DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508

FORMULATION AND APPLICATION OF RUSSELL'S METHOD

By

Jean Win Hou, Principal Investigator

Final Report
For the period ending March 31, 1985

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
Research Grant NAG-1-501
Dr. L. Keith Barker, Technical Monitor
FDCD-Automation Technology Branch

October 1985

OLD DOMINION UNIVERSITY RESEARCH FOUNDATION

DEPARTMENT OF MECHANICAL ENGINEERING AND MECHANICS
SCHOOL OF ENGINEERING
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23508

FORMULATION AND APPLICATION OF RUSSELL'S METHOD

By

Jean Win Hou, Principal Investigator

Final Report
For the period ending March 31, 1985

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
Research Grant NAG-1-501
Dr. L. Keith Barker, Technical Monitor
FDCD-Automation Technology Branch

Submitted by the
Old Dominion University Research Foundation
P.O. Box 6369
Norfolk, Virginia 23508

October 1985

ABSTRACT

It is shown that the numerical technique of Russell's momentum approach can be derived by using Hamilton's principle and Vance's numerical scheme. It results in a set of first order difference equations for solving the angular velocities. The method is simple and easily programmed. The numerical examples show that the method is also reliable.

The algorithm is modified next to perform the analysis of N-body systems with closed-loop topology. To increase the formulation flexibility, the equations of motion are represented by using Cartesian coordinates and Lagrange multipliers. The algorithm consists of two parts, Vance's scheme and an unconstrained minimization. The Vance's scheme is used to find the angular velocities, and the unconstrained minimization is applied to provide the correct angular displacements.

The proposed scheme is further extended to find the design sensitivity of an N-body system with closed-loop configuration, and to carry out the design optimization as well. The numerical example of a small-scaled mechanical system is presented to verify the proposed formulation. Some aspects of future study are discussed to enhance the capability of the proposed scheme.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

TABLE OF CONTENTS - Continued

LIST OF FIGURES - Continued

FORMULATION AND APPLICATION OF RUSSELL'S METHOD

By

Jean Win Hou*

1. INTRODUCTION

The major thrust of the Russell's method [1] for the dynamic analysis of multibody is twofold. Firstly, Russell constructed a set of first order differential equations, uncoupled in terms of primed angular momentum. Secondly, the constraint forces due to joints are eliminated in his formulation. In general, the number of first order differential equations needed to be integrated are less than the number of bodies. After integration, one is left with a set of simultaneous equations for solving the angular velocities. Russell [2] recommended the SOR (Successive Overrelaxation Iteration) scheme as a solver for angular velocities.

The Russell's method will be reformulated by using the Hamilton's principle and the rule of Lagrange multipliers in this report. To derive the Hamilton's equation for a constrained dynamic system, the variations of generalized coordinates and generalized velocities are treated independently and the constraints are introduced into the derivation through the rule of Lagrange multipliers. The Lagrange multipliers can be identified as constraint forces. Note that the constraints for the revolute and spherical joints are holonomic. Then, in section 3, the Russell's formulations for the N-body systems with open-loop topology are derived, and while deriving, the Lagrange multipliers associated with constraints are eliminated. In order to facilitate the development of a computer code, the equations are

*Assistant Professor, Department of Mechanical Engineering and Mechanics, Old Dominion University, Norfolk, Virginia 23508.

given in the matrix and vector forms. Numerical examples of a double and triple pendulum are presented in section 4 to verify the aforementioned algorithm.

For the N-body system with close-loop topology, the Lagrange multipliers are no longer easily eliminated. Nevertheless, the concept of Vance's scheme along with an unconstrained minimization scheme provides a very simple algorithm which is capable of not only performing the analysis, but also carrying out the optimal design of such a system. The detailed formulation and application of this algorithm can be found in Appendix B.

2. HAMILTON'S EQUATIONS

For a N-degree holonomic system, the classical approach is to derive Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i=1,\ldots N \tag{1}$$

from the Hamilton's principle

$$\delta \int_{t_0}^{t_1} L \, dt = 0 \tag{2}$$

where the Lagrangian function L is equal to the sum of kinetic energy T and external work W, i.e., $L = T + W$, and δ represents a contemporaneous variation. During the derivation, however, it is assumed that operations δ and $\frac{d}{dt}$ are exchangeable, i.e.,

$$\frac{d}{dt}\delta q_i = \delta \dot{q}_i, \qquad i=1,\ldots,N \tag{3}$$

In other words, the virtual velocity is obtained by taking the time derivative of the virtual displacement. Therefore, Eqs. 2 and 3 show that $\int L(q, \dot{q}, t)dt$ is stationary in the family of configurations satisfying the N differential equations

$$\frac{dq_i}{dt} = \dot{q}_i \tag{4}$$

To make q's and $\dot{q}$'s vary independently, the rule of multipliers asserts that

$$\int_{t_0}^{t_1} \left[L(q, \dot{q}, t) + \sum_{1}^{N} \lambda_i \left(\frac{dq_i}{dt} - \dot{q}_i\right)\right] dt$$

is stationary for arbitrary variations of q's and $\dot{q}$'s, the λ's being certain functions of t which are to be determined. The necessary conditions for a stationary value are given by the 2N equations

$$\frac{d\lambda_i}{dt} = \frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\partial L}{\partial \dot{q}_i} = \lambda_i, \qquad i=1,\ldots,N. \tag{5}$$

Note that q's and the time are fixed at t_0 and t_1, but not the $\dot{q}$'s. It can be readily verified that λ's correspond to the generalized momentum defined in the Hamilton's principle.

The same procedure can be extended to obtain Hamilton's equations asso-

ciated with constrained dynamic systems. Supposing a dynamic system is consistent with the following holonomic constraints

$$f_\ell(q_i,t) = 0, \qquad \begin{matrix} i=1,\ldots,N \\ \ell=1,\ldots,M,\ M<N \end{matrix} \tag{6}$$

Then, according to the rule of multipliers, there should exist ℓ functions of $\alpha(t)$ such that the functional

$$\int_{t_0}^{t_1} [L(q_i, \dot{q}_i, t) + \sum_1^N \lambda_i \left(\frac{dq_i}{dt} - \dot{q}_i\right) + \sum_1^M \alpha_\ell f_\ell (q_i, t)]\, dt$$

is stationary for arbitrary variations of q's and $\dot{q}$'s. The above conditions yield

$$\frac{d\lambda_i}{dt} = \frac{\partial L}{\partial q_i} + \sum_1^M \alpha_\ell \frac{\partial f_\ell}{\partial q_i}, \qquad i=1,\ldots,N \tag{7}$$

and

$$\lambda_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad i=1,\ldots,N \tag{8}$$

The 2N unknowns q's and $\dot{q}$'s as well as M functions α's are to be determined by solving Eqs. 6-8 together.

As to the system with nonholonomic constraints,

$$f_\ell\ (q_i,\dot{q}_i,\ t) = 0 \qquad \begin{matrix} i=1,\ldots,N \\ \ell=1,\ldots,M\ \ M<N \end{matrix} \tag{9}$$

the kinematically admissible variation δq_i has to satisfy the following equalities.

$$\sum_{1}^{N} \frac{\partial f_\ell}{\partial \dot{q}_i} \delta q_i = 0, \qquad \ell=1,\ldots,M \tag{10}$$

where δ denotes a contemporaneous variation. It is evident that Eq. 2 is no longer true for the nonholonomic system [3]. Instead one has to use the following equation

$$\int_{t_0}^{t_1} \delta L \, dt = 0 \tag{11}$$

Considering Eqs. 3 and 10 as the variations of constraints, the Farkas' lemma [4] ensures that there should exist functions λ's and α's such that

$$\int_{t_0}^{t_1} [\delta L + \sum_{1}^{N} \lambda_i \left(\frac{d\delta q_i}{dt} - \delta \dot{q}_i\right) + \sum_{1}^{M} \alpha_\ell \left(\sum_{1}^{N} \frac{\partial f_\ell}{\partial \dot{q}_i} \delta q_i\right)] \, dt = 0$$

for arbitrary variations of q's and $\dot{q}$'s. It follows from the above condition

$$\frac{d\lambda_i}{dt} = \frac{\partial L}{\partial q_i} + \sum_{1}^{M} \alpha_\ell \frac{\partial f_\ell}{\alpha \dot{q}_i}, \qquad i=1,\ldots,N \tag{12}$$

and

$$\lambda_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad i=1,\ldots,N \tag{13}$$

Using the Eqs. 12 and 13 in conjunction with Eq. 9, the 2N unknowns, q's and $\dot{q}$'s, as well as M function α's can be determined.

From the standpoint of computational efficiency and accuracy, formulations like Eqs. 7-8 or 12-13, for the equations of motion, are desirable. Note that only N first order differential equations appear in the above formulations. Thus, not only the number of differential equations remains as N, but the potential source of error of numerical integration is also minimized. However, only limited numerical schemes associated with Eqs. 7-8 or 12-13 had been developed to solve the equations of motion for dynamic systems. J. M. Vance [5, 6] replaced Eqs. 4 by a finite difference form and derived a set of finite difference equations to solve Eqs. 5.

Regarding the constrained dynamic systems, however, very few publications are available. Numerical difficulties arise in solving q's and $\dot{q}$'s satisfying the constraints and in determining the corresponding multipliers.

There are several techniques available currently for solving the equations of motion consistent with constraints, such as, numerical stabilization [7, 8] and coordinate partition [9]. These methods start with the second order derivatives $\ddot{f}_\ell$ (q,t) = 0, or first order derivative $\dot{f}_\ell$ (q, $\dot{q}$, t) = 0, so that $\ddot{q}$ can be a variable. In other words, an extra second order differential equation is generated from each constraint. Of course, these approaches do not convey the original intention of using Hamilton's principle which consists of first order differential equations only. To avoid the above difficulties, a simple way would be to eliminate

the multipliers from the formulation. This is exactly what Russell did in his work [1] for a dynamic system with open-loop topology.

3. HAMILTON'S EQUATIONS AND RUSSELL'S METHOD

3.1 N-Body System with Cluster Configuration

A N-body system with cluster configuration is shown in Fig. 1. The X_I-Y_I-Z_I coordinate system denotes the inertial frame. The body-fixed coordinate system, X_i-Y_i-Z_i for body i located at its center of mass.

For the sake of easy programming, a skew-symmetric matrix $\tilde{\underline{a}}$ associated with a vector $\underline{a}$ is defined as

$$\tilde{\underline{a}} \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

such that $\hat{a} \times \hat{b} = \tilde{\underline{a}}\,\underline{b}\,\hat{e}$ where $\hat{e}$ is a unit vector parallel to $\hat{a} \times \hat{b}$.

It is easy to prove the following identities:

$$\tilde{\underline{a}}^T = -\tilde{\underline{a}} \tag{14}$$

$$\tilde{\underline{a}}\,\underline{b} = -\tilde{\underline{b}}\,\underline{a} \tag{15}$$

$$(\tilde{\underline{a}}\,\underline{b}) = \tilde{\underline{a}}\,\tilde{\underline{b}} - \tilde{\underline{b}}\,\tilde{\underline{a}} \tag{16}$$

Furthermore, if A is a transformation matrix such that $\underline{a} = A\,\underline{a}'$ and $\underline{w}$ is a vector of angular velocities, it can be shown that

$$\dot{A} = \tilde{\underline{w}}\, A \tag{17}$$

$$\tilde{\underline{a}} = A\, \tilde{\underline{a}}'\, A^T \tag{18}$$

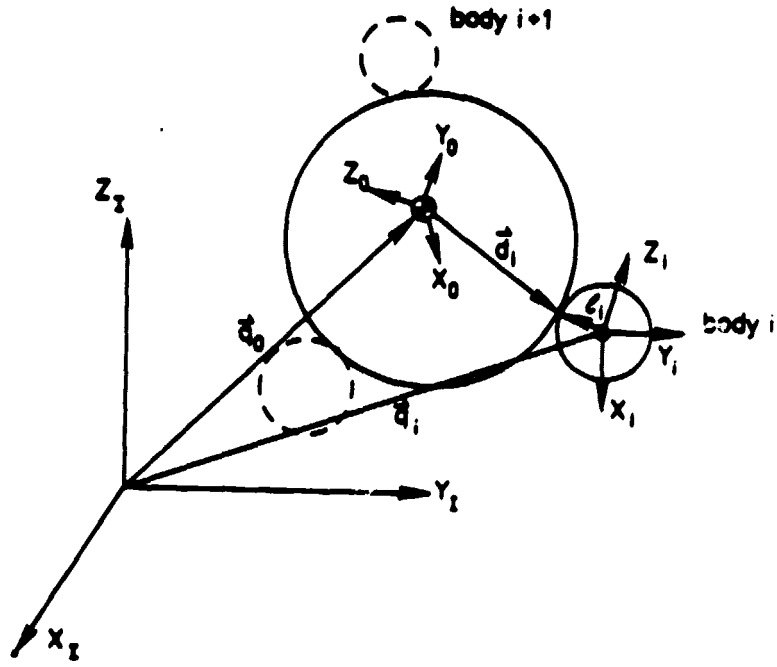


Figure 1. N-Body Cluster Geometry

Generalized Coordinates

The position vectors of the mass centers of bodies and angular displacements of the body-fixed coordinates are taken as the generalized coordinates. The components of generalized coordinates are in terms of the inertial frame.

