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Multigrid Solutions to Quasi-Elliptic Schemes

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Abstract Quasi-elliptic schemes arise from central differencing or finite element discretization of elliptic systems with odd order derivatives on non-staggered grids They are somewhat unstable and less accurate then corresponding staggered-grid schemes When usual multigrid solvers are applied to them, the asymptotic algebraic convergence is necessarily slow Nevertheless, it is shown by mode analyses and numerical experiments that the usual FMG algorithm is very efficient in solving quasi-elliptic equations to the level of truncation errors Also a new type of multigrid algorithm is presented, mode analyzed and tested, for which even the asymptotic algebraic convergence is fast The essence of that algorithm is applicable to other kinds of problems, including highly indefinite ones

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Content

1. Introduction

Quasi-elliptic schemes arise, for example, when central differencing is used to approximate odd-order derivatives in elliptic systems of partial differential equations, such as the Cauchy-Riemann, Stokes and Navier-Stokes systems Usual finite element approximations to such systems also lead to quasi-elliptic schemes. Such schemes are in some sense unstable certain highly-oscillating components are amplified in the discretized solution much more than in the differential solution

Instead of the quasi-elliptic schemes, other discretizations of the same system can usually be constructed which are h-elliptic, hence fully stable, and which are also more accurate than the quasi-elliptic schemes Sometimes, however, these fully elliptic schemes are inconvenient to use In case of elliptic systems with odd-order derivatives, for example, full ellipticity is obtained by grid staggering, i e., by approximating different functions on different grids (cf [3] and [8]) This is inconvenient, especially near curved boundaries. Also the instability of quasi-elliptic approximations seldom really hurts, since the unstable components have very small amplitudes, which are still small even in the discrete solution The inaccuracy is modest The error in the quasi-elliptic solution is typically twice to four times larger than the error in an elliptic solution using the same grid size Thus, quasi-elliptic schemes are often preferred and are widely used

The instability of quasi-elliptic schemes does seem to hurt when multigrid solvers are applied· The asymptotic convergence turns out to be slow, and a simple mode analysis traces this slowness to the unstable modes. One approach, perhaps the best, to deal with this difficulty is simply to ignore it· the algebraic slowness does not matter because it occurs in modes whose amplitudes in the algebraic solution are erroneous anyway, bearing no relation to their amplitudes in the true differential solution One should only take care not to initially admit large unstable amplitudes, and to average them out in case they must latter enter We show, by mode analyses and numerical experiments, that the usual FMG algorithm is very effective in solving quasi-elliptic problems to truncation level (i.e., to the point where algebraic errors are dominated by discretization errors) Sometimes the FMG solution may even be better than the exact solution of the discrete equations, because the unstable components of the latter are slow to enter

Although this is the easiest approach for obtaining fast *differential* convergence (convergence to the differential solution through a sequence of grids), another algorithm is presented below which does provide fast *algebraic* convergence for quasi-elliptic schemes This algorithm, based on multiple coarse-grid corrections, is interesting in its own right, since it is the simplest example of a new kind of algorithms for solving problems with highly-oscillating solutions, including highly indefinite problems (see [1, §3.2], [8] and a subsequent article) Smoothing rate analysis, for one quasi-elliptic example, suitably modified to account for the

multiple coarse-grid corrections, shows that the new algorithm should *algebraically* be as efficient as usual multigrid cycles are for fully elliptic schemes Numerical experiments exactly yield these expected convergence rates (see §8 6, tests of such algorithms were also reported in [6])

The significance of the present studies goes beyond elliptic PDE systems many non-elliptic systems, such as all subsonic steady-state flow problems, have determinants with at least one elliptic factor Most discretizations of such systems provide quasi-elliptic approximation to that factor, leading to troubles and requiring cures similar to those reported here

Moreover, the techniques described in this article illustrate the following *general multigrid approaches to general non-elliptic problems* (i) Differential not algebraic convergence is sought and usually easily obtained Modified methods for apriori analyzing and aposteriori measuring such a convergence have been developed (ii) With considerably more effort, fast *algebraic* convergence can also be obtained (iii) The analysis of difference schemes, and the derivation of efficient smoothers, for any PDE system is based on the factors of the h-principal part of the operator determinant

We thank Ruth Golubev for some of the calculations reported in §8

2. Definitions and Examples

In the following L^h will represent a system of q real difference operators on q grid functions where h, the meshsize of the grid, is for simplicity assumed to be uniform and the same in all directions That is, L^h is a $q \times q$ matrix of real polynomials in $T_1, \quad , T_d, T_1^{-1}, \quad , T_d^{-1}$, where T_i are the grid translation operators, defined by

$$T_1^{\nu_1} \quad T_d^{\nu_d} u(\underline{x}) = u(\underline{x} + \underline{\nu} h),$$

with $\underline{x} = (x_1, \quad , x_d)$, $\underline{\nu} = (\nu_1, \quad , \nu_d)$ and d being the dimension of the Euclidean space housing the grid (In case of staggered grids there may appear non-integral powers of T_j and L^h will most usually be a matrix of polynomials in $T_j^{1/2}$ and $T_j^{-1/2}$, $j = 1 \quad , d$)

Three common examples of difference operator are

(i) The five-point (compact) Laplacian

$$\Delta^h = \frac{1}{h^2}(T_{0,1} + T_{1,0} + T_{0,-1} + T_{-1,0} - 4T_{0,0}) = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}, \qquad (2\ 1)$$

where $T_{\alpha,\beta} = T_1^\alpha T_2^\beta$ and the array on the left is the usual pictorial description

of the weights of the operator This is the simplest approximation to the two-dimensional Laplace operator $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$

(ii) The central non-staggered approximation to the Cauchy-Riemann operator

$$L_{CR}^h = \begin{pmatrix} \partial_1^c & \partial_2^c \\ -\partial_2^c & \partial_1^c \end{pmatrix}, \tag{2.2}$$

where $\partial_i^c = \frac{1}{2h}(T_i - T_i^{-1})$

(iii) The central non-staggered approximation to the Stokes operator in two dimensions

$$L_S^h = \begin{pmatrix} -\Delta^h & 0 & \partial_1^c \\ 0 & -\Delta^h & \partial_2^c \\ \partial_1^c & \partial_2^c & 0 \end{pmatrix} \tag{2.3}$$

For simplicity we will deal in this article only with constant-coefficient operators L^h In this case the *symbol* $\hat{L}^h(\underline{\theta})$ of L^h is defined by

$$L^h A e^{i\underline{\theta}\,\underline{x}/h} = \hat{L}^h(\underline{\theta}) A e^{i\underline{\theta}\,\underline{x}/h}, \qquad (|\underline{\theta}| \le \pi)$$

for any q-vector A, where $\underline{\theta} = (\theta_1, \quad, \theta_d)$, $\underline{\theta}\ \underline{x} = \theta_1 x_1 + \cdots + \theta_d x_d$ and $|\underline{\theta}| = \max(|\theta_1|, \ldots, |\theta_d|)$ Thus, $\hat{L}^h(\underline{\theta})$ is a $q \times q$ matrix of polynomials in $e^{\pm i\theta_j}$, $j = 1, \ldots, d$, obtained from L^h by replacing each T_j with $e^{i\theta_j}$.

