbf{H}(\omega)$, defined as the 3×3 frequency response matrix, is the inverse of $\sum_{n=0}^{N} (i\omega)^n \mathbf{P}_n$. It is desired to estimate $\mathbf{H}(\omega)$ from observations in dual focal plane (DFP) coordinates.

Let ϵ_M (eq. (73)) be sufficiently small, so that mappings D_A and D, defined in equations (21) and (28), are approximated well by

$$\delta \mathbf{w}(t) = D_A[\delta \mathbf{W}(t)] \approx \mathbf{Z}^{-1} \mathbf{A} \, \delta \mathbf{W}(t) \tag{129}$$

and

$$\delta \mathbf{W}(t) = D[\delta \mathbf{w}(t)] \approx \mathbf{P_A Z} \, \delta \mathbf{w}(t) \tag{130}$$

Although $\mathbf{P_A}$ is only a left inverse of $\mathbf{A}$, it follows from equation (30) that

$$\mathbf{Z}^{-1} \mathbf{A} \mathbf{P_A} \mathbf{Z} \, \delta \mathbf{w}(t) = \delta \mathbf{w}(t) \tag{131}$$

whenever $\delta \mathbf{w}(t)$ is contained in the column space of $\mathbf{Z}^{-1}\mathbf{A}$, or equivalently in the range of mapping D_A. Since all $\delta \mathbf{w}(t)$ and $\delta \mathbf{u}(t)$ to be considered are contained in the range of D_A, $\mathbf{P_A Z}$ and $\mathbf{Z}^{-1}\mathbf{A}$ are hereafter treated as two-sided inverses of each other. Therefore $\delta \mathbf{U}(t)$ and $\delta \mathbf{u}(t)$ are related by

$$\delta \mathbf{U}(t) \approx \mathbf{P_A Z} \, \delta \mathbf{u}(t) \tag{132}$$

$$\delta \mathbf{u}(t) \approx \mathbf{Z}^{-1} \mathbf{A} \, \delta \mathbf{U}(t) \tag{133}$$

Let $\delta \mathbf{U}(t)$ be a zero-mean multivariate stochastic process which is stationary in the wide sense (ref. 8). Then $\delta \mathbf{W}(t)$, $\delta \mathbf{u}(t)$, and $\delta \mathbf{w}(t)$ are also zero-mean wide-sense stationary processes. The cross covariance matrix between $\delta \mathbf{W}(t)$ and $\delta \mathbf{U}(t)$ is defined as

$$\mathbf{R_{WU}}(\tau) = E \left[\delta \mathbf{W}(t) \, \delta \mathbf{U}^{*^T}(t - \tau) \right] \tag{134}$$

and the autocovariance matrix of $\delta \mathbf{U}(t)$ is defined as

$$\mathbf{R_{UU}}(\tau) = E\left[\delta \mathbf{U}(t)\delta \mathbf{U}^{*^T}(t - \tau)\right] \tag{135}$$

where E denotes expected value.

The cross-spectral matrix $\mathbf{S_{WU}}(\omega)$ and the autospectral matrix $\mathbf{S_{UU}}(\omega)$ are defined as

$$\mathbf{S_{WU}}(\omega) = F\left[\mathbf{R_{WU}}(\tau)\right] \tag{136}$$

$$\mathbf{S_{UU}}(\omega) = F\left[\mathbf{R_{UU}}(\tau)\right] \tag{137}$$

Similarly define in DFP coordinates

$$\mathbf{S_{uu}}(\omega) = F\left\{E\left[\delta \mathbf{u}(t)\,\delta \mathbf{u}^{*^T}(t - \tau)\right]\right\} \tag{138}$$

$$\mathbf{S_{wu}}(\omega) = F\left\{E\left[\delta \mathbf{w}(t)\,\delta \mathbf{u}^{*^T}(t - \tau)\right]\right\} \tag{139}$$