Generalized Forces

As shown in Fig. 2, a force $\underline{F}_i$ acts on the body i at the point P with the position vector $\underline{P}_i$:

$$\underline{p}_i = \underline{q}_i + \underline{a}_i$$

where $\underline{a}_i$ is a vector measured from the center of mass to the point P. Thus,

$$\begin{aligned}\dot{\underline{p}}_i &= \dot{\underline{q}}_i + \dot{\underline{a}}_i \\ &= \dot{\underline{q}}_i + \tilde{\underline{w}}_i \underline{a}_i \\ &= \dot{\underline{q}}_i - \tilde{\underline{a}}_i \underline{w}_i\end{aligned}$$

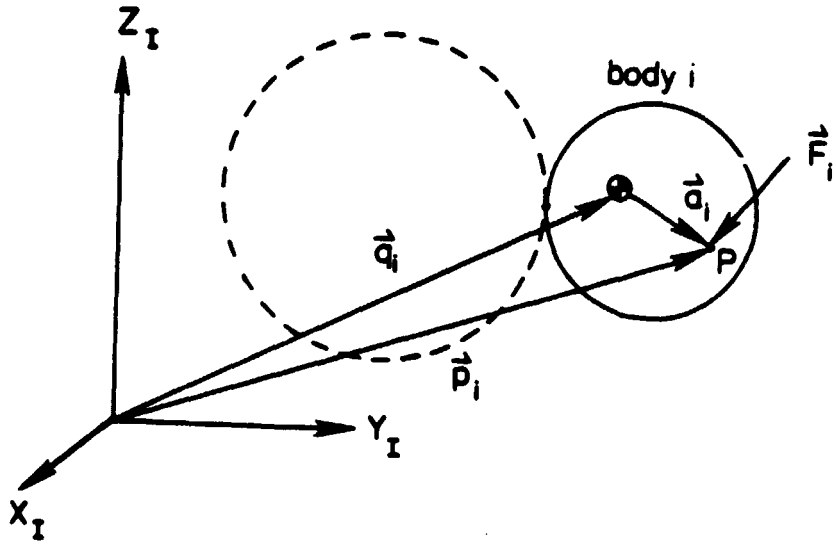


Figure 2. Generalized Forces

Replacing the differential notation d by a small increment Δ, one has

$$\frac{\Delta \underline{p}_i}{\Delta t} = \frac{\Delta \underline{q}_i}{\Delta t} - \tilde{\underline{a}}_i \, \underline{w}_i$$

or

$$\begin{aligned} \Delta \underline{p}_i &= \Delta \underline{q}_i - \tilde{\underline{a}}_i (\underline{w}_i \, \Delta t) \\ &= \Delta \underline{q}_i - \tilde{\underline{a}}_i \, \Delta \underline{\theta}_i \end{aligned}$$

where the definition of $\underline{w}_i = \dfrac{d\underline{\theta}_i}{dt}$ is used. Finally, using the variational notation δ, instead of Δ, the foregoing relation yields

$$\begin{aligned} \underline{F}_i^T \, \delta \underline{p}_i &= \underline{F}_i^T \, \delta \underline{q}_i - \underline{F}_i^T \, \tilde{\underline{a}}_i \, \delta \underline{\theta}_i \\ &= \underline{F}_i^T \, \delta \underline{q}_i + (\tilde{\underline{a}}_i \, \underline{F}_i)^T \, \delta \underline{\theta}_i \end{aligned} \tag{19}$$

The term $\tilde{\underline{a}}_i \, \underline{F}_i$ is the momentum at the mass center induced by the force $\underline{F}_i$. Note that even though the angular displacement $\underline{\theta}_i$ is not a vector, but the infinitesimal angular displacement $\delta \underline{\theta}_i$ is still a vector.

Total Kinetic Energy

The total kinetic energy of the system is given as

$$T = \sum_{0}^{N-1} \left(\frac{1}{2} M_i \, \dot{\underline{q}}_i^T \, \dot{\underline{q}}_i + \frac{1}{2} \underline{w}_i^T \, J_i \, \underline{w}_i \right)$$

where J_i is the inertia tensor of body i with respect to the inertial frame. The J_i can be found equal to $A_i J_i^C A_i^T$ where J_i^C is defined as the inertia tensor with respect to the body-fixed coordinate system and A_i is the transformation matrix between the body-fixed coordinate system and the

inertial frame. Thus, the variation of the total kinetic energy is simply written as

$$\delta T = \sum_{0}^{N-1} (M_i \dot{\underline{q}}_i)^T \delta\dot{\underline{q}}_i + (J_i \underline{w}_i)^T \delta\underline{w}_i \qquad (20)$$

Constraints

There are two sets of constraints needed to derive the Hamilton's equations for the constrained dynamic system.

The first one is a holonomic constraint which defines the nature of connection between the bodies. In the following derivation, only a revolute joint (2 dimensional) and a spherical joint (3 dimensional) are considered. According to Fig. 1, the spherical (or revolute) joint provides a holonomic constraint which can be expressed mathematically as

$$\phi_i = \underline{q}_i - \underline{q}_0 - \underline{d}_i - \underline{\ell}_i = 0, \qquad i=1,\ldots,N-1 \qquad (21)$$

Note that all the components of the vectors in the last equation are referred to the inertial frame.

The time derivative of ϕ_i yields

$$\dot{\phi}_i = 0 = \dot{\underline{q}}_i - \dot{\underline{q}}_0 - \tilde{\underline{w}}_0 \underline{d}_i - \tilde{\underline{w}}_i \underline{\ell}_i$$

After replacing the differential notation $\frac{d}{dt}$ by incremental notation $\frac{\Delta}{\Delta t}$ and by multiplying each term with Δt, one has

$$\Delta \underline{q}_i - \Delta \underline{q}_0 + \tilde{\underline{d}}_i \, \underline{w}_0 \, \Delta t + \tilde{\underline{\ell}}_i \, \underline{w}_i \, \Delta t = 0$$

Thus, the variation of the constraint $\delta\theta_i = 0$ will yield the relation

$$\delta \underline{q}_i - \delta \underline{q}_0 + \tilde{\underline{d}}_i \, \delta\underline{\theta}_0 + \tilde{\underline{\ell}}_i \, \delta\underline{\theta}_i = 0 \qquad (22)$$

The second set of constraints corresponds to the relations between displacement and velocity variations,

$$\frac{d\delta \underline{q}_i}{dt} = \delta \dot{\underline{q}}_i \qquad i=0,\ldots,N-1 \qquad (23)$$

and

$$\frac{d\delta\underline{\theta}_i}{dt} = \delta \underline{w}_i \qquad i=0,\ldots,N-1 \qquad (24)$$

Again note that $\delta\theta_i$ represents infinitesimal angular displacement which is a vector.

For the variations of the systems, Eqs. 19, 20, 22, 23, and 24, Farkas' Lemma asserts that there should exist λ_i $(i=1,\ldots,N-1)$, α_i and β_i $(i=0,\ldots,N-1)$ such that for arbitrary variations $\delta\underline{q}_i$, $\delta\dot{\underline{q}}_i$, $\delta\underline{\theta}_i$ and $\delta\underline{w}_i$:

$$0 = \int_{t_0}^{t_1} \{ \sum_{0}^{N-1} [(M_i \, \dot{\underline{q}}_i)^T \, \delta\dot{\underline{q}}_i + (J_i \underline{w}_i)^T \, \delta \underline{w}_i] + \sum_{0}^{N-1} [\underline{F}_i^T \, \delta \underline{q}_i + (\tilde{\underline{a}}_i \underline{F}_i)^T \, \delta\underline{\theta}_i]$$

$$+ \sum_{1}^{N-1} \underline{\lambda}_i^T \, (\delta \underline{q}_i - \delta \underline{q}_0 + \tilde{\underline{d}}_i \delta\underline{\theta}_0 + \tilde{\underline{\ell}}_i \delta\underline{\theta}_i) + \sum_{0}^{N-1} \underline{\alpha}_i^T \, [\frac{d}{dt} (\delta \underline{q}_i) - \delta \dot{\underline{q}}_i]$$

$$+ \sum_{0}^{N-1} \underline{\beta}_i^T \, [\frac{d}{dt} (\delta\underline{\theta}_i) - \delta \underline{w}_i] \} \, dt$$

Analogous to the Hamilton's principle, $\underline{q}_i$ and $\underline{\theta}_i$ are assumed to be fixed at time $t = t_0$ and $t = t_1$. That means $\delta\underline{q}_i = \delta\underline{\theta}_i = 0$ at $t = t_0$ and $t = t_1$. After integrating by parts and collecting the corresponding terms, the following identities can be obtained

$$M_i \dot{\underline{q}}_i = \underline{\alpha}_i, \qquad i=0,\ldots,N-1 \tag{25}$$

$$J_i \underline{w}_i = \underline{\beta}_i, \qquad i=0,\ldots,N-1 \tag{26}$$

where $\underline{\alpha}_i$ and $\underline{\beta}_i$ can be identified as linear momentum and angular momentum (with respect to the mass center of body i), respectively. Furthermore, one has

$$\dot{\underline{\alpha}}_i - \underline{F}_i - \underline{\lambda}_i = 0, \qquad i=1,\ldots,N-1 \tag{27}$$

$$\dot{\underline{\beta}}_i - \tilde{\underline{a}}_i \underline{F}_i + \tilde{\underline{\ell}}_i \underline{\lambda}_i = 0, \qquad i=1,\ldots,N-1 \tag{28}$$

and

$$\dot{\underline{\alpha}}_0 - \underline{F}_0 + \sum_{1}^{N-1} \underline{\lambda}_i = 0 \tag{29}$$

$$\dot{\underline{\beta}}_0 - \tilde{\underline{a}}_0 \underline{F}_0 + \sum_{1}^{N-1} \tilde{\underline{d}}_i \underline{\lambda}_i = 0 \tag{30}$$

To reduce the number of degrees of freedom, two steps are necessary: (1) eliminating the constraint forces $\underline{\lambda}_i$, and (2) substituting the constraints of Eq. 21, into the formulation explicitly.

Center of Mass

The relations for the mass center of the whole system are defined here.

It can be shown that

$$\sum_{0}^{N-1} M_i \underline{q}_i = M \underline{R}, \tag{31}$$

$$\sum_{0}^{N-1} M_i \dot{\underline{q}}_i = \sum_{0}^{N-1} \underline{\alpha}_i = M \dot{\underline{R}}, \tag{32}$$

and

$$\sum_{0}^{N-1} M_i \ddot{\underline{q}}_i = \sum_{0}^{N-1} \dot{\underline{\alpha}}_i = M \ddot{\underline{R}} = \sum_{0}^{N-1} \underline{F}_i \tag{33}$$

where M and $\underline{R}$ denote the total mass and the position vector for the mass center of the whole system. The second equality in Eq. 32 follows the definition of $\underline{\alpha}_i$ as given in Eq. 25 and the last equation is derived by adding Eqs. 27 and 29. Further,

$$\underline{r}_i = \underline{q}_i - \underline{R}, \qquad i=0,\ldots,N-1$$

or

$$\underline{q}_i = \underline{R} + \underline{r}_i, \qquad i=0,\ldots,N-1$$

where $\underline{r}_i$ is the vector between the mass centers of body i and the whole system. Eq. 21 can then be written as

$$\underline{r}_i - \underline{r}_0 - \underline{d}_i - \underline{\ell}_i = 0, \qquad i=1,\ldots,N-1 \tag{34}$$

Then multiplying Eq. 34 with M_i, summing over all the i and in view of the fact that $\sum_{0}^{N-1} M_i \underline{r}_i = 0$, it can be shown that

$$\underline{r}_0 = -\frac{1}{M} \sum_{1}^{N-1} M_i (\underline{d}_i + \underline{\ell}_i). \tag{35}$$

After substituting for $\underline{r}_0$, Eq. 34 can be rewritten as

$$\underline{r}_i = -\frac{1}{M}\sum_{1}^{N-1} M_i\,(\underline{d}_i + \underline{\ell}_i) + \underline{d}_i + \underline{\ell}_i, \qquad i=1,\ldots,N-1. \tag{36}$$

Furthermore, taking the time derivatives of Eq. 35 and 36, one gets

$$\dot{\underline{r}}_0 = -\frac{1}{M}\sum_{1}^{N-1} M_i\,(\tilde{\underline{w}}_0\,\underline{d}_i + \tilde{\underline{w}}_i\,\underline{\ell}_i) \tag{37}$$

and

$$\dot{\underline{r}}_i = -\frac{1}{M}\sum_{1}^{N-1} M_i\,(\tilde{\underline{w}}_0\,\underline{d}_i + \tilde{\underline{w}}_i\,\underline{\ell}_i) + \tilde{\underline{w}}_0\,\underline{d}_i + \tilde{\underline{w}}_i\,\underline{\ell}_i, \qquad i=1,\ldots,N-1 \tag{38}$$

Note that $\tilde{\underline{r}}_0$ and $\dot{\underline{r}}_i$ are functions of angular velocities only.

