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FOR NAVIER-STOKES EQUATIONS

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ABSTRACT

The Navier-Stokes equations can be viewed as an incompletely elliptic perturbation of the Euler equations. By using the entropy function for the Euler equations as a measure of ´energy´ for the Navier-Stokes equations, we are able to obtain nonlinear ´energy´ estimates for the mixed initial boundary value problem. These estimates are used to derive boundary conditions which guarantee L^2 boundedness even when the Reynolds number tends to infinity. Finally, we propose a new difference scheme for modelling the Navier-Stokes equations in multidimensions for which we are able to obtain discrete energy estimates exactly analogous to those we obtained for the differential equation.

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INTRODUCTION

For computational problems involving the Navier-Stokes equations, it is necessary to limit the domain of computation and introduce artificial boundary conditions. Naturally, we would like these boundary conditions to be stable, compatible with weak boundary layers, and to remain valid even when the Reynolds number tends to infinity. Such a set of boundary conditions were proposed by Gustaffson and Sundström in [4]. They used energy estimates on the linearized Navier-Stokes equations to obtain boundary conditions of maximal dissipative type. In this report we define an ´energy´ in terms of the entropy function for the Euler equations and obtain fully nonlinear ´energy´ estimates from which we are able to extract a family of boundary conditions with the above properties. An attractive feature of these boundary conditions is that they are easy to implement and can be expressed in terms of the physics of the problem.

The Navier-Stokes equations are an incompletely elliptic perturbation of the Euler equations -- which are themselves a hyperbolic system of conservation laws with entropy functions. It was observed by Mock [5] that by introducing the gradient of the entropy as a new variable a system of hyperbolic conservation laws can be reduced to a symmetric, hyperbolic system in terms of this new variable. Further, Harten [5] showed that if the dissipative terms in the Navier-Stokes equations are rewritten in terms of this new variable then the matrix coefficients of the dissipative terms have certain symmetry properties. We are able to show that the augmented matrix formed from these matrix coefficients is, in fact, negative semidefinite. This observation is crucial to the energy estimates we obtain for the Navier-Stokes equations.

This leads us to propose a new difference scheme for modelling Navier-Stokes equations in multidimensions. We are able to obtain discrete ´energy´ estimates — which are exact analogs of the ´energy´ estimates we obtained for the differential equation — at the semidiscrete level, even for meshes with unequal mesh widths. Thus we are able to propose boundary conditions and a difference scheme for the Navier-Stokes equations which give a priori boundedness of ´energy´ for all time.

This report is organized as follows: In Section 2 we define the Navier-Stokes equations and obtain the necessary results to derive the ´energy´ estimates of Section 3. In Section 4 we propose a family of ´stable´ boundary conditions and relate them to the physics of the problem. In Section 5 we propose a new method for differencing the Navier-Stokes equations in multidimensions and obtain discrete ´energy´ estimates for our difference scheme. Finally in Section 6 we obtain stable boundary conditions for the difference scheme and conclude by displaying some numerical simulations in Section 7.

2. PRELIMINARIES

We consider systems of hyperbolic conservation laws of the form:

$$q_t + \sum_{i=1}^{d} f^i(q)_{x_i} = 0. \tag{2.1}$$

Here $q(x,t)$ is an n column vector of unknowns, $f^i(q)$ is a vector valued function of n components, $x = (x_1,\cdots,x_d)$ and $f = (f^1,\cdots,f^d)$.

We can rewrite (2.1) in matrix form:

$$q_t + \sum_{i=1}^{d} A^i(q) q_{x_i} = 0 \tag{2.2}$$

where $A^i(q) = f^i_q$. The system (2.1) is called hyperbolic if the matrix

$$\sum_{i=1}^{d} \omega_i A^i(q) \tag{2.3}$$

has real eigenvalues and a complete set of eigenvectors for all real ω_i.

Following Mock, a scalar function $V(q)$ is an entropy function for (2.1) if:

i) V satisfies

$$V_q f^i_q = F^i_q \tag{2.4}$$

where $F^i(q)$ is some scalar function called entropy flux in the x_i direction.

ii) V is a convex function of q.

It follows from (2.4) upon multiplying (2.1) by V^T_q that every smooth solution of (2.1) also satisfies:

$$V_t + \sum_{i=1}^{d} F^i_{x_i} = 0 \tag{2.5}$$

where $F = (F^1,\dots,F^d)$.

The Euler Equations of Gas Dynamics

Description of variables:

ρ denotes density,

u denotes velocity in the x direction,

v denotes velocity in the y direction,

w denotes velocity in the z direction,

m_u, m_v, and m_w are the components of momentum in the x, y and z directions respectively,

T is the temperature,

p is the pressure,

U is the thermodynamic entropy,

E is the energy,

R is the universal gas constant,

γ is the ratio of specific heats.

Note that we shall use (x, y, z) and (x_1, x_2, x_3) interchangably to denote the spatial vector $\vec{x}$.

We shall also need the following thermodynamic relations:

$$T = \frac{(\gamma - 1)}{R}\left(\frac{E}{\rho} - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho^2}\right)$$

$$p = (\gamma - 1)\left(E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho}\right)$$

$$U = \log\left(E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho}\right) - \gamma \log \rho = \log p - \gamma \log \rho.$$

up to an additive constant.

T, p and ρ will always be restricted to be positive because of obvious physical considerations. q will always denote the vector:

$$q = \begin{bmatrix} \rho \\ m_u \\ m_v \\ m_w \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}.$$

The Euler equations are of the form:

$$q_t + f^1_x + f^2_y + f^3_z = 0$$

where

$$f^1 = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E + p)u \end{bmatrix}, \quad f^2 = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho vw \\ (E + p)v \end{bmatrix}, \quad f^3 = \begin{bmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (E + p)w \end{bmatrix}.$$

We shall write the Euler equations in operator notation as:

(2.6) $$Eq = 0.$$

The Euler equations have a family of strictly convex entropy functions defined by

$$V(q) = -\rho h(U).$$

The preferred entropy function in most physical applications is:

$$V(q) = -\rho U = -\rho \log \left(E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho} \right) + \gamma\rho \log \rho.$$

The entropy flux functions turn out to be:

$$F^1 = -m_u U = -m_u \log \left(E - \frac{(m_u^2 + m_v^2 + m_w^2)}{2\rho} \right) + \gamma m_u \log \rho$$

$$F^2 = -m_v U$$

$$F^3 = -m_w U.$$

It should be noted that the entropy function $V(q)$ is strictly convex but may be nonpositive in general.

Navier-Stokes Equations

We shall denote the Navier-Stokes equations in operator notation as:

$$Nq = 0 \tag{2.7}$$

where

$$Nq = Eq + (-D)q,$$

where $(-D)q = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} A^{ij}(q) q_{x_j} \right)_{x_i}.$

We can represent (2.7) in the alternative form:

$$Nq = Eq + \sum_{i=1}^{3} (h^i)_{x_i}$$

where

$$h^i = \sum_{j=1}^{3} A^{ij}(q)q_{x_j}$$

and h^1, h^2, h^3 are as follows:

$$h^1 = \begin{bmatrix} 0 \\ \theta_{xx} \\ \theta_{yx} \\ \theta_{zx} \\ \theta_{xx} u + \theta_{yz} v + \theta_{zx} w - k \frac{\partial T}{\partial x} \end{bmatrix}, \quad h^2 = \begin{bmatrix} 0 \\ \theta_{xy} \\ \theta_{yy} \\ \theta_{zy} \\ \theta_{xy} u + \theta_{yy} v + \theta_{zy} w - k \frac{\partial T}{\partial y} \end{bmatrix}$$

$$h^3 = \begin{bmatrix} 0 \\ \theta_{xz} \\ \theta_{yz} \\ \theta_{zz} \\ \theta_{xz} u + \theta_{yz} v + \theta_{zz} w - k \frac{\partial T}{\partial z} \end{bmatrix}$$

where:

$$\theta_{xx} = -2\mu \frac{\partial u}{\partial x} - \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\theta_{yy} = -2\mu \frac{\partial v}{\partial y} - \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\theta_{zz} = -2\mu \frac{\partial w}{\partial z} - \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$\theta_{xy} = \theta_{yx} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\theta_{yz} = \theta_{zy} = -\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\theta_{zx} = \theta_{xz} = -\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right).$$

Here the parameters μ, λ and k are as follows:

μ is the shear coefficient of viscosity,

λ is the second coefficient of viscosity,

k is the coefficient of thermal conductivity.

Clearly

$$(-D)q = h_x^1 + h_y^2 + h_z^3.$$

We write the Navier-Stokes equations as

$$Nq = Eq + (-D)q \tag{2.8}$$

where E is the differential operator corresponding to the Euler equations and D is the elliptic perturbation (depending on the parameters μ, λ and k) due to the dissipative terms in the Navier-Stokes equations.

The entropy function $V(q) = -\rho U$ for the Euler equations is not positive valued in general. Since we wish to interpret the entropy as a measure of 'energy' of the system, we can normalize $V(q)$ and define a new entropy $\tilde{V}(q)$ which has the properties:

(i) $\tilde{V}(q) > 0 \ \forall q$

(ii) $\tilde{V}(q) = 0 \Leftrightarrow q = \bar{q}$ for some fixed $\bar{q}$.

For the entropy V is not altered by adding to it an arbitrary inhomogeneous linear function. Hence we define $\tilde{V}(q)$ as follows:

(2.9) $$\tilde{V}(q) = V(q) - V(\bar{q}) - \sum_{i=1}^{5} \frac{\partial V}{\partial q_i}(\bar{q})(q_i - \bar{q}_i).$$

The associated entropy flux functions are given by

(2.10) $$\tilde{F}^i(q) = F^i(q) - F^i(\bar{q}) - \sum_{j=1}^{5} \frac{\partial V}{\partial q_j}(\bar{q})(f^i(q) - f^i(\bar{q}))_j.$$

We choose as our rest state $\bar{q} = (\bar{\rho}, \bar{m}_u, \bar{m}_v, \bar{m}_w, \bar{E})$ such that:

$$\bar{\rho} > 0$$
$$\bar{u} = \bar{v} = \bar{w} = 0$$
$$\bar{T} > 0.$$

In a later section we shall put further restrictions on $\bar{\rho}$ and $\bar{T}$. With this choice of $\bar{q}$ we obtain:

$$\tilde{V}(q) = \rho\left[\bar{U} - U + \frac{(\gamma - 1)}{2R\bar{T}}(u^2 + v^2 + w^2) - \gamma + \frac{T}{\bar{T}}\right] + (\gamma - 1)\bar{\rho}$$

$$\tilde{F}^1(q) = \rho u\left[\bar{U} - U + \frac{(\gamma - 1)}{2R\bar{T}}(u^2 + v^2 + w^2) - \gamma + \frac{T}{\bar{T}}\right].$$

Or more compactly:

$$\tilde{F}^1(q) = u(\tilde{V}(q) - (\gamma - 1)\bar{\rho})$$

$$\tilde{F}^2(q) = v(\tilde{V}(q) - (\gamma - 1)\bar{\rho})$$

$$\tilde{F}^3(q) = w(\tilde{V}(q) - (\gamma - 1)\bar{\rho}).$$

We make the change of variables $v = \tilde{V}_q$ and rewrite the operator Dq in terms of v. Here

$$v = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} .$$

Substituting these relations in

$$(-D)q = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} A^{ij} \, q_{x_j} \right)_{x_i}$$

yields

$$(-D)q = (-\tilde{D})v = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} C^{ij}(v) v_{x_j} \right)_{x_i}$$

where $C^{ij}(v) = A^{ij}(q)q_v$. Harten [5] observed that the matrix coefficients $C^{ij}(v)$ satisfy the symmetry relation:

$$C^{ij} = (C^{ji})^T.$$

For the energy estimates we shall derive in Section 3 we need to show that

$$(2.11) \qquad \sum_{i=1}^{3} \sum_{j=1}^{3} (\xi^i)^T \, C^{ij}(v)(\xi^j) \leq 0 \;\; \forall \; \xi^i, \; \xi^j \in \mathbb{R}^5.$$

Clearly this is equivalent to proving that the augmented matrix $\hat{C}$ defined by

$$\hat{C} = \begin{bmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{bmatrix}$$

is negative semidefinite.

Observe that $C^{ij} = (C^{ji})^T$ $\hat{C}$ is symmetric. Hence to check that $\hat{C}$ is negative semidefinite it is enough to perform an LU decomposition of $\hat{C}$ and then check that the diagonal elements of the upper triangular matrix U are negative. Thus we obtain:

(2.12) $$(2.11) \text{ holds } \Longleftrightarrow \lambda + \mu \geq 0.$$

Typically, for physical fluids $\lambda \leq 0$ and $\mu \geq 0$. Further for most fluids under nonextreme flow conditions the following relation holds:

$$\frac{\gamma\mu}{Pr} \geq \lambda + 2\mu \geq \mu.$$

Here Pr denotes the Prandtl number. So (2.11) holds under these conditions.

3. NONLINEAR ENERGY ESTIMATES

Let Ω denote a bounded domain in $\mathbb{R}^3$ and let $\partial\Omega$ denote the boundary of Ω. Consider the mixed initial boundary value problem in Ω:

(3.1) $$Nq = 0 \quad \forall \vec{x} \in \Omega,\ t \geq 0$$

where N is the Navier-Stokes differential operator with initial condition:

$$(3.2) \qquad q(\vec{x}, t)|_{t=0} = q_0(\vec{x}) \ \forall \ \vec{x} \in \Omega$$

and boundary condition:

$$(3.3) \qquad Bq = 0 \quad \forall \ \vec{x} \in \partial\Omega, \ t \geq 0$$

where B is a boundary operator.

Define:

$$(3.4) \qquad S(t) = \int_{\Omega} \tilde{V}(q(\vec{x}, t))d\vec{x}.$$

We claim that $S(t)$ gives us an estimate of the energy of the system at time t. Note that $\tilde{V}(q)$ is a strictly convex, non-negative function of q, i.e.,

i) $\tilde{V}(q) \geq 0$

ii) $\tilde{V}(q) = 0 \iff q = \overline{q}$

iii) $\tilde{V}qq > cI$ where $c > 0$, at least in some appropriate physical domain.

In particular, (iii) implies:

$$(3.5) \qquad \tilde{V}(q) \geq \alpha\|q - \overline{q}\|^2$$

for some $\alpha > 0$. Hence we conclude that $S(t)$ has the following properties:

i) $S(t) \geq 0$

ii) $S(t) = 0 \iff q(\vec{x}, t) \equiv \overline{q} \ \ \forall \ x \in \Omega$

iii) $S(t) \leq k$, where k is a constant $\Rightarrow$

$$\left(\int_{\Omega} \| q(\vec{x}, t) - \bar{q} \|^2 \, d\vec{x} \right)^{1/2} \leq \left(\frac{k}{\alpha}\right)^{1/2}$$

if $q(\vec{x}, t)$ lies in a domain where an inequality of the form (3.5) holds.

We wish to study the time evolution of $S(t)$. Recollect that the Navier-Stokes equations are:

$$Nq = Eq + (-D)q = 0$$

where

$$(-D)q = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(A^{ij}(q) q_{x_j}\right)_{x_i} .$$

Making the change of variables $v = \tilde{V}_q$ gives:

$$(-D)q = (-\tilde{D})v = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(C^{ij}(v) v_{x_j}\right)_{x_i}$$

where the matrix coefficients $C^{ij}(v)$ satisfy the condition:

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \xi^{iT} C^{ij}(u) \xi^j \leq 0 \ \forall \ \xi^i, \xi^j \in \mathbb{R}^5 . \tag{3.6}$$

This leads us to the following theorem.

Theorem 3.1: Consider the mixed initial boundary value problem (3.1)-(3.3). Let $(\zeta^1, \ldots, \zeta^d)$ denote the outward unit normal to the boundary $\partial\Omega$ and let $\tilde{F}^1, \ldots, \tilde{F}^d$ denote the entropy flux functions as in (2.10). Then any piecewise smooth solution of (3.1)-(3.3) satisfies the energy estimate:

$$\frac{dS}{dt} \leq \int_{\partial\Omega} \left(\sum_{i=1}^{3} \tilde{F}^i \zeta^i + \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{V}_q \zeta^i A^{ij}(q) q_{x_j} \right) d\sigma . \tag{3.7}$$

<u>Here</u> $d\sigma$ <u>is an element of surface area.</u>

<u>Proof</u>: We prove it here for the case $q(\vec{x}, t)$ is 'smooth enough.' We have

$$(3.8) \qquad Nq = q_t + \sum_{i=1}^{3} f^i(q)_{x_i} + \sum_{i=1}^{3} \left(\sum_{j=1}^{3} A^{ij}(q) q_{x_j} \right)_{x_i} = 0.$$

Premultiplying (3.8) by $\tilde{V}_q$ we get

$$\tilde{V}_q \, q_t + \sum_{i=1}^{3} \tilde{V}_q (f^i(q))x_i + \sum_{i=1}^{3} \tilde{V}_q \left(\sum_{j=1}^{3} A^{ij}(q) q_{x_j} \right)_{x_i} = 0.$$

By (2.4), this reduces to

$$(3.9) \qquad \tilde{V}_t + \sum_{i=1}^{3} \tilde{F}^i_{x_i} + \sum_{i=1}^{3} \tilde{V}_q \left(\sum_{j=1}^{3} A^{ij}(q) q_{x_j} \right)_{x_i} = 0.$$

From (3.9) we obtain

$$\frac{dS}{dt} = - \int_\Omega \left\{ \sum_{i=1}^{3} \tilde{F}^i_{x_i} + \sum_{i=1}^{3} \tilde{V}_q \left(\sum_{j=1}^{3} A^{ij}(q) q_{x_j} \right)_{x_i} \right\} d\vec{x}.$$

or

$$\frac{dS}{dt} = - \int_\Omega \left\{ \sum_{i=1}^{3} \tilde{F}^i_{x_i} + \sum_{i=1}^{3} v^T \left(\sum_{j=1}^{3} C^{ij}(v) v_{x_j} \right)_{x_i} \right\} d\vec{x}.$$

By the divergence theorem

$$\frac{dS}{dt} = - \int_{\partial\Omega} \left(\sum_{i=1}^{3} \tilde{F}^i \zeta^i + \sum_{i=1}^{3} \sum_{j=1}^{3} v^T \zeta^i C^{ij}(v) v_{x_j} \right) d\sigma$$

$$+ \int_{\Omega} \left(\sum_{i=1}^{3} \sum_{j=1}^{3} v_{x_i}^T \; C^{ij}(v) v_{x_j} \right) d\vec{x}.$$

But by (3.6)

$$\int_{\Omega} \left(\sum_{i=1}^{3} \sum_{j=1}^{3} v_{x_i}^T \; C^{ij}(v) v_{x_j} \right) d\vec{x} \leq 0.$$

Hence we obtain

$$\frac{dS}{dt} \leq - \int_{\partial\Omega} \left\{ \sum_{i=1}^{3} \tilde{F}^i \zeta^i + \sum_{i=1}^{3} \sum_{j=1}^{3} v^T \zeta^i \; C^{ij}(v) v_{x_j} \right\} d\sigma.$$

This finally yields

$$\frac{dS}{dt} \leq - \int_{\partial\Omega} \left(\sum_{i=1}^{3} \tilde{F}^i \zeta^i + \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{V}_q \; \zeta^i \; A^{ij}(q) q_{x_j} \right) d\sigma. \tag{3.7}$$

■

<u>Remark</u>: Estimate (3.7) is a fully nonlinear ´energy´ estimate which holds for a broad class of solutions of the mixed initial boundary value problem (3.1)-(3.3). In fact (3.7) is simpler to obtain and broader in scope than the linearized energy estimates in current use.

4. STABLE BOUNDARY CONDITIONS FOR THE NAVIER-STOKES EQUATIONS

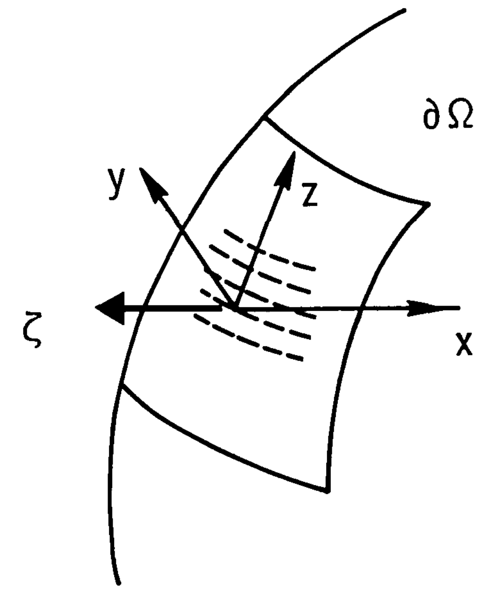


Figure 4.1

Since the Navier-Stokes equations are rotationally invariant, we can consider a moving coordinate frame (x, y, z) where x points in the direction of the inward normal and y and z are tangential to $\partial\Omega$. Of course, we reorient u, v, and w so that u is the component of velocity in the x direction, v in the y direction, and w in the z direction. Let ζ denote the outward unit normal to $\partial\Omega$. Clearly $\zeta = (-1, 0, 0)$.

