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**The Nature of Symmetric  
Instability and Its  
Similarity to Convective  
and Inertial Instability**

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# The Nature of Symmetric Instability and Its Similarity to Convective and Inertial Instability

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## 1. Introduction

As one of the theoretical possibilities for the initiation of frontal rainbands, SI (Symmetric Instability), moist SI, and especially CSI (Conditional SI) have been shown to have many attractive features (Hoskins, 1974; Raymond, 1978; Bennetts and Hoskins, 1979; Emanuel, 1979, 1983; Parsons and Hobbs, 1983, Xu and Zhou, 1982; Xu, 1984). However in comparison with BI (Buoyancy or Convective Instability) (cf. Chandraseckhar, 1961 and Joseph, 1976a,b), the SI problem has not been explored sufficiently. Obviously the presence of basic wind shear and, by the thermal wind relationship, a horizontal temperature gradient makes SI problems generally more difficult than BI problems.

It has been noted, however, that the most unstable motion of SI may be locally described as II (Inertial Instability) on isentropic surfaces in an hydrostatic atmosphere (Hoskins, 1974) and that a linear viscous SI problem can be transformed into a form of linear viscous BI or II (Xu and Zhou, 1982). Thus the similarity and intrinsic relationship among SI, BI and II for nonlinear viscous flows becomes more perceptible. In the following section we will see that the governing SI equations can be changed via a coordinate transformation into a form similar to those of the BI problem. Thus the well-studied BI theory can be utilized to improve our understanding of SI problems.

Since a similarity among SI, BI and II exists, it will be expected that SI has the same basic nature of BI and II and thus the physical picture of SI may be illustrated almost as intuitively as that of BI or II. In fact, by treating buoyancy and inertial effects separately, an intuitive analysis for

the parcel dynamics of SI has been attempted by Emanuel (1983). A similar idea was also used by McIntyre (1970) to explain the diffusive destabilization mechanism in a viscous SI problem where the Prandtl number differed from unity. Referring to Emanuel's analysis, we will emphasize the analogy between SI, BI and II. To illustrate the properties of SI, we shall analyze the dynamics and energetics of the parcel motions in Section 3. We shall also examine to what extent the results for dry SI may be applied to moist SI. Moist SI will be discussed in Section 4, where a qualitative analysis for CSI will also be involved.

## 2. Similarity Between SI, BI, and II

Assume that  $V = (0, V(x, z), 0)$  and  $\Theta_0(x, z)$  are the basic wind and potential temperature fields respectively (cf., Fig. 1). By the Boussinesq approximation we have the following nonlinear viscous equations for symmetric perturbations

$$\frac{du}{dt} - fv + \alpha_0 \frac{\partial p}{\partial x} = \alpha_0 \nabla \cdot \rho_0 v \nabla u, \quad (2.1a)$$

$$\frac{dw}{dt} - \theta + \alpha_0 \frac{\partial p}{\partial z} = \alpha_0 \nabla \cdot \rho_0 v \nabla w, \quad (2.1b)$$

$$f \frac{dv}{dt} + F^2 u + S^2 w = \alpha_0 \nabla \cdot \rho_0 v \nabla fv, \quad (2.1c)$$

$$\frac{d\theta}{dt} + S^2 u + N^2 w = \alpha_0 \nabla \cdot \rho_0 \nabla \theta \quad , \quad (2.1d)$$

$$\frac{\partial}{\partial x}(\rho_0 u) + \frac{\partial}{\partial z}(\rho_0 w) = 0 \quad . \quad (2.1e)$$

where  $f$  is the Coriolis parameter (assumed constant) and  $\rho_0(x, z) \equiv \alpha_0^{-1}(x, z)$  is the density of the background atmosphere. Here  $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z}$ ,  $(u, w, v, \theta, p) \equiv (u', w', v', g \frac{\theta'}{\theta_0}, p')$  and the primes represent perturbation quantities. In (2.1b) the effect of perturbation pressure on buoyancy is neglected. The symbol  $\nu$  is the kinematic molecular or eddy diffusivity.  $N$ ,  $S$  and  $F$  the Brunt Väisälä, baroclinic and inertial frequencies respectively, are defined as follows:

$$N^2 \equiv \frac{g}{\theta_0} \frac{\partial \theta_0}{\partial z} \quad , \quad S^2 \equiv \frac{g}{\theta_0} \frac{\partial \theta_0}{\partial x} = f \frac{\partial V}{\partial z} \quad , \quad F^2 \equiv f(f + \frac{\partial V}{\partial x}) \quad , \quad (2.2)$$

where thermal wind relation has been used.