Constraint Forces Elimination

The constraint forces $\underline{\lambda}_i$ can be eliminated from Eq. 28, by substituting Eq. 27,

$$\dot{\underline{\beta}}_i + \tilde{\underline{\ell}}_i\,\dot{\underline{\alpha}}_i = (\tilde{\underline{a}}_i + \tilde{\underline{\ell}}_i)\,\underline{F}_i, \qquad i=1,\ldots N-1$$

Combining the definition of $\underline{\alpha}_i$ and $\underline{r}_i$, as well as Eq. 33, one has

$$\begin{aligned}\dot{\underline{\alpha}}_i &= M_i\,\ddot{\underline{q}}_i \\ &= M_i\,\ddot{\underline{R}} + M_i\,\ddot{\underline{r}}_i \\ &= \frac{M_i}{M}\sum_{0}^{N-1}\underline{F}_i + M_i\,\ddot{\underline{r}}_i, \qquad i=1,\ldots N-1\end{aligned} \tag{40}$$

Then Eq. 39 can be rewritten in terms of $\ddot{\underline{r}}_i$ as,

$$\dot{\underline{\beta}}_i + \tilde{\underline{\ell}}_i\,(M_i\,\ddot{\underline{r}}_i) = (\tilde{\underline{a}}_i + \tilde{\underline{\ell}}_i)\,\underline{F}_i - \frac{M_i}{M}\,\tilde{\underline{\ell}}_i\,\left(\sum_0^{N-1}\underline{F}_i\right) \qquad i=1,\ldots N-1$$

Since $\tilde{\underline{\ell}}_i\,(M_i\,\ddot{\underline{r}}_i) = \dot{\tilde{\underline{\ell}}}_i\,(M_i\,\dot{\underline{r}}_i) - M_i\,\dot{\tilde{\underline{\ell}}}_i\,\dot{\underline{r}}_i$, the above equation becomes

$$\dot{\underline{\beta}}_i + M_i\,\dot{\tilde{\underline{\ell}}}_i\,\dot{\underline{r}}_i = (\tilde{\underline{a}}_i + \tilde{\underline{\ell}}_i)\underline{F}_i - \frac{M_i}{M}\,\tilde{\underline{\ell}}_i\,\left(\sum_0^{N-1}\underline{F}_i\right) + M_i\,\dot{\tilde{\underline{\ell}}}_i\,\dot{\underline{r}}_i, \qquad (41)$$

$$i=1,\ldots,N-1$$

where $\dot{\tilde{\underline{\ell}}}_i = \dot{\underline{\ell}}_i = \widetilde{\underline{w}_i\,\underline{\ell}_i} = \tilde{\underline{w}}_i\,\tilde{\underline{\ell}}_i - \tilde{\underline{\ell}}_i\,\tilde{\underline{w}}_i$, and $\dot{\underline{r}}_i$ is a linear combination of angular velocities as given by Eq. 36.

The term $\underline{\beta}_i + M_i\,\tilde{\underline{\ell}}_i\,\dot{\underline{r}}_i$ is called the primed angular momentum by Russell. It is worthwhile mentioning that the right hand side of Eq. 41 is a linear combination of angular velocities and the left hand side is a quadratic function of angular velocities.

Since only N-1 equations are available in Eq. 41, one more equation has to be established for angular velocity $\underline{w}_0$ of the central body. Making use of Eq. 40, the constraint force can be obtained by

$$\begin{aligned}\underline{\lambda}_i &= -\dot{\underline{\alpha}}_i + \underline{F}_i \\ &= -\frac{M_i}{M}\left(\sum_0^{N-1}\underline{F}_i\right) + M_i\,\ddot{\underline{r}}_i + \underline{F}_i, \qquad i=1,\ldots,N-1\end{aligned}$$

Substituting the above equality into Eq. 30, one obtains

$$\dot{\underline{\beta}}_0 = \tilde{\underline{a}}_0 \underline{F}_0 + \sum_1^{N-1} \tilde{\underline{d}}_i \left[M_i \ddot{\underline{r}}_i + \frac{M_i}{M} \left(\sum_0^{N-1} \underline{F}_i\right) - \underline{F}_i\right]$$

or

$$\frac{\dot{}}{\underline{\beta}_0 - \sum_1^{N-1} \tilde{\underline{d}}_i (M_i \dot{\underline{r}}_i)} = \tilde{\underline{a}}_0 \underline{F}_0 - \sum_1^{N-1} \tilde{\underline{d}}_i (M_i \dot{\underline{r}}_i) + \sum_1^{N-1} \tilde{\underline{d}}_i \left[\frac{M_i}{M} \left(\sum_0^{N-1} \underline{F}_i\right) - \underline{F}_i\right]$$

or

$$\frac{\dot{}}{\underline{\beta}_0 - \sum_1^{N-1} M_i d_i \dot{\underline{r}}_i} = \tilde{\underline{a}}_0 \underline{F}_0 + \sum_1^{N-1} (\tilde{\underline{d}}_i \tilde{\underline{w}}_0 - \tilde{\underline{w}}_0 \tilde{\underline{d}}_i)\dot{\underline{r}}_i + \sum_1^{N-1} \tilde{\underline{d}}_i \left[\frac{M_i}{M} \left(\sum_0^{N-1} \underline{F}_i\right) - \underline{F}_i\right] \qquad (42)$$

The above equation uses the primed angular momentum of the central body to calculate the angular velocity $\underline{w}_0$. This approach is different from Russell's formulation in that the angular momentum of the whole system is established to calculate $\underline{w}_0$.

3.2 N-Body System with Tree Configuration

An N-body system with tree configuration may be considered as many legs appended to the central body as shown in Fig. 3. Note that a leg represents a structure with open-loop topology and there is only a point joint connecting a pair of bodies.

So far, the equations of motion are derived only for an N-body system with cluster configuration. However, the derivation of equations of motion, discussed in the previous section, can be extended and applied to the N-body system with tree configuration. The derivation procedure is illustrated

best by deriving the equations of motion for a simple example as shown in Fig. 4. It is shown that a leg which consists of four bodies is connected to the central body through a single joint. In addition to the constraints given in Eq. 21, the constraints associated with the generalized coordinate of each of the legs can be expressed in terms of generalized coordinates of the central body as given below. By investigating the kinematic relations indicated in Fig. 4, one has

$$\underline{q}_i - \underline{q}_0 - \underline{d}_i - \underline{\ell}_i = 0, \qquad i=1,\ldots,N-2 \tag{43}$$

and

$$\underline{q}_j - \underline{q}_0 - \underline{d}_{N-1} - \underline{h}_j = 0, \qquad j=N-1,\ldots,N+2 \tag{44}$$

where

$$\underline{h}_{N-1} = \underline{\ell}_{01},$$
$$\underline{h}_N = \underline{\ell}_{32} - \underline{\ell}_{23} + \underline{\ell}_{13} - \underline{\ell}_{31} + \underline{\ell}_{01},$$
$$\underline{h}_{N+1} = \underline{\ell}_{13} - \underline{\ell}_{31} + \underline{\ell}_{01},$$

and

$$\underline{h}_{N+2} = \underline{\ell}_{34} - \underline{\ell}_{43} + \underline{\ell}_{13} - \underline{\ell}_{31} + \underline{\ell}_{01}$$

In Eqs. 43-44, $\underline{q}_i$ and $\underline{q}_0$ are the position vectors of body i and the central body with respect to the inertial frame.

Note that $\underline{h}$ is a function of angular displacements. Thus, the variations of $\underline{h}$ can be written as follows:

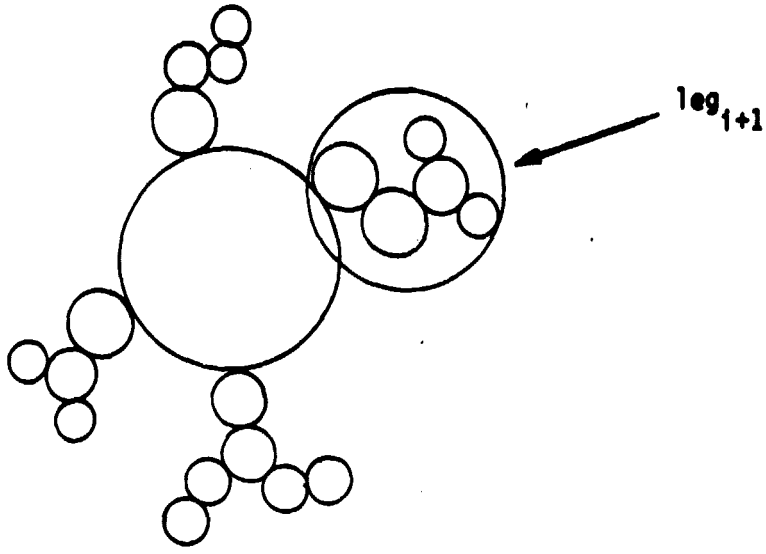


Figure 3. N-Body Tree Configuration.

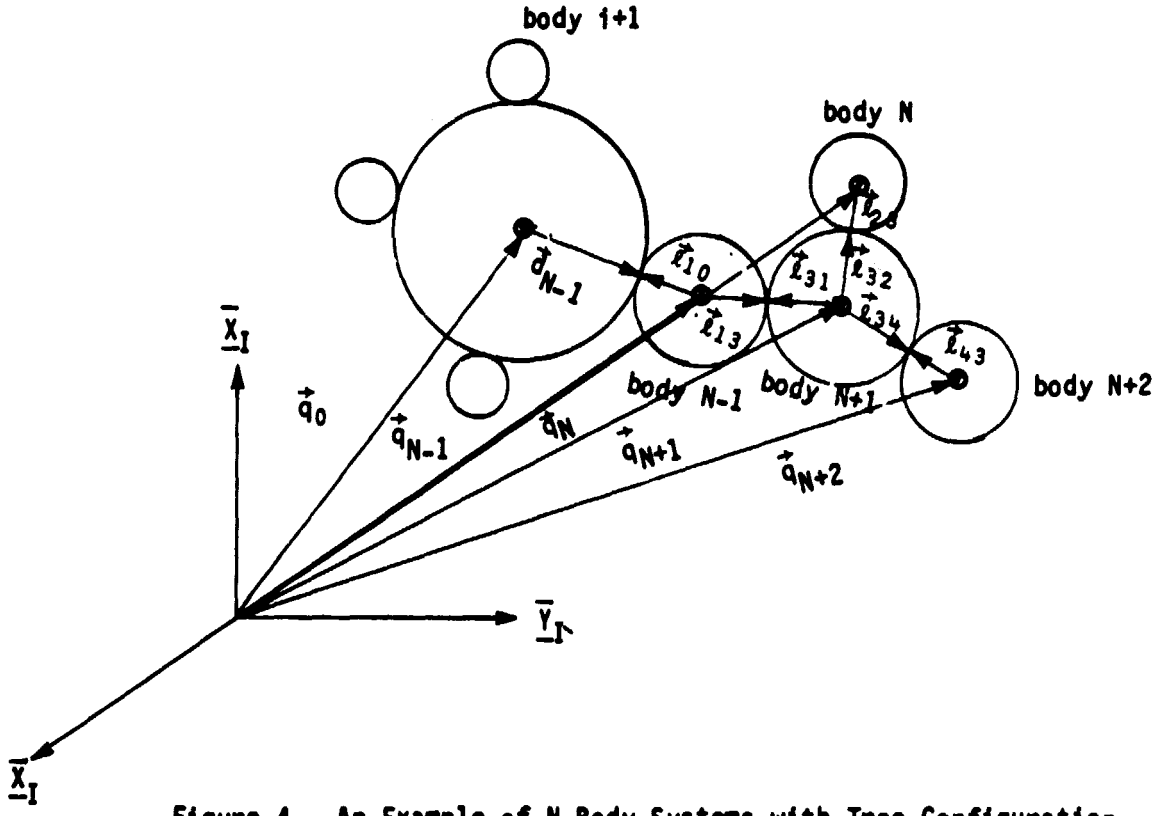


Figure 4. An Example of N-Body Systems with Tree Configuration.