It can be shown from equations (130) and (132) that

$$\mathbf{S_{UU}}(\omega) = \mathbf{P_A}\,\mathbf{Z}\,\mathbf{S_{uu}}(\omega)\,\mathbf{Z}\mathbf{P}_\mathbf{A}^T \tag{140}$$

$$\mathbf{S_{WU}}(\omega) = \mathbf{P_A}\,\mathbf{Z}\,\mathbf{S_{wu}}(\omega)\,\mathbf{Z}\mathbf{P}_\mathbf{A}^T \tag{141}$$

It is shown in reference 9 that if $\delta \mathbf{W}(\omega)$ and $\delta \mathbf{U}(\omega)$ are related by equation (128), then $\mathbf{S_{WU}}(\omega)$ and $\mathbf{S_{UU}}(\omega)$ are related by

$$\mathbf{S_{WU}}(\omega) = \mathbf{H}(\omega)\,\mathbf{S_{UU}}(\omega) \tag{142}$$

Thus, if $\mathbf{S_{WU}}(\omega)$ and $\mathbf{S_{UU}}(\omega)$ are known, the solution of equation (142) determines $\mathbf{H}(\omega)$. It can be shown (ref. 9) that if $\mathbf{W}(t)$ contains additive noise, then the solution of equation (142) furnishes a minimum-mean-square-error estimate of $\mathbf{H}(\omega)$.

Since experimental data are recorded in DFP coordinates, it is advantageous computationally (ref. 2) to estimate $\mathbf{S_{uu}}(\omega)$ and $\mathbf{S_{wu}}(\omega)$ and to then estimate $\mathbf{H}(\omega)$ by using equations (140) through (142). Smoothed estimates of $\mathbf{S_{uu}}(\omega)$ and $\mathbf{S_{wu}}(\omega)$ are computed as averaged conjugate outer products of the Fourier transforms of N observed records of length T of $\delta \mathbf{w}(t)$ and $\delta \mathbf{u}(t)$. Thus the Fourier transforms of $\delta \mathbf{w}(t)$ and $\delta \mathbf{u}(t)$ for the nth record are

$$\delta \hat{\mathbf{w}}_n(\omega) = \int_{(n-1)T}^{nT} \delta \mathbf{w}(t)\,e^{-i\omega t}\,dt \tag{143}$$

$$\delta \hat{\mathbf{u}}(\omega) = \int_{(n-1)T}^{nT} \delta \mathbf{u}(t)\,e^{-i\omega t}\,dt \tag{144}$$

and the smoothed estimated autospectral and cross-spectral matrices are

$$\hat{\mathbf{S}}_{\mathbf{uu}}(\omega) = \frac{1}{N}\sum_{n=1}^{N} \delta \hat{\mathbf{u}}_n(\omega)\,\delta \hat{\mathbf{u}}_n^{*^T}(\omega) \tag{145}$$

$$\hat{\mathbf{S}}_{\mathbf{wu}}(\omega) = \frac{1}{N}\sum_{n=1}^{N} \delta \hat{\mathbf{w}}_n(\omega)\,\delta \hat{\mathbf{u}}_n^{*^T}(\omega) \tag{146}$$

Smoothed estimates of $\mathbf{S_{UU}}(\omega)$ and $\mathbf{S_{WU}}(\omega)$ may be obtained from $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$ and $\hat{\mathbf{S}}_{\mathbf{wu}}(\omega)$ by means of transformations (140) and (141) to yield

$$\hat{\mathbf{S}}_{\mathbf{UU}}(\omega) = \mathbf{P_A}\mathbf{Z}\,\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)\,\mathbf{Z}\mathbf{P}_\mathbf{A}^T \tag{147}$$

$$\hat{\mathbf{S}}_{\mathbf{WU}}(\omega) = \mathbf{P_A}\mathbf{Z}\,\hat{\mathbf{S}}_{\mathbf{wu}}(\omega)\,\mathbf{Z}\mathbf{P}_\mathbf{A}^T \tag{148}$$