Consider now the local behaviour of (2.1) by assuming that  $N^2$ ,  $S^2$  and  $F^2$  are constants. We can introduce a rotational transformation  $R_{\hat{\alpha}} \equiv \begin{pmatrix} \cos \hat{\alpha} & -\sin \hat{\alpha} \\ \sin \hat{\alpha} & \cos \hat{\alpha} \end{pmatrix}$  where positive  $\hat{\alpha}$  corresponds to anticlockwise rotation, i.e., see Fig. 1,

$$\begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{w} \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \cos \hat{\alpha} & -\sin \hat{\alpha} \\ \sin \hat{\alpha} & \cos \hat{\alpha} \end{pmatrix} \begin{pmatrix} x & u & fv \\ z & w & \theta \end{pmatrix} \quad , \quad (2.3)$$

where

$$\hat{\alpha} = \frac{1}{2} \text{Arctan} \left( \frac{-2S^2}{N^2 - F^2} \right) . \quad (2.4)$$

With (2.3) and (2.4), (2.1) can be transformed into

$$\left( \frac{\partial}{\partial t} + \hat{\mathbf{y}} \cdot \nabla - \alpha \nabla \cdot \rho_0 \mathbf{v} \nabla \right) \hat{\mathbf{y}} - \hat{\mathbf{h}} + \alpha \nabla p = 0 , \quad (2.5a)$$

$$\left( \frac{\partial}{\partial t} + \hat{\mathbf{y}} \cdot \nabla - \alpha \nabla \cdot \rho_0 \mathbf{v} \nabla \right) \hat{\mathbf{h}} + \hat{\Pi} \cdot \hat{\mathbf{y}} = 0 , \quad (2.5b)$$

$$\nabla \cdot \rho_0 \hat{\mathbf{y}} = 0 , \quad (2.5c)$$

where  $\nabla = (\partial/\partial \hat{x}, \partial/\partial \hat{z})$ ,  $\hat{\mathbf{y}} = (\hat{u}, \hat{w})$ ,  $\hat{\mathbf{h}} = (\hat{a}, \hat{b})$ , and  $\hat{\Pi} = \begin{pmatrix} \hat{F}^2 & 0 \\ 0 & \hat{N}^2 \end{pmatrix}$  is the diagonal form of stability tensor  $\Pi = \begin{pmatrix} F^2 & S^2 \\ S^2 & N^2 \end{pmatrix}$ . Here

$$\begin{aligned} \hat{N}^2 &= \frac{1}{2}(F^2 + N^2 + B) , \\ \hat{F}^2 &= \frac{1}{2}(F^2 + N^2 - B) , \\ B &= \text{sgn}(N^2 - F^2) [(N^2 - F^2)^2 + 4S^4]^{1/2} . \end{aligned} \quad (2.6)$$

It has been shown by Xu and Zhou (1982) that (2.4), or equivalently

$$\tan \hat{\alpha} = \frac{N^2 - F^2}{2S^2} - \frac{[(N^2 - F^2)^2 + 4S^4]^{1/2}}{2S^2} , \quad (2.7)$$

gives the most unstable or least stable slope angle for linear SI, i.e., a disturbance which induces parcel displacements at this angle in the  $(x, z)$  plane will be most unstable. Now with (2.5) it is easy to see that the most unstable angle  $\hat{\alpha}$  is indeed only a property of the basic flow and not changed by including nonlinearity. From (2.5) we can see that if  $N^2 < F^2$ , then  $\hat{z}$  is in the direction of  $\hat{\alpha}$  with  $\hat{N}^2 = \min_{\alpha} \omega^2(\alpha)$  and  $\hat{F}^2 = \max_{\alpha} \omega^2(\alpha)$  where

$$\omega^2(\alpha) = N^2 \sin^2 \alpha + 2S^2 \sin \alpha \cos \alpha + F^2 \cos^2 \alpha \quad (2.8)$$

(cf. Ooyama, 1966, Hoskins, 1974, or Xu et al., 1982). If  $N^2 > F^2$ , then  $\hat{x}$  is in the direction of  $\hat{\alpha}$  where  $\hat{N}^2 = \max_{\alpha} \omega^2(\alpha)$  and  $\hat{F}^2 = \min_{\alpha} \omega^2(\alpha)$ .

In fact, the above two cases, corresponding to the sign of  $N^2 - F^2$ , share a similar nature symmetrically because one can change into the other simply by interchanging  $(\hat{x}, \hat{u}, \hat{a})$  and  $(\hat{z}, \hat{w}, \hat{b})$ . For  $N^2 < F^2$  and  $F^2 > 0$ , Eqs. (2.5) have a similar form to the governing equations of nonlinear BI problems in a stratified ( $\hat{N}^2$ ), rotating ( $\hat{F}$ ) fluid. Thus, even in the nonlinear sense, SI is similar to BI and  $\hat{z}$  is the most unstable (if  $\hat{N}^2 < 0$ ) or least stable (if  $\hat{N}^2 > 0$ ) direction, i.e., the direction of  $\hat{\alpha}$ . For  $N^2 > F^2$  and  $N^2 > 0$ , Eqs. (2.5) indicate that SI is similar to II (Inertial Instability). However, as we mentioned above, these two cases corresponding to the sign of  $N^2 - F^2$  are similar. Thus, BI, II and SI are similar, at least, if we only consider the transverse circulation  $(\hat{u}, \hat{w})$  and take  $(\hat{a}, \hat{b})$  as two scalars.