$$\delta \underline{h}_{N-1} = - \tilde{\underline{r}}_{01} \, \delta \underline{\theta}_{N-1},$$

$$\delta \underline{h}_{N} = - \tilde{\underline{r}}_{32} \, \delta \underline{\theta}_{N} + (\tilde{\underline{r}}_{23} - \tilde{\underline{r}}_{13}) \, \delta \underline{\theta}_{N+1} - \tilde{\underline{r}}_{01} \, \delta \underline{\theta}_{N-1},$$

$$\delta \underline{h}_{N+1} = - \tilde{\underline{r}}_{13} \, \delta \underline{\theta}_{N+1} + (\tilde{\underline{r}}_{31} - \tilde{\underline{r}}_{01}) \, \delta \underline{\theta}_{N-1},$$

and

$$\delta \underline{h}_{N+2} = - \tilde{\underline{r}}_{34} \, \delta \underline{\theta}_{N+2} + (\tilde{\underline{r}}_{43} - \tilde{\underline{r}}_{13}) \, \delta \underline{\theta}_{N+1} + (\tilde{\underline{r}}_{31} - \tilde{\underline{r}}_{01}) \, \delta \underline{\theta}_{N-1}$$

With the application of Hamilton's principle and Farka's lemma, the equations of motion for the system shown in Fig. 4 can then be derived as

$$M_i \, \dot{\underline{q}}_i = \underline{\alpha}_i, \qquad i=0,\ldots,N+2 \tag{45}$$

$$J_i \, \underline{w}_i = \underline{\beta}_i, \qquad i=0,\ldots,N+2 \tag{46}$$

Furthermore, one also has

$$\dot{\underline{\alpha}}_i - \underline{F}_i - \underline{\lambda}_i = 0, \qquad i=0,\ldots,N+2 \tag{47}$$

$$\dot{\underline{\beta}}_i - \tilde{\underline{a}}_i \, \underline{F}_i + \tilde{\underline{r}}_i \, \underline{\lambda}_i = 0, \qquad i=0,\ldots,N-2 \tag{48}$$

$$\dot{\underline{\beta}}_{N-1} - \tilde{\underline{a}}_{N-1} \, \underline{F}_{N-1} + \tilde{\underline{r}}_{01} \, \underline{\lambda}_{N-1} + \tilde{\underline{r}}_{01} \, \underline{\lambda}_{N} + (\tilde{\underline{r}}_{31} - \tilde{\underline{r}}_{01}) \, \underline{\lambda}_{N+1} - (\tilde{\underline{r}}_{31} - \tilde{\underline{r}}_{01}) \, \underline{\lambda}_{N+2} = 0, \tag{49}$$

$$\dot{\underline{\beta}}_{N} - \tilde{\underline{a}}_{N} \, \underline{F}_{N} + \tilde{\underline{r}}_{32} \, \underline{\lambda}_{N} = 0, \tag{50}$$

$$\dot{\underline{\beta}}_{N+1} - \tilde{\underline{a}}_{N+1} \, \underline{F}_{N+1} - (\tilde{\underline{r}}_{23} - \tilde{\underline{r}}_{13}) \, \underline{\lambda}_{N} + \tilde{\underline{r}}_{13} \, \underline{\lambda}_{N+1} - (\tilde{\underline{r}}_{43} - \tilde{\underline{r}}_{13}) \, \underline{\lambda}_{N+2} = 0, \tag{51}$$

$$\dot{\underline{\beta}}_{N+2} - \tilde{\underline{a}}_{N+2}\ \underline{F}_{N+2} + \tilde{\underline{r}}_{34}\ \underline{\lambda}_{N+2} = 0, \tag{52}$$

and

$$\dot{\underline{a}}_0 - \underline{F}_0 + \sum_{1}^{N+2} \underline{\lambda}_i = 0, \tag{53}$$

$$\dot{\underline{\beta}}_0 - \tilde{\underline{a}}_0\ \underline{F}_0 + \sum_{1}^{N-2} \tilde{\underline{d}}_i\ \underline{\lambda}_i + \tilde{\underline{d}}_{N-1} \left(\sum_{N-1}^{N+2} \underline{\lambda}_i \right) = 0. \tag{54}$$

Comparing Eqs. 45-54 with Eqs. 25-30, they are essentially identical except that there are extra Lagrange multipliers involved in Eqs. 45-54.

Center of Mass

The center of mass of the N-body system with tree configuration can be derived in the same manner as the one done in the previous section for the N-body system with cluster configuration. It is not difficult to show that

$$\sum_{0}^{n+2} \dot{\underline{a}}_i = M \ddot{\underline{R}} = \sum_{0}^{n+2} \underline{F}_i \tag{55}$$

where M and $\underline{R}$, as defined before, denote the total mass and the position vector for the mass center of the entire system. By introducing the vector $\underline{r}_i$ for the relative distance between the mass center of body i and the entire system and considering the constraints, Eqs. 43-44, the relation between the displacements of translation and rotation can be obtained for each body as

$$\underline{r}_0 = -\frac{1}{M}\left[\sum_{1}^{N-2} M_i\,(\underline{d}_i + \underline{\ell}_i) + \sum_{N-1}^{N+2} M_i\,(\underline{d}_{N-1} + \underline{h}_i)\right], \tag{56}$$

$$\underline{r}_i = -\frac{1}{M}\left[\sum_{1}^{N-2} M_i\,(\underline{d}_i + \underline{\ell}_i) + \sum_{N-1}^{N+2} M_i\,(\underline{d}_{N-1} + \underline{h}_i)\right] + \underline{d}_i + \underline{\ell}_i, \qquad i=1,\ldots N-2, \tag{57}$$

and

$$\underline{r}_i = -\frac{1}{M}\left[\sum_{1}^{N-2} M_i\,(\underline{d}_i + \underline{\ell}_i) + \sum_{N-1}^{N+2} M_i\,(\underline{d}_{N-1} + \underline{h}_i)\right] + \underline{d}_{N-1} + \underline{h}_i, \qquad i=N-1,\ldots,N+2 \tag{58}$$

Furthermore, taking the time derivative of the above equations, one gets

$$\dot{\underline{r}}_0 = -\frac{1}{M}\left[\sum_{1}^{N-2} M_i\,(\tilde{\underline{w}}_0\,\underline{d}_i + \tilde{\underline{w}}_i\,\underline{\ell}_i) + \sum_{N-1}^{N+2} M_i\,(\tilde{\underline{w}}_i\,\underline{d}_i + \dot{\underline{h}}_i)\right] \tag{59}$$

$$\dot{\underline{r}}_i = \dot{\underline{r}}_0 + \tilde{\underline{w}}_0\,\underline{d}_i + \tilde{\underline{w}}_i\,\underline{\ell}_i, \qquad i=1,\ldots,N-2 \tag{60}$$

and

$$\dot{\underline{r}}_i = \dot{\underline{r}}_0 + \tilde{\underline{w}}_0\,\underline{d}_{N-1} + \underline{h}_i, \qquad i=n-1,\ldots N+2 \tag{61}$$

where it is known that $\dot{\underline{h}}_i$ is a given function of angular displacements and velocities. As an example, $\dot{\underline{h}}_i$ is obtained as

$$\dot{\underline{h}}_{n+1} = \tilde{\underline{w}}_{N+2}\,\underline{\ell}_{34} - \tilde{\underline{w}}_{N+1}\,(\underline{\ell}_{43} - \underline{\ell}_{13}) - \tilde{\underline{w}}_{N+1}\,(\underline{\ell}_{31} - \underline{\ell}_{01})$$

Constraint Force Elimination

Note that, among Eqs. 47-54, there are N+2 Lagrange multipliers in

total which can be completely replaced by $\dot{\underline{q}}$'s using Eq. 47. Furthermore, those $\dot{\underline{\alpha}}$'s can be expressed in terms of $\ddot{\underline{r}}$'s, because it can be shown by using Eqs. 40 and 55, that

$$\dot{\underline{\alpha}}_i = \frac{M_i}{M} \sum_{0}^{N+2} \underline{F}_i + M_i \ddot{\underline{r}}_i, \qquad i=1,\ldots,N+2 \tag{62}$$

In addition, it is indicated in Eqs. 58-60 that $\ddot{\underline{r}}_i$ is a function of angular displacements and derivatives of angular displacements. Hence, both are ready to be calculated in accordance with Eqs. 47 and 62, as long as the values of $\underline{R}$, $\underline{\theta}_i$, and $\underline{\lambda}_i$ are valid. The position of mass center of the entire system $\underline{R}$, can be integrated from Eq. 55. As to the angular displacements and their derivatives, they also can be integrated based on the equations associated with primed angular momentums. Since the derivations of those equations are similar to the ones presented in the last section, only the primed angular momentum of the central body is derived here as an illustration.

Substituting Eq. 47 into Eq. 54 for the Lagrange multipliers, one could obtain

$$\dot{\underline{\beta}}_0 - \tilde{\underline{a}}_0 \underline{F}_0 - \sum_{1}^{N-2} \tilde{\underline{d}}_i \underline{F}_i - \tilde{\underline{d}}_{N+1} \left(\sum_{N-1}^{N+2} \underline{F}_i \right) + \sum_{1}^{N-2} \tilde{\underline{d}}_i \dot{\underline{\alpha}}_i + \tilde{\underline{d}}_{N-1} \left(\sum_{N-1}^{N+2} \dot{\underline{\alpha}}_i \right) = 0$$

Furthermore, the term $\dot{\underline{\alpha}}_i$ can be replaced by $\ddot{\underline{r}}_i$ with the application of Eq. 62,

$$\underline{\beta}_0 - \tilde{\underline{a}}_0\underline{F}_0 - \sum_{1}^{N-2} \tilde{\underline{d}}_i\underline{F}_i - \tilde{\underline{d}}_{N-1}\left(\sum_{N-1}^{N+2} \underline{F}_i\right) + \sum_{1}^{N-2} \tilde{\underline{d}}_i \left(\frac{M_i}{M} \sum_{0}^{N+2} \underline{F}_i\right)$$

$$+ \tilde{\underline{d}}_{N-1}\left[\sum_{N-1}^{N+2} \frac{M_i}{M}\left(\sum_{0}^{N+2} \underline{F}_i\right)\right] + \sum_{1}^{N-2} M_i \tilde{\underline{d}}_i \ddot{\underline{r}}_i + \sum_{N-1}^{N+2} M_i \tilde{\underline{d}}_{N-1} \ddot{\underline{r}}_i = 0$$

Using the equality, $\dot{\overline{uv}} = \dot{u}\dot{v} + u\ddot{v}$, the above equation can be rearranged to provide an equality for the primed angular momentum of the central body as:

$$\begin{aligned}
&\dot{\left(\underline{\beta}_0 + \sum_{1}^{N-2} M_i \tilde{\underline{d}}_i\dot{\underline{r}}_i + \sum_{N-1}^{N+2} M_i \tilde{\underline{d}}_{N-1}\dot{\underline{r}}_i\right)} \\
&= \tilde{\underline{a}}_0\underline{F}_0 + \sum_{1}^{N-2} \tilde{\underline{d}}_i\underline{F}_i + \tilde{\underline{d}}_{N-1}\left(\sum_{N-1}^{N+2} \underline{F}_i\right) \\
&\quad - \sum_{1}^{N-2} \tilde{\underline{d}}_i \left(\frac{M_i}{M} \sum_{0}^{N+2} \underline{F}_i\right) - \tilde{\underline{d}}_{N-1}\left[\sum_{N-1}^{N+2} \frac{M_i}{M}\left(\sum_{0}^{N+2} \underline{F}_i\right)\right] \\
&\quad + \sum_{1}^{N-2} M_i \left(\tilde{\underline{d}}_i\tilde{\underline{w}}_0 - \tilde{\underline{w}}_0\tilde{\underline{d}}_i\right)\dot{\underline{r}}_i \\
&\quad + \sum_{N-1}^{N+2} M_i \left(\tilde{\underline{d}}_{N-1}\tilde{\underline{w}}_0 - \tilde{\underline{w}}_0\tilde{\underline{d}}_{N-1}\right)\dot{\underline{r}}_i
\end{aligned} \tag{63}$$

where the left side of the equality denotes the total derivative of the primed angular momentum of the central body. Note that the preceding equation is a perfect form suitable for applying Vance's numerical scheme [5, 6]. The numerical implementation of such a scheme for the dynamic systems of concern is to be discussed in the next section.