Also for simplicity we will deal here only with *homogeneous operators* L^h, i e , operators for which all terms in det L^h (the determinant of L^h) have the same power in h (This means that L^h approximates a homogeneous differential operator L, i e., det L is a homogeneous polynomial in $\partial/\partial x_1, \ldots, \partial/\partial x_d$ All examples above are homogeneous) For homogeneous difference operators, the general notion of ellipticity measure on a given scale (cf [2, Sec. 3 1] or [3, Sec. 2.1]) is not needed, and we can use the following simpler definition.

Definition The homogeneous difference operator L^h is *elliptic of order* $2m$ iff

$$|\det \hat{L}^h(\underline{\theta})| \ge Ch^{-2m} \sum_{j=1}^{d} \theta_j^{2m} \quad \text{for all } |\underline{\theta}| \le \pi, \tag{2.4}$$

where C is positive and independent of $\underline{\theta}$.

Ellipticity of differential operators is defined in the same way (The parameter h is arbitrary then, and the range of $\underline{\theta}$ is unrestricted. It is thus more natural in the continuous case to replace $\underline{\theta}/h$ by another phase variable, $\underline{\omega} = \underline{\theta}/h$ say.) It is easy to see that both Δ and Δ^h are second-order elliptic. Generally, simplest central approximations to second-order scalar ($q = 1$) elliptic operators are themselves

elliptic. But not all central approximations are For example, the "skew Laplacian"

$$\Delta^{\times} = \frac{1}{2h^2}(T_{1,1} + T_{1,-1} + T_{-1,1} + T_{-1,-1} - 4T_{0,0}) = \frac{1}{2h^2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad (2\,5)$$

or the "long Laplacian"

$$\Delta^{2h} = \frac{1}{4h^2}(T_{2,0} + T_{0,2} + T_{-2,0} + T_{0,-2} - 4T_{0,0}) \qquad (2\,6)$$

both approximating Δ, have the symbols

$$\hat{\Delta}^{\times}(\underline{\theta}) = \frac{1}{h^2}[(\cos\theta_1 - \cos\theta_2)^2 + \sin^2\theta_1 + \sin^2\theta_2]$$

$$\hat{\Delta}^{2h}(\underline{\theta}) = \frac{1}{h^2}(\sin^2\theta_1 + \sin^2\theta_2)$$

which clearly fail to satisfy (2 4) Indeed, $\hat{\Delta}^{\times}(\pi,\pi) = 0$ and $\hat{\Delta}^{2h}(\pi,0) = \hat{\Delta}^{2h}(0,\pi)$ $= \hat{\Delta}^{2h}(\pi,\pi) = 0$ Whereas these examples seem somewhat artificial (although the skew Laplacian does naturally arise in various situations, e g., in semi-implicit Lagrange codes [4, § IV] and for some kinds of finite elements [7]) non-elliptic operators are very common in approximations to elliptic *systems* ($q > 1$) The discrete Cauchy-Riemann (2.2) and Stokes (2 3) operators well represent this situation· They are the simplest (non-staggered) central approximations to elliptic operators, but $\det L^h_{CR} = \Delta^{2h}$ and $\det L^h_S = \Delta^h\Delta^{2h}$, hence they do not satisfy (2 4), their symbol vanishing wherever $\hat{\Delta}^{2h}$ does Note that taking the determinant commutes with passing to the symbol, hence ellipticity of L^h is equivalent to ellipticity of $\det L^h$, which in turn is equivalent to ellipticity of all factors of $\det L^h$.

Finite element discretizations of the same elliptic systems, with uniform non-staggered partitions, give rise to similarly non-elliptic difference operators This is not usually recognized because finite element discretizations are seldom Fourier-analyzed as uniform-grid operators

In all the above examples, even when L^h fails to satisfy (2 4), it still satisfies the weaker condition

$$|\det \hat{L}^h(\underline{\theta})| \geq Ch^{-2m}\sum_{j=1}^{d}\sin^{2m}(\theta_j), \quad \text{for all } |\underline{\theta}| \leq \pi, \qquad (2\,7)$$

where C is positive and independent of $\underline{\theta}$ The term *quasi-elliptic* was introduced in [8] to describe such operators

Perhaps all reasonable approximations to homogeneous elliptic equations satisfy (2.7), but for the purpose of including some additional, not-so-reasonable approximations, we can extend the class of operators, and admit any homogeneous

operators L^h for which det $\hat{L}^h(\underline{\theta})$ vanishes only at a finite number of points. This class includes for example $\Delta^{2h\times} = (T_{2,2} + T_{2,-2} + T_{-2,2} + T_{-2,-2} - 4T_{0,0})/(8h^2)$, which satisfies neither (2.4) nor (2.7), but for which the methods described below are still applicable.

More generally when inhomogeneous operators are also admitted, our methods will extend to any operator L^h with $O(1)$ "measure of quasi-ellipticity", defined by

$$E^{h,\alpha}(L^h) = \min_{\alpha \geq |\underline{\theta}| \geq |\underline{\theta}'|} |\det L^h(\underline{\theta})| / |\det L^h(\underline{\theta}')|, \tag{2.8}$$

for some reasonable $\alpha > 0$. $E^{h,\pi}$ is the usual measure of ellipticity E^h described in [3]. The methods here will in principle work for any positive α, although they will gradually deteriorate with the decrease of α for which $E^{h,\alpha}(L^h)$ is still $O(1)$.

For clarity, we discuss below only homogeneous operators, and the strict quasi-ellipticity (2.7) is assumed.