So far we can see that, although BI and II are particular cases of SI, the local SI or SI problem with a uniform basic state can be transformed into the form of BI or II. Furthermore both linear (Xu and Zhou, 1982) and nonlinear theories (cf. Chapter IV, Xu, 1984) of generalized energetics indicate that the nonuniformity of the basic state does not change the basic nature of BI, II and SI. Therefore, it is indeed feasible either to induce their common natures from already known II, SI, and especially BI theory, or to deduce certain properties for a particular stability problem, especially the SI problem which we are particularly concerned with from their common natures. Based on the previous theories of BI, II, and SI given by the formerly referenced studies and on the similarity among BI, II and SI discussed above, the basic nature of SI, can be summarized as follows:

- i) For an inviscid fluid, the instability simply depends on the external constraints, i.e., the stability tensor  $\Pi = \begin{pmatrix} F^2 & S^2 \\ S^2 & N^2 \end{pmatrix}$  which determines the orientation and strength of the instability, and has no scale selection. The sufficient condition for instability is

$$\min (\hat{N}^2, \hat{F}^2) < 0, \text{ i.e., } N^2 F^2 - S^4 < 0, \quad (2.9)$$

where the unusual case of both  $N^2 < 0$  and  $F^2 < 0$  is not included in the second formula. The most unstable slope angle is given by (2.4) or (2.7), which with (2.9) leads to

$$|k_v| < |\tan \hat{\alpha}| < |k_\theta|. \quad (2.10)$$

Here  $k_v = -F^2/S^2$  is the slope of iso-momentum (i.e.,  $fx + V = \text{const}$ ) surfaces and also the slope of the background absolute vorticity vector, while  $k_\theta = -S^2/N^2$  is the slope of background isentropic surfaces.

- ii) For a viscous fluid, the instability results from a trade-off between two energy transport regimes - the conductive regime (no perturbation) and circulation regime. When the latter becomes more effective, instability occurs and the circulation depends on two other competitive factors: the external effect of the stability tensor and internal effect of the stress tensor due to viscosity and pressure gradient. In this case, a scale selection mechanism exists and depends on the competition of the above two factors. Since the Prandtl number = 1 here, by the already known theory (Chandrasekhar, 1961 and Veronis, 1966) which asserts that the primary bifurcation point must be a steady one unless both rotation (measured by the Coriolis parameter) is strong enough and the Prandtl number is small enough (much smaller than unity), we may expect that the viscous SI motion with certain boundary condition will first occur as a steady circulation.

### 3. Parcel Dynamics and Energetics

We shall for simplicity omit the effect of viscosity and thermal conductivity. In this case ( $\nu = 0$ ), (2.1c-d) give

$$\frac{d}{dt} (v + M) = 0 \quad , \quad (3.1a)$$

$$\frac{d}{dt} (\theta + B_o) \equiv \frac{d}{dt} (\theta + g \ln \theta_o) = 0 , \quad (3.1b)$$

where  $M \equiv fx + V$  is the absolute momentum of the basic state. If a parcel (here a parcel in  $\underline{x} = (x, z)$  plane is a tube along  $y$  direction in the 3-dimensional space) initially with  $v = \theta = 0$  at  $\underline{x}^o$  moves to  $\underline{x}$ , then the integration of (3.1a-b) following the parcel gives

$$v = M(\underline{x}^o) - M(\underline{x}) , \quad (3.2a)$$

$$\theta = \theta_o(\underline{x}^o) - \theta_o(\underline{x}) . \quad (3.2b)$$

Now we rewrite (2.1a-b) as ( $v = 0$ )

$$\ddot{\xi} = fv - \alpha_o \frac{\partial p}{\partial x} , \quad (3.3a)$$

$$\ddot{\zeta} = \theta - \alpha_o \frac{\partial p}{\partial z} , \quad (3.3b)$$

where  $(\dot{\phantom{x}}) \equiv \frac{d}{dt}$ . Since the pressure perturbations only redistribute the perturbation energy (cf. Chapter IV, Xu, 1984), we may ignore the effect of pressure perturbations and thus follow the methods of Emanuel (1983), to obtain physical insights about SI.

Alternatively, Eq. (2.5) with  $v = 0$  suggests that the above two step analysis, i.e., (3.1) - (3.3), can be simplified by locally introducing a "thermal-inertial potential"

$$\hat{A} \equiv f(fx + V) \cos \alpha - B_0 \sin \alpha$$

for the basic state. It is easy to prove that  $\hat{A} = \hat{A}(x)$  is locally independent of  $z$ , i.e., an iso- $\hat{A}$  line is locally parallel to  $z$  coordinate (see Fig. 3.3), and that  $\partial \hat{A} / \partial x = F^2$ . The quantity  $\hat{a}$  defined in (2.3) is the perturbation of  $\hat{A}$ . Thus, similarly to (3.2) - (3.3), we have

$$\hat{a} = \hat{A}(x^0) - \hat{A}(x) ,$$

and

$$\ddot{\xi} = \hat{a} - \alpha_0 \frac{\partial p}{\partial \hat{x}} .$$

If we consider  $\hat{a}$ ,  $\hat{A}$ ,  $x$  and  $\xi$  as  $\theta$ ,  $\theta_0$ ,  $z$  and  $\zeta$  respectively, then we have pure BI instability.