Then the nonlinear energy estimate (3.7) takes the simpler form:

$$\frac{dS}{dt} \leq \int_{\partial\Omega} \left(\tilde{F}^1 + \sum_{j=1}^{3} \tilde{V}_q A^{1j} q_{x_j} \right) d\sigma \tag{4.1}$$

or

$$\frac{dS}{st} \leq \int_{\partial\Omega} (\tilde{F}^1 + \tilde{V}_q\ h^1) d\sigma \tag{4.2}$$

where

$$h^1 = \sum_{j=1}^{3} A^{1j}\ q_{x_j}.$$

Substituting the relations obtained in Section 2 we obtain

$$\tilde{V}_q\ h^1 = -\frac{(\gamma-1)}{R}\left[0,\ \frac{u}{T},\ \frac{v}{T},\ \frac{w}{T},\ -\left(\frac{1}{T}-\frac{1}{\bar{T}}\right)\right]\begin{bmatrix} 0 \\ 2\mu\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \left\{\begin{array}{l} 2\mu u\frac{\partial u}{\partial x} + \lambda u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \\ + \mu v\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \mu u\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) + k\frac{\partial T}{\partial x}\end{array}\right. \end{bmatrix},$$

or,

$$\tilde{V}_q\ h^1 = \frac{k(\gamma-1)}{R}\left(\frac{1}{T}-\frac{1}{\bar{T}}\right)\frac{\partial T}{\partial x} - \frac{(\gamma-1)}{R\bar{T}}\left\{(2\mu+\lambda)u\frac{\partial u}{\partial x} + \lambda u\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \mu v\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \mu w\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right\}. \tag{4.3}$$

So finally we obtain the needed 'energy' estimate:

$$\frac{dS}{dt} \leq \int_{\partial\Omega} u(\tilde{V} - (\gamma - 1)\bar{\rho})d\sigma + \int_{\partial\Omega} \frac{k(\gamma - 1)}{R\bar{T}} \left(\frac{\bar{T} - T}{\bar{T}}\right) \frac{\partial T}{\partial x}\, d\sigma$$

$$(4.4) \qquad - \int_{\partial\Omega} \frac{(\gamma - 1)}{R\bar{T}}\, u\left((2\mu + \lambda)\, \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right) d\sigma$$

$$- \int_{\partial\Omega} \frac{(\gamma - 1)}{R\bar{T}}\, v\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)d\sigma - \int_{\partial\Omega} \frac{(\gamma - 1)}{R\bar{T}}\, w\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)d\sigma$$

where

$$(4.5) \qquad u(\tilde{V} - (\gamma - 1)\bar{\rho}) = \rho u\left[\bar{U} - U + \frac{(\gamma - 1)}{2R\bar{T}}\, (u^2 + v^2 + w^2) - \gamma + \frac{T}{\bar{T}}\right]$$

and

$$(4.6) \qquad \tilde{V} = \rho\left[\bar{U} - U + \frac{(\gamma - 1)}{2R\bar{T}}\, (u^2 + v^2 + w^2) - \gamma + \frac{T}{\bar{T}}\right] + (\gamma - 1)\bar{\rho}.$$

So far we have left $\bar{T}$ and $\bar{\rho}$ vaguely defined. We are now going to specify $\bar{T}$ and $\bar{\rho}$, but first we have to make a few assumptions about the solution $q(\vec{x}, t)$ to the mixed initial boundary value problem (3.1)-(3.3).

Assumptions:

$$(4.7) \qquad \left.\begin{array}{l} \text{i)} \;\; \exists\, \rho_{min} > 0 \ni \rho(\vec{x}, t) \geq \rho_{min} \quad \forall\, \vec{x} \in \Omega,\ t \geq 0 \\[2ex] \text{ii)} \;\; \exists\, \rho_{max} > \rho_{min} \ni \rho(\vec{x}, t) < \rho_{max} \quad \forall\, \vec{x} \in \Omega,\ t \geq 0 \\[2ex] \text{iii)} \;\; \exists\, T_{min} > 0 \ni T(\vec{x}, t) \geq T_{min} \quad \forall\, \vec{x} \in \Omega,\ t \geq 0. \end{array}\right\}$$

Assumptions (i)-(iii) exclude cavitation, freezing of the fluid, and such esoteric phenomena as the formation of black holes. Further, it should be noted that if (4.7) is valid $\tilde{V}(q)$ is strictly convex and hence, then we are able to obtain boundary conditions which ensure that $S(t)$ remains bounded, i.e.,

$S(t) \leq k \quad \forall\, t \geq 0$ where k is a constant; then by the discussion following (3.5)

$$\Rightarrow \left(\int_\Omega \| q(\vec{x}, t) - \bar{q} \|^2 \, d\vec{x} \right)^{1/2} \leq C \quad \text{(another constant)} \quad \forall\, t \geq 0.$$

Hence we obtain L^2 boundedness of the solution for all time.

Choose $0 < \bar{\rho} < \rho_{min}$ such that

$$\tilde{V} - (\gamma - 1)\bar{\rho} > 0 \tag{4.8}$$

and $0 < \bar{T} < T_{min}$ such that

$$\bar{U} - U < 0 \tag{4.9}$$

$\forall\, q$ staisfying (4.7).

It is easy to see that this can always be done. For by (4.6)

$$\tilde{V}(q) \geq \rho \left[\bar{U} - U - \gamma + \frac{T}{\bar{T}} \right] + (\gamma - 1)\bar{\rho}$$

or

$$\tilde{V}(q) \geq \rho \left[(\gamma - 1) \log \left(\frac{\rho}{\bar{\rho}}\right) - \gamma + \frac{T}{\bar{T}} - \log \left(\frac{T}{\bar{T}}\right) \right] + (\gamma - 1)\bar{\rho}.$$

Clearly

$$\frac{T}{\bar{T}} - \log\left(\frac{T}{\bar{T}}\right) \geq 1 \;\forall\; T \geq 0.$$

Hence we get

$$\tilde{V}(q) \geq \rho\left[(\gamma - 1)\log\left(\frac{\bar{\rho}}{\rho}\right) - \gamma + 1\right] + (\gamma - 1)\bar{\rho}.$$

Thus if $\rho \geq \rho_{min} > 0$ we can choose $\bar{\rho} < \rho_{min}$ such that

$$\tilde{V}(q) - (\gamma - 1)\bar{\rho} > 0.$$

Hence (4.8) holds.

Now

$$\bar{U} - U = (\gamma - 1)\log\left(\frac{\rho}{\bar{\rho}}\right) - \log\left(\frac{T}{\bar{T}}\right).$$

If (4.7) is valid then

$$\bar{U} - U < (\gamma - 1)\log\left(\frac{\rho_{max}}{\bar{\rho}}\right) - \log\left(\frac{T_{min}}{\bar{T}}\right).$$

We can choose $0 < \bar{T} < T_{min}$ such that

$$(\gamma - 1)\log\left(\frac{\rho_{max}}{\bar{\rho}}\right) - \log\left(\frac{T_{min}}{\bar{T}}\right) < 0.$$

Hence (4.9) holds.

The Navier-Stokes equations are:

$$Nq = Eq + (-D)q = 0 \qquad (4.10)$$

where E is the hyperbolic differential operator corresponding to the Euler equations and D is the incompletely elliptic perturbation, depending on the parameters, μ, λ, and k, due to the dissipative terms in the Navier-Stokes equations. It is assumed that μ, λ, and k are proportional to a small parameter $\varepsilon > 0$. Thus the Navier-Stokes equations may be viewed as an incompletely elliptic perturbation of a system of hyperbolic equations.

The question we are concerned with is: for which boundary conditions are the solutions of the perturbed problem (i.e., $\varepsilon > 0$) well defined in some time interval $0 \leq t \leq \tau_0$ and are bounded in an appropriate norm as $\varepsilon \to 0$.

Michelson [8] suggests boundary conditions of the type given below for the singular perturbation problem:

$$\text{(4.11a)} \qquad Bq = S\left(\{(\varepsilon D)^{\alpha}\}_{\alpha \varepsilon A};\ \vec{x}\ \varepsilon \partial\Omega,\ t \geq 0,\ \varepsilon > 0\right) = 0$$

where

$$(\varepsilon D)^{\alpha} = \varepsilon^{|\alpha|}\, D_{x_1}^{\alpha_1}\, D_{x_2}^{\alpha_2}\, D_{x_3}^{\alpha_3}\, D_{t}^{\alpha_4}$$

and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a multi-index in some finite set $A \subseteq \mathbb{Z}^4$.

It is well known that a singular perturbation problem of the type we are considering exhibits a boundary layer phenomenon. We wish to choose boundary conditions in such a way that a strong boundary layer does not develop. For the boundary conditions to be compatible with a weak boundary layer, we need a condition of the sort given by:

$$\text{(4.11b)} \qquad B|_{\varepsilon=0}\, q = S\left(\{(\varepsilon D)^{\alpha}\}_{\alpha \varepsilon A};\ \vec{x}\ \varepsilon \partial\Omega,\ t \geq 0,\ \underline{\varepsilon = 0}\right) = 0$$

gives a well-posed boundary value problem for the hyperbolic part of the Navier-Stokes equation:

$$N\big|_{\varepsilon=0}\, q = Eq = 0$$

(see [8]).

Further, from Strikwerda's work on initial boundary value problems for incompletely parabolic systems, we know that the number of boundary conditions which should be imposed for the Navier-Stokes equations to obtain a well-posed problem are:

5 boundary conditions for inflow boundary and

4 boundary conditions for outflow boundary.

At the same time the unperturbed hyperbolic system requires:

5 boundary conditions for supersonic inflow,

4 boundary conditions for subsonic inflow,

1 boundary condition for subsonic outflow, and

none for supersonic outflow.

The boundary conditions we are going to impose will be of the form:

$$\varepsilon\, R \frac{\partial q}{\partial x} + Sq = g \tag{4.12}$$

where R is a matrix of rank at most 4.

To get a set of boundary conditions that also works for $\varepsilon = 0$ S must be chosen in such a way that $Sq = g$ is a proper set of boundary conditions for the unperturbed hyperbolic problem. If $Sq = g$ gives too many boundary conditions for the unperturbed hyperbolic problem, the solution will contain boundary layer components of the form $\exp(-x/\varepsilon)$ for $\varepsilon \to 0$; (see [4]).

We now proceed to suggest a set of maximal dissipative boundary conditions for the Navier-Stokes equations which are compatible with weak boundary layers.

Outflow Boundary Conditions

We need to specify 4 boundary conditions for the perturbed problem. For supersonic outflow $(-u > c > 0)$ we need not specify any boundary conditions for the unperturbed hyperbolic problem, while for subsonic outflow $(c > -u > 0)$ we need to specify only one. From (4.4) we have

$$\frac{dS}{dt} \leq \int_{\partial\Omega} u(\tilde{V} - (\gamma - 1)\bar{\rho})d\sigma + \int_{\partial\Omega} \frac{k(\gamma - 1)}{R\bar{T}} \left(\frac{\bar{T} - T}{T}\right) \frac{\partial T}{\partial x}\, d\sigma$$

$$- \int_{\partial\Omega} \frac{(\gamma - 1)}{R\bar{T}}\, u\left((2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)d\sigma$$

$$- \int_{\partial\Omega} \frac{(\gamma - 1)}{R\bar{T}}\, v\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)d\sigma - \int \frac{(\gamma - 1)}{R\bar{T}}\, w\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)d\sigma.$$

By (4.8)

$$\int_{\partial\Omega} u(\tilde{V} - (\gamma - 1)\bar{\rho})d\sigma \leq 0.$$

The boundary conditions we suggest which give a decay of energy are:

$$\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_1 T = g_1, \qquad \alpha_1 \geq 0$$

$$(4.13) \qquad (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \alpha_2 u = g_2, \quad \alpha_2 \geq 0$$

$$\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \alpha_3 v = g_3, \qquad \alpha_3 \geq 0$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \alpha_4 w = g_4, \qquad \alpha_4 \geq 0.$$

It should be noted that if we put $\alpha_i = 0$ we must also put $g_i = 0$ for $i = 1,\cdots,4$. Since we want (4.13) to be compatible with the preceding discussion, we end up with the following set of maximal dissipative′ boundary conditions:

Supersonic outflow

$$\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} = 0$$

$$(4.14) \qquad (2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0$$

Subsonic Outflow

$$\frac{k(\gamma - 1)}{R}\frac{\partial T}{\partial x} = 0$$

$$(2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \alpha_2 u = g_2 \qquad \alpha_2 \geq 0$$

(4.15)

$$\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0$$

Inflow Boundary Conditions

By (4.9)

$$\int_{\partial\Omega} \rho u\left[U - \overline{U}\right]d\sigma \leq 0$$

Subsonic Inflow

Since $c > u$ we have $u^2 < \gamma RT$. Hence

$$\int_{\partial\Omega} \frac{(\gamma - 1)}{2R\overline{T}}(\rho u)u^2\, d\sigma \leq \int_{\partial\Omega} \frac{(\gamma - 1)}{2R\overline{T}}(\rho u)\gamma RT d\sigma$$

$$\Rightarrow \frac{dS}{dt} \leq -\int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}}\, u\left((2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right)d\sigma$$

$$+ \int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}}\, v\left(\frac{(\rho u)}{2} v - \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)d\sigma$$

$$+ \int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}}\, w\left(\frac{(\rho u)w}{2} - \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right)d\sigma$$

$$+ \int_{\partial\Omega} \frac{1}{\overline{T}}\left[\frac{(\overline{T} - T)}{T}\frac{k(\gamma - 1)}{R}\frac{\partial T}{\partial x} + \frac{(\gamma^2 - \gamma + 2)}{2}(\rho u)T\right]d\sigma.$$

The boundary conditions we could specify are:

$$
\begin{aligned}
&\rho u = g_1 \\
&(2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \\
&\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \alpha_3 v = g_3 \\
&\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \alpha_4 w = g_4 \\
&\frac{k(\gamma - 1)}{R}\frac{\partial T}{\partial x} - \alpha_5 T = g_5.
\end{aligned}
\tag{4.16}
$$

Here we must impose the conditions

$$\alpha_3 > g_1/2$$

$$\alpha_4 > g_1/2$$

$$\alpha_5 > \frac{(\gamma^2 - \gamma + 2)}{2} g_1 + \varepsilon \quad \text{for some suitable} \quad \varepsilon > 0$$

depending on $\overline{T}$.

Supersonic Inflow

$$\frac{dS}{dt} < \int_{\partial\Omega} \rho u[\overline{U} - U]d\sigma + \int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}} u\left[\frac{(\rho u)}{2} u - \left\{(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right\}\right]d\sigma$$

$$+ \int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}} v\left[\frac{(\rho u)}{2} v - \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)d\sigma\right] + \int_{\partial\Omega} \frac{(\gamma - 1)}{R\overline{T}} w\left[\frac{(\rho u)}{2} w - \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]d\sigma$$

$$+ \int_{\partial\Omega} \frac{1}{\overline{T}}\left[\frac{(\overline{T} - T)}{T} \frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} + (\rho u)T\right]d\sigma.$$

The boundary conditions we could specify are:

$$\rho u = g_1$$

$$(2\mu + \lambda) \frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \alpha_2 u = g_2$$

$$\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \alpha_3 v = g_3 \tag{4.17}$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \alpha_4 w = g_4$$

$$\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_5 T = g_5.$$

Here the following conditions must hold:

$$\alpha_2, \alpha_3, \alpha_4 > \frac{g_1}{2}$$

and $\alpha_5 > g_1 + \varepsilon$ for some suitable $\varepsilon > 0$.

Discussion of the Boundary Conditions

The outflow boundary conditions we have proposed are of the following type:

$$\frac{\partial T}{\partial x} = 0. \tag{4.18}$$

(4.18) says that the computational boundary corresponds to an insulated wall and there is no conduction of heat across it.

$$\theta_{xx} = \theta_{yx} = \theta_{zx} = 0 \tag{4.19}$$

(4.19) asserts that there is no shearing of the fluid either in the normal or tangential direction against the computational boundary.

For an inflow boundary one of the boundary conditions we have specified is:

$$\rho u = g. \tag{4.20}$$

In other words, we specify the momentum influx across the computational boundary.

For the temperature component of the state vector, we have the boundary condition:

$$\frac{k(\gamma - 1)}{R} \frac{\partial T}{\partial x} - \alpha_5 T = g_5 \tag{4.21}$$

(4.21) has an interesting physical interpretation. For injection through a porous wall into the main stream (sometimes called transpiration) the injected fluid may be, say, a coolant at a temperature considerably different from the wall, and one needs to consider an energy balance at the wall. A good approximation for coolant injection is to use the boundary condition proposed by Roberts [in Truit (1960, chapter 11)].

$$(4.22) \quad \text{Injection:} \qquad k \frac{\partial T}{\partial x} \simeq \rho_w \, u_w \, c_p (T_w - T_{coolant})$$

where $\rho_w \, u_w$ is the momentum flow of coolant per unit area through the wall, T_w is the temperature of the wall, $T_{coolant}$ is the temperature of the coolant and $\left.\frac{\partial T}{\partial x}\right|_w$ is the temperture gradient at the wall. (For more information refer to [12, chapter 1.4].)

By choosing α_5 and g_5 appropriately (4.21) can be geared to satisfy (4.22). The physical effect of such a boundary condition would be to make the temperature of the fluid within the computational boundary stabilize at $T_{coolant}$ over time. When we discretize the boundary conditions, however, we can choose g_5 so that the discretized version of (4.21) is of extrapolation type. This has the effect of allowing the system to evolve as if there were no boundary.

For the velocity component of the state vector, the boundary conditions we have specified are:

$$(2\mu + \lambda)\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \alpha_2 u = g_2$$

$$\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \alpha_3 v = g_3 \tag{4.23}$$

$$\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \alpha_4 w = g_4.$$

For one-dimensional fluid flow the boundary conditions (4.23) would take the simple form:

$$(2\mu + \lambda)\frac{\partial u}{\partial x} - \alpha_2 u = g_2. \tag{4.24}$$

For perfectly diffuse reflection of a gas or fluid against a solid wall we have the boundary condition (after Maxwell in 1879):

$$u_w \simeq \ell\left(\frac{\partial u}{\partial x}\right)_w \tag{4.25}$$

where u_w is the velocity of the fluid at the wall, ℓ is the mean free path of the gas or fluid at the wall and $\left(\frac{\partial u}{\partial x}\right)_w$ is the velocity gradient at the wall (see [12]).

By choosing g_2 and α_2 appropriately we can choose (4.24) to satisfy (4.25). Since we want the boundary conditions to be nonreflecting, however, this would be a bad choice indeed. If we let the computational boundary move with the same velocity as that of the fluid at the boundary interface, we should not have any diffuse reflection. And, in fact, we can discretize (4.24) to achieve this exactly. The discrete boundary conditions so obtained are of extrapolation type and the effect of the computational boundary is minimized.

A last remark is that the boundary conditions proposed are of maximal dissipative type and such boundary conditions are intrinsically radiative.

5. STABLE SEMI-DISCRETE DIFFERENCE SCHEMES FOR THE DIFFERENTIAL EQUATION

The stable difference scheme we are going to propose is valid for the Navier-Stokes equations in multidimensions. However, to avoid getting drowned in notation we restrict the discussion that follows to Navier-Stokes equations in one space dimension.

Consider the incompletely parabolic system obtained from a hyperbolic system of conservation laws with entropy:

$$q_t + f(q)_x + (A(q)q_x)_x = 0. \tag{5.1}$$

Let $\tilde{V}(q)$ be a normalized entropy function as in (2.9) for the hyperbolic system of conservation laws:

$$q_t + f(q)_x = 0 \tag{5.2}$$

and let $v^T = \tilde{V}_q$.

We can rewrite (5.1) as:

$$q_t + f(q)_x + (C(v)v_x)_x = 0 \tag{5.3}$$

where we assume $C(v) \leq 0$.

Divide the x axis into a mesh of points $\{x_i\}_{i\varepsilon \mathbb{N}}$ where $x_i = i\Delta x$ and Δx is the mesh width.

Let $q_j(t) = q(x_j, t)$. Consider the semidiscrete difference approximation to the hyperbolic system of conservation laws (5.2):

$$\frac{dq_j}{dt} + \frac{\Delta_-}{\Delta x} (h(q_{j-r}, \cdots, q_{j+s}) = 0 \tag{5.4}$$

where r and s are integers ≥ 0 and $h(q_{j-r}, \cdots, q_{j+s})$ is the numerical flux function corresponding to the flux function $f(q)$, i.e., $h(q_{j-r}, \cdots, q_{j+s})$ satisfies the consistency condition:

$$h(q, \cdots, q) = f(q). \tag{5.5}$$

Let the order of accuracy of the semidiscrete difference approximation be α where we can take $\alpha \geq 1$.