3. Instability and Inaccuracy

Quasi-elliptic operators do meet some general stability requirements even if they do not satisfy (2.4). For example, the skew Laplacian (2.5) is a positive type operator, hence satisfying the maximum principle. The associated matrix has a dominant diagonal. Nevertheless, in a certain sense such operators are not quite stable. Namely, since det $\hat{L}^h(\underline{\theta}) = 0$ for some $\underline{\theta} \neq 0$, in an infinite space, or under periodic boundary conditions, there exists a highly-oscillating function $v^h(\underline{x}) = A\exp(i\underline{\theta}\ \underline{x}/h)$ which satisfies the homogeneous equation $L^h v^h(\underline{x}) \equiv 0$. Hence the solution, unlike the corresponding differential solution, is not unique (upto an additive constant); it contains an undetermined highly-oscillating component. Similarly, in any bounded domain with any boundary conditions, functions $w^h(\underline{x})$ close to $v^h(\underline{x})$ (e.g. $w^h = \varphi_1 v^h + \varphi_2$, φ_j being smooth) exist which satisfy the boundary conditions and for which $L^h w^h$ is everywhere very small. Such w^h therefore forms an unstable mode. A small change in the equation can introduce a large change proportional to w^h. This is a kind of numerical instability, since a corresponding large change in the differential solution cannot occur.

This numerical instability need not hurt much. If the differential system is $LU = F$ and the discrete system is $L^h U^h = F^h$, all one has to do is to define $F^h = I^h F$, say, through an averaging operator I^h which liquidates the unstable modes, i.e. $\hat{I}^h(0) = 1$ and the ratio $\hat{I}^h(\underline{\theta})/\hat{L}^h(\underline{\theta})$ is uniformly bounded for all $|\underline{\theta}| \geq \varepsilon > 0$. For example, one can take $I^h = S^h I'^h$, where I'^h is any F averaging suitable for the fully elliptic case and S^h is like the solution averaging S^h described below. Even this is unnecessary in the usual, smooth case (in the same way that the above rule for I^h is frequently neglected for fully elliptic L^h), because the

unstable modes, even when unduly magnified by the discretization, are usually still small

Generally, the main disadvantage of quasi-elliptic operators is a certain loss of accuracy compared to corresponding truely elliptic operators, which is simply due to the larger differencing steps taken in certain terms of the quasi-elliptic scheme In some cases this is particularly obvious, since the grid is locally decoupled into several subgrids which are not connected to each other by the quasi-elliptic operator For example, the skew-Laplacian (2.5) introduces no coupling between red and black points (in the usual sense of checkerboard coloring, one color being associated with gridpoints where $(x_1 + x_2)/h$ is odd, the other with even) On each subgrid the discretization looks like the compact Laplacian (2 1) on a rotated grid with meshsize $h_1 = \sqrt{2}h$ Similarly, in case of (2 2), the grid is decoupled into 4 *staggered* subgrids with meshsizes $2h$, on each of which the operator has good ellipticity (being in fact equivalent to the staggered-grid approximation described in [3, §17.2] or [5, §5 2]) Thus, since the approximation is $O(h^2)$, the error in case of (2 5) is on the average twice larger, and in case of (2 2) four times larger than the errors in corresponding fully elliptic approximations (assuming other discretization errors, related for example to the representation of right-hand sides or boundary conditions, behave similarly) In these cases, in other words, each of the subgrids can produce the resulting accuracy by itself, other subgrids only add work

When derivatives are calculated from the solution, however, the approximating difference quotients may show much greater loss of accuracy, because they involve differences between values belonging to different subgrids The error in ℓ-order derivatives will generally be $O(h^{-\ell})$ times the errors in the function itself This excessive error can be avoided by taking differences only from one subgrid at a time, or, more generally, by using only difference operator D^h such that $\hat{D}^h(\underline{\theta})$ vanishes wherever $\hat{L}^h(\underline{\theta})$ does In case of (2 3), for example, derivatives of the third unknown function (the pressure) should be approximated by long differences such as ∂_j^c, $\partial_j^c \partial_k^c$, etc.

The instability described above can also be removed, and the inaccuracy in derivatives proportionally reduced, by *averaging the solution*, that is, by replacing the computed solution u^h by $S^h u^h$, where S^h is an averaging operator which removes all the unstable components In other words, $\hat{S}^h(0) = 1$ and outside a neighborhood of $\underline{\theta} = 0$ the ratio $\hat{S}^h(\underline{\theta})/\hat{L}^h(\underline{\theta})$ should be uniformly bounded (wherever defined) For the quasi-elliptic L^h satisfying (2.7) there always exists such an averaging operator of the form

$$S^h = \Pi_{j=1}^d \left(\tfrac{1}{2} T_j^{1/2} + \tfrac{1}{2} T_j^{-1/2}\right)^{m_j}, \tag{3.1}$$

with integral $m_j \leq m$. On the other hand, the averaging may further reduce the accuracy of the solution. With the averaging (3 1) the lost accuracy is $O(h^2)$ One

can make that loss $O(h^{2s})$ by taking for example

$$S^h = \Pi_{j=1}^{d}\left(1 - \left(\tfrac{1}{2} - \tfrac{1}{4}T_j - \tfrac{1}{4}T_j^{-1}\right)^s\right)^{m_j}. \qquad (3\ 2)$$

Another slight difficulty typical to quasi-elliptic approximations is the need to define extra boundary relations This can satisfactorily be done by extrapolation (cf , e.g., §8 2)

In summary. although quasi-elliptic discretizations are in principle inferior to fully elliptic ones (obtainable for systems by grid staggering), they *can* be used Since many programmers consider grid staggering a serious complication, especially near general boundaries, quasi-elliptic schemes and their fast solution become important

4. Multigrid Troubles and Their Implications

Usual multigrid solvers yield *poor asymptotic convergence rates* when applied to quasi-elliptic schemes (see [4] and §8 4 below). The reason is simple. Slow to converge are the unstable modes, such as v^h or w^h above. They cannot significantly converge by coarse-grid corrections, since they are high-frequency modes. essentially invisible on coarser levels Neither can they significantly converge by any type of relaxation, since an error like w^h shows a very small residual function $L^h w^h$ (compared with residuals shown by other modes with comparable amplitude) and the corrections introduced by any relaxation scheme are proportional to the size of the residuals (cf. [3, Sec 1 1]) In particular, the amplification factor $\mu(\underline{\theta})$ of the error mode $\exp(i\underline{\theta}\ \underline{x}/h)$ per relaxation sweep must be 1 when $\hat{L}^h(\underline{\theta}) = 0$, and since the latter equality holds for some $|\underline{\theta}| = \pi$, the smoothing factor $\bar{\mu} = \max_{\pi/2 \le |\theta| \le \pi} |\mu(\underline{\theta})|$ cannot be smaller than 1.