In principle $\mathbf{H}(\omega)$ can be estimated by equation (142), provided $\hat{\mathbf{S}}_{\mathbf{UU}}(\omega)$ is nonsingular, as

$$\hat{\mathbf{H}}(\omega) = \hat{\mathbf{S}}_{\mathbf{WU}}(\omega)\,\hat{\mathbf{S}}_{\mathbf{UU}}^{-1}(\omega) \tag{149}$$

If the three components of $\mathbf{U}(t)$ are mutually independent stochastic processes, then $\mathbf{S_{UU}}(\omega)$ is a nonsingular diagonal matrix, from which $\mathbf{H}(\omega)$ is readily obtained via equation (149). If $\mathbf{S_{UU}}(\omega)$ is nonsingular but nondiagonal, calculation of its inverse in equation (149) for each value of ω may be excessively expensive computationally. If $\mathbf{U}(t)$ is confined to a line or to a plane in space, so that

$$\mathbf{U}(t) = g_1(t)\,\mathbf{U}_1 + g_2(t)\,\mathbf{U}_2 \tag{150}$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are fixed, then $\mathbf{S_{UU}}(\omega)$ is singular, and equation (149) cannot be employed. Two alternative procedures for estimation of $\mathbf{H}(\omega)$ are now proposed.

Estimation of Diagonal Frequency Response Matrix

The observed driving function, $\delta\mathbf{u}(t)$, and the observed response function, $\delta\mathbf{w}(t)$, in DFP coordinates are related through equations (126), (130), and (132). Take Fourier transforms of $\delta\mathbf{W}(t)$ and $\delta\mathbf{U}(t)$ in equations (130) and (132) and premultiply equation (128) by $\mathbf{Z}^{-1}\mathbf{A}$ to obtain

$$\delta\mathbf{w}(\omega) = \mathbf{Z}^{-1}\mathbf{A}\,\mathbf{H}(\omega)\,\mathbf{P_A}\mathbf{Z}\,\delta\mathbf{u}(\omega) \tag{151}$$

It follows from equation (151) that $\mathbf{S_{wu}}(\omega)$ and $\mathbf{S_{uu}}(\omega)$ are related by

$$\mathbf{S_{wu}}(\omega) = \mathbf{Z}^{-1}\mathbf{A}\,\mathbf{H}(\omega)\,\mathbf{P_A}\mathbf{Z}\,\mathbf{S_{uu}}(\omega) \tag{152}$$

so that

$$\mathbf{P_A}\mathbf{Z}\,\mathbf{S_{wu}}(\omega) = \mathbf{H}(\omega)\,\mathbf{P_A}\mathbf{Z}\,\mathbf{S_{uu}}(\omega) \tag{153}$$

If it is known that $\mathbf{H}(\omega)$ is diagonal (interactions between the X_c, Y_c, and Z_c components are absent), then $\mathbf{H}(\omega)$ is easily obtained from equation (153) as follows. Let

$$\Sigma_{\mathbf{wu}}(\omega) = \mathbf{P_A}\mathbf{Z}\,\mathbf{S_{wu}}(\omega) \tag{154}$$

and

$$\Sigma_{\mathbf{uu}}(\omega) = \mathbf{P_A}\mathbf{Z}\,\mathbf{S_{uu}}(\omega) \tag{155}$$

so that equation (153) becomes

$$\Sigma_{\mathbf{wu}}(\omega) = \mathbf{H}(\omega)\,\Sigma_{\mathbf{uu}}(\omega) \tag{156}$$

Then $h_{ii}(\omega)$ can be estimated by using equation (156) as

$$\hat{h}_{ii,j}(\omega) = \frac{\hat{\sigma}_{\mathbf{wu}_{ij}}(\omega)}{\hat{\sigma}_{\mathbf{uu}_{ij}}(\omega)} \tag{157}$$

for $i = 1, 2, 3$, and $j = 1, 2, 3, 4$, where $\sigma_{\mathbf{wu}_{ij}}(\omega)$ and $\sigma_{\mathbf{uu}_{ij}}(\omega)$ are elements of $\Sigma_{\mathbf{wu}}(\omega)$ and $\Sigma_{\mathbf{uu}}(\omega)$ and $\hat{}$ denotes values obtained from the estimated matrices $\hat{\mathbf{S}}_{\mathbf{wu}}(\omega)$ and $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$. If $\mathbf{H}(\omega)$ is diagonal, then $\hat{h}_{ii,j}(\omega)$ will be independent of j. Conversely, if equation (157) is independent of j for arbitrary