4. NUMERICAL ALGORITHM AND EXAMPLES

The numerical implementation of Russell's formulation is discussed

hereafter. Numerical examples are also presented to verify the derived formulation's and numerical algorithm.

For simplicity, Eqs. 33, 38, 41-42 and Eq. 63 can be represented symbolically by

$$\dot{\underline{h}}\ (\underline{\theta},\ \underline{p},\ \underline{w}) = \underline{G}\ (\underline{\theta},\ \underline{p},\ \underline{w}) \tag{64}$$

where angular velocity $\underline{w} = \dot{\underline{\theta}}$ and $\underline{p}$ denotes position vector such as $\underline{\ell}$, $\underline{d}$ or $\underline{a}$ defined with respect to the inertia frame. It is evident that $\underline{p}$ depends on $\underline{\theta}$. Furthermore, the primed angular momentum $\underline{h}$ is a linear function of $\underline{w}$, and $\underline{G}$, on the other hand, is a nonlinear function of $\underline{w}$. To implement Eq. 64 numerically, the major step is to approximate the differential operator by a difference operator [6]. In the numerical examples discussed below, the trapezoidal rule is used, i.e.,

$$\underline{h}_{n+1} = \frac{\Delta t}{2}\left[G\ (\underline{\theta}_n,\ \underline{p}_n,\ \underline{w}_n) + G\ (\underline{\theta}_{n+1},\ \underline{p}_{n+1},\ \underline{w}_{n+1})\right] \tag{65}$$

where the subscript denotes the time grid point. Since Eq. 65 represents a set of nonlinear algebraic equations for $\underline{w}_{n+1}$, an iterative scheme is required to find $\underline{w}_{n+1}$. The detailed numerical algorithm is described in Fig. 5.

There are two examples, a double pendulum and a triple pendulum. The results obtained according to the Russell's formulation are compared with those calculated by directly employing Lagrange's equations of motion without introducing Lagrange multipliers. The latter can be done when the minimum set of generalized coordinates are selected to describe the system. For

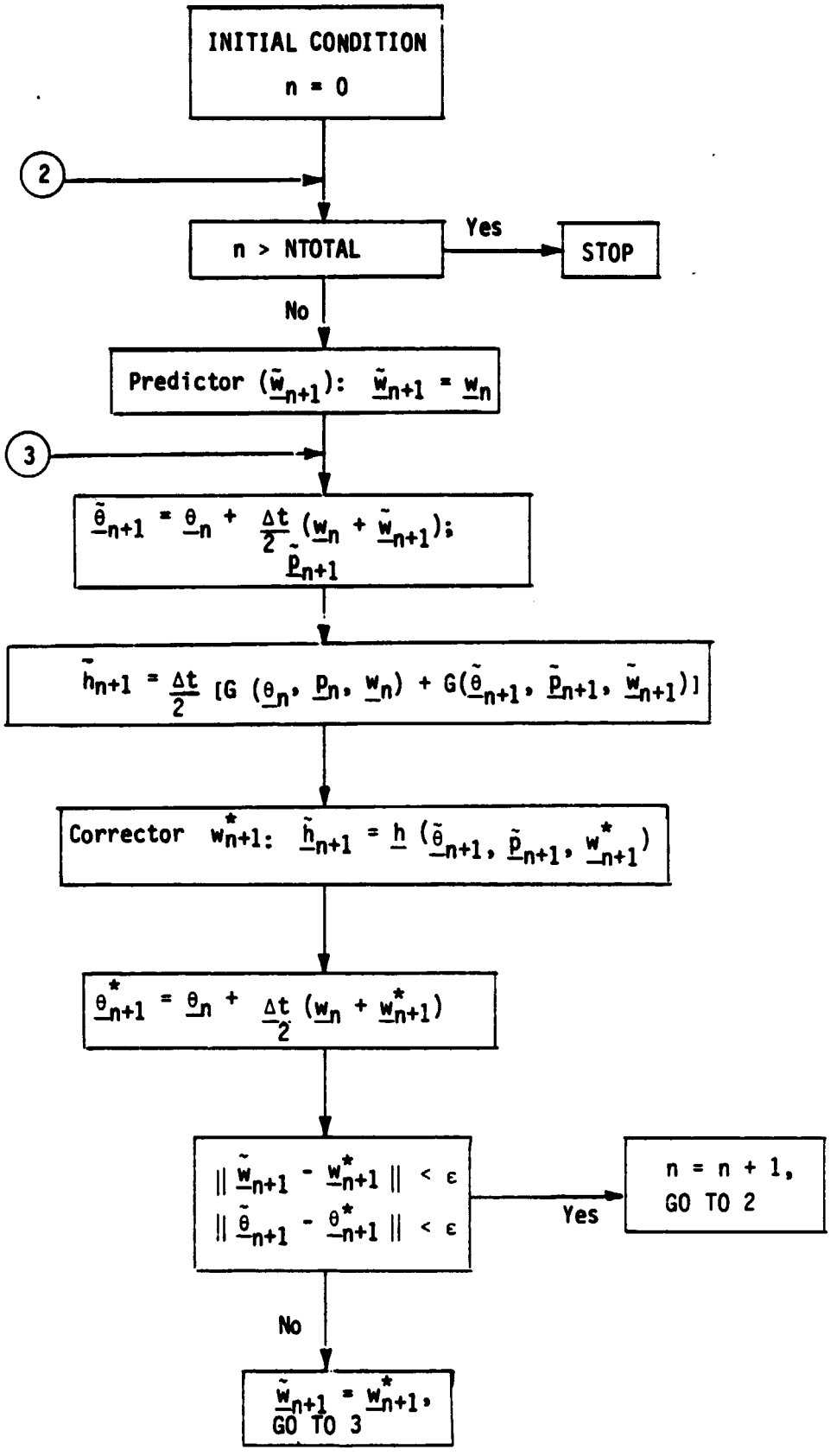


Figure 5. Flowchart for the Numerical Algorithm.

the examples studied, the angular displacement of each pendulum is defined as a generalized coordinate. The Lagrange's equations of motion constitute a set of simultaneous second-order equations which are generally nonlinear:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right) = \frac{\partial T}{\partial \underline{q}} + \underline{Q}\,(\underline{q}, \underline{\dot{q}}, t)$$

Note that the term $\frac{\partial T}{\partial \underline{\dot{q}}}$ is subjected to the total differentiation. Thus, the Lagrange's equations of motion can be replaced by a set of finite difference equations which are used to solve $\underline{\dot{q}}$, following the same numerical implementation as used in the Russell's formulation. The detailed formulation of Lagrange's equations of motion, as well as the Russell's formulation, for a double and a triple pendulum are give in Appendix A.

The mass, length and the moment of inertia at the mass center of each pendulum are assigned as 1, 1 and 0.08333 units, respectively. A two unit force is applied at the free end of the systems and is always normal to the pendulum. The initial configuration of the pendulum is straight in the vertical position as shown in Figures 6-7. The numerical results calculated with $\Delta t = 0.05$ seconds are listed in Tables 1 and 2 for the double and triple pendulum, respectively. The history of motion is depicted in Figures 6-7 as well. The algorithm performs well. In general, it takes two to four iterations for $\underline{w}$ and $\underline{\theta}$ to get convergence ($\varepsilon = 10^{-5}$) at each time grid point. For the examples studied, it is indicated that the Russell's formulation and the Lagrange's equations of motion provide essentially the same results. Nevertheless, the Russell's formulation requires more CPU time because the introduction of joint reactions (Lagrange multipliers) and the

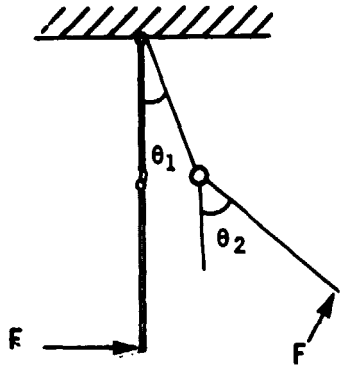


Figure 6a. Initial Configuration

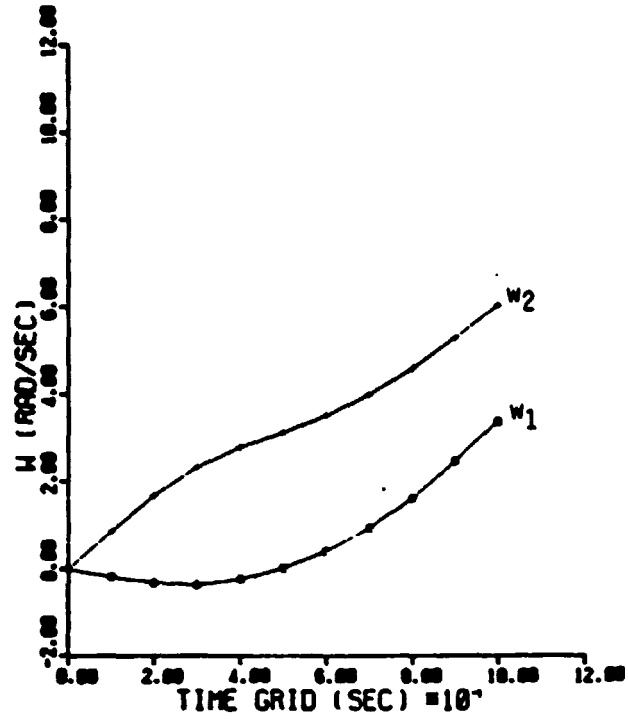


Figure 6b. Changes of Angular Velocities

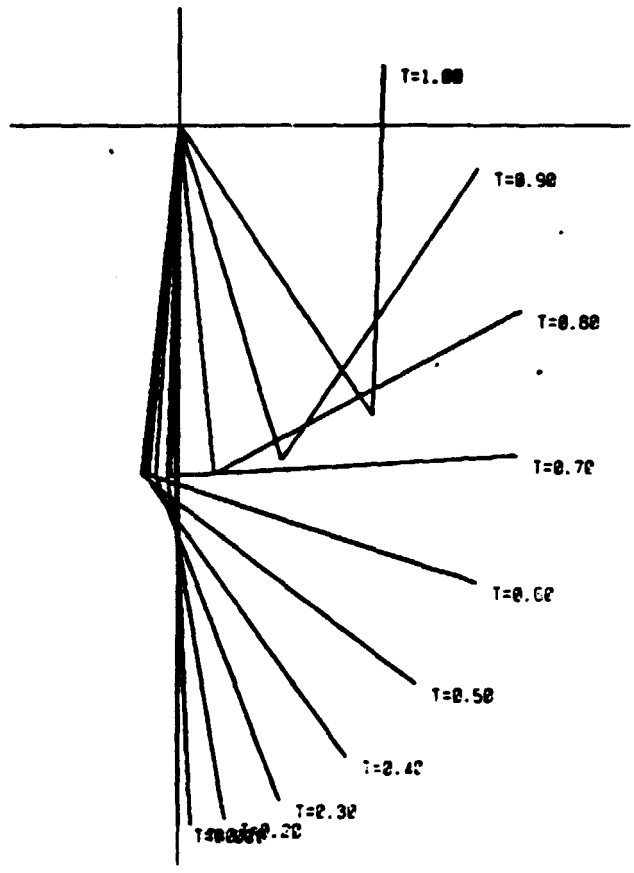


Figure 6c. Changes of Positions

Figure 6. The Motion of a Double Pendulum.