Further, suppose (5.4) satisfies a semidiscrete entropy inequality:

$$\frac{d\tilde{V}(q_j)}{dt} + \frac{\Delta_-}{\Delta x} [H(q_{j-r}, \cdots, q_{j+s})] \leq 0 \tag{5.6}$$

where $H(q_{j-r}, \cdots, q_{j+s})$ is the numerical entropy flux function corresponding to the entropy flux function $\tilde{F}(q)$, i.e., $H(q_{j-r}, \cdots, q_{j+s})$ satisfies the consistency relation:

$$H(q, \cdots, q) = \tilde{F}(q). \tag{5.7}$$

We propose the following semidiscrete approximation for the Navier-Stokes equation (5.1):

$$\text{(5.8)} \qquad \frac{dq_j}{dt} + \frac{\Delta-}{\Delta x}\,(h(q_{j-r},\cdots,q_{j+s})) + \frac{\Delta-}{\Delta x}\left(C(v_{j+1/2})\,\frac{\Delta_+ v_j}{\Delta x}\right) = 0$$

where

$$v_{j+1/2} = \frac{v_j + v_{j+1}}{2}\,.$$

It is easy to see that the order of accuracy β of (5.8) is given by $\beta = \min(\alpha, 2)$. Hence if (5.4) is a second-order accurate approximation to (5.2) then (5.8) is a second-order accurate approximation to (5.1). In any case, (5.7) is at least a first-order accurate approximation to (5.1).

<u>Remark</u>: The only second-order accurate semidiscrete difference scheme for a hyperbolic system of conservation laws which satisfies an entropy condition of the type (5.6) of which we are aware of is one proposed by Osher in [10].

Lemma 5.1: <u>The semidiscrete difference approximation (5.8) to the Navier-Stokes equations (5.1) satisfies the entropy inequality:</u>

$$\text{(5.9)} \qquad \frac{d\tilde{V}}{dt}\,(q_j) + \frac{\Delta-}{\Delta x}\,[H(q_{j-r},\cdots,q_{j+s})] + \frac{v_j^T}{\Delta x}\,\Delta - \left(C(v_{j+1/2})\,\frac{\Delta_+ v_j}{\Delta x}\right) \le 0.$$

Fix $\Delta t > 0$. Let

$$q_j^*(\Delta t) = q_j - \frac{\Delta t}{\Delta x}\,\Delta - (h(q_{j-r},\cdots,q_{j+s})).$$

Let $q^E(t)$ denote the solution to the semidiscrete difference scheme

$$\frac{dq^E}{dt} + \frac{\Delta-}{\Delta x} h\left(q^E_{j-r},\cdots,q^E_{j+s}\right) = 0 \tag{5.10}$$

with $q^E_j(0) = q_j$. Then $q^*_j(\Delta t) = q^E_j(\Delta t) + 0(\Delta t^2)$. Since (5.10) satisfies the entropy inequality:

$$\frac{d\tilde{V}}{dt}\left(q^E_j\right) + \frac{\Delta-}{\Delta x}\left[H(q_{j-r},\cdots,q_{j+s})\right] \leq 0$$

$\forall$ sequence $\{\Delta t_k\} \ni \Delta t_k \to 0$ $\exists$ a sequence $\{\varepsilon_k\} \ni \varepsilon_k \to 0$ and the discrete entropy inequality:

$$\frac{\tilde{V}\left(q^*_j(\Delta t_k)\right) - \tilde{V}(q_j)}{\Delta t_k} + \frac{\Delta-}{\Delta x}\left[H(q_{j-r},\cdots,q_{j+s})\right] \leq \varepsilon_k \tag{5.11}$$

holds.

We now define a discrete difference approximation for the Navier-Stokes equation:

$$\overline{q}_j(\Delta t) = q^*_j(\Delta t) - \frac{\Delta t}{\Delta x}\Delta - \left(C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right).$$

Let $q_j(t)$ denote the solution to the semidiscrete difference scheme:

$$\frac{dq_j}{dt} + \frac{\Delta-}{\Delta x}\left(h(q_{j-r},\cdots,q_{j+s})\right) + \frac{\Delta-}{\Delta x}\left(C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right) = 0.$$

Then it is obvious that $\overline{q}_j(\Delta t) = q_j(\Delta t) + 0(\Delta t^2)$. Define an approximate entropy function

$$W(\overline{q}_j(\Delta t)) = \tilde{V}\left(q^*_j(\Delta t)\right) - v^T_j\frac{\Delta t}{\Delta x}\Delta - \left(C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right).$$

By (5.11)

$$W(q_j(\Delta t_k)) < \tilde{V}(q_j) - \frac{\Delta t_k}{\Delta x} \Delta - [H(q_{j-r},\cdots,q_{j+s})]$$

$$- v_j^T \frac{\Delta t_k}{\Delta x} \left(C(v_{j+1/2}) \frac{\Delta_+ v_j}{\Delta x}\right) + \varepsilon_k \Delta t_k,$$

or

$$\frac{W(\overline{q}_j(\Delta t_k) - \tilde{V}(q_j)}{\Delta t_k} \leq - \frac{\Delta-}{\Delta x} [H(q_{j-r},\cdots,q_{j+s})] \tag{5.12}$$

$$- \frac{v_j^T}{\Delta x} [C(v_{j+1/2}) \frac{\Delta_+ v_j}{\Delta x}] + \varepsilon_k.$$

By a Taylor series argument:

$$W(\overline{q}_j(\Delta t)) = \tilde{V}(q_j(\Delta t)) + O(\Delta t^2).$$

And since $\overline{q}_j(\Delta t) = q_j(\Delta t) + O(\Delta t^2) \Rightarrow$

$$\frac{d\tilde{V}}{dt}(q_j) \leq - \frac{\Delta-}{\Delta x} [H(q_{j-r},\cdots,q_{j+s})] - \frac{v_j^T}{\Delta x} \Delta - \left(C(v_{j+1/2}) \frac{\Delta_+ v_j}{\Delta x}\right)$$

by letting $\Delta t_k \to 0$ in (5.12).

Define a discrete version of the energy $S(t)$ by:

$$S(t) = \sum_{j=1}^{N} \Delta x \tilde{V}(q_j). \tag{5.13}$$

We can now easily obtain a discrete version of the energy estimates of Section 2. (Henceforth we shall denote $h(q_{j-r},\cdots,q_{j+s})$ by h_j and $H(q_{j-r},\cdots,q_{j+s})$ by H_j.)

Theorem 5.1: Consider the semidiscrete difference approximation:

$$\frac{dq_j}{dt} = -\frac{\Delta_-}{\Delta x}[h_j] - \frac{\Delta_-}{\Delta x}\left[C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right] \tag{5.8}$$

and let

$$S(t) = \sum_{j=1}^{N} \Delta x \tilde{V}(q_j).$$

Then the following 'energy' estimate holds:

$$\frac{dS}{dt} \le \left[H_0 + v_1^T C(v_{1/2})\frac{\Delta_+ v_0}{\Delta x}\right] - \left[H_N + v_{N+1}^T C(v_{N+1/2})\frac{\Delta_+ v_N}{\Delta x}\right]. \tag{5.14}$$

Proof: By Lemma 5.1

$$\frac{d\tilde{V}(q_j)}{dt} \le -\frac{\Delta_-}{\Delta x}[H_j] - v_j^T \frac{\Delta_-}{\Delta x}\left[C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right].$$

Hence

$$\frac{dS}{dt} = \sum_{j=1}^{N} \Delta x \frac{d\tilde{V}(q_j)}{dt} \le \sum_{j=1}^{N} \Delta - [H_j] - \sum_{j=1}^{N} v_j^T \Delta - \left[C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}\right]$$

$$\Rightarrow \frac{dS}{dt} \le H_0 - H_N + \sum_{j=1}^{N} \Delta + v_j^T C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x}$$

$$+ v_1^T C(v_{1/2})\frac{\Delta_+ v_0}{\Delta x} - v_{N+1}^T C(v_{N+1/2})\frac{\Delta_+ v_N}{\Delta x}$$

using summation by parts. Since

$$C \le 0 \Rightarrow \sum_{j=1}^{N} \Delta + v_j^T C(v_{j+1/2})\frac{\Delta_+ v_j}{\Delta x} \le 0.$$

Hence

$$\frac{dS}{dt} \leq \left[H_0 + v_j^T\, C(v_{1/2})\, \frac{\Delta_+ v_0}{\Delta x}\right] - \left[H_N + v_{N+1}^T\, C(v_{N+1/2})\, \frac{\Delta_+ v_N}{\Delta x}\right].$$

Remark: The generalization of this theorem to multidimensions is obvious.

We now extend these results to uneven meshes, though restricting (5.4) to the Godunov scheme.

The Godunov Scheme

Consider a grid of points $\{x_j\}_{j\varepsilon \mathbb{N}}$ and define $\Delta x_j = x_{j+1} - x_j$. Define $x_{j-1/2} = \dfrac{x_{j-1} + x_j}{2}$ and $\delta x_j = \dfrac{\Delta x_{j-1} + \Delta x_j}{2}$.

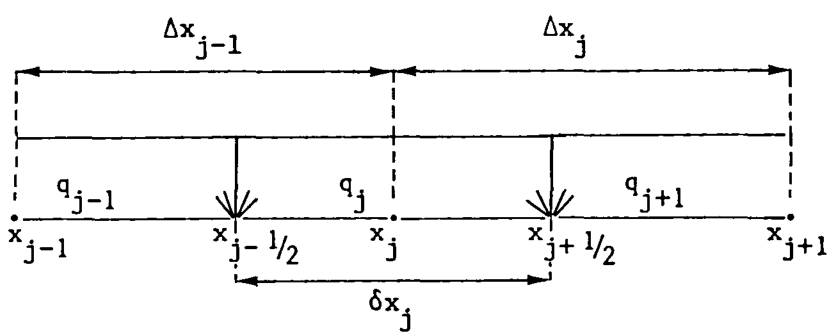


Figure 5.1

$$
(5.15)\quad \begin{cases} \text{Consider the hyperbolic system of conservation laws:} \\ \qquad q_t + f(q)_x = 0 \\ \text{with Riemann initial data:} \\ \qquad q(x,\ t)\big|_{t=0} = \begin{cases} q_j & \text{for}\quad x < 0 \\ q_{j+1} & \text{for}\quad x > 0\ . \end{cases} \end{cases}
$$

Then the Godunov flux $f^G(q_j,\ q_{j+1}) = f_j^G$ is defined as:

$$
(5.16)\qquad f^G(q_j,\ q_{j+1}) = f(q(x = 0,\ t = 0^+))
$$

where $q(x,\ t)$ is the solution to the Riemann problem (5.15).

The semidiscrete version of the Godunov scheme is:

$$
(5.17)\qquad \frac{dq_j}{dt} + \frac{\Delta-}{\Delta x}\left[f^G(q_j,\ q_{j+1})\right] = 0
$$

where $f^G(q_j,\ q_{j+1})$ is defined in (5.16). The semidiscrete Godunov scheme satisfies the entropy inequality:

$$
(5.18)\qquad \frac{d\tilde{V}(q_j)}{dt} + \frac{\Delta-}{\Delta x}\left[F^G(q_j,\ q_{j+1})\right] \le 0
$$

where $F^G(q_j,\ q_{j+1}) = F_j^G$ is the numerical entropy flux for the Godunov scheme and is defined as:

$$
F^G(q_j,\ q_{j+1}) = \tilde{F}(q(x = 0,\ t = 0^+))
$$

where $q(x, t)$ is the solution to the Riemann problem (5.15).

A natural modification of (5.8) for uneven meshes is:

$$\frac{dq_j}{dt} + \frac{\Delta_-}{\delta x_j} \left(f^G(q_j, q_{j+1})\right) + \frac{\Delta_-}{\delta x_j} \left(C(v_{j+1/2}) \frac{\Delta_+ v_j}{\Delta x_j}\right) = 0. \tag{5.19}$$

Clearly (5.19) is a first-order accurate semidiscrete difference approximation to (5.1). We claim that (5.19) satisfies the entropy inequality:

$$\frac{d\tilde{V}(q_j)}{dt} + \frac{\Delta_-}{\delta x_j} \left(F^G(q_j, q_{j+1})\right) + v_j^T \frac{\Delta_-}{\delta x_j} \left(C(v_{j+1/2}) \frac{\Delta_+ v_j}{\Delta x_j}\right) \leq 0. \tag{5.20}$$

The proof is exactly the same as in Lemma 5.1.

Define a discrete version of the 'energy' $S(t)$ for the uneven mesh $\{x_j\}$ by:

$$S(t) = \sum_{j=1}^{N} \delta x_j \tilde{V}(q_j). \tag{5.21}$$

Then we obtain the following theorem.

Theorem 5.2: Consider the semidiscrete difference approximation:

$$\frac{dqj}{dt} = -\frac{\Delta_-}{\delta x_j} (f_j^G) - \frac{\Delta_-}{\delta x_j} \left(C(v_{j+1/2}) \frac{\Delta + v_j}{\Delta x_j}\right) \tag{5.19}$$

and let

$$S(t) = \sum_{j=1}^{N} \delta x_j \tilde{V}(q_j).$$

Then the following ´energy´ estimate holds:

$$(5.22)\quad \frac{dS}{dt} \le \left[F_0^G + v_1^T\, C(v_{1/2})\, \frac{\Delta_+ v_0}{\Delta x_0}\right] - \left[F_N^G + v_{N+1}^T\, C(v_{N+1/2})\, \frac{\Delta_+ v_N}{\Delta x_N}\right].$$

Proof: The proof is exactly as in Theorem 5.1.

Remark: Theorem 5.2, though a simple extension of Theorem 5.1, will prove very useful in proposing stable boundary conditions for semidiscrete difference approximations to the Navier-Stokes equations.

6. STABLE BOUNDARY CONDITIONS FOR THE DIFFERENCE SCHEMES

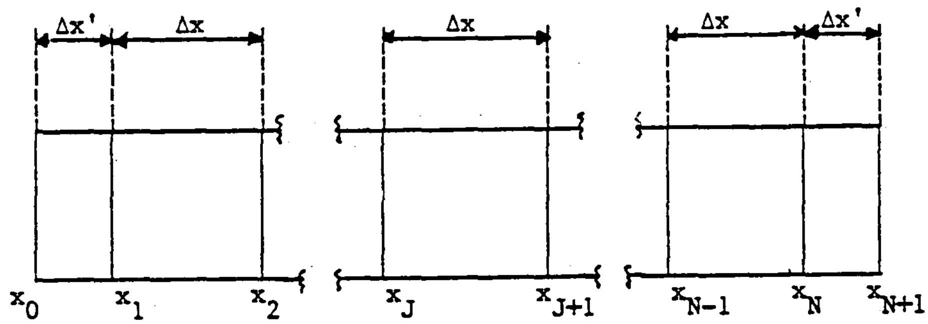


Figure 6.1

Suppose we have the boundary condition at $x = 0$:

$$\beta \frac{\partial T}{\partial x} - T = g \quad \text{where} \quad \beta \text{ can be arbitrarily small.}$$

Then a discrete version of this boundary condition would be:

$$\beta \frac{(T_1 - T_0)}{\Delta x_0} - T_0 = g$$

$$\Rightarrow T_0 = \frac{\beta T_1 - g\Delta x_0}{(\beta + \Delta x_0)} .$$

So if $\Delta x_0 \gg \beta \Rightarrow T_0 \to -g$ and hence could give rise to a steep 'numerical boundary layer.' To avoid this we should have Δx_0 of the same order of magnitude as β or smaller. If, however, we choose an even mesh this becomes computationally infeasible since β can be arbitrarily small.

To overcome this difficulty we propose the following: Divide the interval $[x_0, x_{N+1}]$ into $N + 2$ points $x_0, x_1, \cdots, x_N, x_{N+1}$. As before let $\Delta x_J = x_{J+1} - x_J$. Choose

$$\Delta x_0 = \Delta x_N = \Delta x'$$

$$\Delta x_i = \Delta x \quad \text{for} \quad i = 1,\cdots,N-1$$

where we may choose $\Delta x' \ll \Delta x$. (See Figure 6.1.) So the mesh points are evenly spaced in the interior, but x_0 is close to x_1 and x_{N+1} is close to x_N.

Define $\delta x = \frac{\Delta x + \Delta x'}{2}$. Henceforth we shall restrict q to be $q = (\rho, m_u, E)^T$.

The semidiscrete difference approximation for the mixed initial boundary value problem is tailored according to (4.19):

$$(6.1)\begin{cases} \dfrac{dq_1}{dt} + \dfrac{\Delta_-}{\delta x}\,[f^G(q_1,q_2)] + \dfrac{1}{\delta x}\left[C(v_{3/2})\,\dfrac{\Delta_+ v_1}{\Delta x} - C(v_{1/2})\,\dfrac{\Delta_+ v_0}{\Delta x'}\right] = 0 \\ \dfrac{dq_J}{dt} + \dfrac{\Delta_-}{\Delta x}\,[f^G(q_J,q_{J+1})] + \dfrac{\Delta_-}{\Delta x}\left[C(v_{J+1/2})\,\dfrac{\Delta_+ v_J}{\Delta x_J}\right] = 0 \quad \text{for} \quad j=1,\cdots,N-1 \\ \dfrac{dq_N}{dt} + \dfrac{\Delta_-}{\delta x}\,[f^G(q_N,q_{N+1})] + \dfrac{1}{\delta x}\left[C(v_{N+1/2})\,\dfrac{\Delta_+ v_N}{\Delta x'} - C(v_{N-1/2})\,\dfrac{\Delta_+ v_{N-1}}{\Delta x}\right] = 0 \\ \text{with initial condition:} \\ \qquad q_J(0) = Q_J \\ \text{and boundary conditions:} \\ \qquad B(q_0, q_1, q_2) = 0 \ \forall t \\ \qquad \tilde{B}(q_{N-1}, q_N, q_{N+1}) = 0 \ \forall t \\ \text{where } B \text{ and } \tilde{B} \text{ are boundary operators.} \end{cases}$$

In general, B and $\tilde{B}$ may be an underdetermined set of boundary conditions. For example, $B(q_0, q_1, q_2)$ may be of the form:

$$u_0 = u_1$$
$$T_0 = T_1$$
and ρ_0 is unspecified.

If B is a ´legitimate´ boundary operator we can connect q_1 to q_0 by just two waves, say a 2- contact and a 3- shock or rarefaction, such that $B(q_0, q_1, q_2) = 0$ holds. We make this statement more precise by examining the boundary Riemann problem.

Suppose we have a boundary (a wall) given by $x = st$. We want to give a number, m, of nonlinear boundary conditions depending on q at the boundary:

(6.2)
$$\begin{cases} h(q)\big|_{x=st} = 0 \\ h\colon \mathbb{R}^n \to \mathbb{R}^m,\ h \varepsilon C^\infty \\ \text{and initial condition:} \\ q(x, t)\big|_{t=0} = q_r \text{ (a constant state) for } x > 0. \end{cases}$$

The underlying differential equation is

(6.3)
$$q_t + f(q)_x = q_t + A(q)q_x = 0$$

where

$$A(q) = \frac{\partial f}{\partial q} .$$

We want to construct similarity solutions for this initial boundary problem. Clearly, for the problem to be well-posed we have to choose q_r so that

$$\lambda_1(q_r) < \cdots < \lambda_k(q_r) < s < \lambda_{k+1}(q_r) < \cdots < \lambda_n(q_r)$$

where $\lambda i(q)$ are the eigenvalues of the matrix $A(q)$ and $m = n - k$.

We can connect q_r to the boundary by $n - k$ waves. The rarefunction and shock curves of the $j^{\underline{th}}$ family determine a wave curve:

$$W^j(q_r, \varepsilon_j) = \begin{cases} R^j(q_r, \varepsilon_r), & \varepsilon_j \geq 0 \\ \\ S^j(q_r, \varepsilon_j), & \varepsilon_j < 0 \end{cases}$$

where W^i is twice continuously differentiable in both its arguments (see [2]).

Given q_r and sufficiently small parameters $\varepsilon_{k+1},\cdots,\varepsilon_n$ we define states $\tilde{q}^n,\cdots,\tilde{q}^{k+1}$ inductively by:

$$\tilde{q}^n = W^n(q_r, \varepsilon_n)$$

$$\tilde{q}^{j-1} = W^j(q^j, \varepsilon_j) \quad \text{for} \quad j = n,\cdots,k + 2.$$

Set:

$$q_\ell = \tilde{q}^{k+1}(q_r, \varepsilon_{k+1},\cdots,\varepsilon_n).$$

Then the boundary condition $h(q) = 0$ becomes m equations in the $n - k$ unknowns $\varepsilon_{k+1},\cdots,\varepsilon_n$:

$$h(q_\ell(q_r, \varepsilon_{k+1},\cdots,\varepsilon_n)) = 0.$$

By the implicit function theorem we will be able to solve for $\varepsilon_{k+1},\cdots,\varepsilon_n$ in terms of q_r in a C^2 manner if $m = n - k$ and the Jacobian $\frac{\partial h}{\partial \varepsilon}$ has rank m. Differentiating we get the local solvability condition:

$$(6.4) \qquad \frac{\partial h}{\partial \tilde{q}} \left(\begin{matrix} | & & | \\ r_{k+1}, & \cdots, & r_n \\ | & & | \end{matrix} \right) \Bigg|_{q=q_r} \quad \text{has rank} \quad m = n - k$$

where r_i in the $i^{\underline{th}}$ eigenvector of the matrix $A(q)$ corresponding to the eigenvalue λ_i. The construction is described in Figure (6.2).