The poor asymptotic rates are *not a real trouble*, though The modes slow to converge are exactly those unstable modes for which algebraic convergence is not really desired, their amplitudes in the algebraic solution being unrelated to their amplitudes in the differential solution The only concern is that these amplitudes will remain suitably small

This situation is *typical to all problems which are not fully elliptic*, including most problems in fluid dynamics. Slow asymptotic convergence of suitable multigrid cycles occur exactly in those components where not much convergence is needed anyway Whenever this situation arises, it is in a sense an absurd to try and fix the algorithm (although we show in Sec. 7 below how to do it), since one would then often end up investing most of his human and computer resources to obtain improvements which are meaningless in terms of solving the original *differential* equations.

Thus, the real objective of multigrid solvers should not be a fast *algebraic convergence* (convergence of the computed solution u^h to the exact discrete solution U^h), but fast *differential convergence* (convergence of u^h to the true differential solution U), using any sequence of meshsizes h and measured directly in terms of the decrease in $\| u^h - U \|$ as function of the overall computational work (cf [3, §13]). This modified objective allows for simpler algorithms, but also calls for some modifications in our approach for analyzing algorithms, for apriori predicting and aposteriori measuring their performance The next two sections will illustrate these modifications for the case of quasi-elliptic schemes

5. Modified Mode Analysis

It was shown in Sec 4 that in case of quasi-elliptic systems $\bar{\mu} \geq 1$, but that this bad smoothing factor is not relevant to our real objective To analyze a given relaxation scheme, assume first that it is as efficient as needed for the *differential* convergence of the highest frequency modes (which should latter be checked by the 2-level FMG mode analysis mentioned below) The question then is what efficiency one should expect from the multigrid cycle (employing the given scheme on all levels) in reducing all other modes As in the conventional smoothing analysis, our simplifying assumption here will be that relaxation on each level should efficiently treat all modes in only one segment of modes, and that the union of these segments should cover all relevant modes Instead of assigning to grid h the conventional segment $\pi/2 \leq |\underline{\theta}| \leq \pi$, however, we can assign to it any segment of the form $\alpha/2 \leq \| \underline{\theta} \| \leq \alpha$, with any norm $\| \underline{\theta} \|$. That would automatically assign to grid $h/2$ the segment $\alpha/4 \leq \| \underline{\theta} \| \leq \alpha/2$, and so on It means that we allow some of the highest frequency components on any intermediate level not to converge efficiently by relaxation on that level, as long as those components efficiently converge by the next-finer-level relaxation This only leaves the highest frequency modes on the finest grid unaccounted for, which is exactly the segment where we do not seek simple algebraic convergence Thus, the modified definition of the smoothing factor relevant for our purpose here is

$$\bar{\mu} = \min \max_{\alpha/2 \leq \|\underline{\theta}\| \leq \alpha} |\mu(\underline{\theta})|, \tag{5 1}$$

where $\mu(\underline{\theta})$ is the amplification factor of $\exp(i\underline{\theta} \cdot \underline{x}/h)$ per relaxation sweep, and the minimum can be taken over all $\alpha > 0$ and over all possible choices of the norm $\| \cdot \|$ (For a generalization of this definition to cases of semi coarsening, cf [3, §12]).

In case of the skew Laplacian (2 5), for example, the lexicographically order Gauss-Seidel relaxation yields the amplification factor

$$\mu(\underline{\theta}) = e^{i\theta_1} \cos\theta_2 / \left(2 - e^{-i\theta_1} \cos\theta_2\right), \tag{5.2}$$

so that $\mu(\pi,\pi) = 1$ and the conventional smoothing factor is 1. But choosing $\alpha = \pi$ and $\| \underline{\theta} \| = \max(|\theta_1 + \theta_2|, |\theta_1 - \theta_2|)$ easily shows that the modified factor (5.1) yields $\bar{\mu} \leq 5$. The same can be shown for the long Laplacian (2.6), by taking $\alpha = \pi/2$ and $\| \underline{\theta} \| = |\underline{\theta}|$.

In case of systems ($q > 1$), the $q \times q$ amplification matrix μ of the mode $A \exp(i\underline{\theta} \cdot \underline{x}/h)$ depends both on $\underline{\theta}$ and on the q-vector A. The modified smoothing factor $\bar{\mu}$ is then defined by

$$\bar{\mu} = \min_{\alpha/2 \leq \|\underline{\theta}\| \leq \alpha} \max \| \mu A \| / \| A \|$$

where α is allowed to depend on both $\underline{\theta}/|\underline{\theta}|$ and $A/|A|$. With these definitions and suitable distributed Gauss-Seidel (DGS) relaxation schemes (see e.g. [3, §18.6]) this again yields $\bar{\mu} \leq .5$, for both the Cauchy-Riemann (2.2) and the Stokes (2.3) operators. In all these cases, still better factors are obtained by four-color ordering, for which definitions (5.1) and (5.2) should further be extended (cf. (3.2) in [3]).

As for *two-level analyses* (cf. [3, §4.1] or [5, §4.6]), they always couple lowest with highest frequency modes. In non-elliptic cases some highest-frequency modes are not expected to converge fast. What the analysis should then tell us is how efficient is the entire multigrid algorithm in reducing the algebraic errors below the truncation errors. This can be done by a *two-level FMG mode analysis*, which Fourier analyzes the N-FMG algorithm described below (usually for $N = 1$) by assuming exact solution of the coarse grid equations (both for obtaining the first approximation and in each of the N cycles) and by comparing for each mode the final algebraic error with the truncation error (see [3, §7.4]).

6. FMG Solution to Truncation Level

Since the multigrid cycling is inefficient in reducing unstable mode errors, the multigrid solver should take care not to start with an initial solution which contains large amplitudes of such errors. The overall initial error in unstable modes should better be smaller than the overall truncation error. This is easily obtained by taking a first approximation from a coarser grid, employing interpolation of suitable order. The usual "Full multigrid" (FMG, also called "nested iteration") algorithm can therefore be used, with slight modifications. The usual algorithm and its modifications are briefly described in the following. For a flowchart, and a detailed discussion of FMG algorithms and the order of the first interpolation, see Secs. 1.6 and 7 in [3]. For simplicity we describe here the Correction Scheme (CS) version of the algorithm, so the problems are assumed linear; it should be converted to Full Approximation Scheme (FAS) to treat nonlinear problems [3, §8].