By considering the energy exchange between a displaced parcel and the environment, some useful insights into the nature of SI can be gained. For a locally uniform basic state, i.e.,  $\Pi = \begin{pmatrix} F^2 & S^2 \\ S^2 & N^2 \end{pmatrix}$  is constant (3.2a-b) can be rewritten as

$$\dot{h} = -\Pi \cdot \underline{\dot{\zeta}} , \quad (3.4)$$

where  $\underline{h} \equiv (fv, \theta)$  and  $\underline{\ell} \equiv (\xi, \zeta) \equiv |\underline{\ell}|(\cos \alpha, \sin \alpha)$  is the parcel displacement in the cross section  $(\hat{x}, \hat{z})$  or  $(x, z)$ . Similarly (3.3a-b) can be rewritten as

$$\frac{d^2}{dt^2} \underline{\ell} = -\underline{\Pi} \cdot \underline{\ell} - \alpha_0 \nabla p \quad (3.5)$$

Eq. (3.5) indicates that a fluid parcel with displacement  $\underline{\ell}$  is accelerated by two forces: (a) related to the stability tensor of the basic flow  $-\underline{\Pi} \cdot \underline{\ell}$ , and (b) due to the perturbation pressure gradient. Thus, the energy absorbed (or released) by a fluid parcel from (or to) the basic flow is

$$\begin{aligned} -A &= -\rho_0 \int \underline{\ell} \cdot \underline{\Pi} \cdot d\underline{\ell} = -\frac{\rho_0}{2} \int d(F^2 \xi^2 + 2S^2 \zeta \xi + N^2 \zeta^2) \\ &= -\frac{\rho_0}{2} \omega^2 |\underline{\ell}|^2, \end{aligned}$$

or

$$= -\frac{\rho_0}{2} (\hat{F}^2 \hat{\xi}^2 + \hat{N}^2 \hat{\zeta}^2) \text{ in } (\hat{x}, \hat{z}) \text{ coordinates of (2.3),} \quad (3.6)$$

where  $\omega^2 \equiv \omega^2(\alpha)$  is given in (2.8). Eq. (3.6) indicates that (a)  $-A$  depends on the parcel's displacement  $\underline{\ell}$  only and not on the particular trajectory; (b) for fixed  $|\underline{\ell}|$ , when  $\underline{\ell}$  is along (or normal to) the

most unstable slope of (2.4), i.e.,  $\tan^{-1}(\zeta/\xi) \equiv \alpha = \hat{\alpha}$  (or  $\tan^{-1}(\zeta/\xi) \equiv \hat{\alpha} \perp \alpha$ ),  $-A$  reaches a maximum (or minimum) where  $\omega^2 = \min_{\alpha} \omega^2 = \min(\hat{F}^2, \hat{N}^2)$  (or  $\omega^2 = \max_{\alpha} \omega^2 = \max(\hat{F}^2, \hat{N}^2)$ ).

Furthermore the instantaneous power gained (or lost) by a fluid parcel with velocity  $\underline{y}$  at  $\underline{l}$  from (or to) the basic flow is

$$-W = -\rho_0 \underline{y} \cdot \underline{\Pi} \cdot \underline{l} = -\rho_0 \omega^2(\alpha, \beta) |\underline{y}| |\underline{l}|, \quad |\alpha - \beta| < \frac{\pi}{2} \quad (3.7)$$

where

$$\omega^2(\alpha, \beta) = F^2 \cos \alpha \cos \beta + S^2 \sin(\alpha + \beta) + N^2 \sin \alpha \sin \beta,$$

$\alpha$  and  $\beta$  are the direction angles of  $\underline{l}$  and  $\underline{y}$ , respectively. Here the constraint  $|\alpha - \beta| < \frac{\pi}{2}$  is required by the fact that for any growing disturbance the angle between  $\underline{y} = \frac{d\underline{l}}{dt}$  and  $\underline{l}$ , i.e.,  $|\alpha - \beta|$ , must be acute. In this case, it is also easy to prove that for any fixed  $|\underline{y}|$  and  $|\underline{l}|$   $-W$  reaches the maximum (minimum) when  $\alpha = \beta = \hat{\alpha}$  ( $\alpha = \beta \perp \hat{\alpha}$ ).

#### 4. Moist SI and CSI

As mentioned before, the parcel dynamics associated with (3.2a-b) and (3.3a-b) does not necessarily require the thermal wind relation between  $\Theta_0$  and  $V$ . Thus by replacing  $\Theta_0$  with  $\Theta_w$  the same analyses and results for the parcel dynamics will be exactly applicable to moist SI. However, to estimate the most unstable slope angle for moist SI, we need to analyze the moist parcel energetics as follows.

For a moist atmosphere, if the basic moisture field is also only a function of  $(x, z)$ , then in addition to (2.2) there are another two basic frequencies  $N_0$  and  $S_0$  defined as

$$N_0^2 = \frac{g}{\Theta_0} \frac{\partial \Theta_w}{\partial z}, \quad S_0^2 = \frac{g}{\Theta_0} \frac{\partial \Theta_w}{\partial x}, \quad (4.1)$$

where  $\Theta_w$  is the wet bulb potential temperature of the basic state.