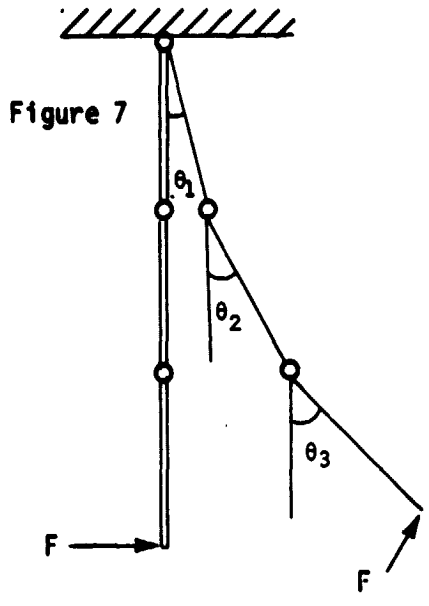


Figure 7a. Initial Configuration

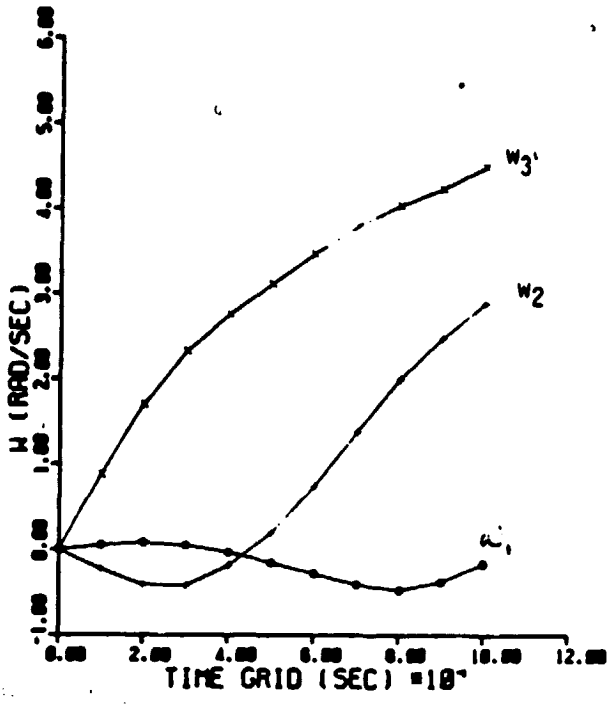


Figure 7b. Changes of Angular Velocities

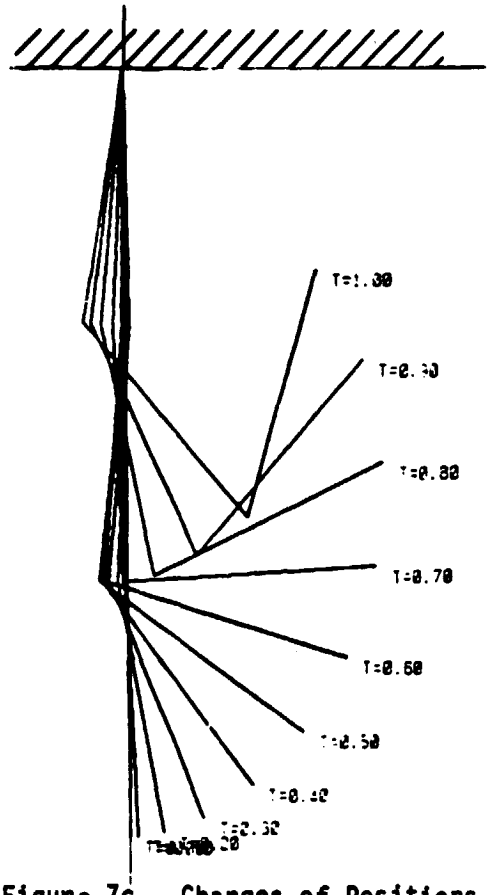


Figure 7c. Changes of Positions

Figure 7. The Motion of a Triple Pendulum.

Table 1. Numerical results for a double pendulum

a. Results based on the Lagrange's equations of motion

(CPU time - 1.17 sec. on DEC 10)

Time (Sec.)	θ_1 (Rad.)	θ_2 (Rad.)	w_1 (Rad/Sec.)	w_2 (Rad/Sec.)
0.00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.10	-0.85545E-02	0.42832E-01	-0.17076E+00	0.85613E+00
0.20	-0.33358E-01	0.16995E+00	-0.31859E+00	0.16755E+01
0.30	-0.68293E-01	0.37174E+00	-0.36078E+00	0.23314E+01
0.40	-0.99210E-01	0.62794E+00	-0.23739E+00	0.27714E+01
0.50	-0.11053E+00	0.92259E+00	0.26391E-01	0.31194E+01
0.60	-0.89484E-01	0.12531E+01	0.40967E+00	0.34990E+01
0.70	-0.23541E-01	0.16261E+01	0.92792E+00	0.39764E+01
0.80	0.10224E+00	0.20531E+01	0.16093E+01	0.45796E+01
0.90	0.30438E+00	0.25461E+01	0.24503E+01	0.52892E+01
1.00	0.59483E+00	0.31115E+01	0.33567E+01	0.60147E+01

b. Results based on the Russell's formulation

(CPU) time = 3.19 sec. on DEC 10)

Time (Sec.)	θ_1 (Rad.)	θ_2 (Rad.)	w_1 (Rad/Sec.)	w_2 (Rad/Sec.)
0.00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.10	-0.85545E-02	0.42832E-01	-0.17076E+00	0.85613E+00
0.20	-0.33358E-01	0.16995E+00	-0.31859E+00	0.16755E+01
0.30	-0.68293E-01	0.37174E+00	-0.36078E+00	0.23314E+01
0.40	-0.99210E-01	0.62794E+00	-0.23739E+00	0.27714E+01
0.50	-0.11053E+00	0.92259E+00	0.26391E-01	0.31194E+01
0.60	-0.89484E-01	0.12531E+01	0.40967E+00	0.34990E+01
0.70	-0.23541E-01	0.16261E+01	0.92792E+00	0.39764E+01
0.80	0.10224E+00	0.20531E+01	0.16093E+01	0.45796E+01
0.90	0.30438E+00	0.25461E+01	0.24503E+01	0.52892E+01
1.00	0.59483E+00	0.31115E+01	0.33567E+01	0.60147E+01

Table 2. Numerical results for triple pendulum

a. Results based on the Lagrange's equations of motion

(CPU time = 1.58 sec. on DEC 10)

Time (Sec.)	θ_1 (Rad.)	θ_2 (Rad.)	θ_3 (Rad.)	w_1 (Rad/Sec.)	w_2 (Rad/Sec.)	w_3 (Rad/Sec.)
0.00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.10	0.22985E-02	-0.11507E-01	0.43812E-01	0.45780E-01	-0.22951E+00	0.87553E+00
0.20	0.86714E-02	-0.44351E-01	0.17337E+00	0.77629E-01	-0.41446E+00	0.17013E+01
0.30	0.15699E-01	-0.87451E-01	0.37659E+00	0.53883E-01	-0.41694E+00	0.23318E+01
0.40	0.17095E-01	-0.11901E+00	0.63199E+00	-0.31714E-01	-0.18951E+00	0.27591E+01
0.50	0.82107E-02	-0.11915E+00	0.92590E+00	-0.14841E+00	0.20472E+00	0.31171E+01
0.60	-0.13139E-01	-0.72917E-01	0.12554E+01	-0.27984E+00	0.73654E+00	0.34728E+01
0.70	-0.47662E-01	0.32297E-01	0.16195E+01	-0.40754E+00	0.13763E+01	0.38010E+01
0.80	-0.92040E-01	0.20179E+00	0.20122E+00	-0.46504E+00	0.20003E+01	0.40404E+01
0.90	-0.13514E+00	0.42684E+00	0.24258E+01	-0.37794E+00	0.24829E+01	0.42337E+01
1.00	-0.16281E+00	0.69541E+00	0.28614E+01	-0.16233E+00	0.28871E+01	0.44908E+01

b. Results based on the Russell's formulation

(CPU time = 1.66 sec. on DEC 10)

Time (Sec.)	θ_1 (Rad.)	θ_2 (Rad.)	θ_3 (Rad.)	w_1 (Rad/Sec.)	w_2 (Rad/Sec.)	w_3 (Rad/Sec.)
0.00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
0.10	0.22985E-02	-0.11507E-01	0.43812E-01	0.45780E-01	-0.22951E+00	0.87553E+00
0.20	0.86714E-02	-0.44351E-01	0.17337E+00	0.77629E-01	-0.41446E+00	0.17013E+01
0.30	0.15699E-01	-0.87451E-01	0.37659E+00	0.53883E-01	-0.41694E+00	0.23318E+01
0.40	0.17095E-01	-0.11901E+00	0.63199E+00	-0.31714E-01	-0.18951E+00	0.27591E+01
0.50	0.82107E-02	-0.11915E+00	0.92590E+00	-0.14841E+00	0.20472E+00	0.31171E+01
0.60	-0.13139E-01	-0.72917E-01	0.12554E+01	-0.27984E+00	0.73654E+00	0.34728E+01
0.70	-0.47662E-01	0.32297E-01	0.16195E+01	-0.40754E+00	0.13763E+01	0.38010E+01
0.80	-0.92040E-01	0.20179E+00	0.20122E+00	-0.46504E+00	0.20003E+01	0.40404E+01
0.90	-0.13514E+00	0.42684E+00	0.24258E+01	-0.37794E+00	0.24829E+01	0.42337E+01
1.00	-0.16281E+00	0.69541E+00	0.28614E+01	-0.16233E+00	0.28871E+01	0.44908E+01

position vector of mass center of the entire system complicates the computation.

5. DISCUSSION AND CONCLUSION

The Vance's scheme has been successfully applied to analyze the dynamics of mechanical systems. The same scheme is also extended to perform the design sensitivity analysis and the optimum design. The proposed algorithm is simple and easily programmed.

There is no need for this algorithm to evaluate the higher order derivatives of equations of constraints. Moreover, if the acceleration of the dynamics system is of no concern, it is also not necessary to find the time derivative of the mass matrix. Thus, the proposed algorithm relieves, to some extent, the complexity of formulation and computation of mechanical system dynamics.

It is noted that the time increment, Δt, in this algorithm plays an important role, not only for the numerical error and stability but also for the convergence rate of the iterative scheme, because the extrapolation of the current state variable with a smaller Δt provides a better estimate of the new state variable for starting the iterative computation at the new time grid.

The example of the slider-crank mechanism discussed in Appendix B shows the success of the application of the proposed algorithm for solving a small-scale problem. In order to analyze the large-scale problem, the proposed algorithm can be upgraded by replacing the unconstrained minimization scheme introduced in Eqs. B.9-11 by a more efficient scheme designed for the large nonlinear problem. As an example, the Fletcher-Reeves's conjugate gradient algorithm could be a promising substitute.

In order to fully develop Russell's momentum approach, the following tasks are also proposed:

1. Extend the derivation to include flexibility, and to extend to systems with 3-D configuration.
2. Extend the derivation to include complicated joint conditions, such as joint friction.
3. Investigate the application of the sparse matrix technique to improve the computational efficiency of solving angular velocities.

In summary, while a simple strategy is proposed and successfully implemented to analyze and optimally design a small-scaled mechanical system with differential/algebraic equations, further study is required to enhance the algorithm's rigorousness and versatility.

REFERENCES

1. Russell, W. J., "On the Formulation of the Equation of Rotational Motion for an N Body Spacecraft," Aerospace Corp. Report, TOR-0200(4133)-2, February 1959.

2. Russell, W. J., "On the Iterative Solution of the Simultaneous Linear Equations in the Rotational Dynamics of Multibody Spacecraft," Aerospace Corp. Report, TOR-0172(2133)-2, July 1971.

3. Pars, L. A., A Treatise on Analytical Dynamics, Heinemann Educational Books Ltd., London, 1965.

4. Fiacco, A. V. and McCormick, G. P., Nonlinear Programming: Sequential Unconstrained Minimization Technique, John Wiley and Sons, New York, 1968.

5. Vance, J. M., "Numerical Solution of Dynamical Systems by Direct Application of Hamilton's Principle," International Journal for Numerical Methods in Engineering, Vol. 4, 1972, pp. 207-216.

6. Vance, J. M. and Sitchin, A., "Derivation of First-Order Difference Equations for Dynamical Systems by Direct Application of Hamilton's Principle," Journal of Applied Mechanics, Vol. 37, 1970, pp. 276-278.

7. Baumgarte, J., "Stabilization of Constraints and Integrals of Motions in Dynamic Systems," Computational Methods in Applied Mechanics and Engineering, Vol. 1, 1972, pp. 1-6.

8. Baumgarte, J., "A New Method of Stabilization for Holonomic Constraints," Journal of Applied Mechanics, Vol. 50, 1983 pp. 869-870.

9. Wehage, R. A. and Haug, E. J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamic Systems," Journal of Mechanical Design, Vol. 104, no. 1, 1982, pp. 247-255.

APPENDIX A

EQUATIONS OF MOTION FOR DOUBLE AND TRIPLE PENDULUM

A.1. Lagrange's Equations of Motion

Double Pendulum

Definitions of notations are given in Figure A.1.