6.1 Multigrid cycle A sequence of grids is given with meshsizes h_k ($k = 1, 2, 3,$), where $h_{k+1} = h_k/2$ On the h_k grid the discrete equations have the form

$$L^k U^k = F^k \tag{6 1}$$

where L^k approximates L^{k+1} Given u_0^k, an approximate solution to (6.1), the multigrid cycle MG for producing an improved approximation u_1^k

$$u_1^k \leftarrow MG(k, u_0^k, F^k) \tag{6.2}$$

is recursively defined as follows

If $k = 1$ solve (6 1) by any direct or iterative method, yielding the final result u_1^k Otherwise do (A) through (D)

(A) Perform ν_1 relaxation sweeps on (6 1), resulting in a new approximation $\bar{u}^k$

(B) Starting with $u_0^{k-1} = 0$ make γ successive cycles

$$u_j^{k-1} \leftarrow MG\big(k-1, u_{j-1}^{k-1}, I_k^{k-1}(F^k - L^k \bar{u}^k)\big), \qquad (j = 1, \ldots, \gamma)$$

where I_k^{k-1} is a transfer ("reduction") of residuals from grid h_k to grid h_{k-1} We have used the "full weighting"

$$\begin{aligned} I_k^{k-1} = {} & \tfrac{1}{4} T_{0,0} + \tfrac{1}{8}\big(T_{0,1} + T_{1,0} + T_{0,-1} + T_{-1,0}\big) \\ & + \tfrac{1}{16}\big(T_{1,1} + T_{1,-1} + T_{-1,1} + T_{-1,-1}\big) \end{aligned} \tag{6 3}$$

(C) Calculate $\tilde{u}^k = \bar{u}^k + I_{k-1}^k u_\gamma^{k-1}$, where I_{k-1}^k is a suitable interpolation ("prolongation") from grid h_{k-1} to grid h_k For problems considered here, bilinear interpolation is used

(D) Perform ν_2 relaxation sweeps on (6 1), starting with $\tilde{u}^k$ and yielding the final result u_1^k

The cycle with $\gamma = 1$ is called *V cycle* or $V(\nu_1, \nu_2)$, and the one with $\gamma = 2$ is called *W cycle* or $W(\nu_1, \nu_2)$

6.2 Full Multigrid (FMG) The N-FMG is an algorithm for calculating an approximate solution

$$u_N^k = FMG(k, F^k, N) \tag{6.4}$$

to equation (6.1), defined recursively as the following two successive steps

(a) Calculating a first approximation u_0^k. If $k = 1$, put $u_0^k = 0$ Otherwise put

$$u_0^k = I_{k-1}^k FMG(k-1, I_k^{k-1} F^k, N), \tag{6.5}$$

where I_{k-1}^{k} is an interpolation of solutions from grid h_{k-1} to grid h_k, and I_k^{k-1} is some transfer ("averaging") from grid k to grid $k-1$, usually a full weighting of the type (6 3). The interpolation I_{k-1}^{k} should usually be of order higher than that of the correction interpolation I_{k-1}^{k} mentioned above [3, §7.1]. In our experiments bicubic interpolation was used

(b) Improve the first approximation by N successive MG cycles

$$u_j^k \leftarrow MG(k, u_{j-1}^k, F^k), \qquad (j = 1, \ldots N)$$

as defined in Sec 6.1.

6.3 Averaging The algorithm above is the conventional one, and for equations with constant coefficients it requires no modifications In case of quasi-elliptic equations with variable coefficients, and in particular in case of nonlinear equations, it is not enough to prevent unstable initial errors, because such errors can also later be introduced due to interaction between modes It is then better to explicitly reduce the unstable modes by averaging, such as (3 1) or (3 2) It may also then be important to replace $I_{k-1}^{k} u^{k-1}$ in step (C) above by $I_{k-1}^{k} S^{h_{k-1}} u^{k-1}$ In fact, experiments with non-staggered Navier-Stokes equations (cf Sec 8 2) gave slowly diverging MG cycles unless this averaging was used

6.4 Measuring convergence In various situations where algebraic convergence is not attempted, as in the present algorithm and double discretization [3, §10 2] and other algorithms, the question is raised how to measure convergence, how to know, in particular, that a solution to the truncation level (i e , with algebraic errors dominated by discretization errors) has been obtained

The answer is that solution to the truncation level is not really the important information when *differential* convergence is our objective (as it should most often be), because (i) Solving to truncation level tells us nothing about the truncation error itself We may for instance be doing good job in solving the algebraic system due to having chosen an easy-to-solve but badly-approximating discretization (ii) A smaller differential error may often be obtained faster by switching to a finer grid before the equations on the present grid have been solved to truncation level

The important information is the differential convergence itself as function of computational work This very information can directly be obtained from the N-FMG algorithm Indeed, the sequence of approximations u_N^k, $(k = 1, 2, \ )$ is a sequence converging to the differential solution, hence the decrease in the sequence of differences $\delta_k = \| \ \bar{I}_k^{k-1} u_N^k - u_N^{k-1} \ \|$ exactly exhibit the speed of differential convergence, where the norm $\| \ \|$ used to measure δ_k can be chosen to exactly represent the sense in which convergence is sought. One only has to check that

the smallness of δ_k is not governed by lack of change from u_0^k to u_N^k It is enough for this purpose to check that the suitable residual norm $r_N^k = \| F^k - L^k u_N^k \|$ is considerably smaller than r_0^k One can usually also verify that the algebraic errors are below truncation level, e g , by confirming that r_N^k/r_0^k is considerably smaller than δ_k/δ_{k-1}

7 Algorithm for Fast Algebraic Convergence

Although fast asymptotic algebraic convergence is not needed for fast differential convergence, it can still be produced by a more involved multigrid algorithm This algorithm (also described in [6]) may be interesting in its own right, since it is the simplest example of a new kind of algorithms (first mentioned in [1, §3.2], and more fully in [8]) for solving problems with highly-oscillatory solutions, including highly indefinite problems

7 1 Multiple coarse grid corrections Let $\underline{\theta}^1, \underline{\theta}^2, \ldots, \underline{\theta}^\ell$ be all the components for which $\hat{L}^h$ vanishes, or, more generally, the centers of all neighborhoods in which $\hat{L}^h(\underline{\theta})$ is small Usually $\underline{\theta}^1 = 0$ Then (by [3, §1 1], for example) there exists a relaxation scheme with fast convergence for all Fourier components except those close to some $\underline{\theta}^j$ The error after few such relaxation sweeps must therefore have the form

$$V^h(\underline{x}) = \sum_{j=1}^{\ell} V_j^h(\underline{x}) \exp(i\underline{\theta}^j \cdot \underline{x}/h), \tag{7.1}$$

where V_j^h are smooth functions. Whereas classical multigrid seeks to approximate V^h on a coarser grid and the algorithm of Sec 6 approximates V_1^h, the new algorithm will separately approximate each of the V_j^h, by successively employing ℓ different coarse-grid corrections