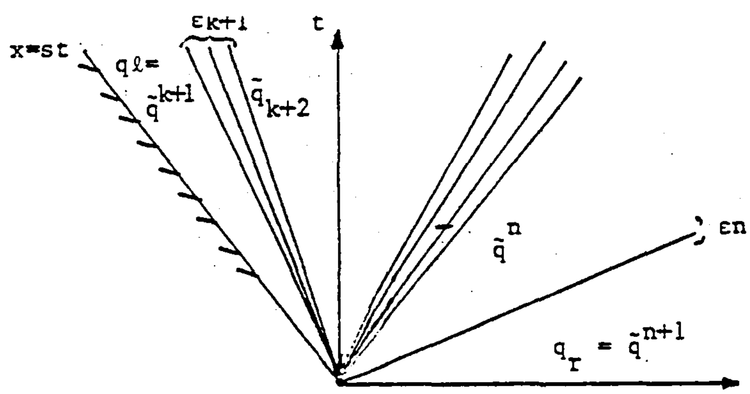


Figure 6.2

(6.4) is also a necessary condition for the linearized initial boundary value problem to be well-posed.

We relate this now to the boundary operator $B(q_0, q_1, q_2) = 0$. Let q_ℓ denote q_0 and q_r denote q_1. Then $B(q_0, q_1, q_2) = 0$ corresponds to the m boundary conditions:

$$h(q)|_{x=0} = 0 \tag{6.5}$$

and the initial condition for the initial boundary Riemann problem (6.2) is:

$$q(x, t)|_{t=0} = q_r = q_1.$$

So $B(q_0, q_1, q_2)$ is a 'legitimate' operator if (6.4) holds. Assuming that it is, we can connect q_1 to a state q_0 such that $B(q_0, q_1, q_2) = 0$ holds.

Returning to the difference scheme (6.1) we can now define the Godunov flux $f^G(q_0, q_1)$ using this construction. Suppose at $t = 0$ $q_1(x, t) = q_1$ and we have the boundary operator $B(q_0, q_1, q_2) = 0$.

Consider the Riemann initial boundary value problem:

$$q(x, t)|_{t=0} = q_1 \quad \text{for} \quad x_{1/2} < x < x_{3/2}$$

and boundary condition:

$$h(q) = 0 \quad \text{for} \quad x = x_0 \quad \text{corresponding to} \quad B(q_0, q_1, q_2) = 0$$

as in (6.5).

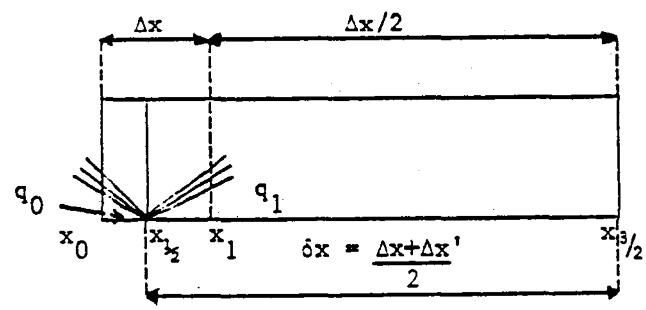


Figure 6.3

Then if (6.4) holds we can join q_1 to a state q_0 in a small enough neighborhood of q_1 so that $B(q_0, q_1, q_2) = 0$.

In general, the number of boundary conditions, m, specified by the boundary operator B will be more than the number of boundary conditions needed for the hyperbolic Euler equations since we are solving the Navier-Stokes equations, i.e., $m > n - k$. Hence, when we connect q_1 to a state q_0 lying to the left of the line $x = x_{1/2}$ some of the waves will lie in the region $x < x_{1/2}$. This corresponds to letting waves be radiated at the boundary.

We know that we can connect q_1 to a state q_0 such that $B(q_0, q_1, q_2) = 0$ if q_0 and q_1 are sufficiently close. We chose an uneven mesh precisely for this reason. We set $\Delta x_0 = \Delta x_N = \Delta x'$ and $\Delta x_i = \Delta x$ for $i = 1,\cdots,N - 1$ where $\Delta x'$ could be arbitrarily small. In fact, by choosing $\Delta x'$ small enough we can make q_0 lie as close to q_1 as we want. This has the effect of making the waves in the cell bounded by $x = x_0$ and $x = x_1$ very weak and hence we conclude that

$$F^G(q_0, q_1) \simeq F^G(q_1, q_1) = \tilde{F}(q_1).$$

Recollect that the energy estimate we obtained for an uneven mesh was of the form:

$$
(6.6) \qquad \begin{aligned} \frac{dS}{dt} \le {} & [F^G(q_0, q_1) + v_1^T C(v_{1/2}) \frac{\Delta + v_0}{\Delta x}] \\ & - [F^G(q_N, q_{N+1}) + v_{N+1}^T C(v_{N+1/2}) \frac{\Delta + v_N}{\Delta x'}]. \end{aligned}
$$

Since $q_0 \to q_1$ as $\Delta x' \to 0$ the estimate (6.6) reduces to:

$$
(6.7) \qquad \frac{dS}{dt} \le [\tilde{F}(q_1) + \tilde{V}_q(q_1)A(q_1) \frac{\Delta + q_0}{\Delta x'}] - [\tilde{F}(q_N) + \tilde{V}_q(q_N)A(q_N) \frac{\Delta + q_N}{\Delta x'}]
$$

for sufficiently small $\Delta x'$.

One further point that should be noted is that when we take a fully discrete version of our semidiscrete scheme the CFL condition we have to impose to prevent wave interactions is:

$$
(6.8) \qquad \Delta t \lambda(A) \le \frac{\Delta x + \Delta x'}{2}
$$

where $\lambda(A)$ is the spectral radius of the matrix $A(q)$. Clearly (6.8) is not an unduly restrictive condition.

For the numerical simulation presented in the next and last section, we nondimensionalize the Navier-Stokes equations.

Let L be a reference length, ρ_f a reference density, and u_f a reference velocity. Define:

$$
\begin{aligned} x^* &= x/L, \\ t^* &= tu_f/L, \\ u^* &= u/u_f, \end{aligned}
$$

$$p^* = p/\rho_f\, u_f^2,$$
$$T^* = TR/u_f^2,$$
$$\rho^* = \rho/\rho_f,$$
$$E^* = E/\rho_f\, u_f^2.$$

We also need the following parameters, which occur extensively in fluid dynamics:

$$Pr = \frac{\mu R \gamma}{k(\gamma - 1)} = \frac{\mu c_p}{k}$$

$$Re = \frac{\rho_f\, u_f\, L}{\mu}\,.$$

Here Pr and Re are abbreviations for the Prandtl number and Reynolds number respectively. Henceforth we drop the '*' for notational convenience. For our numerical simulation we use Stokes' assumption:

$$\lambda = -2/3\mu.$$

The nondimensionalized Navier-Stokes equations then take the form:

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_t + (\rho u^2 + p)_x = \frac{4}{3Re}\frac{\partial^2 u}{\partial x^2}$$

$$(E)_t + \left[(E + p)u\right]_x = \frac{4}{3Re}\frac{\partial}{\partial x}\left(u\,\frac{\partial u}{\partial x}\right) + \frac{\gamma}{(\gamma - 1)Pr\;Re}\frac{\partial^2 T}{\partial x^2}\,.$$

Rewriting the dissipative term $(-D)q$ in terms of our new variable $v = \tilde{V}_q$, we obtain:

$$(C(v)v_x)_x = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-4T}{3Re} & \frac{-4uT}{3Re} \\ 0 & \frac{-4uT}{3Re} & \frac{-4u^2 T}{3Re} - \frac{\gamma T^2}{(\gamma - 1)Re\ Pr} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} 0 \\ \frac{u}{T} \\ \frac{-1}{T} \end{bmatrix} \right).$$

Our nondimensionalized boundary conditions take the form:

<u>Supersonic Outflow</u>

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial T}{\partial x} = 0$$

<u>Subsonic Outflow</u>

$$\frac{4}{3Re} \frac{\partial u}{\partial x} - \alpha_2 u = g_2 \quad \text{where} \quad \alpha_2 \geq 0$$

$$\frac{\partial T}{\partial x} = 0$$

<u>Subsonic Inflow</u>

$$\rho u = g_1$$

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\gamma}{Re\ Pr} \frac{\partial T}{\partial x} - \alpha_3 T = g_3 \quad \text{where} \quad \alpha_3 > \frac{(\gamma^2 - \gamma + 2)}{2} g_1$$

Supersonic Inflow

$$\rho u = g_1$$

$$\frac{4}{3Re} \frac{\partial u}{\partial x} - \alpha_2 u = g_2$$

$$\frac{\gamma}{Re\ Pr} \frac{\partial T}{\partial x} - \alpha_3 T = g_3 \quad \text{where} \quad \alpha_2 > \frac{g_1}{2}, \ \alpha_3 > g_1.$$

We now propose a discrete version of these boundary conditions which are of extrapolation type and hence minimize the effect of the computational boundary.

Supersonic Outflow

$$u_0 = u_1$$
$$T_0 = T_1$$

We can connect q_0 to q_1 without any waves at all by putting

$$\rho_0 = \rho_1.$$

Subsonic Outflow

The boundary conditions proposed for supersonic outflow would work equally well for subsonic outflow. This corresponds to choosing $\alpha_2 = 0$ in the subsonic outflow boundary conditions. If we choose $\alpha_2 > 0$ the discretized boundary conditions would be:

$$\frac{4}{3Re} \frac{(u_1 - u_0)}{\Delta x} - \alpha_2 u_1 = g_1$$

$$T_0 = T_1$$

where we could put

$$g_1 = \frac{4}{3Re} \frac{(u_2 - u_1)}{\Delta x} - \alpha_2 u_2.$$

Then

$$u_0 = \frac{(4/3Re) - \alpha_2 \Delta x')u_1 - g_1 \Delta x'}{(4/3Re)} .$$

Clearly $u_0 \to u_1$ as $\Delta x' \to 0$. We choose ρ_0 by joining q_1 to q_0 by two wave interactions.

Subsonic Inflow

$$(\rho u)_0 = g_1 = (\rho u)_1 - \frac{\Delta x'}{\Delta x} [(\rho u)_2 - (\rho u)_1]$$

$$u_0 = u_1$$

$$\frac{\gamma}{Re\ Pr} \frac{(T_1 - T_0)}{\Delta x'} - \alpha_3 T_1 = g_3 = \frac{\gamma}{Re\ Pr} \frac{(T_2 - T_1)}{\Delta x} - \alpha_3 T_2.$$

Here $\alpha_3 > (\gamma^2 - \gamma + 2)g_1/2$.

Supersonic Inflow

$$(\rho u)_0 = g_1 = (\rho u)_1 - \frac{\Delta x'}{\Delta x} [(\rho u)_2 - (\rho u)_1]$$

$$\frac{4}{3Re} \frac{(u_1 - u_0)}{\Delta x'} - \alpha_2 u_1 = g_2 = \frac{4}{3Re} \frac{(u_2 - u_1)}{\Delta x} - \alpha_2 u_2$$

$$\frac{\gamma}{Re\ Pr} \frac{(T_1 - T_0)}{\Delta x'} - \alpha_3 T_1 = g_3 = \frac{\gamma}{Re\ Pr} \frac{(T_2 - T_1)}{\Delta x} - \alpha_3 T_2.$$

Here $\alpha_2 > (g_1\ )/2$, $\alpha_3 > g_1$.

As mentioned earlier all the boundary conditions are such that $q_0 \to q_1$ as $\Delta x' \to 0$ and they are a discretized version of the boundary conditions for the differential equations, i.e.,

$$\varepsilon R \frac{\partial q}{\partial x} + Sq = g.$$

It is easy to verify from (6.7) that for $\Delta x'$ small enough we get bounded growth of the discrete version of the energy $S(t)$ in time by choosing α_2 and α_3 appropriately.

7. NUMERICAL RESULTS

Throughout the simulations we use the following parameter values:

$$Pr = 0.7$$

$$\gamma = 1.4$$

$$\Delta x = 0.1$$

$$\Delta x' = 0.000001$$

$$\Delta t = 0.01.$$

<u>Simulation 1</u>: For the first simulation we use a very large value of the Reynolds number

$$Re = 10^6.$$

We run the program with Riemann initial data:

$$\rho_L = 1.0$$
$$u_L = 3.0$$
$$p_L = 1.0$$

and

$$\rho_R = 0.4734821$$
$$u_R = 2.1393370$$
$$p_R = 0.3333333.$$

At the left boundary our boundary conditions correspond to supersonic inflow. At the right boundary we have supersonic outflow boundary conditions.

The solution to the Euler equations with initial data corresponding to the above Riemann problem is a 3 shock moving to the right with a speed = 3.774567.

The numerical simulations bear this out very well. The boundary conditions are radiative and allow the shock to pass through. Further for long time the solution stabilizes to a constant state.

ACKNOWLEDGEMENTS

The author is grateful to Professor S. Osher for his valuable help and encouragement and would like to thank Professor Ralston and Professor Bube for helpful discussions.

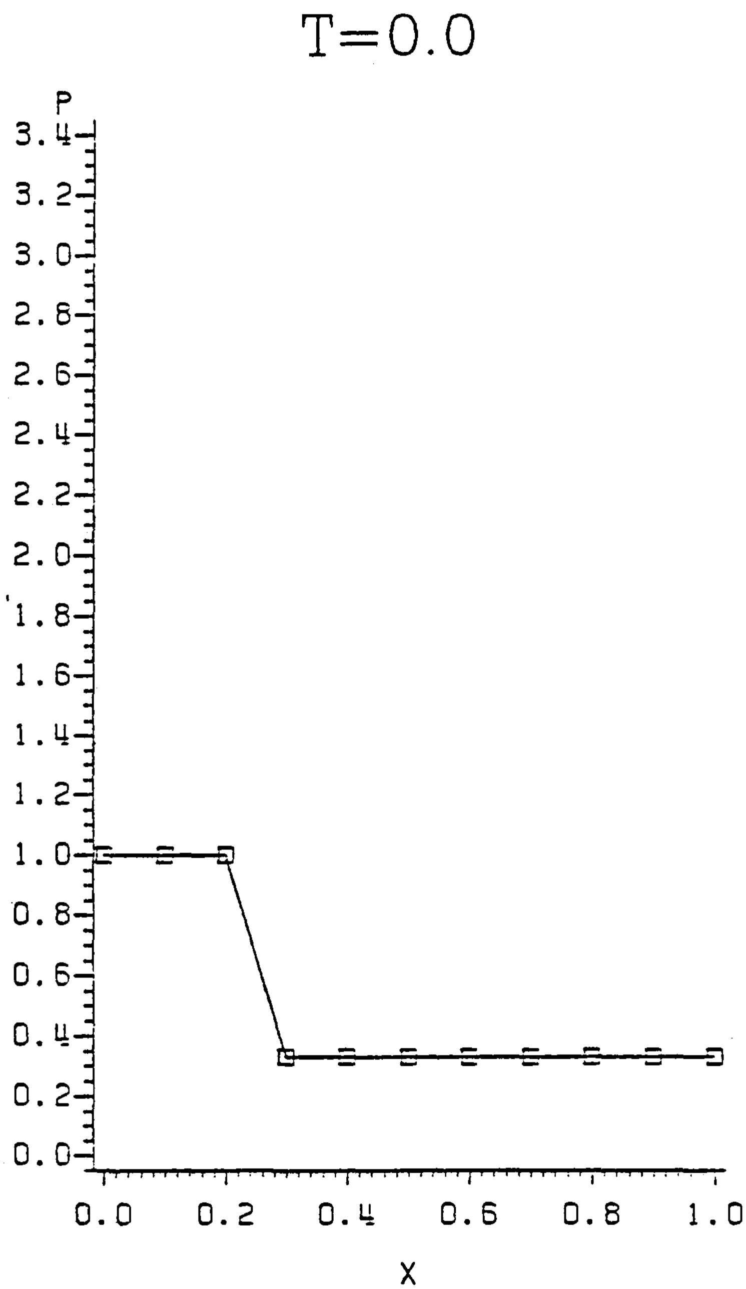
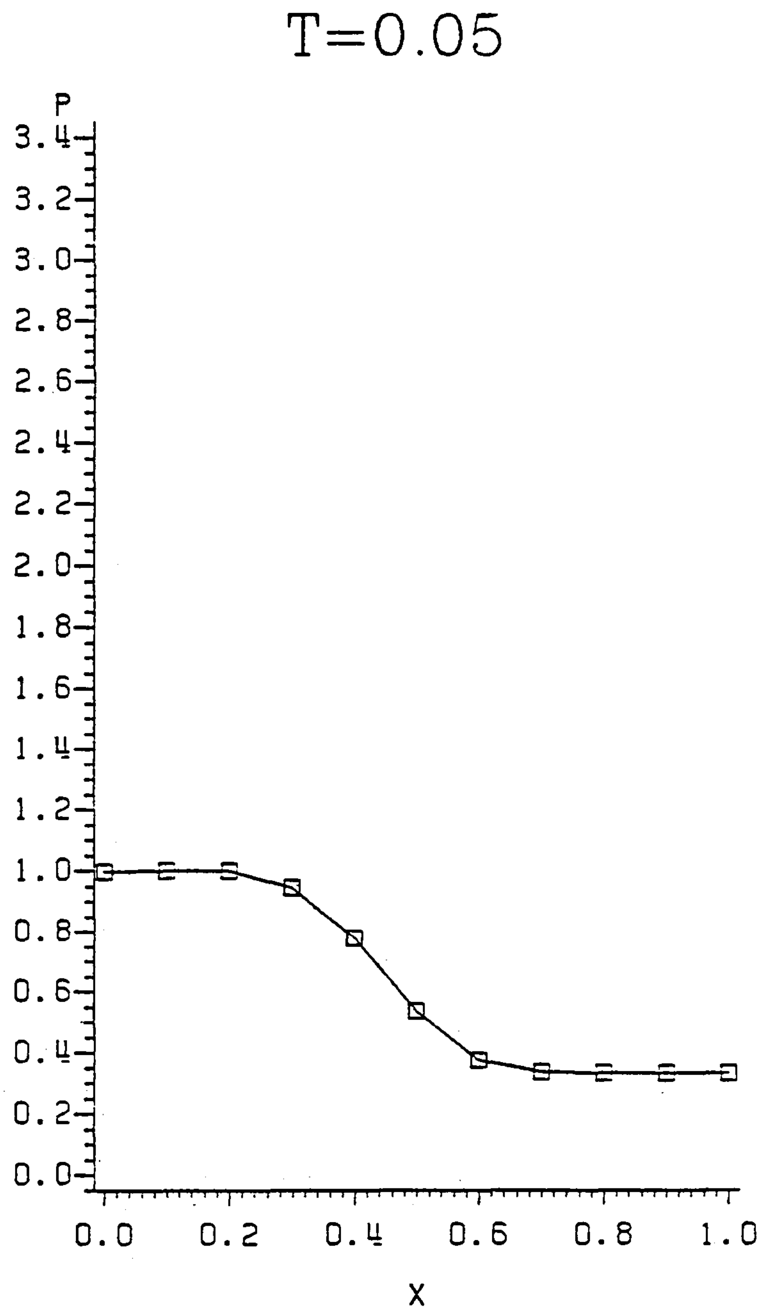