If the atmosphere is saturated everywhere, i.e., there is enough condensed liquid (or solid) water in downdrafts, then  $S^2$  and  $N^2$  in (2.1d) should be replaced by  $N_o^2$  and  $S_o^2$  respectively while  $S^2$  in (2.1c) is unchanged. Since  $\Theta_w$  does not usually satisfy the thermal wind relation, the moist stability tensor  $\begin{pmatrix} F^2 & S^2 \\ S_o^2 & N_o^2 \end{pmatrix}$  is generally nonsymmetric and cannot be diagonalized by an orthogonal transformation such as (2.3). However, the moist stability tensor always can be transformed into either a diagonal or Jordan form by a proper affine transformation. The  $\nabla$ -derivative terms, especially the nonlinear terms, will generally become more complicated. There is no general way to transform the moist governing equations into a simple form like (2.5). Another difficulty with moist SI is that the integrand  $\underline{l} \cdot \Pi \cdot d\underline{l}$  in (3.6) is no longer a total differential if  $\Pi$  is nonsymmetric, i.e.,  $\Pi \neq \Pi^T$ . This prevents us from following the parcel energetic analysis of (3.6). However, notice that (3.4) and (3.5) are still valid for nonsymmetric constant  $\Pi$ . Thus we can analyze moist SI with the instantaneous parcel energetics of (3.7).

For nonsymmetric constant  $\Pi$ , the  $\omega^2(\alpha, \beta)$  in (3.7) becomes

$$\begin{aligned} \omega^2(\alpha, \beta) = & F^2 \cos \alpha \cos \beta + S^2 \cos \alpha \sin \beta + S_o^2 \sin \alpha \cos \beta \\ & + N_o^2 \sin \alpha \sin \beta, \quad |\alpha - \beta| < \frac{\pi}{2} \end{aligned} \quad (4.2)$$

It is easy to prove that  $\omega^2(\alpha, \beta)$  reaches its minimum at  $\alpha = \hat{\alpha}_0$  and  $\beta = \hat{\beta}_0$ , where  $\hat{\alpha}_0$  and  $\hat{\beta}_0$  are the roots of

$$\tan(\alpha - \beta) = (S_0^2 - S^2)/(F^2 + N_0^2) \quad , \quad (4.3a)$$

$$\tan(\alpha + \beta) = (S_0^2 + S^2)/(F^2 - N_0^2) \quad . \quad (4.3b)$$

For  $S_0^2 \neq S^2$ , and thus  $\hat{\alpha}_0 \neq \hat{\beta}_0$ , maximum power (corresponding to  $\min_{\alpha, \beta} \omega^2(\alpha, \beta)$ ) can only be gained by the parcels with velocity  $\underline{v} = (\cos \hat{\beta}_0, \sin \hat{\beta}_0)|\underline{v}|$  and displacement  $\underline{z} = (\cos \hat{\alpha}_0, \sin \hat{\alpha}_0)|\underline{z}|$ . Since the velocity  $\underline{v}$  generally (except for  $\hat{\alpha}_0 = \hat{\beta}_0$  in the case of  $S^2 = S_0^2$ ) deviates the parcel from  $\hat{\alpha}_0$  when the maximum power is reached, this maximum power can only be gained instantly by the parcel. In order to keep the parcel position angle  $\alpha$  even temporarily constant, the parcel velocity must be in the same direction as  $\alpha$ , i.e.,  $\alpha = \beta$ . In this case the most unstable slope angle  $\hat{\alpha}$  is given by (4.3b) with  $\alpha = \beta$ , or equivalently by (2.4) with  $S_0^2 + S^2$  and  $N_0^2$  instead of  $2S^2$  and  $N^2$  respectively. Thus,  $\hat{N}^2$  and  $\hat{F}^2$  are given by (2.6) with  $S_0^2 + S^2$  and  $N_0^2$  instead of  $2S^2$  and  $N^2$  respectively.

We can also show that above analysis and results are consistent with following inviscid linear analysis. For a free mode where the stream function  $\psi \sim e^{i(\omega t - m x - n z)}$ , it is easy to show that

$$\omega^2 = F^2 \cos^2 \alpha + (S^2 + S_0^2) \cos \alpha \sin \alpha + N_0^2 \sin^2 \alpha \quad , \quad (4.4)$$



where  $\alpha$  is the slope angle of stream line, i.e.,

$$\tan \alpha = \frac{z}{x} \Big|_{\psi = \text{const}} = - \frac{m}{n} .$$

Eq. (4.4) is just (2.8) with  $S^2 + S_0^2$  and  $N_0^2$  replacing  $2S^2$  and  $N^2$  respectively. Thus, the linear most unstable (or least stable) slope angle and the corresponding value of  $\min_{\alpha} \omega^2$  are consistent with the above parcel energetics result.