Generally, denoting by H the coarser-grid meshsize ($H = 2h$), the equations for V_j^H, the coarse-grid approximation to V_j^h, should have the form

$$L_j^H V_j^H = I_{h,j}^H R^h \tag{7 2}$$

where $\hat{L}_j^H(\underline{\theta}) \approx \hat{L}_j^h(\underline{\theta}^j + \underline{\theta})$ for small $\underline{\theta}$, and $\hat{I}_{h,j}^H(\underline{\theta}^k) \approx \delta_{jk}$ (=0 except for $\delta_{jj} = 1$) The boundary conditions may couple V_j^H and V_k^H on any piece of boundary along which $\exp(i(\underline{\theta}^j - \underline{\theta}^k) \cdot \underline{x}/h)$ is a smooth function There are various ways, variational ones and more direct ones to derive L_j^H, $I_{h,j}^H$ and the boundary conditions There also exist various ways for solving (7.2). In highly indefinite problems the latter leads to creating more components on grid $4h$, etc., so that on increasingly coarser grids the representation tends to a Fourier representation

Here we give only the very simple example of solving for the skew Laplacian (2.5). (For a more general case, see [6].) In this case $\ell = 2$, $\underline{\theta}^1 = (0,0)$ and $\underline{\theta}^2 = (\pi,\pi)$, and one may simply take $L_1^H = L_2^H$ to be any H-approximation to the Laplace operator. In some situations, where the same mechanism creates both the fine grid and the coarse grid equations, these L_j^H may again be skew-Laplacians. As transfer operators one can use

$$I_{h,1}^H = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad I_{h,2}^H = \frac{1}{16}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \tag{7.3}$$

Considering the case that the fine-grid boundary conditions are Dirichlet conditions identically satisfied by any fine-grid approximation, the coarse-grid boundary conditions for both V_1^H and V_2^H are the homogeneous Dirichlet conditions. For solving the coarse-grid equations (7.2) the MG cycle of Sec. 6.1 can be used, *even in the case that L_j^H are themselves quasi-elliptic*, because, for the purpose of accelerating the fine-grid algebraic convergence, equations (7.2) need to be solved each time only to their truncation level (i.e., only to the level of the error $V_j^H - V_j^h$). In case of similar equations but with non-constant coefficient, averaging as in Sec. 6.3 should better be used.

7.2 The modified algorithm. Given an approximate solution u_0^k to (6.1), the modified multigrid cycle MMG for producing the improved solution u_1^k

$$u_1^k \leftarrow MMG(k, u_0^k, F^k) \tag{7.4}$$

is defined *non*-recursively as follows.

If $k = 1$ solve (6.1) by any direct or iterative method, yielding the final u_1^k. Otherwise, perform ν relaxation sweeps on (6.1), resulting in a new approximation $u^{k,0}$, and then, for $j = 1, 2, \ldots, \ell$, calculate

$$v^{k-1,j} \leftarrow MG\big(k-1, 0, I_{h_k,j}^{2h_k}(F^k - L^k u^{k,j-1})\big)$$

$$u^{k,j} = u^{k,j-1} + \exp(i\underline{\theta}^j \ \underline{x}/h)\, I_{k-1}^k v^{k-1,j}$$

with $u_1^k = u^{k,\ell}$ being the final result. I_{k-1}^k again denotes linear interpolation. MG is the cycle defined in Sec. 6.1, with a choice of γ, ν_1, ν_2.

With this MMG cycle replacing the MG cycle, the modified FMG algorithm is defined in the same way as FMG in Sec. 6.2.

7.3 Modified smoothing analysis. The smoothing factor for the above MMG cycle, i.e., the ideal factor of convergence one can expect from such a cycle per relaxation sweep on the finest grid is defined by

$$\bar{\mu} = \max_{\pi/2 \le |\underline{\theta}-\underline{\theta}^j| \text{ for one } j,\ |\underline{\theta}| \le \pi} |\mu(\underline{\theta})|. \tag{7.5}$$

where $|\mu(\underline{\theta})|$ is the spectral radius of the amplification matrix (or the absolute value of the amplification factor, if $q = 1$) Note that for $\ell \geq 2^d$, the domain of $\underline{\theta}$ over which the maximum is taken may be empty. In such a situation convergence can in principle be obtained without any relaxation on the finest grid This does not mean that the algorithm is more efficient than a conventional multigrid, because it employs at least ℓ times as many relaxation sweeps on each coarser grid

A more precise *two-level analysis* can of course be made here in the conventional way [3, §4 1]

For the skew-Laplacian and the algorithm described above, the lexicographic Gauss-Seidel amplification factor (5 2) attains its maximum (7.5) at $(\pm\pi/2, 0)$ and at $(\pm\pi/2, \pi)$, yielding $\bar{\mu} = .447$

8 Numerical Experiments

8 1 The skew Laplacian problem Our main experimental studies were conducted with the skew Laplacian scheme (2.5) in the rectangle $\{0 \leq x_1 \leq 2, 0 \leq x_2 \leq 3\}$ with Dirichlet boundary conditions These conditions and the right-hand side of the differential equation $\Delta U = F$ were chosen so that the solution U of the differential equations is known, to allow direct measurements of discretization errors The sequence of grids have meshsizes $h_k = 2^{1-k}$ $(k = 1, 2, \ldots)$, each positioned so that the boundaries of Ω coincide with grid lines On every level L^k is the skew Laplacian, and the relaxation is lexicographic Gauss-Seidel The algorithms were those described in Secs 6 and 7.