T=0.0
P
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.1



T=0.05
P
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.2

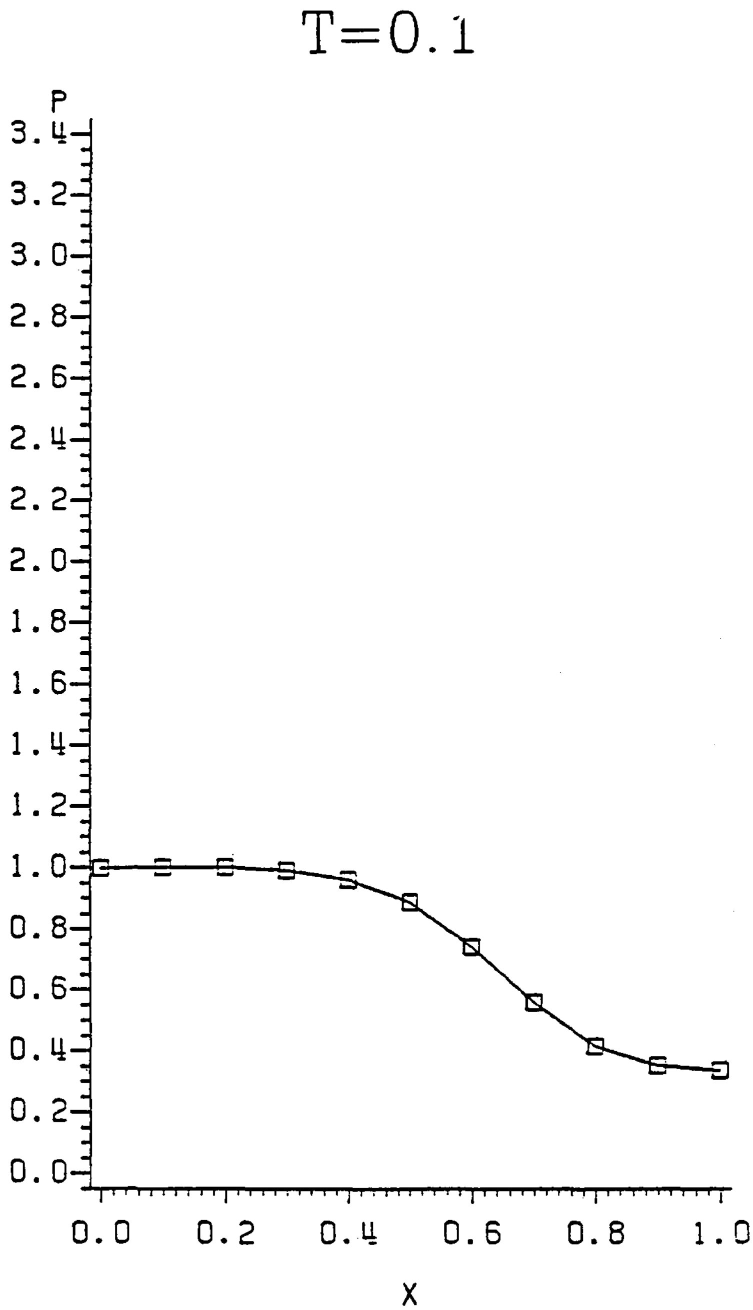

T=0.1
P
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.3

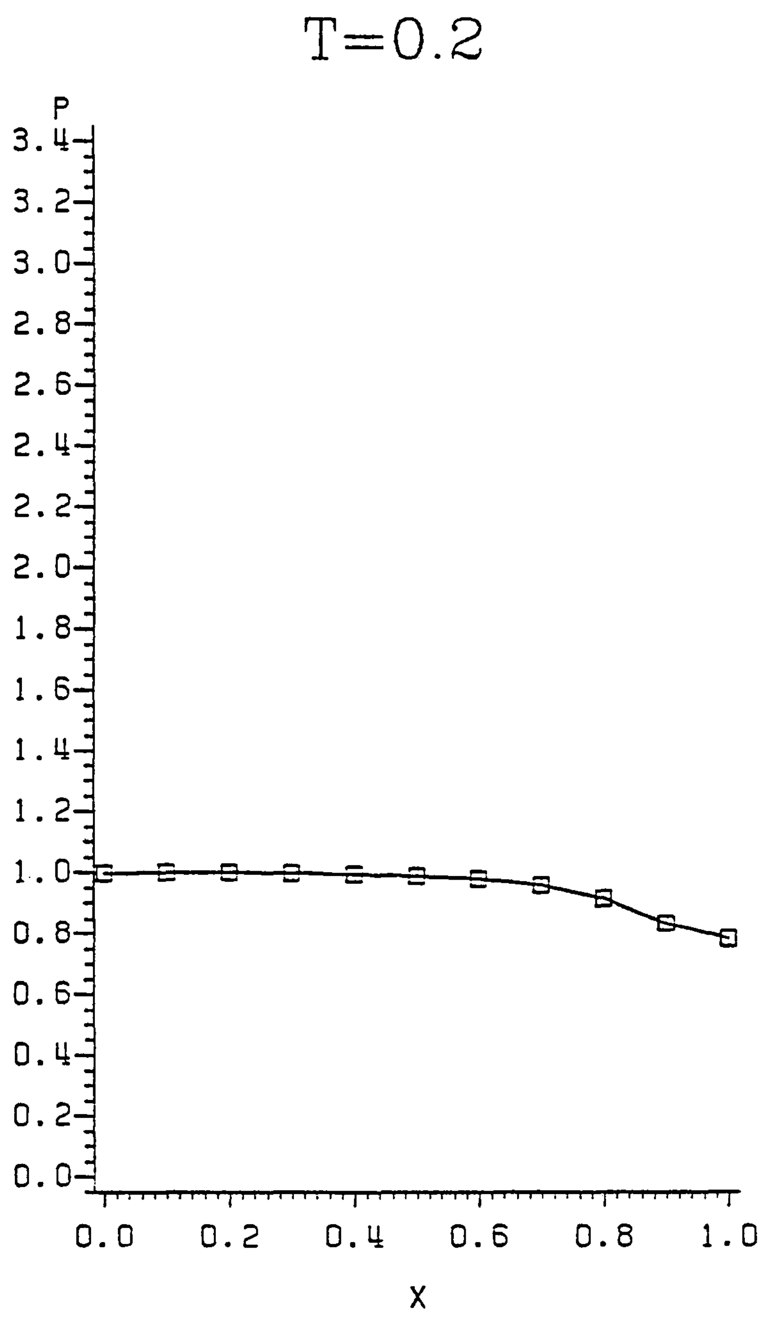

T=0.2
P
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.4

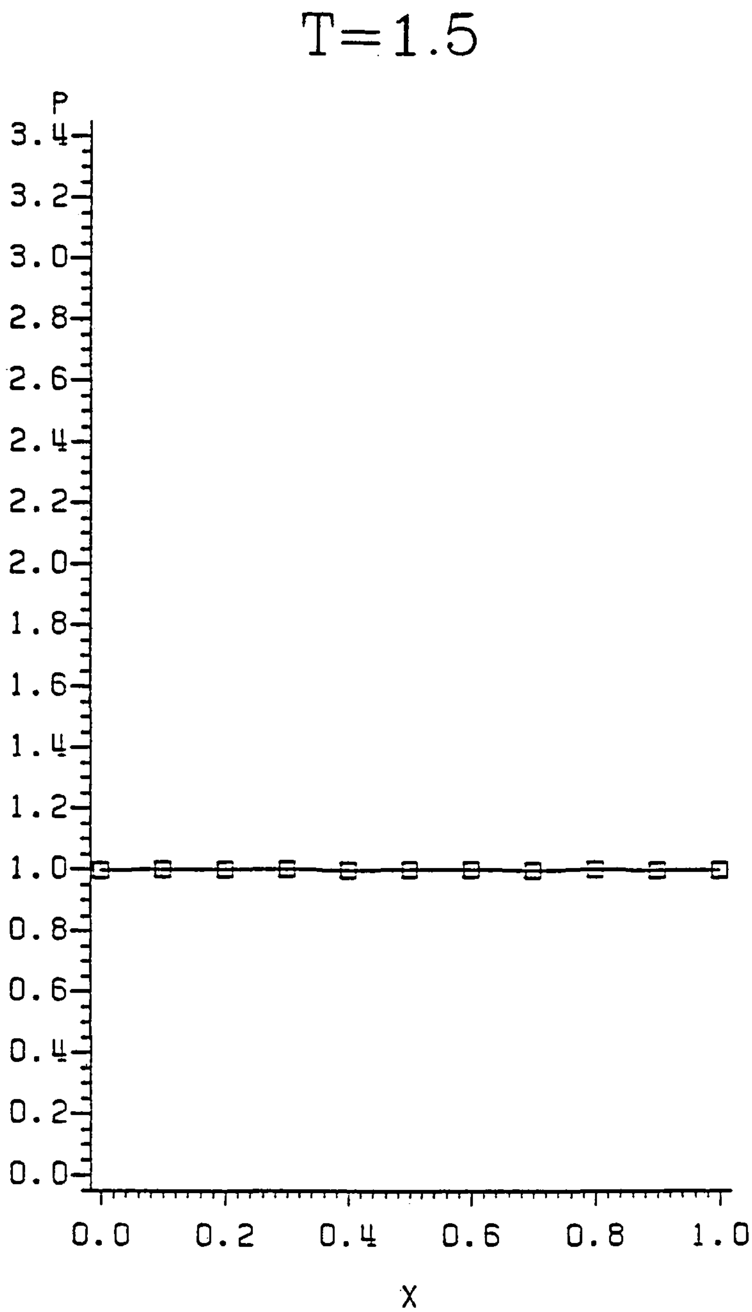

T=1.5
P
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.5

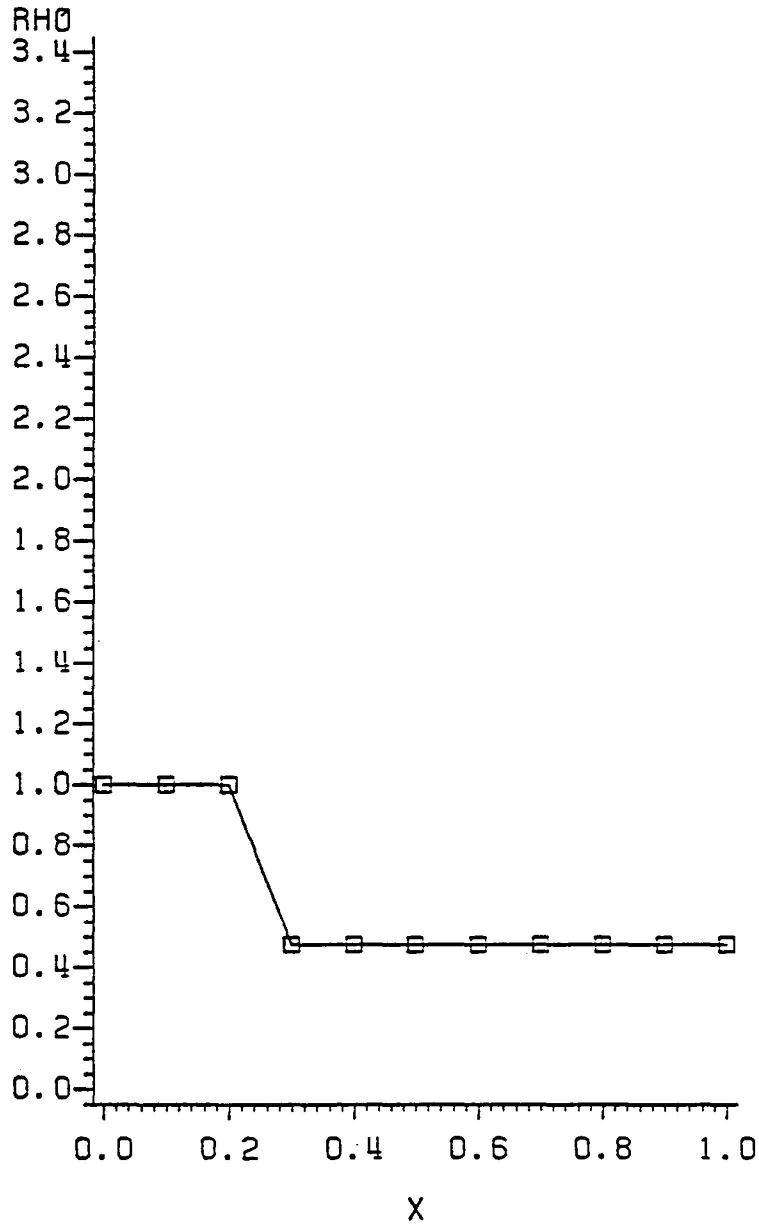

RHO
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.6

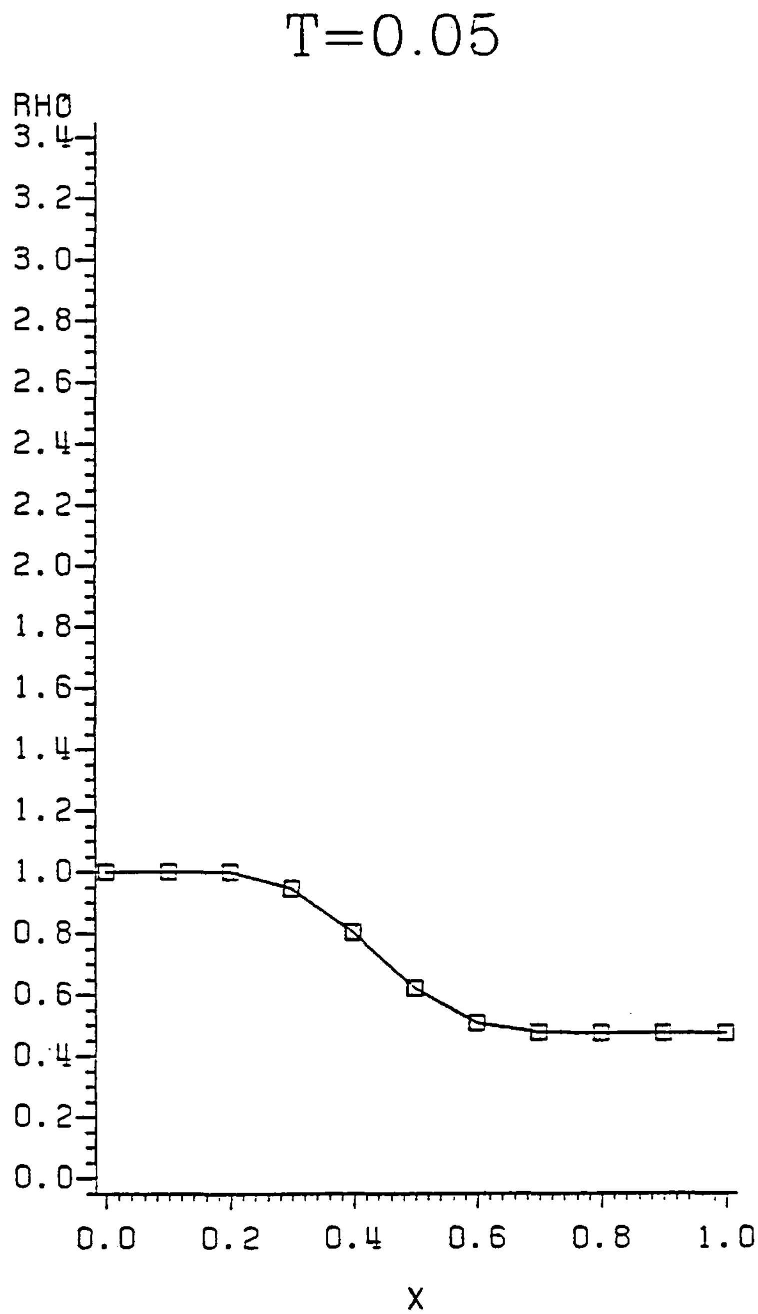

T=0.05
RHO
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.7

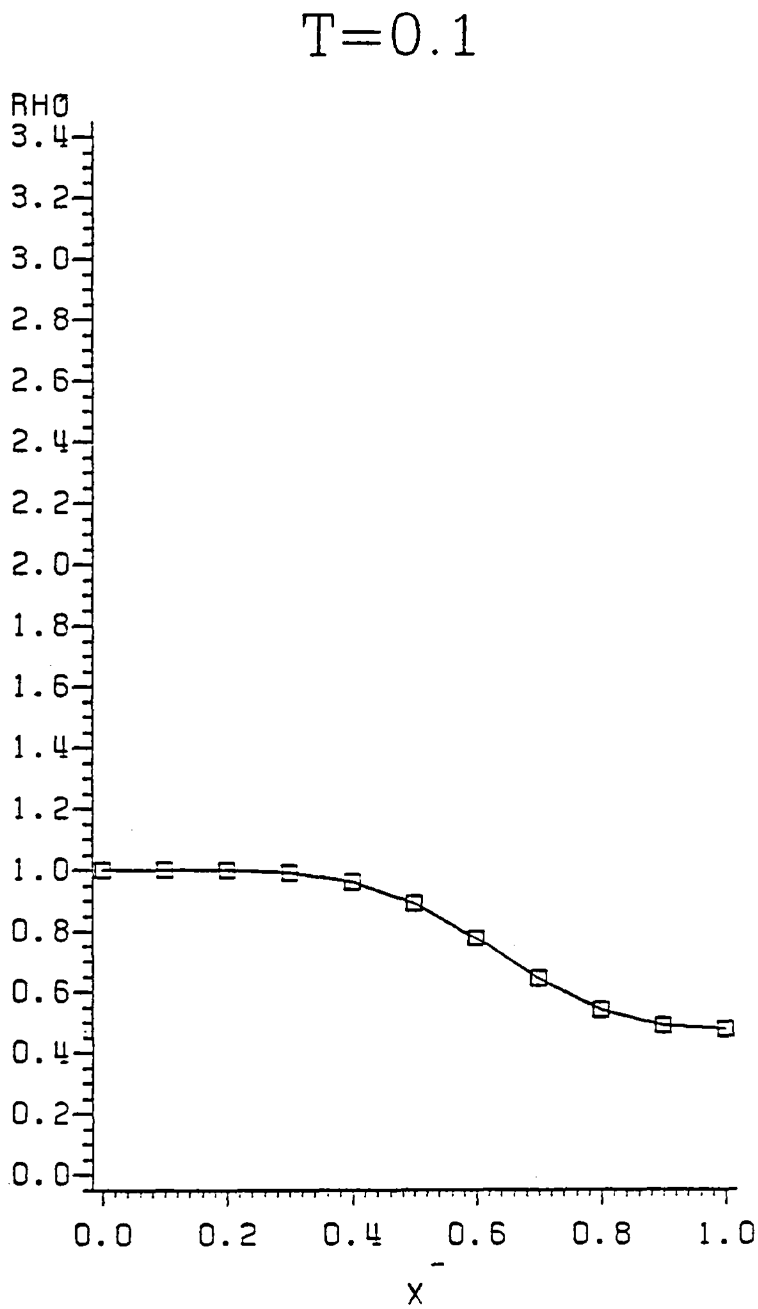

T=0.1
RHO
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.8

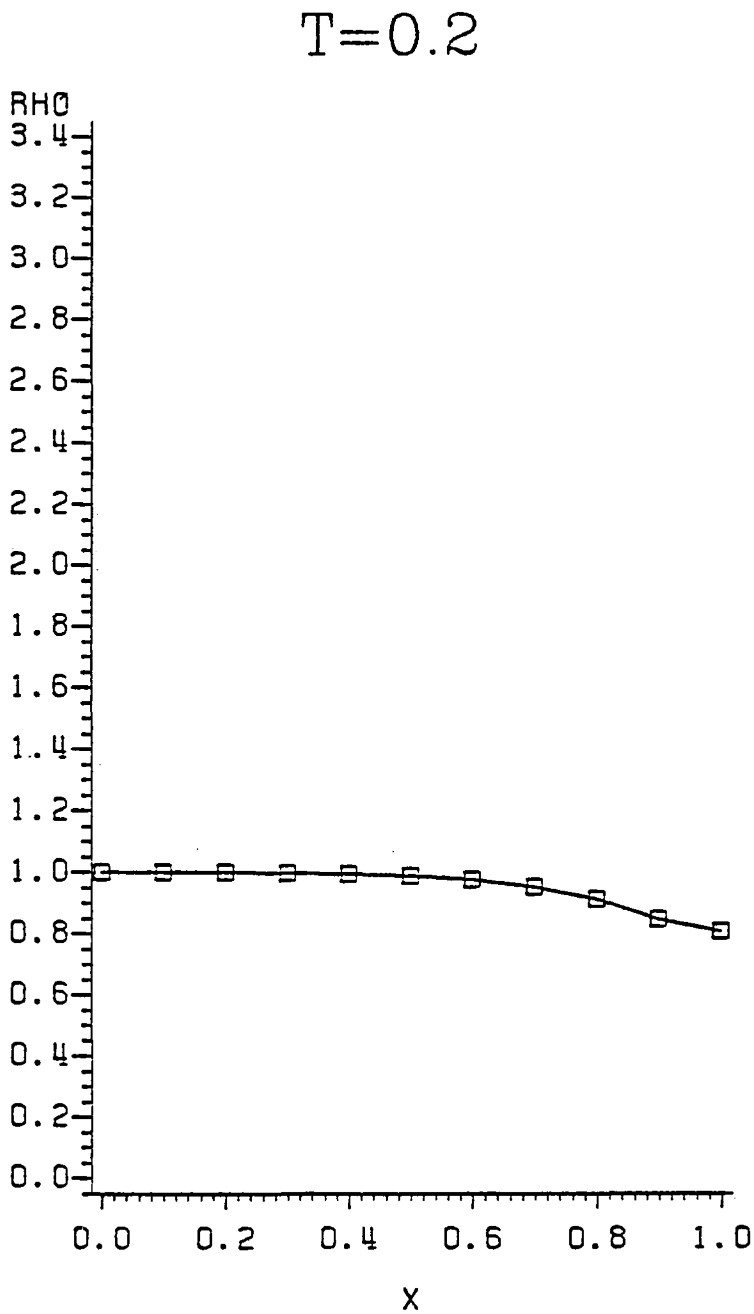

T=0.2
RHO
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.9

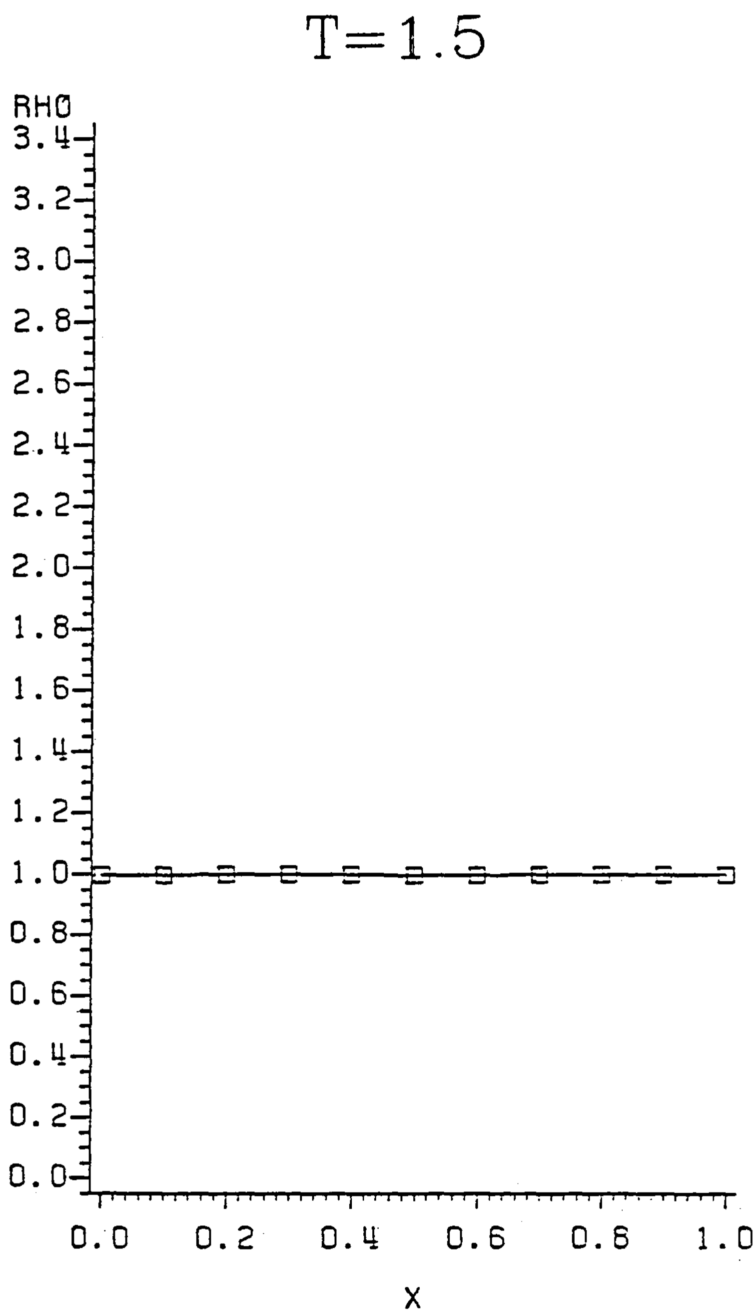
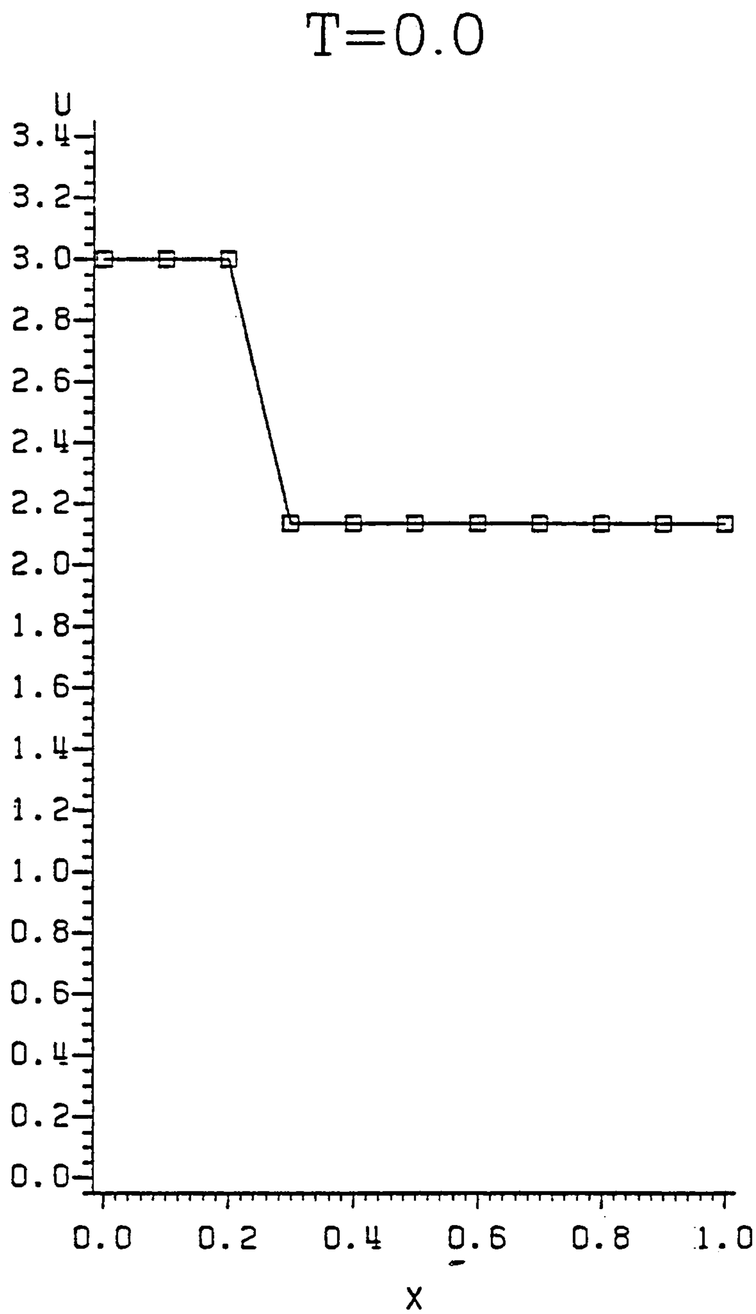

T=1.5
RHO
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.10


T=0.0
U
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3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.11