$\mathbf{\Sigma_{wu}}(\omega)$ and $\mathbf{\Sigma_{uu}}(\omega)$, then $\mathbf{H}(\omega)$ must be diagonal. Thus, the diagonality of $\mathbf{H}(\omega)$ may be confirmed by computations of $\hat{h}_{ii,j}(\omega)$ (eq. (157)) for all 12 combinations of indices for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ and verification that $\hat{h}_{ii,j}(\omega)$ is independent of j.

Estimation of Nondiagonal Frequency Response Matrix

In the general case where $\mathbf{H}(\omega)$ is a full nondiagonal matrix, let a unidirectional driving function $g(t)$ be applied in three independent measurements in the X_c-, Y_c-, and Z_c-directions separately. One column of $\mathbf{H}(\omega)$ may then be estimated from each measurement, in DFP coordinates, of the observed input $\delta\mathbf{u}(t)$ and the observed response $\delta\mathbf{w}(t)$. Thus apply $g(t)$ in the X_c-direction. Input vector $\delta\mathbf{U}(t)$ is then

$$\delta\mathbf{U}(t) = g(t) \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{158}$$

Expressed in DFP coordinates, the input vector is

$$\delta\mathbf{u}(t) = g(t)\,\mathbf{Z}^{-1}\mathbf{A} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \tag{159}$$

Autospectral matrix $\mathbf{S_{uu}}(\omega)$ is then

$$\mathbf{S_{uu}}(\omega) = s_{gg}(\omega)\mathbf{Z}^{-1}\mathbf{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{A}^T\mathbf{Z}^{-1} \tag{160}$$

where $s_{gg}(\omega)$ is the autopower spectrum of $g(t)$. Equation (160) may be rewritten as

$$\mathbf{P_A}\mathbf{Z}\,\mathbf{S_{uu}}(\omega) = s_{gg}(\omega)\,\mathbf{\Gamma}_k \tag{161}$$

where (for $k = 1$)

$$\mathbf{\Gamma}_1 = \begin{bmatrix} \dfrac{a_{11}}{\varsigma_1} & \dfrac{a_{21}}{\varsigma_1} & \dfrac{a_{31}}{\varsigma_2} & \dfrac{a_{41}}{\varsigma_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{162}$$

and a_{ij} is the element in the ith row and jth column of $\mathbf{A}$. From equations (153) and (161), it follows that

$$\mathbf{P_A}\mathbf{Z}\,\mathbf{S_{wu}}(\omega) = s_{gg}(\omega)\,\mathbf{H}(\omega)\,\mathbf{\Gamma}_k \tag{163}$$

Autopower spectrum $s_{gg}(\omega)$ can be estimated from $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$ by means of equation (161). Column 1 of $\mathbf{H}(\omega)$ is then estimated from the estimated $\hat{s}_{gg}(\omega)$ and cross-spectral matrix $\hat{\mathbf{S}}_{\mathbf{wu}}(\omega)$ by means of equation (163).

Columns 2 and 3 of $\mathbf{H}(\omega)$ are estimated in a similar manner. Apply $g(t)$ in the Y_c-direction for $k = 2$ to give

$$\delta\mathbf{U}(t) = g(t) \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \tag{164}$$

and

$$\mathbf{\Gamma}_2 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ \dfrac{a_{12}}{\varsigma_1} & \dfrac{a_{22}}{\varsigma_1} & \dfrac{a_{32}}{\varsigma_2} & \dfrac{a_{42}}{\varsigma_2} \\ 0 & 0 & 0 & 0 \end{Bmatrix} \tag{165}$$