Table 1 shows the maximal differential error (maximal differences between computed and differential solutions) on various grids In addition, columns headed by ∂ or ∂^c show maximal error in first derivatives, approximated either at grid midpoint by short difference quotients (the ∂ columns), or at gridpoints by ∂_j^c (the ∂^c columns). The upper part of the table gives these errors for the exact discrete solution, the lower part – for the solution obtained by a 1-FMG algorithm with $V(2,1)$ cycles For grid 5 an additional result (5a) is sometimes given It shows errors measured after the solution is averaged by $(T_1^{1/2} + T_1^{-1/2})/2$ (cf. (3 1)) The table compares skew-Laplacian with usual (compact) Laplacian (using the same meshsize and the same relaxation), and a case of smooth solution with a highly-oscillatory case The latter is shown in order to emphasize how bad quasi-elliptic schemes *can* be. In practice such highly oscillatory components have very small amplitudes If their amplitudes are bigger than $O(h^2)$ (here $h_5^2 = .001$), then second-order approximations cannot be obtained by *any* discretization In the highly oscillating case it was of course necessary to use the full weighting (6 3) for I_k^{k-1} in (6.5), this was started with $k = 7$. In the smooth case, however, injection of F was used, in order to obtain a clearer picture, clean of F-averaging

errors

TABLE 1		$U = \sin(3x + 2y)$					$U = x(2-x)y(3-y)\cos\frac{\pi(x+y)}{h_5}$				
		$\Delta^\times$			Δ^h		$\Delta^\times$			Δ^h	
grid			∂	∂^c		∂		∂	∂^c		∂
Exact	3	1703	410	271	.0517	108	2.25	3 93	3 37	0152	030
	4	0417	115	092	0129	033	2 25	4 22	3 93	0038	.008
	5	.0104	031	.027	.0032	009	608	19494	979	.0009	002
	5a	0084	.027	031			30 6	979	38 1		
1-FMG	3	1709	436	.276	0606	151	2 25	3 93	3 37	.0198	.039
	4	.0418	109	.092	0169	048	2 25	4.22	3 93	0055	.012
	5	0105	030	027	0045	014	21 5	752	64 7	0014	003
	5a	0085	.027	.031			2 13	64 7	6 41		

8 2 The Stokes and Navier-Stokes problems We have also conducted experiments with the Stokes operator (2.3), described in detail in [3, §18.6] (with slight improvements, to be described in the new edition) The unknown grid functions of this operator are U^h, V^h and P^h – the discrete horizontal velocity, vertical velocity and pressure, respectively

In the differential problem only velocities are normally given on the boundary. In the non-staggered discretization (2 3) some boundary conditions for P^h should be introduced (which is a disadvantage typical to many quasi-elliptic operators) For clarity of exposition we here avoid this issue by showing results for *periodic boundary conditions* (adjusting undetermined additive constants before measuring errors).

The exact treatment of boundary conditions is important only in measuring *asymptotic* convergence rates It does not much affect results of 1-FMG Therefore we will show such results also for the Dirichlet boundary conditions In these experiments P^h at each boundary point is taken equal to the nearest interior value of P^h, and it changes whenever the latter does This does not correspond to Neumann boundary conditions, but to coupling the four subgrids into which the P^h grid decouples. A partial relaxation sweep near Dirichlet boundaries is performed before each full relaxation sweep

The relaxation employed is *distributed Gauss-Seidel* (DGS), a special case of a scheme for relaxing general PDE systems, explained in [3. §3.7] Briefly, it is equivalent to writing $U^h = \varphi_1^h - \partial_1^c \varphi_3^h$, $V^h = \varphi_2^h - \partial_2^c \varphi_3^h$ and $P^h = -\Delta^h \varphi_3^h$,

and relaxing by usual Gauss-Seidel the resulting equations in φ_j^h. The changes in the latter imply changes in U^h, V^h and P^h, which define the actual changes performed by the DGS relaxation. The relaxation ordering is 4-colored, relaxing the four mentioned subgrids one at a time.

The domain for this problem is the square $\{0 \le x_j \le 2\pi\}$. The meshsizes are $h_k = 2^{-k}\pi$. The right-hand side and the boundary conditions are chosen so as to give the prescribed solution $U = V = P = \sin\bigl(\cos(x_1 + 2x_2)\bigr)$, a periodic solution which includes many Fourier modes. The discrete right-hand sides were calculated by $F^{k-1} = I_k^{k-1} F^k$, using (6.3), starting at $k = 8$.

Some experiments were conducted with averaging (cf. Sec. 6.3). In the present case this means averaging of P^h only, since U^h and V^h vanish in the unstable modes. When used, this *P-averaging* employed (3.1), with $m_1 = m_2 = 2$, performed on P^h in any solution or correction just before interpolating it to a finer grid.

Also mentioned below are experiments with non-stagered *incompressible Navier-Stokes* (*INS*) *equations*, with procedures similar to those for Stokes. For details see [3, §19]; the modification from staggered to non-staggered formulation and processing are the same as for the Stokes equations. Results are given for the Dirichlet problem (U and V given on the boundary, P on the boundary treated as above), for the case $U = V = P = 1 + .2\sin\bigl(\cos(x_1 + 2x_2)\bigr)$. We have experimented with small and large Reynolds numbers, Re. In the latter case anisotropic artificial viscosity was used in relaxation, its magnitude being 1.4 times the viscosity introduced by upstream differencing. Central differencing without artificial viscosity was used for the fine-to-coarse residual calculations, allowing $O(h^2)$ solutions to be obtained. The large Re PDE problem is not elliptic (more precisely, it has small ellipticity measure), so its detailed discussion is beyond the scope here. Indeed, the present example is not fully typical for large Re, because it has no boundary layers and no gridline-streamline alignments.

Table 2 summarizes four numerical experiments. Three with the Stokes ($Re = 0$) problem (exact solutions for the periodic ("Per.") boundary conditions, 1-FMG solutions with $W(2,1)$ cycles for the same problem, and similar 1-FMG solutions for the Dirichlet ("Dir.") problem), and one experiment for "infinite" Re, i.e., with viscosity completely dominated by artificial viscosity. The latter experiment uses 2-FMG algorithm with $W(2,0)$ cycles, because double discretization (different artificial viscosities at different stages) is involved (cf. [3, §10.2]). For each experiment and each grid k, the three numbers shown in the first column are $\max(\| u^k - U \|, \| v^k - V \|)$, $\| p^k - P \|$ and $\| \bar{p}^k - P \|$, where (u^k, v^k, p^k) is the solution obtained for that grid, $\bar{p}^k = \frac{1}{4}\Pi_{j=1}^2 (T_j^{1/2} + T_j^{-1/2}) p^k$ and $\| \ \|$ is the discrete L_1 norm per unit area. The three numbers in the next column (headed by "∂") are $\max_{j=1,2} \max(\| \partial_j^k u^k - \partial_j U \|, \| \partial_j^k v^k - \partial_j V \|)$, $\| \partial_j^k p^k - \partial_j P \|$, and $\| \partial_j^k \bar{p}^k - \partial_j P \|$, where $\partial_j = \partial/\partial x_j$ and $\partial_j^k = (T_j^{1/2} - T_j^{-1/2})/h_k$. In the next col-