Now we can consider the CSI problem qualitatively (cf., Fig. 2). In dry subsidence regions the downward motion tends to be along the least (dry) stable slope  $\tan \hat{\alpha}_1$  which is given by (2.4) with  $N^2 = N_1^2$  and  $S^2 = S_1^2$  where the basic state is dry symmetric stable, i.e.,

$$N_1^2 F^2 - S_1^4 > 0 . \quad (4.5)$$

Condition (4.5) implies that the downward motion is passive and must be forced in compensation to the moist updrafts. Thus, the basic state in the moist updrafts must satisfy moist SI condition, i.e.,

$$N_0^2 F^2 - (S_1^2 + S_0^2)^2 / 4 < 0 . \quad (4.6)$$

Here the updrafts tend to slant along the most (moist) unstable slope  $\tan \hat{\alpha}_0$  which is given by (2.4) or (2.7) with  $S_1^2 + S_0^2$  and  $N_0^2$  replacing  $2S^2$  and  $N^2$  respectively, while the dry subsidence tends to follow the slope  $\tan \hat{\alpha}_1$ , thus the averaged slope (angle  $\phi$ ) is between the two, i.e., (see Fig. 2)

$$|\hat{\alpha}_1| < |\phi| < |\hat{\alpha}_0| . \quad (4.7)$$

Furthermore, under conditions (4.6) and (4.7), it is easy to prove that

$$\begin{aligned} |\phi_0| < \frac{\pi}{2} , \text{ as } N_0^2 > 0 , \\ 0 < |\phi_1| < |\hat{\alpha}_1| < |\phi_v| < |\hat{\alpha}_0| < \left\{ \begin{array}{l} \frac{\pi}{2} < |\phi_0| < \pi , \text{ as } N_0^2 < 0 , \end{array} \right. \end{aligned} \quad (4.8)$$

where  $\phi_1$  and  $\phi_0$  are the slope angles of dry and moist isentropic lines (in  $x$  plane),  $\phi_v$  is the slope angle of iso-momentum line. Thus, if  $\phi$  equals to the slope angle of a moist updraft, then the estimation of (4.7) can be improved as

$$|\phi_v| < |\phi| < |\hat{\alpha}_0| , \quad (4.9)$$

because the unstable slope for a rising moist parcel must be steeper than the iso-momentum lines (cf. Section 3). Obviously, for a given scale height  $H$ , (4.7) or (4.9) renders a broad mesoscale range of permissible values for the intrinsic horizontal scale  $L = H \tan \phi$  for CSI circulations (see Fig. 2).

## 5. Concluding Remarks

Via a proper rotational coordinate transformation in a transverse section of the basic flow, we have shown that the nonlinear viscous

equations with a SI (Symmetric Instability) basic state can be locally transformed into equations with a BI (Buoyancy or Convective Instability) or II (Inertial Instability) basic state. This indicates the common nature of SI, BI and II with respect to their local dynamics. However, it is worth to note that for BI the horizontal stability, due solely to Coriolis force, is generally much weaker than the vertical instability. For SI a stable stratification can render the "horizontal" (along  $x$ ) stability much stronger than the "vertical" (along  $z$ ) instability. This can make for a large difference between BI and SI (especially when the Prandtl number  $\neq 1$ , cf. Miller, 1984).

The similarity between SI, BI and II is conducive to a better understanding of various SI problems and suggests qualitative results for these problems without even solving them analytically. However, this analogy does not mean that a SI problem can always be treated as easily as a BI or II problem. The additional difficulties with SI problems often arise due to the following factors: a) when the basic state is not uniform, i.e., the stability tensor is a function of position, then the SI problem generally cannot be transformed into a BI or II problem; (b) even if the basic state is uniform and thus the SI problem can be globally transformed into a BI problem, the boundary will usually become more irregular and the problem may still not be amenable to analytical treatment. Thus, as far as the global dynamics are concerned, SI is not generally equivalent to BI and II.

Furthermore, when the instability is conditional, the stratification and thus stability tensor will change discontinuously at the interfaces between the moist updrafts and dry subsidences. Thus, the differences between CSI and CBI (Conditional BI) will be even more than those cited above. For CBI, the stability tensor does not have a sudden rotation at

the interfaces as it does for CSI. Since the most unstable direction for the moist stability tensor is steeper than the least stable direction for the dry stability tensor, the CSI circulations will be slantwise between these two directions. If the slope of the moist updraft is considered as a measure of the slope of a CSI circulation, then it must be also steeper than the iso-momentum surfaces of the basic state. Associated with the slope of a CSI circulation, the intrinsic horizontal scale may cover a broad mesoscale range. These qualitative results are confirmed by linear CSI solutions (Chapter III, Xu, 1984).

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## Figure Captions

Figure 1 Schematic of uniform basic state and rotational coordinate transformation  $R_{\hat{\alpha}}$ .