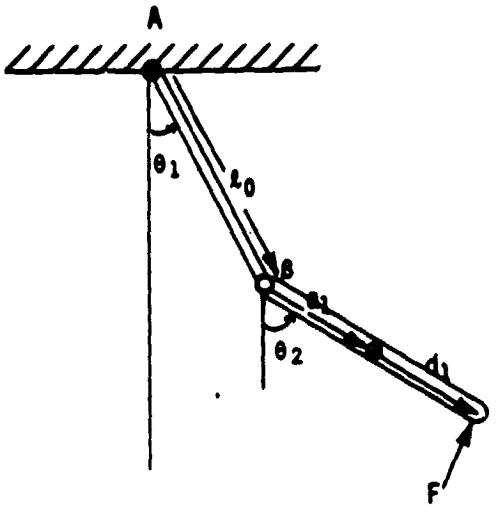


Figure A.1. A Double Pendulum.

The equations of motion for a double pendulum are

$$\dot{J_0 \underline{w}_0} - m_1 \underline{\tilde{r}}_0 \underline{\tilde{r}}_0 \underline{w}_0 - m_1 \underline{\tilde{r}}_0 \underline{\tilde{a}}_1 \underline{w}_1 = m_1 \underline{\tilde{r}}_0 \underline{\tilde{w}}_0 \underline{\tilde{a}}_1 \underline{w}_1 + \underline{\tilde{r}}_0 \underline{F}$$

and

$$\dot{J_1 \underline{w}_1} - m_1 \underline{w}_0^T \underline{\tilde{r}}_0 \underline{\tilde{a}}_1 = m_1 \underline{\tilde{a}}_1 \underline{\tilde{w}}_1 \underline{\tilde{r}}_0 \underline{w}_0 + \underline{\tilde{d}}_1 \underline{F}$$

where J_0 and J_1 are the moments of inertia of pendulums 0 and 1 with respect to points A and B, respectively.

Triple Pendulum

Definitions of notations are given in Figure A.2.

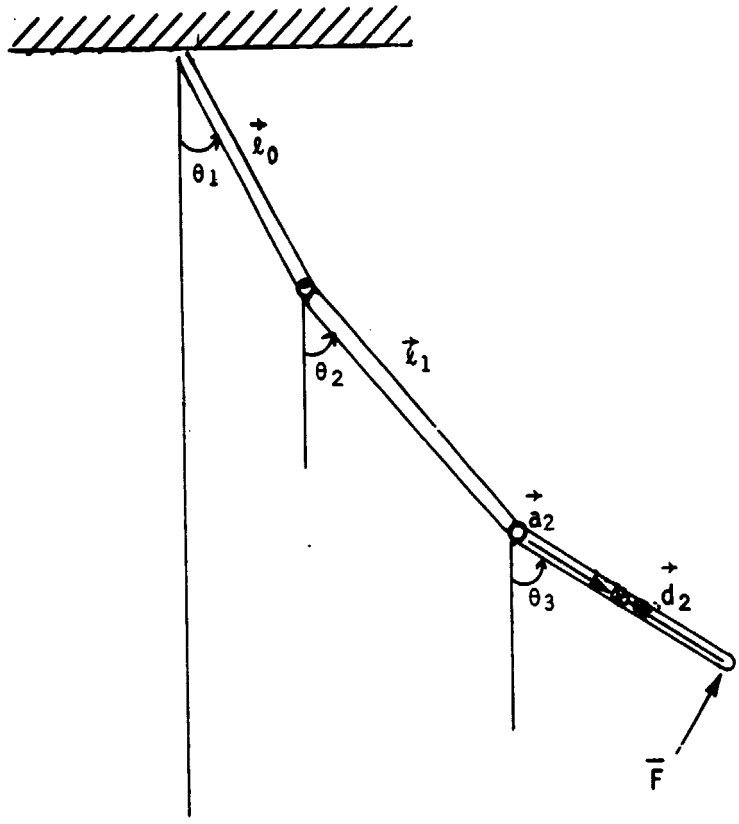


Figure A.2. A Triple Pendulum.

The equations of motion for a triple pendulum are

$$\dot{J_0 \underline{w}_0} - (m_1+m_2)\,\tilde{\underline{\ell}}_0\tilde{\underline{\ell}}_0\underline{w}_0 - m_1\tilde{\underline{\ell}}_0\tilde{\underline{a}}_1\underline{w}_1 - m_2\tilde{\underline{\ell}}_0\tilde{\underline{a}}_2\underline{w}_2 - m_2\tilde{\underline{\ell}}_0\tilde{\underline{\ell}}_1\underline{w}_1$$
$$= \tilde{\underline{\ell}}_0\tilde{\underline{w}}_0\,(m_1\tilde{\underline{a}}_1\underline{w}_1 + m_2\tilde{\underline{\ell}}_1\underline{w}_1 + m_2\tilde{\underline{a}}_2\underline{w}_2) + \tilde{\underline{\ell}}_0\underline{F},$$

$$\dot{J_1 \underline{w}_1} - (m_1\tilde{\underline{a}}_1\tilde{\underline{\ell}}_0 + m_2\tilde{\underline{\ell}}_1\tilde{\underline{\ell}}_0)\,\underline{w}_0 - m_2\tilde{\underline{\ell}}_1\tilde{\underline{\ell}}_1\underline{w}_1 - m_2\tilde{\underline{\ell}}_1\tilde{\underline{a}}_2\underline{w}_2$$
$$= m_1\tilde{\underline{a}}_1\tilde{\underline{w}}_1\tilde{\underline{\ell}}_0\underline{w}_0 + m_2\tilde{\underline{\ell}}_1\tilde{\underline{w}}_1\,(\tilde{\underline{\ell}}_0\underline{w}_0 + \tilde{\underline{a}}_2\underline{w}_2) + \tilde{\underline{\ell}}_1\underline{F},$$

and

$$\dot{J_2 \underline{w}_2} - m_2\tilde{\underline{a}}_2\tilde{\underline{\ell}}_0\underline{w}_0 - m_2\tilde{\underline{a}}_2\tilde{\underline{\ell}}_1\underline{w}_1$$
$$= m_2\tilde{\underline{a}}_2\tilde{\underline{w}}_2\,(\tilde{\underline{\ell}}_0\underline{w}_0 + \tilde{\underline{\ell}}_1\underline{w}_1) + \tilde{\underline{\ell}}_2\,\underline{F}$$

where J_0, J_1 and J_2 are the moments of inertia of pendulums defined with respect to the points A, B and C, respectively.

A.2. Russell's Equations of Motion

A.2.1. Double Pendulum

Definitions of notations are given in Figure A.3.

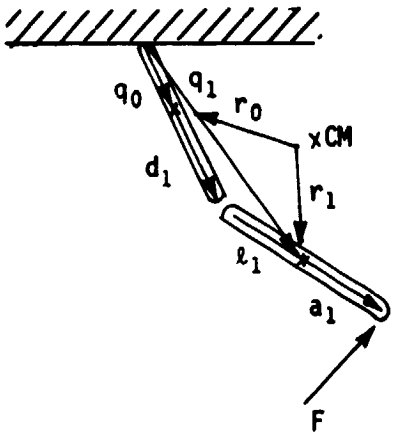


Figure A.3. A Double Pendulum.

Russell's equations of motion for a double pendulum are

$$\frac{\cdot}{J_0\underline{w}_0 + \tilde{\underline{d}}_1 (m_1\dot{\underline{r}}_1)} = \tilde{\underline{a}}_0\underline{F}_0 + m_1 (\tilde{\underline{w}}_0\tilde{\underline{d}}_1 - \tilde{\underline{d}}_1\tilde{\underline{w}}_0)\,\dot{\underline{r}}_1 - \tilde{\underline{d}}_1\left[\frac{m_1}{m_1+m_2}\,\underline{F}-\underline{F}\right],$$

and

$$\frac{\cdot}{J_1\underline{w}_1 + \tilde{\underline{\ell}}_1 (m_1\dot{\underline{r}}_1)} = (\tilde{\underline{a}}_0+\tilde{\underline{\ell}}_1)\,\underline{F} - \frac{m_1}{m_1+m_2}\,\tilde{\underline{\ell}}_0\,\underline{F} + m_1\dot{\tilde{\underline{\ell}}}_1\dot{\underline{r}}_1.$$

where

$$\dot{\underline{r}}_1 = -\frac{m_1}{m_1+m_2}\,(\tilde{\underline{w}}_0\underline{d}_1 + \tilde{\underline{w}}_1\underline{\ell}_1) + \tilde{\underline{w}}_0\underline{d}_1 + \tilde{\underline{w}}_1\underline{\ell}_1,$$

and

$$\dot{\tilde{\underline{\ell}}}_1 = \dot{\tilde{\underline{\ell}}} = \widetilde{\underline{w}_1\underline{\ell}_1} = \tilde{\underline{w}}_1\tilde{\underline{\ell}}_1 - \tilde{\underline{\ell}}_1\tilde{\underline{w}}_1$$

Triple Pendulum

Definitions of notations are given in Figure A.4.

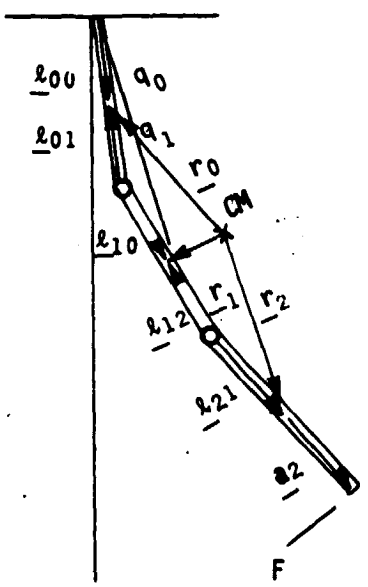


Figure A.4. A Triple Pendulum.

Russell's equations of motion for a triple pendulum are

$$\dot{\left(J_2\underline{w}_2 + \tilde{\underline{\ell}}_{21}\underline{a}_2\right)} = \tilde{\underline{a}}_2\underline{F} + \tilde{\underline{\ell}}_{21}\underline{F} + \dot{\tilde{\underline{\ell}}}_{21}\underline{a}_2,$$

$$\dot{\left(J_1\underline{w}_1 + \tilde{\underline{\ell}}_{10}\underline{a}_1 - (\tilde{\underline{\ell}}_{12}-\tilde{\underline{\ell}}_{10})\underline{a}_2\right)} = (\tilde{\underline{\ell}}_{10}-\tilde{\underline{\ell}}_{12})\underline{F} + \dot{\tilde{\underline{\ell}}}_{10}\underline{a}_1 - (\dot{\tilde{\underline{\ell}}}_{12}-\dot{\tilde{\underline{\ell}}}_{10})\underline{a}_2,$$

and

$$\dot{\left(J_0\underline{w}_0 - \tilde{\underline{\ell}}_{00}\underline{a}_0 - (\tilde{\underline{\ell}}_{00}+\tilde{\underline{\ell}}_{01})\underline{a}_1 - (\tilde{\underline{\ell}}_{00}+\tilde{\underline{\ell}}_{01})\underline{a}_2\right)} = -\dot{\tilde{\underline{\ell}}}_{00}\underline{a}_0 - (\dot{\tilde{\underline{\ell}}}_{00}+\dot{\tilde{\underline{\ell}}}_{01})\underline{a}_1 - (\dot{\tilde{\underline{\ell}}}_{00}+\dot{\tilde{\underline{\ell}}}_{01})\underline{a}_2 - (\tilde{\underline{\ell}}_{00}+\tilde{\underline{\ell}}_{01})\underline{F},$$

where

$$\underline{a}_0 = m_0\dot{\underline{g}}_0 = -m_0\tilde{\underline{w}}_0\underline{\ell}_{00},$$

$$\underline{a}_1 = -m_1\dot{\underline{\ell}}_{00} + \left(\frac{m_1^2}{m_0} + m_1\right)\dot{\underline{r}}_1 + \frac{m_1m_2}{m_0}\dot{\underline{r}}_2,$$

$$\underline{a}_2 = -m_2\dot{\underline{\ell}}_{00} + \frac{m_1m_2}{m_0}\dot{\underline{r}}_1 + \left(\frac{m_2^2}{m_0} + m_2\right)\dot{\underline{r}}_2;$$

and

$$\dot{\tilde{\underline{\ell}}}_{00} = \tilde{\underline{w}}_0\underline{\ell}_{00} = \tilde{\underline{w}}_0\tilde{\underline{\ell}}_{00} - \tilde{\underline{\ell}}_{00}\tilde{\underline{w}}_0,$$

$$\dot{\tilde{\underline{\ell}}}_{01} = \tilde{\underline{w}}_0\underline{\ell}_{01} = \tilde{\underline{w}}_0\tilde{\underline{\ell}}_{01} - \tilde{\underline{\ell}}_{01}\tilde{\underline{w}}_0,$$

$$\dot{\tilde{\underline{\ell}}}_{10} = \tilde{\underline{w}}_1\underline{\ell}_{10} = \tilde{\underline{w}}_1\tilde{\underline{\ell}}_{10} - \tilde{\underline{\ell}}_{10}\tilde{\underline{w}}_1,$$

$$\dot{\tilde{\underline{\ell}}}_{12} = \tilde{\underline{w}}_1\underline{\ell}_{12} = \tilde{\underline{w}}_1\tilde{\underline{\ell}}_{12} - \tilde{\underline{\ell}}_{12}\tilde{\underline{w}}_1,$$

$$\dot{\tilde{\underline{\ell}}}_{21} = \tilde{\underline{w}}_2\underline{\ell}_{21} = \tilde{\underline{w}}_2\tilde{\underline{\ell}}_{21} - \tilde{\underline{\ell}}_{21}\tilde{\underline{w}}_2;$$

$$\dot{\underline{r}}_0 = -\frac{1}{m_1+m_2+m_3}\left[m_1\left(\tilde{\underline{w}}_1\underline{\ell}_{10}-\tilde{\underline{w}}_0\underline{\ell}_{01}\right)+m_2\left(\tilde{\underline{w}}_2\underline{\ell}_{21}-\tilde{\underline{w}}_1\underline{\ell}_{12}+\tilde{\underline{w}}_1\underline{\ell}_{10}-\tilde{\underline{w}}_0\underline{\ell}_{01}\right)\right].$$

$$\dot{\underline{r}}_1 = \tilde{\underline{w}}_1\,\underline{\ell}_{10} - \tilde{\underline{w}}_0\underline{\ell}_{01} + \dot{\underline{r}}_0.$$

$$\dot{\underline{r}}_2 = \tilde{\underline{w}}_2\underline{\ell}_{21} - \tilde{\underline{w}}_1\underline{\ell}_{12} + \tilde{\underline{w}}_1\underline{\ell}_{10} - \tilde{\underline{w}}_0\underline{\ell}_{01} + \dot{\underline{r}}_0.$$