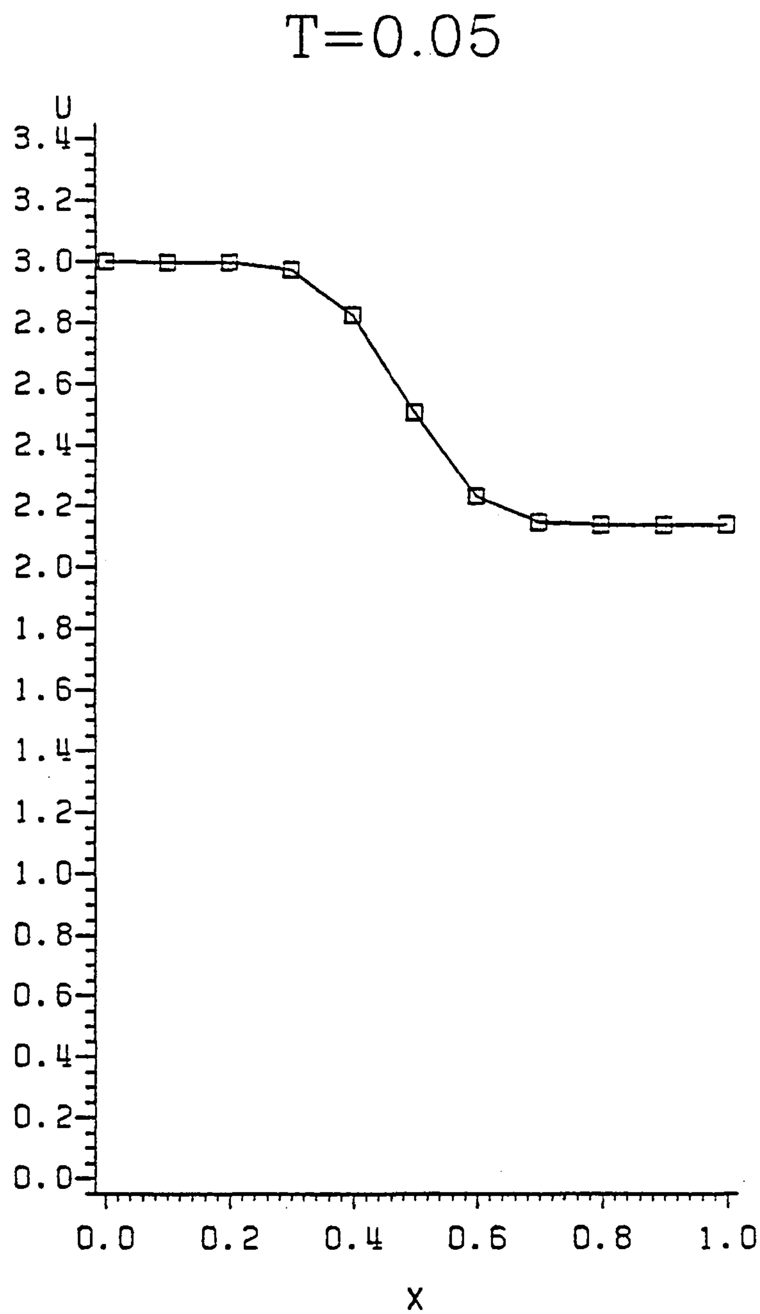
T=0.05
U
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.12

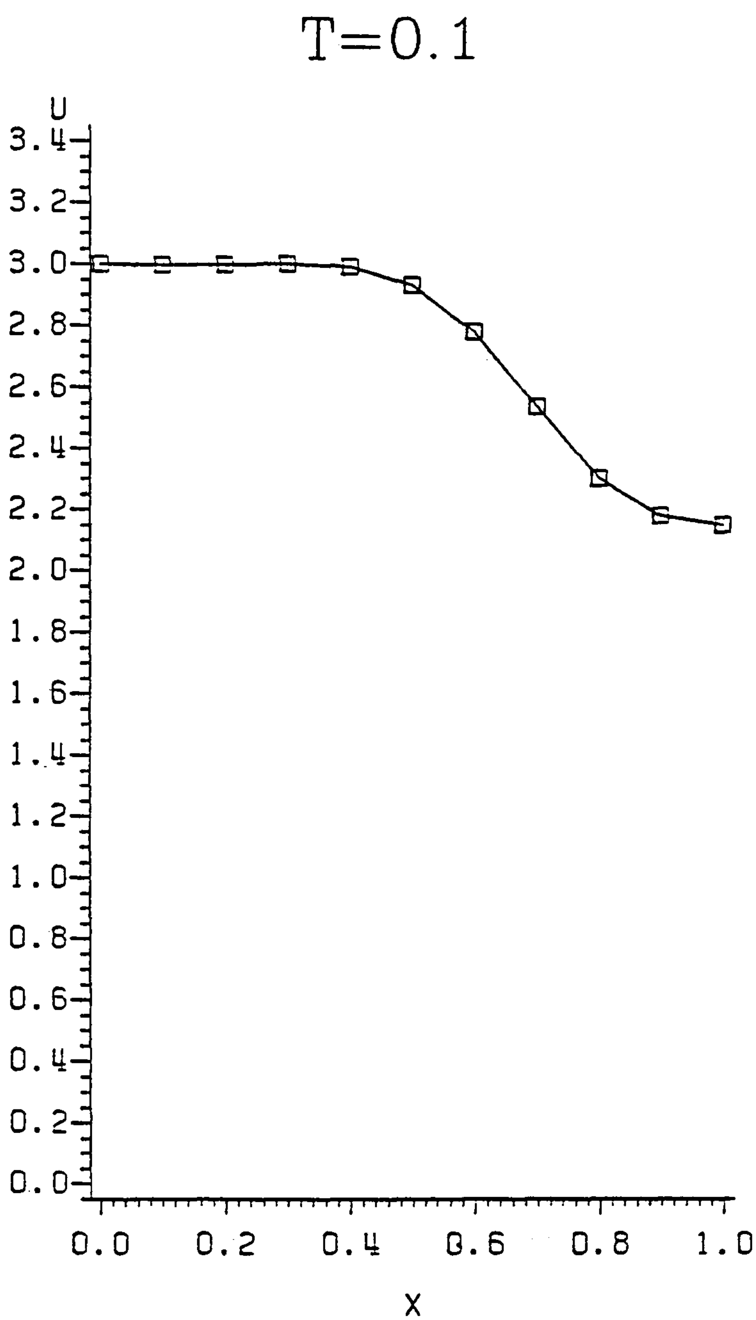
T=0.1
U
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.13

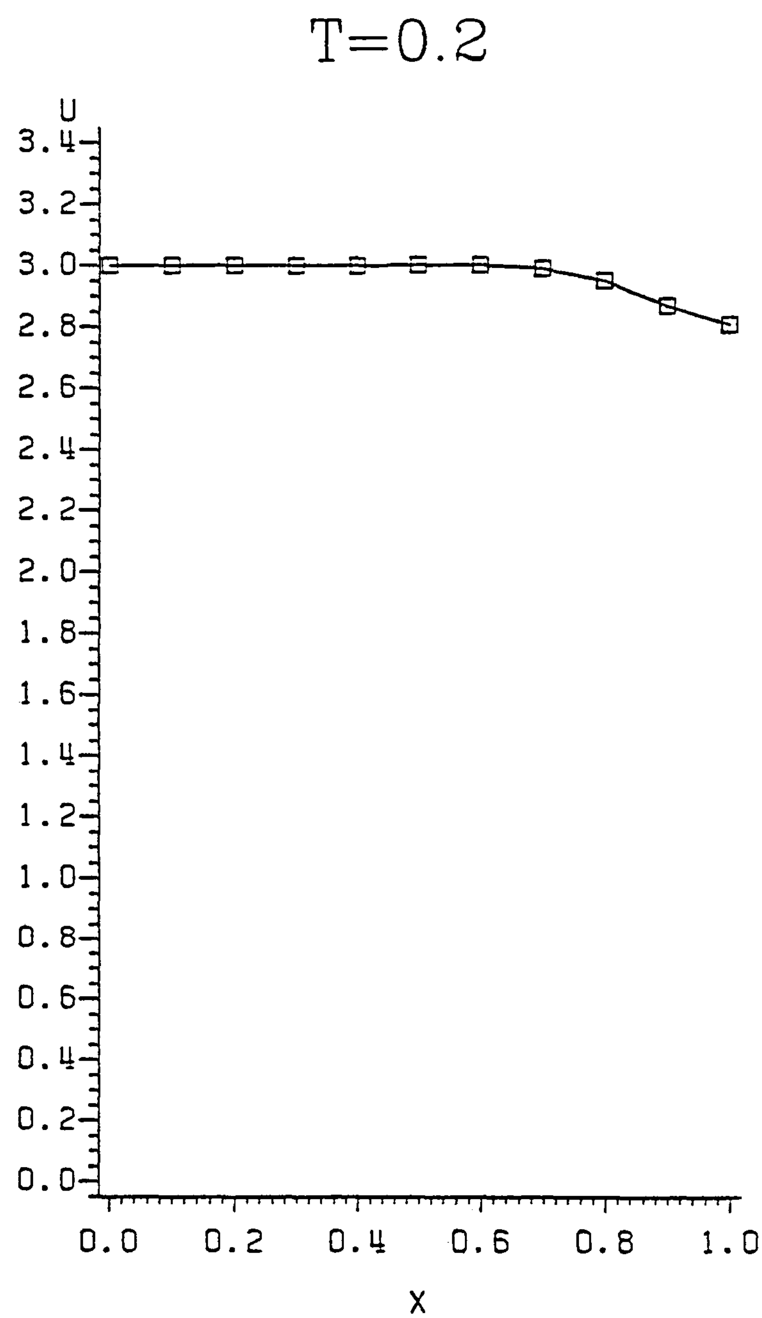
T=0.2
U
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.14

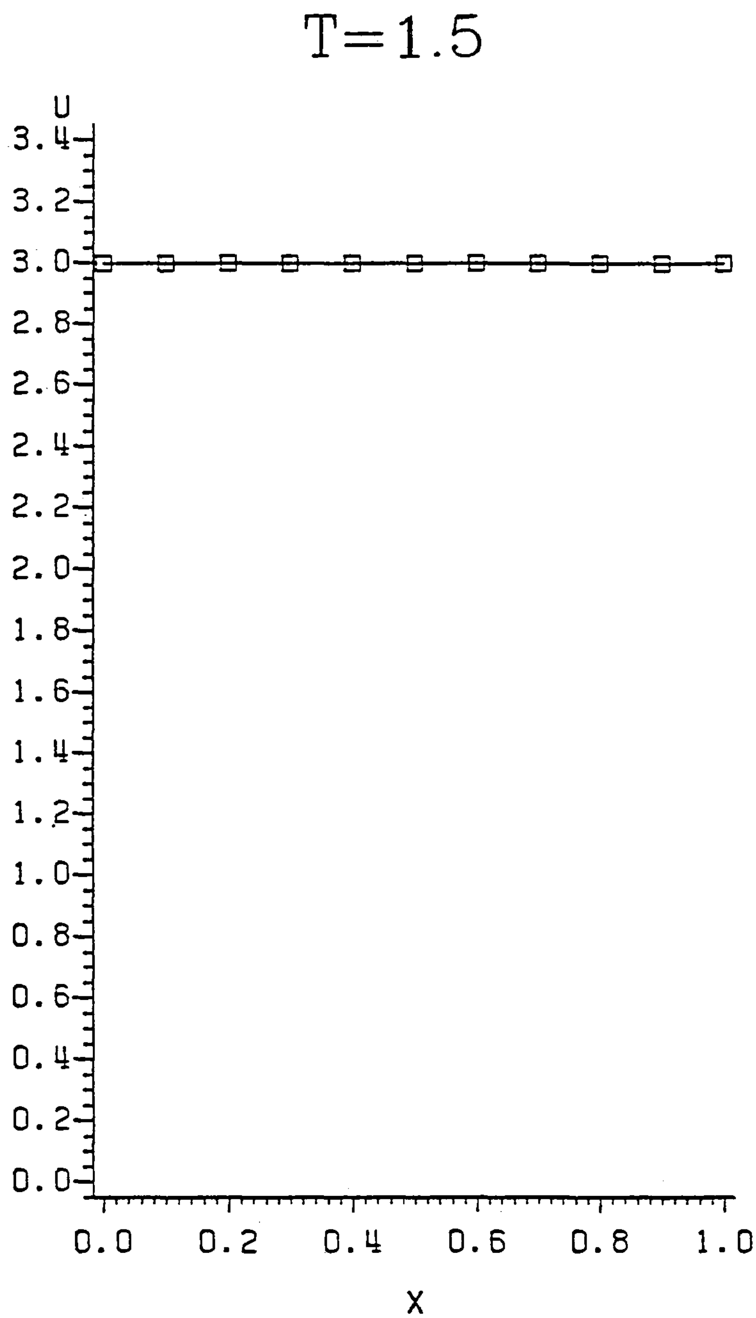
T=1.5
U
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
X

Figure 7.15

Simulation 2: We use a low Reynolds number:

$$Re = 500.$$

Once more we run the program with Riemann initial data:

$$\rho_L = 1.0$$

$$u_L = 1.0$$

$$p_L = 1.0$$

and

$$\rho_R = 1.625000$$

$$u_R = 0.3798263$$

$$p_R = 2.000000.$$

At the left boundary we have subsonic inflow boundary conditions and at the right boundary the boundary conditions correspond to subsonic outflow.

The solution to the Euler equations with the above Riemann initial data is a 1 shock moving to the left with a speed = 0.6124516.

The numerical results, once again, have all the desirable properties we observed in Simulation 1.