TABLE 2		Non-staggered Navier-Stokes						Staggered Nav -Stokes	
		No P-averaging			P-averaging				
grid			∂	∂^c		∂	∂^c		∂
$Re = 0$	5	00084	0030	0108	.00084	.0030	0108	00079	.0055
Per		00500	.0394	0392	.00500	0394	0392	00150	.0094
		00542	.0395	.0395	.00542	0395	0395	00513	0212
Exact	6	00024	.0007	0026	00024	0007	0026	00018	.0013
Sol		00123	0098	0098	00123	0098	.0098	00037	0025
		00135	0100	0108	00135	0100	0108	00127	0054
$Re = 0$	5	00090	0036	0113	00097	0031	0113	00080	0052
Per		.00661	2086	0555	00978	.3452	.0682	00163	.0215
		00562	.0447	0445	00977	0670	.0691	.00540	0243
$W(2,1)$	6	.00024	.0008	0027	.00025	.0008	.0027	00018	0013
1-FMG		.00136	0536	0146	00216	1346	.0181	00036	.0036
		00136	0119	0117	00209	0180	0189	00129	0057
$Re = 0$	5	00104	0041	0111	00097	0035	0109	00076	0055
Dir		01285	3715	.0851	01480	3946	0763	00198	.0176
		.00712	0665	.0530	.00971	0719	.0649	.00544	0246
$W(2,1)$	6	00027	0011	.0028	.00026	.0008	.0027	00017	0013
1-FMG		.00337	1586	0451	00371	.1776	0264	00047	0038
		00191	.0348	.0271	.00223	.0252	.0191	00132	.0059
$Re = \infty$	5	00272	0433	.0253	.00215	.0250	0097	00168	0180
Dir		.01515	.4536	.2382	.00637	.2098	.0256	.00242	0832
		.00957	.1945	.1594	.00273	.0168	.0147	.00106	0076
$W(2,0)$	6	00138	0547	.0357	00088	.0154	.0046	.00039	.0064
2-FMG		.01517	.8227	.4913	.00142	.0830	.0085	00066	.0436
		.01051	4062	.3413	.00074	.0048	.0033	.00038	.0016

umn (headed by ∂^c), similar numbers are given, with the long difference quotient $\bar{\partial}_j^k = \partial_j^c = (T_j - T_j^{-1})/(2h_k)$ replacing ∂_j^k The next 3 columns show similar sets of results for the case that P-averaging is used The remaining 2 columns give for comparison results obtained on a *staggered* grid with the same meshsize, without P-averaging (Using $\bar{p}^k$ for approximation, especially of derivatives, still may pay if $\nu_2 = 0$)

8 3 Accuracy and Stability Tables 1 and 2 clearly show that the exact quasi-elliptic solutions ($\Delta^\times$ and the non-staggered Stokes, the latter mainly in terms of P) are several times less accurate than the corresponding fully elliptic ones (Δ^h and staggered Stokes, respectively), but they are still $O(h^2)$ Errors in the highly oscillating case, exhibiting instability, could of course all be reduced to $O(1)$ (or $O(h_s^{-1})$ in derivatives) by enough F-averaging (see §3) Averaging the solution (row 5a, or the $\bar{p}^k$ results), or taking suitable long difference quotients, cure the worst behavior too but also somewhat further reduce the smooth-component accuracy, which nevertheless remains $O(h^2)$

8 4 Poor asymptotic algebraic convergence Denote by λ the asymptotic convergence factor per multigrid cycle, i e , $\lambda = (r_\ell/r_m)^{1/(\ell-m)}$ for sufficiently large ℓ, m and $\ell - m$, where r_ℓ is any error (or residual) norm measured at any fixed stage of the ℓ-th cycle As expected (see §4 1), the usual cycles $MG(k, \quad)$ yielded poor λ for quasi-elliptic schemes

In case of the Skew Laplacian and $V(2\ 1)$ cycles, our experiments exhibited $\lambda = .845$ and $\lambda = \ 96$ for levels $k = 4$ and $k = 5$, respectively The convergence rate $\log 1/\lambda$ is clearly $O(h^2)$, as the rate of a simple Gauss-Seidel solver for the compact Laplacian Δ^h Indeed, on each subgrid (red or black) the relaxation does look like Gauss-Seidel for Δ^{h_1}, and the coarse grid corrections are no help in case the black residuals cancel the red ones in the transfer to grid $k-1$ For comparison $V(2,1)$ cycles for the compact Laplacian Δ^h with lexicographic Gauss-Seidel yield $\lambda \approx \ 12$ on all grids

Similarly, for the periodic Stokes problem and $W(2.1)$ cycles, $\lambda = \ 80$ and $\lambda = \ 945$ were obtained on levels 4 and 5, respectively, exhibiting again $O(h^2)$ rate. The rates were almost identically the same whether P averaging was used or not. For comparison, for staggered-grid Stokes discretizations the red-black DGS relaxation gives $\lambda = \ 30$ and $\lambda = \ 20$ for the $W(1,0)$ and the $W(2,0)$ cycles, respectively These same excellent rates are obtained both for the periodic and the Dirichlet boundary conditions (provided some local relaxation near boundaries is added in the latter case) The same results are obtained for the Navier-Stokes problem with small Re For large Re, divergence occur unless P-averaging is used (cf §6 3).

8 5 FMG results Despite the bad asymptotic convergence, Tables 1 and 2 clearly show that results obtained for the quasi-elliptic cases by short FMG algorithms are very good In smooth cases they yield differential errors practically as small as in the exact discrete solutions Moreover, in case of the unstable mode, the FMG results are visibly much better than the exact solution (precisely *because* the bad behavior is slow to enter) In case of non-linear equations (Table 2, $Re = \infty$) proper averaging (Sec 6 3) is evidently necessary for good FMG results

8 6 Asymptotic convergence with new algorithm The $MMG(5\quad)$ cycle of §7 2 has been employed to solve the skew Laplacian problem with $\nu = 3$ relaxation sweeps per cycle and with $V(2,1)$ used as the $MG(4,\quad)$ inner cycle For many cycles the convergence factor per cycle was steadily between 07 and 08, or a convergence factor of 425 per fine-grid relaxation close to the value 447 expected by the smoothing mode analysis (§7 3)

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