Figure 2 Schematic comparison of the slope angles:  
 $\hat{\alpha}$  - most unstable slope angle in moist basic state,  
 $\hat{\alpha}^0$  - least stable slope angle in dry basic state,  
 $\phi^1$  - slope angle of a CSI circulation,  
 $H$  - scale height,  
 $L$  - intrinsic horizontal scale,

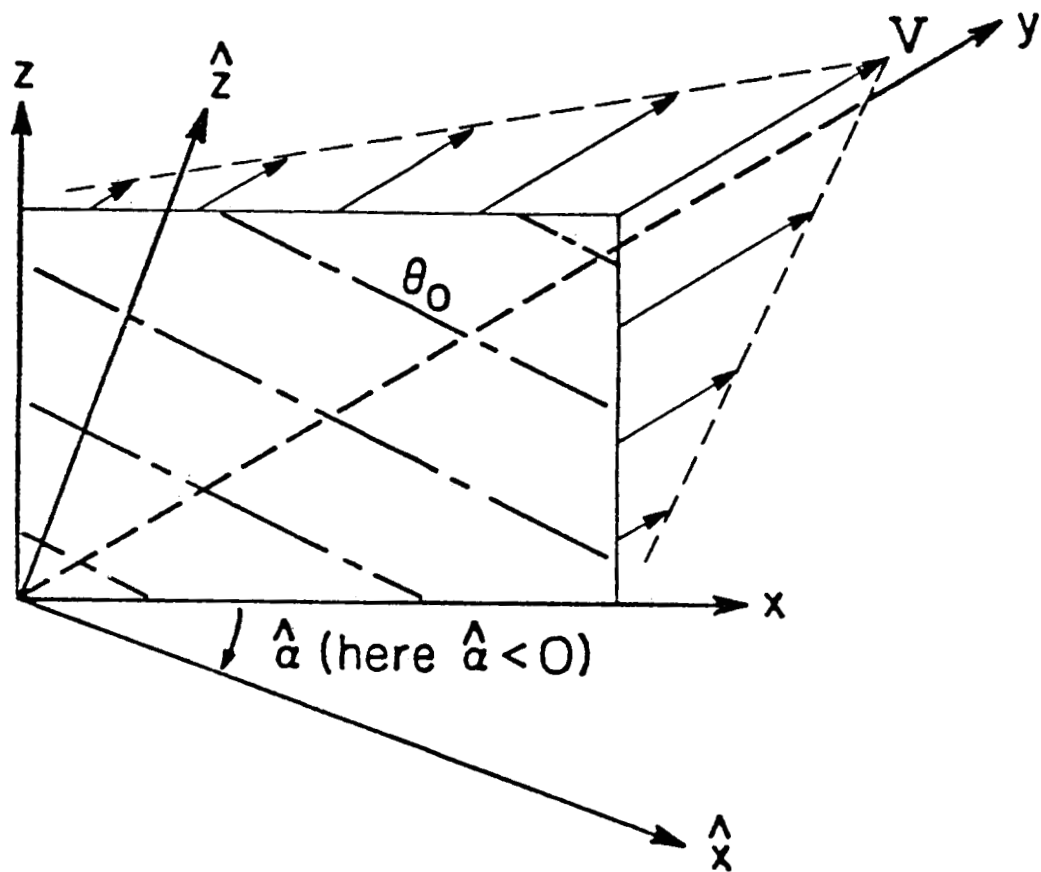


Figure 1 Schematic of uniform basic state and rotational coordinate transformation  $R_{\hat{\alpha}}$ .



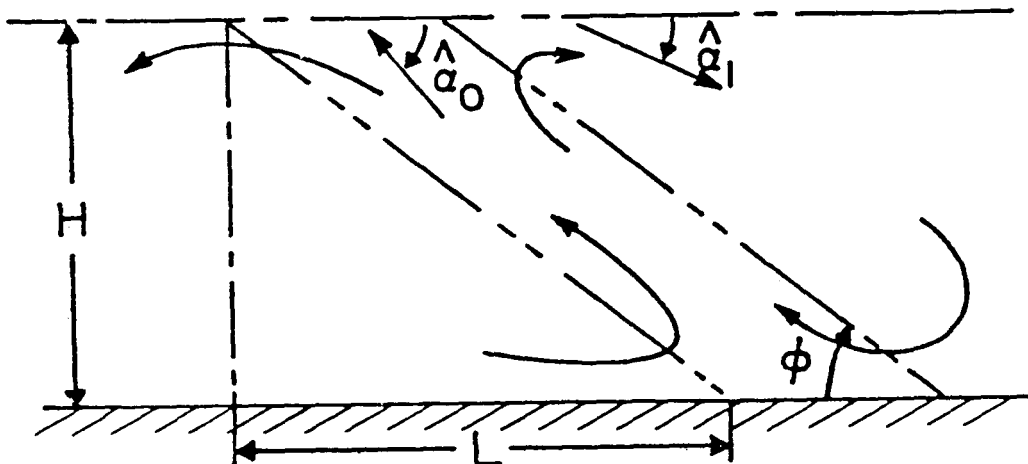


Figure 2 Schematic comparison of the slope angles:

- $\hat{\alpha}_0$  - most unstable slope angle in moist basic state,
- $\hat{\alpha}_1$  - least stable slope angle in dry basic state,
- $\phi^1$  - slope angle of a CSI circulation,
- $H$  - scale height,
- $L$  - intrinsic horizontal scale,

## I. Summary of Accomplishments

The work performed under this contract can roughly be divided into two parts: the first was mainly theoretical in nature and aimed at understanding the dynamics of convectively-driven meso- and synoptic-scale processes and the second was principally observational and sought to observe the effects of latent heat release associated with slow quasi-geostrophic ascent on a March, 1978 storm system.