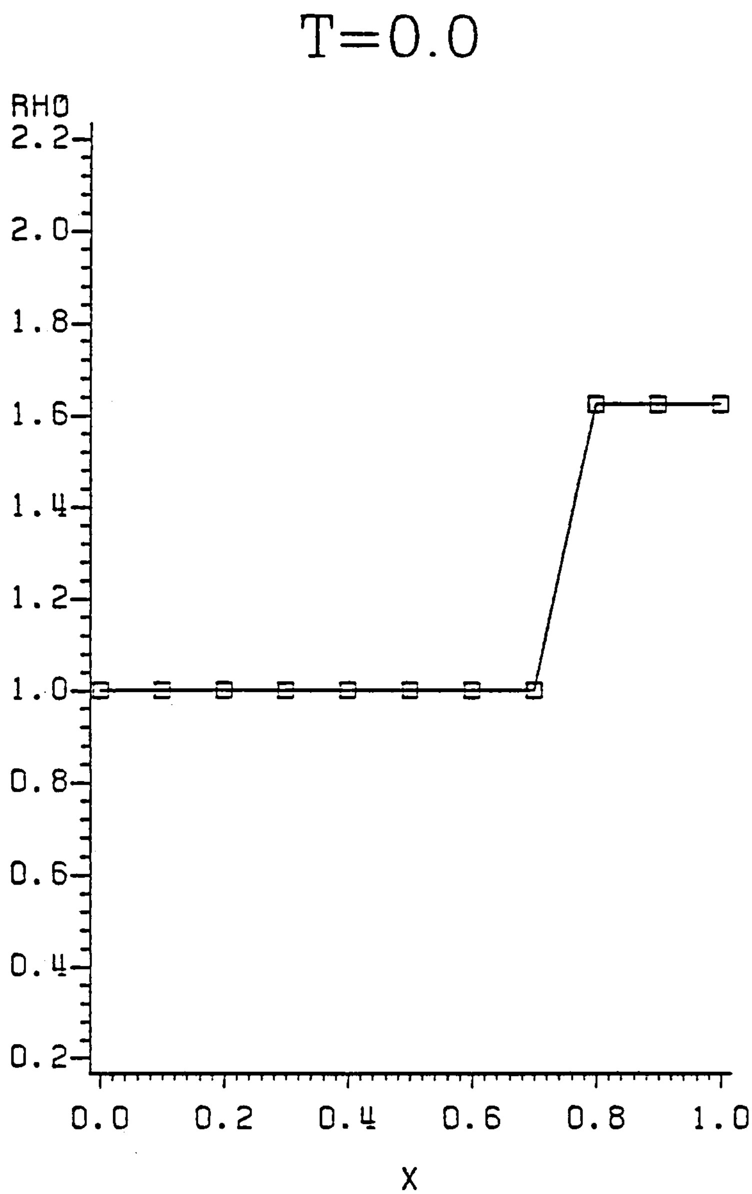

Figure 7.16

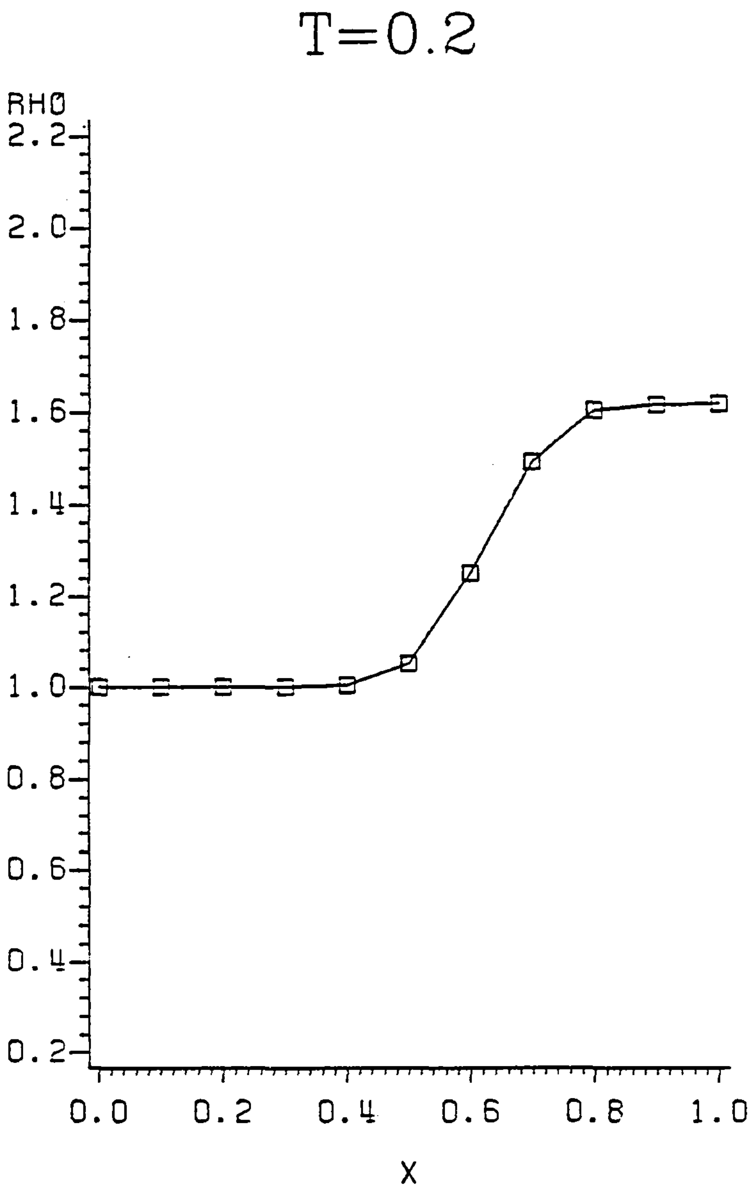

Figure 7.17

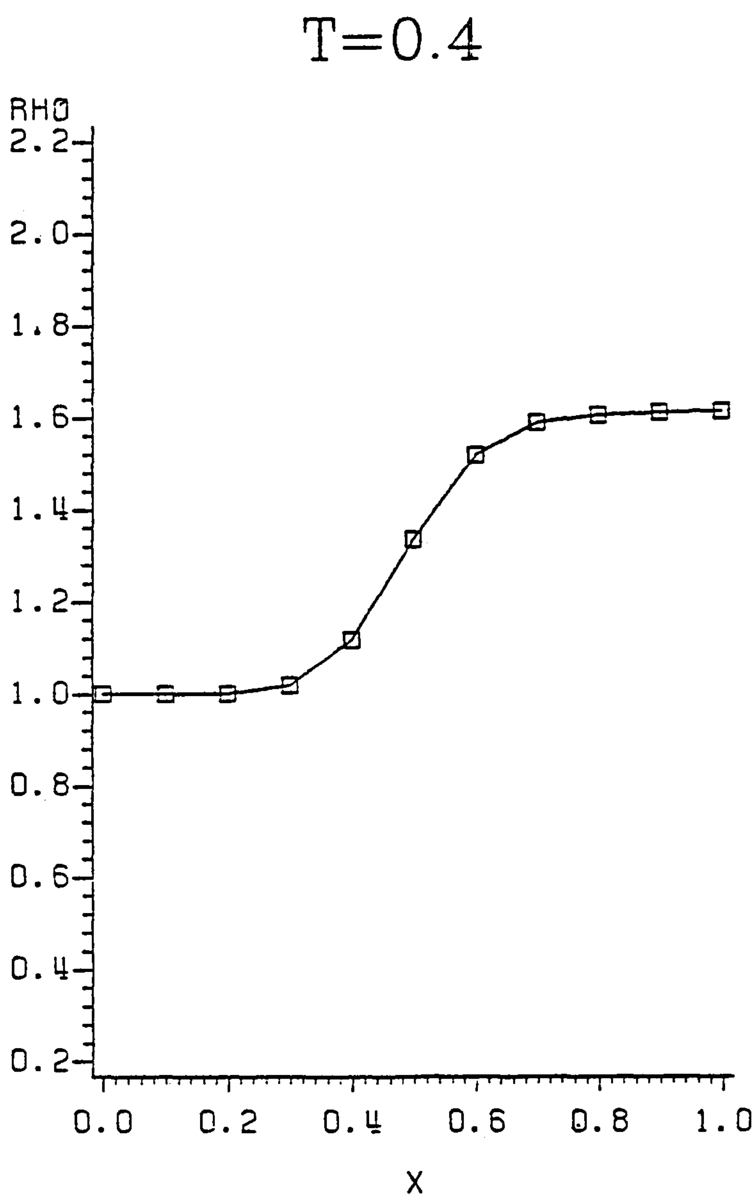

Figure 7.18

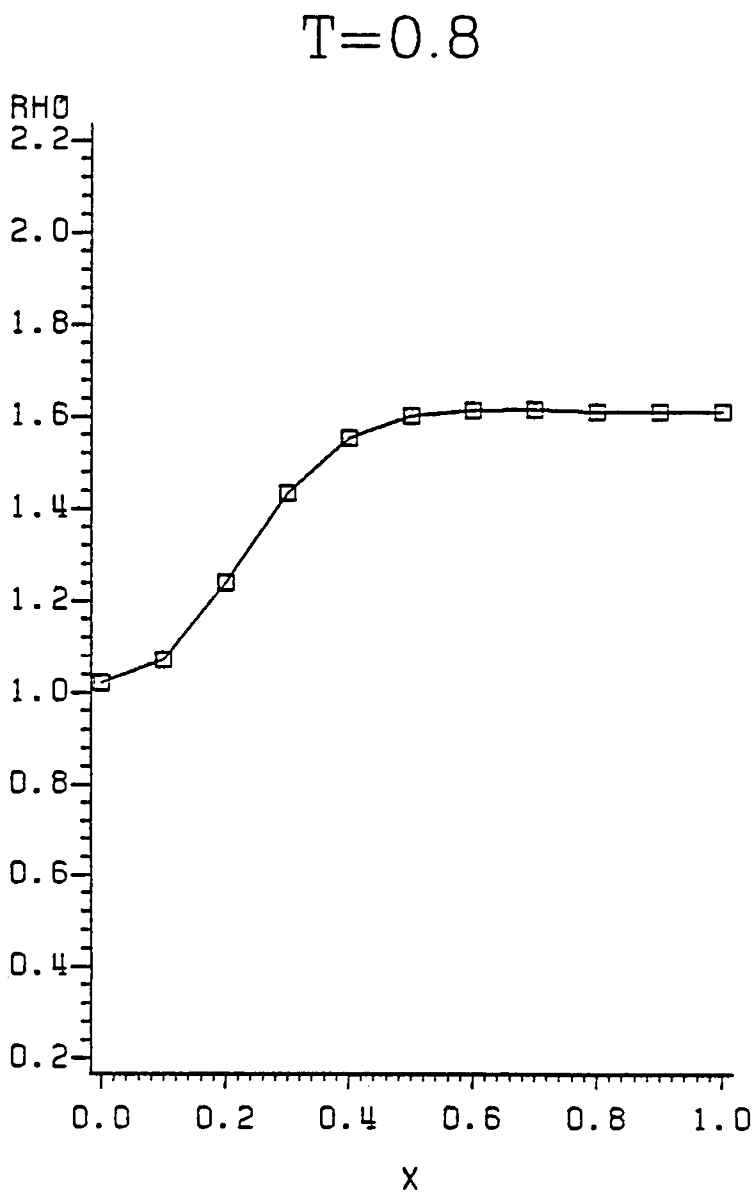

Figure 7.19

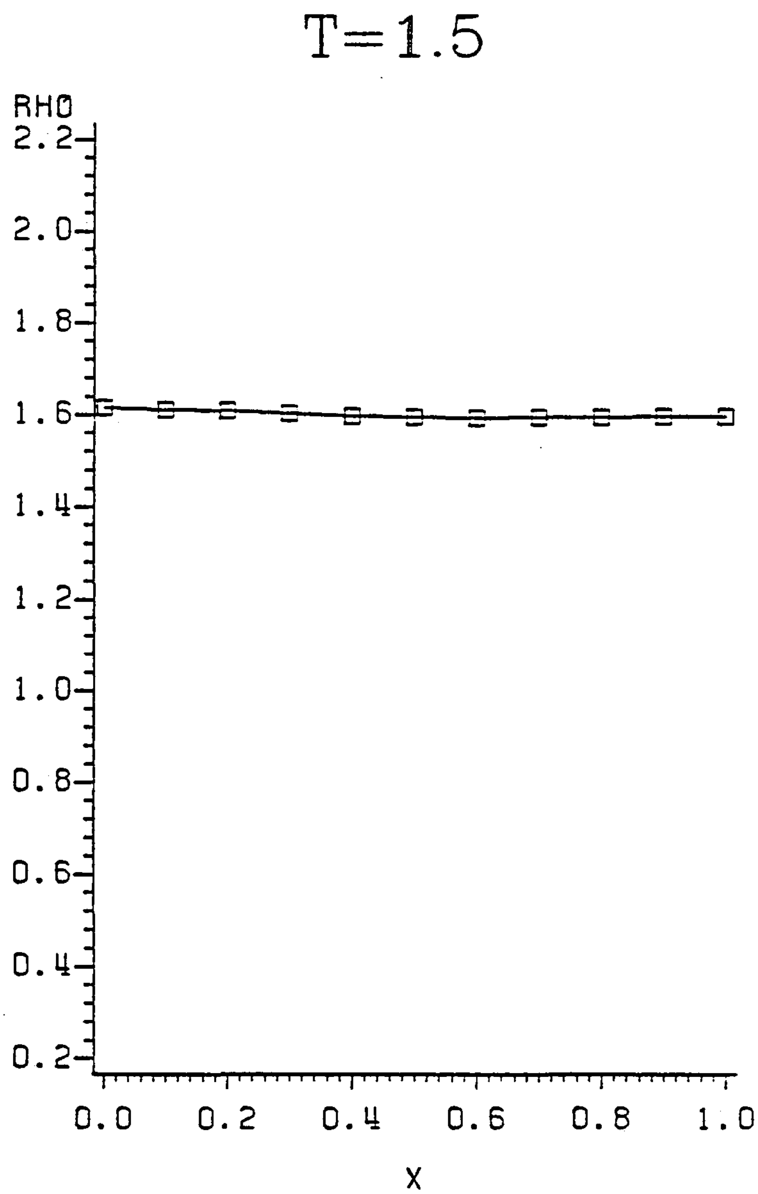

Figure 7.20

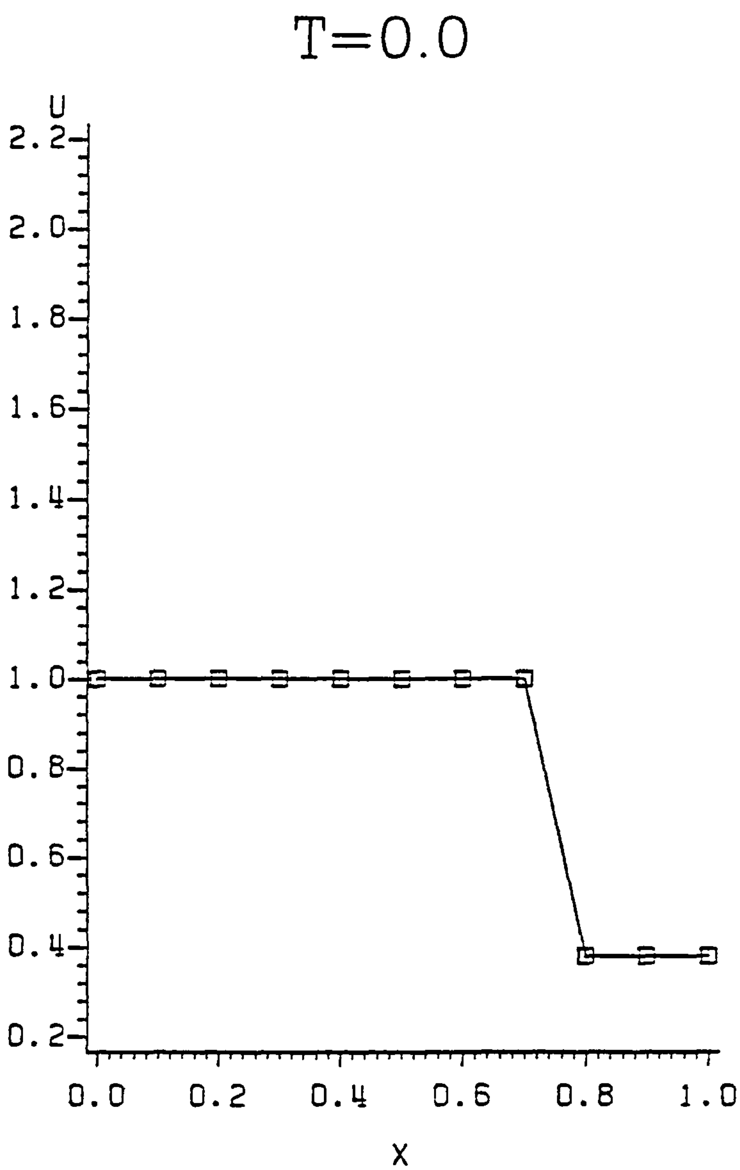

Figure 7.21

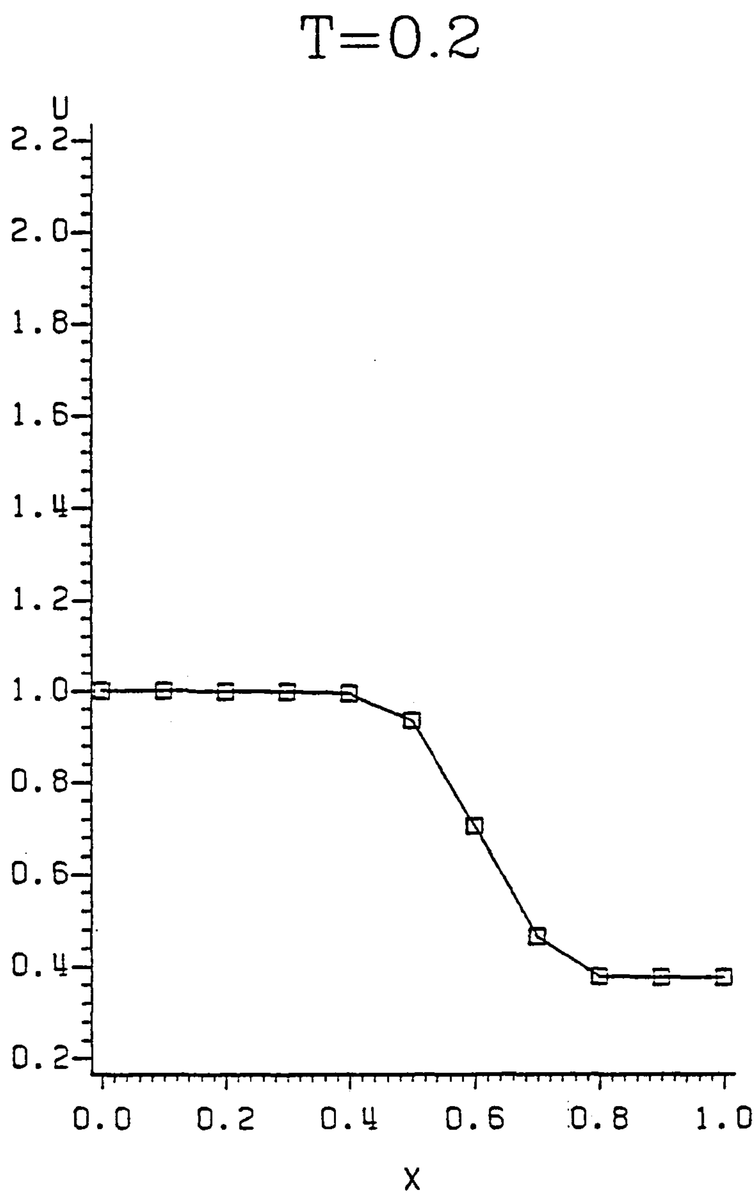

Figure 7.22

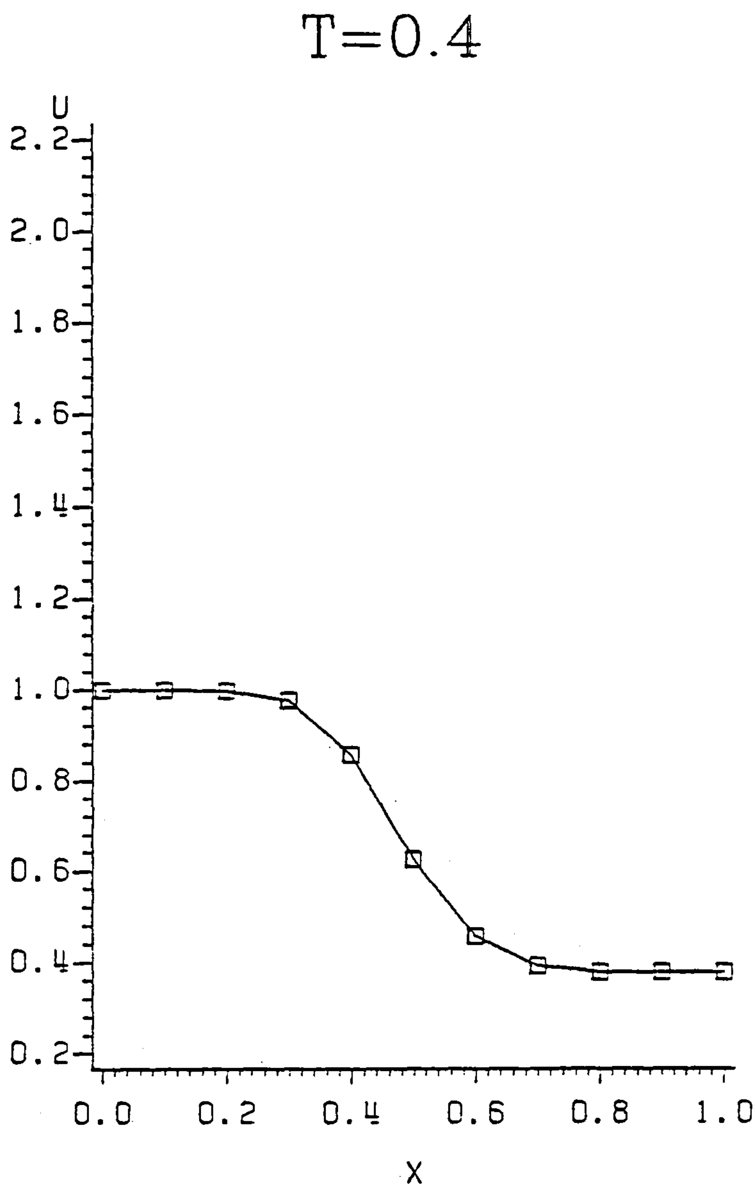

Figure 7.23

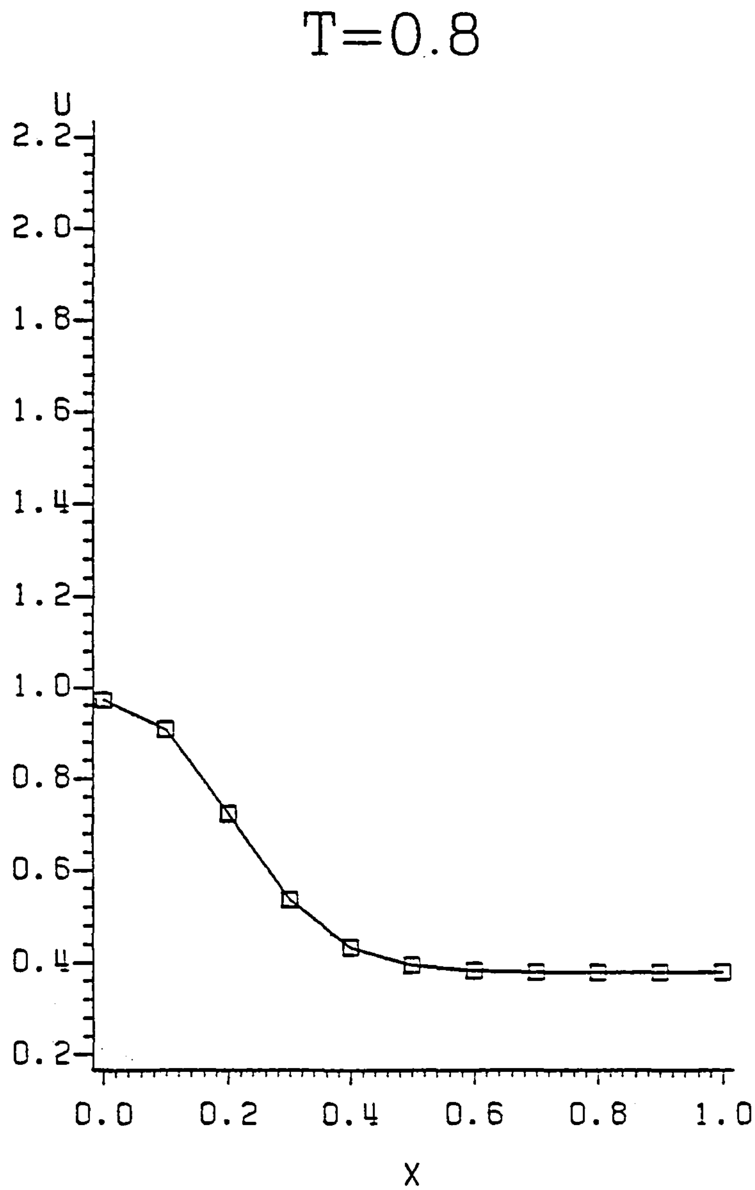

Figure 7.24

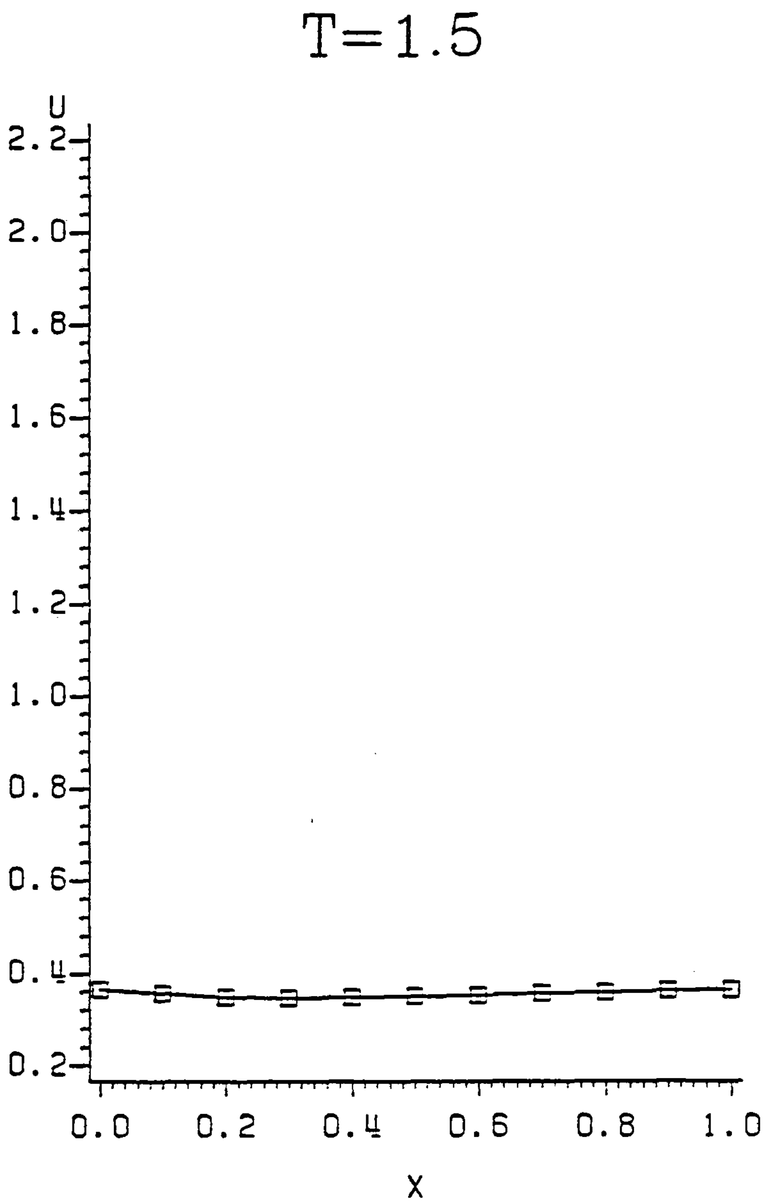

Figure 7.25

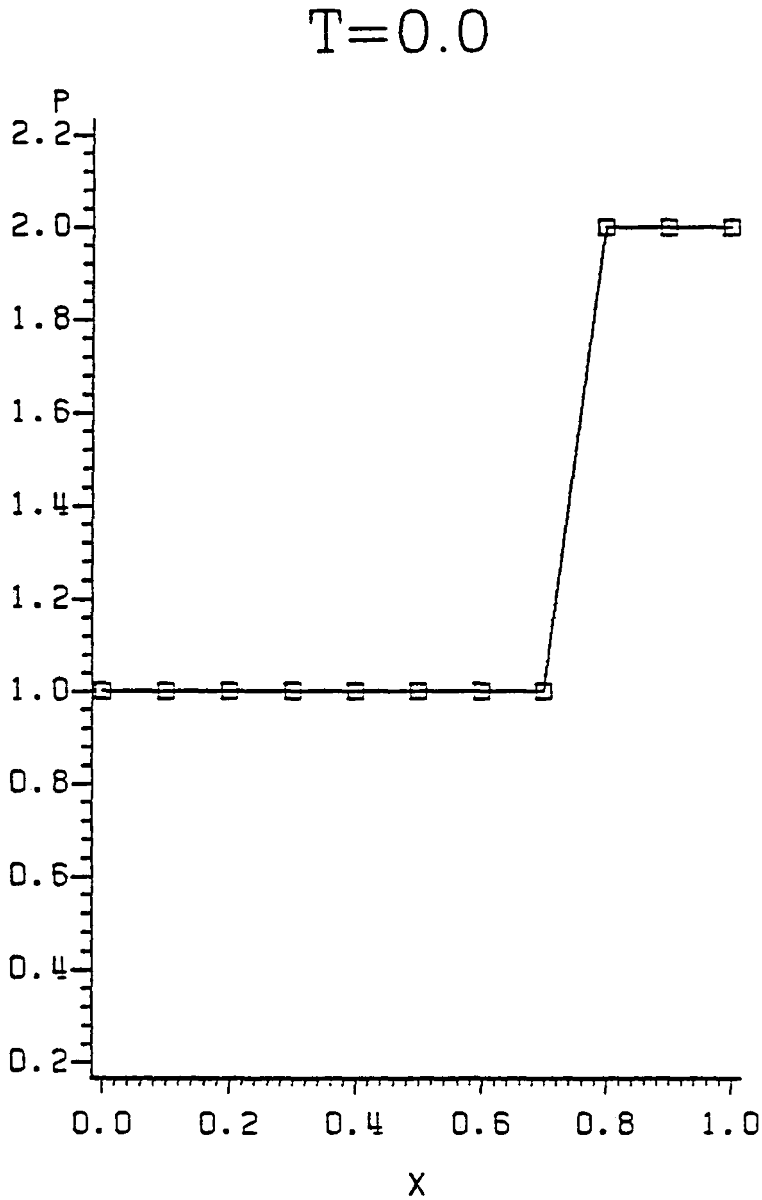

Figure 7.26

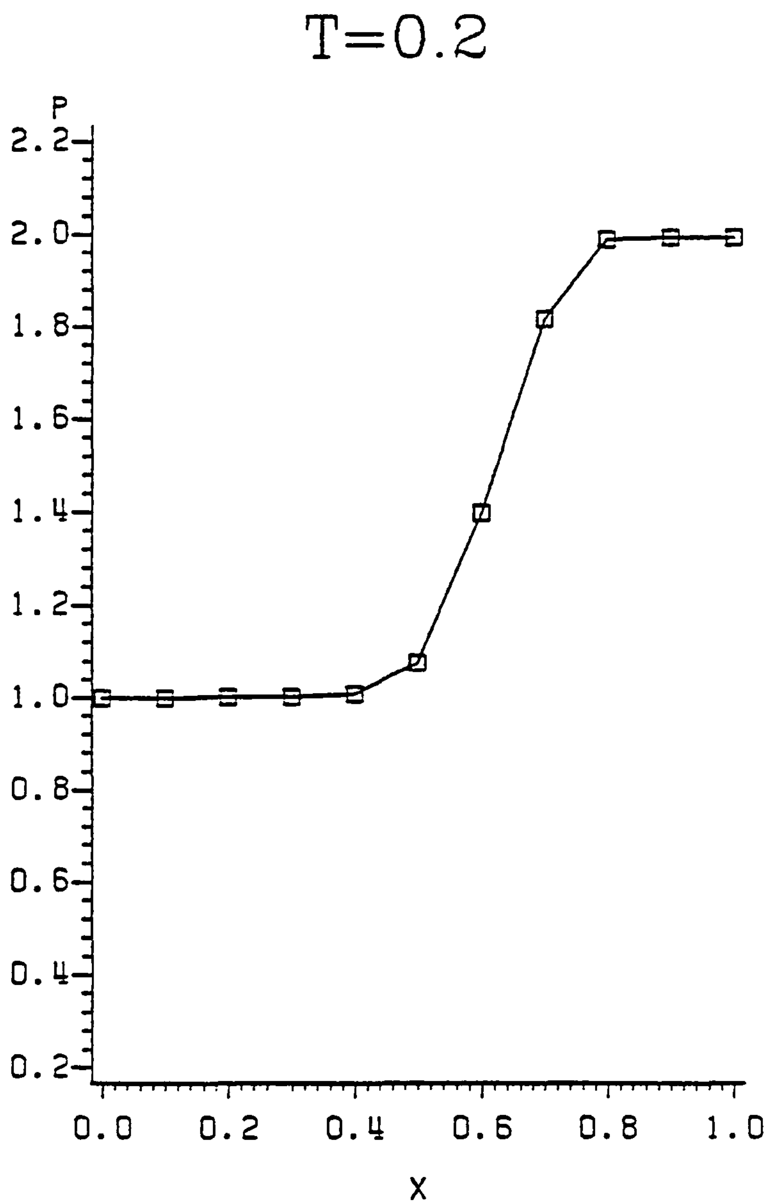

Figure 7.27

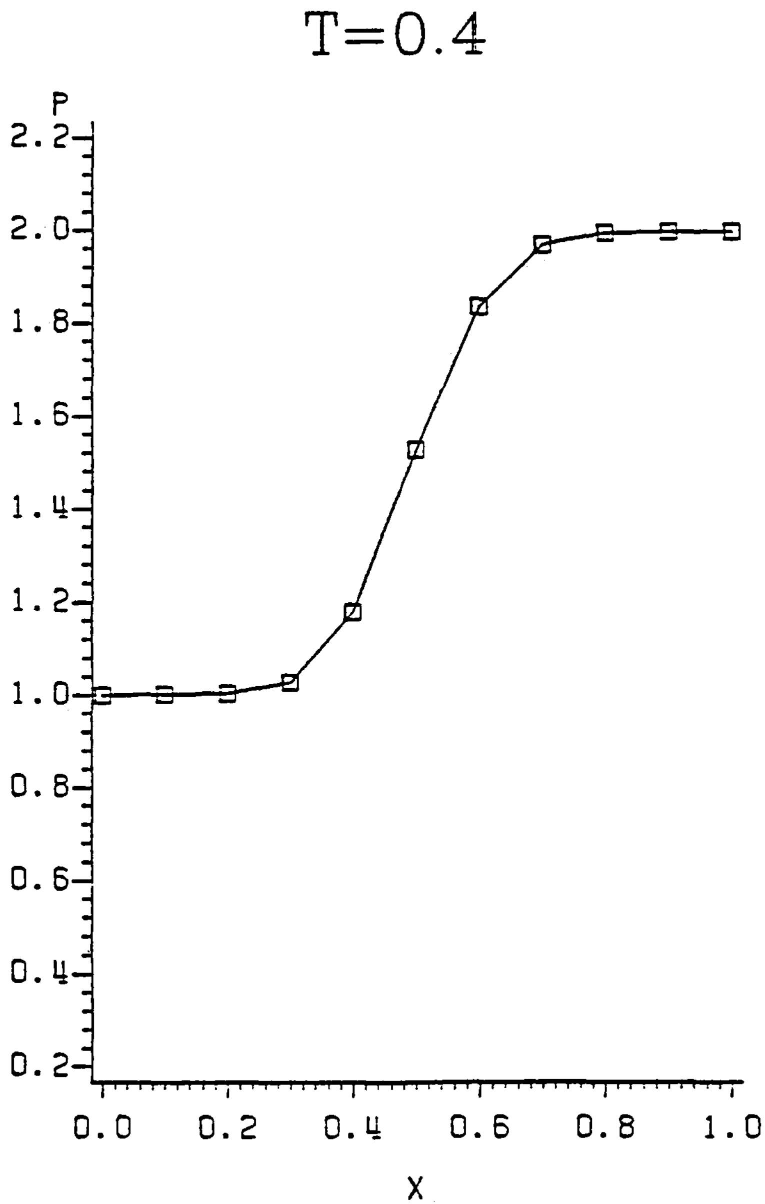

Figure 7.28

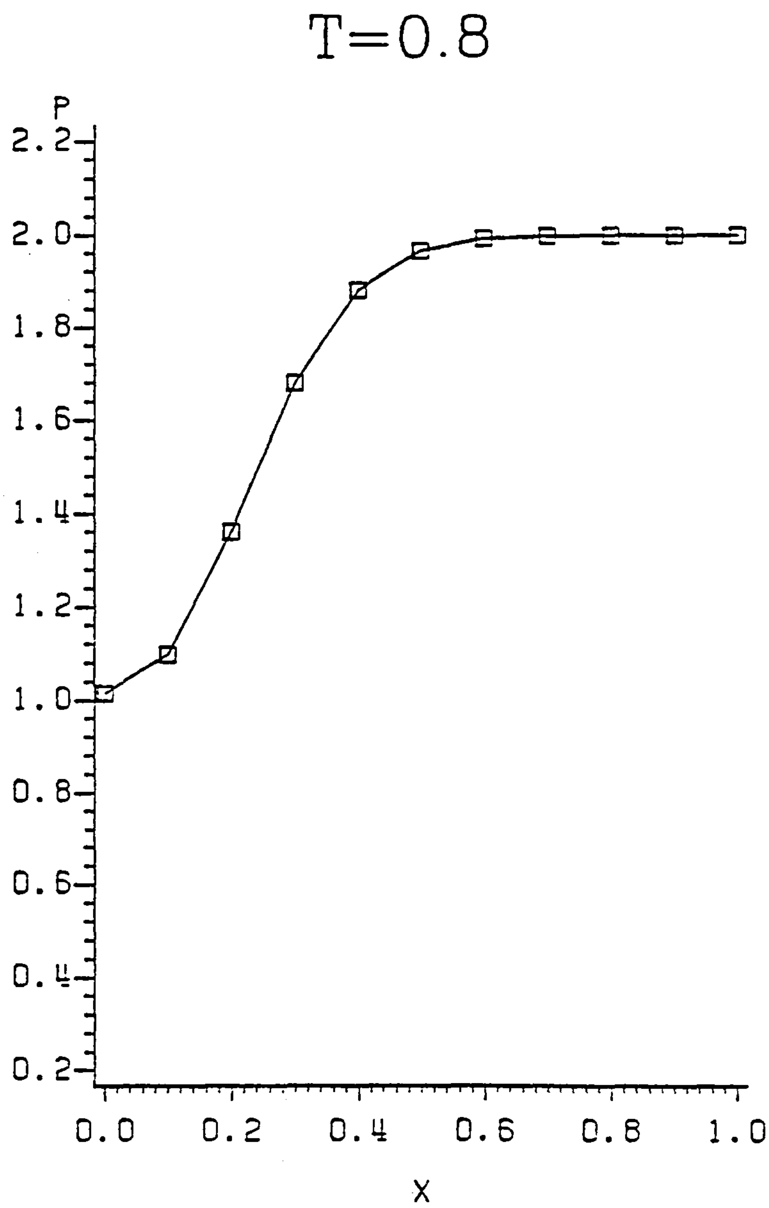

Figure 7.29

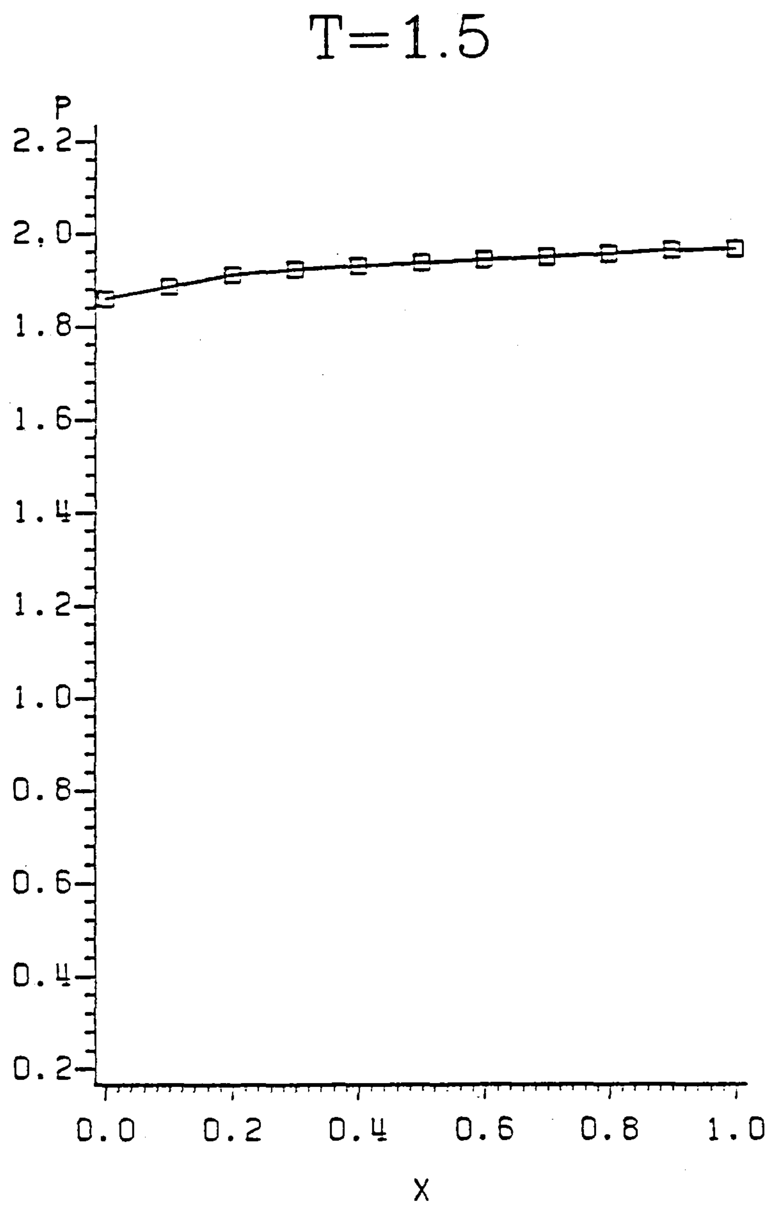

Figure 7.30

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16. Abstract

The Navier-Stokes equations can be viewed as an incompletely elliptic perturbation of the Euler equations. By using the entropy function for the Euler equations as a measure of ´energy´ for the Navier-Stokes equations, we are able to obtain nonlinear ´energy´ estimates for the mixed initial boundary value problem. These estimates are used to derive boundary conditions which guarantee L^2 boundedness even when the Reynolds number tends to infinity. Finally, we propose a new difference scheme for modelling the Navier-Stokes equations in multidimensions for which we are able to obtain discrete energy estimates exactly analogous to those we obtained for the differential equation.

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