Use equations (161) and (163) to estimate $s_{gg}(\omega)$ and column 2 of $\mathbf{H}(\omega)$. Finally apply $g(t)$ in the Z_c-direction for $k = 3$ to obtain

$$\delta \mathbf{U}(t) = g(t) \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{166}$$

and

$$\boldsymbol{\Gamma}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_{13}}{\varsigma_1} & \frac{a_{23}}{\varsigma_1} & \frac{a_{33}}{\varsigma_2} & \frac{a_{43}}{\varsigma_2} \end{bmatrix} \tag{167}$$

Use equations (161) and (163) as before to estimate $s_{gg}(\omega)$ and column 3 of $\mathbf{H}(\omega)$.

Least-Squares Estimation of Frequency Response Matrix

A method is developed for least-squares estimation of $\mathbf{H}(\omega)$ from equations (161) and (163). Since $\mathbf{S_{uu}}(\omega)$ and $\mathbf{S_{wu}}(\omega)$ are estimated by means of equations (145) and (146) and since measurements of $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are noisy, no values of $\hat{s}_{gg}(\omega)$ and $\hat{\mathbf{H}}(\omega)$ exist in general which will satisfy equations (161) and (163) for estimated matrices $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$ and $\hat{\mathbf{S}}_{\mathbf{wu}}(\omega)$. Thus, in general for every value of $\hat{s}_{gg}(\omega)$ in equation (161)

$$\hat{\mathbf{S}}_{\mathbf{uu}}(\omega) \neq \hat{s}_{gg}(\omega) \, \boldsymbol{\Lambda}_k \tag{168}$$

where

$$\boldsymbol{\Lambda}_k = \mathbf{Z}^{-1} \mathbf{A} \boldsymbol{\Gamma}_k \tag{169}$$

and for every $\hat{\mathbf{H}}(\omega)$ in equation (163)

$$\mathbf{P_A} \mathbf{Z} \, \hat{\mathbf{S}}_{\mathbf{wu}}(\omega) \neq \hat{s}_{gg}(\omega) \hat{\mathbf{H}}(\omega) \, \boldsymbol{\Gamma}_k \tag{170}$$

Optimum estimates of $s_{gg}(\omega)$ and column k of $\mathbf{H}(\omega)$, denoted by $\hat{\mathbf{h}}_{.k}(\omega)$, can be determined by choosing $\hat{s}_{gg}(\omega)$ and $\hat{\mathbf{h}}_{.k}(\omega)$ such that the sum of the squares of the Euclidian distances between corresponding columns is minimized in inequalities (168) and (170). Thus, from equation (168) define

$$
\begin{aligned}
\epsilon_u^2 &= \sum_{j=1}^{4} \| \hat{\mathbf{s}}_{\mathbf{uu}.j}(\omega) - \hat{s}_{gg}(\omega) \, \boldsymbol{\lambda}_{.j} \|^2 \\
&= \sum_{j=1}^{4} \left[\hat{\mathbf{s}}_{\mathbf{uu}.j}(\omega) - \hat{s}_{gg}(\omega) \, \boldsymbol{\lambda}_{.j} \right]^T \left[\hat{\mathbf{s}}_{\mathbf{uu}.j}(\omega) - \hat{s}_{gg}(\omega) \, \boldsymbol{\lambda}_{.j} \right]
\end{aligned} \tag{171}
$$

where $\hat{\mathbf{s}}_{\mathbf{uu}.j}(\omega)$ is the jth column of $\hat{\mathbf{S}}_{\mathbf{uu}}(\omega)$, and $\boldsymbol{\lambda}_{.j}$ is the jth column of $\boldsymbol{\Lambda}_k$.