First of all, a list of degrees earned and thesis titles supported by this contract is presented:

1. Christine A. Staver, M.S., 1981: "The Effects of Latent Heat Release on the Development of a Middle Latitude Cyclone."
2. C.-L. Shie, M.S., 1985: "The Effect of Latent Heat on Cyclone Development."
3. Qin Xu, Ph.D., 1984: "Conditional Symmetric Instability and Mesoscale Rainbands."

Secondly, the papers published are:

1. Xu, Q. and J.H.E. Clark, 1984: Wave CISK and Mesoscale Convective Systems. J. Atmos. Sci., 41, 2089-2107.
2. Tang, C.-M. and G.H. Fichtl, 1984: Non-Quasi-Geostrophic Effects in Baroclinic Waves with Latent Heat Release. J. Atmos. Sci., 41, 1498-1512.\*
3. Tang, C.-M. and G.H. Fichtl, 1983: The Role of Latent Heat Release in Baroclinic Waves. J. Atmos. Sci., 40, 53-72.\*

\*These papers were supported under an associated contract sent directly to NASA MSFC which was originally meant to be part of the original contract NAS8-33797.

Papers submitted or to be submitted are:

Q. Xu: Conditional Symmetric Instability and Mesoscale Rainbands,  
Quart. J. Roy. Meteor. Soc..

Q. Xu: Generalized Energetics for Linear and Nonlinear Symmetric  
Instability, J. Atmos. Sci..

C.A. Staver, C.-L. Shie, J.H.E. Clark: Ageostrophic Motions as  
Diagnosed by a Semi-Geostrophic Model and their Influence on  
Cyclone Development, Mon. Wea. Rev..

## II. Major Findings

1. We carried out a number of diagnostic studies of the effect of latent heat release on a March, 1978 storm. In the M.S. thesis by Staver, we found that the large-scale latent heating had a small influence on enhancing the growth of the storm. The pattern of quasi-geostrophic vertical motion was enhanced in the vicinity of the cold front by the latent heating. We estimated the areas of heating from sample measurements and satellite infrared and visible imagery.

2. In the ensuing M.S. thesis by Shie and the paper by Staver, Shie and Clark, we extended our diagnostic scheme to a semi-geostrophic formalism. In this scheme, advection of heat and vorticity by the ageostrophic as well as geostrophic part of the horizontal wind field was accounted for. In the thesis, the equations were solved in transformed geostrophic coordinates but this proved to be inaccurate because of the many smoothing and interpolation operations involved in the coordinate change. In the paper we develop an accurate iterative scheme which can be used in physical space. The method involves an initial guess of the ageostrophic winds using the quasi-geostrophic formalism and then a reevaluation of the advection terms with the total wind. Convergence was found to be rapid. We noted only small

differences in the large-scale vertical motion field with and without latent heating between the semi- and quasi-geostrophic diagnostics. However, with the semi-geostrophic total winds we are able to evaluate the tendency of the synoptic-scale motion field to produce conditions favorable for symmetric instability. This would manifest itself in the form of mesoscale rainbands oriented parallel to the low-level vertical wind shear. We noted that within a period of less than 12 hours, differential, horizontal and vertical advection by the total wind field would render the atmosphere convectively unstable along iso-momentum surfaces in the vicinity of the warm front in the March, 1978 storm. We thus concluded that latent heat release on the convective and mesoscale probably plays an important role in the development of this winter storm.

3. We proceeded to investigate some models to account for the interaction of convective scale motions and the larger scale motion field in which it is embedded. In the paper by Xu and Clark, we considered a wave-CISK model with the important feature that the convective motions would not respond instantaneously to large scale forcing. Rather they would be triggered by the forcing but then would go through a life cycle that depended on

- a. the intrinsic mean cloud lifetime,
- b. the amount of available convective potential energy in the environment, and
- c. eddy-dissipation.

We were able to produce wave-like disturbances which strongly resembled traveling waves associated with convection in the vicinity of cold and warm fronts and in warm sectors where the background shear is weak.

We have also carried out a number of theoretical studies of conditional symmetric instability (CSI). In the enclosed paper by Xu and Clark, we

develop an analytical technique for solving the linear problem with and without latent heat release. In ensuing papers we investigate some implications of nonlinearities and viscosity on the growth of CSI disturbances in an unstable environment.

1. REPORT NO. NASA CR-3945		2. GOVERNMENT ACCESSION NO.		3. RECIPIENT'S CATALOG NO.	
4. TITLE AND SUBTITLE The Nature of Symmetric Instability and Its Similarity to Convective and Inertial Instability				5. REPORT DATE October 1985	
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7. AUTHOR(S) Qin Xu and J. H. E. Clark				8. PERFORMING ORGANIZATION REPORT #	
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16. ABSTRACT  It is shown that there exists a local similarity among SI (Symmetric Instability), BI (Buoyancy or Convective Instability), and II (Inertial Instability) even for fully nonlinear viscous motion. The most unstable slope angles for SI and Moist SI motions are analyzed based on parcel energetics. These considerations also suggest qualitatively that CSI (Conditional SI) circulations will be slantwise and lie between the moist most unstable slope and dry least stable slope of the basic state.					
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