To minimize ϵ_u^2, differentiate equation (171) with respect to $\hat{s}_{gg}(\omega)$, set the result equal to zero, and solve for $\hat{s}_{gg}(\omega)$ to obtain

$$\hat{s}_{gg}(\omega) = \frac{\sum_{j=1}^{4} \boldsymbol{\lambda}_{.j}^T \hat{\mathbf{s}}_{\mathbf{uu}.j}(\omega)}{\sum_{j=1}^{4} \boldsymbol{\lambda}_{.j}^T \boldsymbol{\lambda}_{.j}} \tag{172}$$

To estimate $\hat{\mathbf{h}}_{.k}(\omega)$, use equation (170) to define

$$\epsilon_h^2 = \sum_{j=1}^{4} \| \hat{\boldsymbol{\sigma}}_{\mathbf{wu}.j}(\omega) - \hat{s}_{gg}(\omega) \, \gamma_{kj} \, \hat{\mathbf{h}}_{.j}(\omega) \|^2 \tag{173}$$

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where $\boldsymbol{\sigma}_{\mathbf{wu}_{.j}}(\omega)$ is the jth column of $\mathbf{P_A Z S_{wu}}(\omega)$ and $\hat{s}_{gg}(\omega)$ is given by equation (172). Proceed as above to minimize ϵ_h^2, and solve for $\hat{\mathbf{h}}_{.k}(\omega)$ to obtain

$$\hat{\mathbf{h}}_{.k}(\omega) = \frac{\sum_{j=1}^{4} \gamma_{kj} \hat{\boldsymbol{\sigma}}_{\mathbf{wu}_{.j}}(\omega)}{\hat{s}_{gg}(\omega) \sum_{j=1}^{4} \gamma_{kj}^2} \tag{174}$$

where γ_{kj} is the element in the kth row and jth column of matrix $\boldsymbol{\Gamma}_k$.

Concluding Remarks

The accuracy of frequency response function estimation from stereoscopic dynamic data by a rapid computational procedure has been computed. It was shown that the error incurred by reversal of the operations of coordinate transformation and Fourier transformation includes terms dependent on the magnitudes of components of motion normal to the camera focal planes (whose effects cannot be eliminated), as well as on the condition of the stereoscopic projection matrix. Additional error terms exist which may be eliminated by proper scaling of the Fourier transforms prior to coordinate transformation. Spectral analysis of the dominant error term, which is dependent on the scale factor, shows the presence of harmonic and cross-modulated spectral peaks caused by the nonlinearity of the perspective projection from spatial coordinates to focal plane coordinates. Only when the camera geometry is such that all motion is parallel to both camera planes is the spatial transformation linear and no error introduced.

The full 3×3 frequency response matrix of a three-dimensional structure can be practicably estimated from one stereoscopic dynamic data set only if the orthogonal components of the driving function are mutually independent stochastic processes. Alternative procedures have been given for estimation of a diagonal frequency response matrix and for least-squares estimation of a full matrix, column-by-column in three independent sets of measurements.

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1. Report No. NASA TP-2471	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Rapid Estimation of Frequency Response Functions by Close-Range Photogrammetry		5. Report Date November 1985
		6. Performing Organization Code 505-31-53-09
7. Author(s) John S. Tripp		8. Performing Organization Report No. L-15955
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No.
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001		13. Type of Report and Period Covered Technical Paper
		14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

The accuracy of a rapid method which estimates the frequency response function from stereoscopic dynamic data is computed. It is shown that reversal of the order of the operations of coordinate transformation and Fourier transformation, which provides a significant increase in computational speed, introduces error. A portion of the error, proportional to the perturbation components normal to the camera focal planes, cannot be eliminated. The remaining error may be eliminated by proper scaling of frequency data prior to coordinate transformation. Methods are developed for least-squares estimation of the full 3×3 frequency response matrix for a three-dimensional structure.

17. Key Words (Suggested by Authors(s)) Frequency response function Cross power spectral matrix Direct linear transformation Close-range photogrammetry Modal identification	18. Distribution Statement Unclassified - Unlimited Subject Category 35

19. Security Classif.(of this report) Unclassified	20. Security Classif.(of this page) Unclassified	21. No. of Pages 29	22. Price A03

**National Aeronautics and
Space Administration
Code NIT-3**

**Washington, D.C.
20546-0001**

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