APPENDIX B

DYNAMIC ANALYSIS AND OPTIMIZATION OF CONSTRAINED MECHANICAL SYSTEMS*

*Submitted to the International Journal for Numerical Methods in Engineering for the consideration of publication.

B.I. INTRODUCTION

For a complex mechanical system, it has been indicated in the literature [1, 2] that the equations of motion may be presented by using Cartesian coordinates and lagrangian multipliers, resulting in a system of mixed differential and algebraic equations. This approach greatly increases the formulation flexibility, because it does not rely on the engineer's intuition to determine a set of independent variables.

Several numerical methods can be found in the literature [1, 2] to solve these differential-algebraic equations. One method is to differentiate the equations of constraints and append them to the equations of motion. This expanded system of equations is solved for the lagrangian multipliers and accelerations. The integrations of accelerations provide good predictions for the velocities which must be subsequently corrected based on the equations of constraints. The velocities and displacements can be treated in a similar way. A large number of state variables is usually encountered in this approach which may become a prohibitive problem to be solved. To overcome this difficulty, a coordinate partition scheme [3] is implemented at each time step to sort out the independent and dependent coordinates based on the equations of constraints. Only the independent coordinates are to be integrated. Recently, the singular value decomposition scheme has been introduced [4, 5] to select a set of composite coordinates as independent coordinates which constitute a hyperplane tangent to the equations of constraints. The numerical study shows that the singular value decomposition method is quite promising.

An easily programmed algorithm is presented in this paper. The algorithm consists of two parts: (a) a scheme introduced by Vance [6, 7] and (b) an

unconstrained minimization. Vance's scheme has been limited to solve unconstrained equations of motion. However, in this study, Vance's scheme, along with an unconstrained minimization algorithm is used to analyze the dynamics of a constrained mechanical system. An unconstrained minimization is implemented to correct the state variables by minimizing the constraint violations. The proposed scheme is further extended to find the design sensitivity of constrained mechanical systems by the adjoint variable technique [8] and to carry out the problems of optimization as well.

B.II. DYNAMICS ANALYSIS

A mechanical system is defined as a system that consists of bodies with inertias and elements without inertias such as control force, damper, etc. To define a mechanical system, one may assign a body-fixed coordinate for each body and introduce the equations of contraints to describe the kinematic relations between the bodies. Each body has either three degrees of freedom corresponding to a two dimensional configuration or six degrees of freedom corresponding to a three dimensional configuration. In this way, the kinetic energy and the external work of each individual body can be easily established. Based on Hamilton's principle and the theorem of lagrangian multipliers, the equations of motion of a whole system can then be derived as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \underline{\dot{q}}}\right) - \frac{\partial T}{\partial \underline{q}} - \left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^T \underline{\lambda} = \underline{Q}\ (\underline{q},\ \underline{\dot{q}},\ t) \tag{B.1}$$

and a set of constraints as

$$\underline{\Phi}\ (\underline{q},\ t) = 0\ . \tag{B.2}$$

The total kinetic energy T, a quadratic form of velocities, is the sum of the kinetic energy of each body. The terms $\underline{q}$, $\underline{\dot{q}}$ and $\underline{Q}$ denote the generalized

coordinates, velocities and forces, respectively. In this study, the constraint vector, $\underline{\Phi}$, is limited to a set of holonomic constraints. Note that the generalized coordinates $\underline{q}$ and largrangian multipliers $\underline{\lambda}$ are the unknowns in the above equations of motion. The numerical implementation of this system of differential/algebraic equations is discussed next.

For simplicity, Eqs. B.1 and B.2, can be represented symbolically by

$$\frac{\cdot}{M\underline{\dot{q}}} - \left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^T \underline{\lambda} = \underline{G}\ (\underline{q},\ \underline{\dot{q}},\ t) \tag{B.3}$$

where the momentum term $\partial T/\partial \underline{\dot{q}}$ is equal to $M\underline{\dot{q}}$. At this stage, one may introduce Vance's scheme to approximate the differential operation by a finite difference operator. As an example, if the trapezoidal method is employed, Eq. B.3 can be replaced by a set of finite difference equations defined at time $(n+1)\,\Delta t$ and $n\Delta t$:

$$M(\underline{q})\ \underline{\dot{q}}\ |_{n+1} = M(\underline{q})\ \underline{\dot{q}}\ |_n + \frac{\Delta t}{2}\left\{\left[\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^T \underline{\lambda} + \underline{G}\right] |_n + \left[\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^T \underline{\lambda} + \underline{G}\right] |_{n+1}\right\}. \tag{B.4}$$

The preceding formula is usually a nonlinear equation with roots, $\underline{q}_{n+1}$ and $\underline{\dot{q}}_{n+1}$. To find them, the simple linear iteration is sufficient and convenient. Let $\underline{q}_{n+1}^{(0)}$ and $\underline{\dot{q}}_{n+1}^{(0)}$ be good initial estimates of solutions $\underline{q}_{n+1}$ and $\underline{\dot{q}}_{n+1}$. After rearrangement, Eq. B.4 may be rewritten as the following recursive form for jth iteration:

$$M(\underline{q}_{n+1}^{(j)})\ \underline{\dot{q}}_{n+1}^{(j+1)} - \left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)_{n+1}^{(j)\,T} \cdot \left(\frac{\Delta t}{2}\,\underline{\lambda}_{n+1}\right)$$

$$= M\ (\underline{q}_n)\ \underline{\dot{q}}_n + \frac{\Delta t}{2}\left[\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)_n^T \underline{\lambda}_n + \underline{G}_n\right] + \frac{\Delta t}{2} \cdot \underline{G}_{n+1}^{(j)} \tag{B.5}$$

Although the above equations become linear equations with $\underline{\dot{q}}_{n+1}^{(j+1)}$ and $\frac{\Delta t}{2}\,\underline{\lambda}_{n+1}$, they are unable to solve both $\underline{\dot{q}}_{n+1}^{(j+1)}$ and $\frac{\Delta t}{2}\,\underline{\lambda}_{n+1}$, because the

number of unknowns is larger than the number of equations. Nevertheless, with the help of equations of constraints, one obtains the following identity by differentiating Eq. B.2,

$$-\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)\dot{\underline{q}} = \frac{\partial \underline{\Phi}}{\partial t}$$

It may also be written in a recursive form defined at the time equal to t_{n+1},

$$\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^{(j)}_{n+1} \dot{\underline{q}}^{(j+1)}_{n+1} = -\left(\frac{\partial \underline{\Phi}}{\partial t}\right)^{(j)}_{n+1} \tag{B.6}$$

The last equation along with Eq. B.5 provides a matrix form to solve $\dot{\underline{q}}^{(j+1)}_{n+1}$ and $\frac{\Delta t}{2}\underline{\lambda}^{(j+1)}_{n+1}$ simultaneously:

$$\begin{bmatrix} M^{(j)}_{n+1} & -\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^{(j)^T}_{n+1} \\ -\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^{(j)}_{n+1} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\underline{q}}^{(j+1)}_{n+1} \\ \frac{\Delta t}{2}\underline{\lambda}^{(j+1)}_{n+1} \end{Bmatrix}$$

$$= \begin{Bmatrix} (M\dot{\underline{q}})_n + \frac{\Delta t}{2}\left[\left(\frac{\partial \underline{\Phi}}{\partial \underline{q}}\right)^T \underline{\lambda} + \underline{G}\right]_n + \frac{\Delta t}{2}\underline{G}^{(j)}_{n+1} \\ \left(\frac{\partial \underline{\Phi}}{\partial t}\right)^{(j)}_{n+1} \end{Bmatrix} \tag{B.7}$$

The leading coefficient matrix can be proved to be positive definite provided that the rows of $\frac{\partial \underline{\Phi}}{\partial \underline{q}}$ are linearly independent [3]. Thus, the existence and uniqueness of $\dot{\underline{q}}^{(j+1)}_{n+1}$ and $\frac{\Delta t}{2}\underline{\lambda}^{(j+1)}_{n+1}$ are ensured. The new value of $\underline{q}^{(j+1)}_{n+1}$ can then be obtained by numerically integrating $\dot{\underline{q}}^{(j+1)}_{n+1}$, for instance, by using the trapezoidal method,

$$\underline{q}^{(j+1)}_{n+1} = \underline{q}_n + \frac{\Delta t}{2}\left(\dot{\underline{q}}_n + \dot{\underline{q}}^{(j+1)}_{n+1}\right) \tag{B.8}$$

Since the generalized velocities $\underline{\dot{q}}_{n+1}^{(j+1)}$ are not independent, the generalized coordinates $\underline{q}_{n+1}^{(j+1)}$ obtained by integrating $\underline{\dot{q}}_{n+1}^{(j+1)}$ are not kinematically permissible. In other words, $\underline{q}_{n+1}^{(j+1)}$ may not satisfy the equations of constraints, i.e., $\underline{\Phi}\,(\underline{q}_{n+1}^{(j+1)}, t) \neq \underline{0}$. To find the $\underline{q}_{n+1}$ consistent with the equations of constraints, an unconstrained minimization scheme is proposed to simply reduce the deviations of $\underline{\Phi}\,(\underline{q}_{n+1}^{(j+1)}, t)$, i.e.,

$$\underset{\underline{q}_{n+1}}{\text{Min}} \quad \phi_0 = \underline{\Phi}\,(\underline{q}, t)^T\, \underline{\Phi}\,(\underline{q}, t) \tag{B.9}$$

where the design variable is $\underline{q}_{n+1}$. The initial estimate of $\underline{q}_{n+1}$ is provided by the direct integration of $\underline{\dot{q}}_{n+1}$, Eq. B.8.

There are many methods available to carry out the unconstrained minimization defined in Eq. B.9. Numerical results presented in section 4 are obtained by a recursive quadratic programming algorithm [9], called the linearization method, which has been proved to be globally convergent. More specifically, the new value of $\underline{q}^*$ is obtained by modifying the current value of $\underline{q}$ in the following way:

$$\underline{q}^* = \underline{q} + 2\alpha\, \underline{\Phi}^T \frac{\partial \underline{\Phi}}{\partial \underline{q}} \tag{B.10}$$

where the parameter α is a step size determined in such a way that the cost ϕ_0 is always reduced for the improved value of $\underline{q}^*$, i.e.,

$$\phi_0\,(\underline{q}^*) < \phi_0\,(\underline{q}) - \alpha\varepsilon \left\| \underline{\Phi}^T \cdot \frac{\partial \underline{\Phi}}{\partial \underline{q}} \right\|^2 \tag{B.11}$$

where ε is a given constant, usually defined as 0.1, and the notation

$||\cdot||$ denotes the L^2 norm. The computations of Eqs. B.10 and B.11 constitute an iterative process to be terminated whenever the value of $||\underline{\Phi}||$ becomes very small. After the value of $\underline{q}_{n+1}^{(j+1)}$ is updated by the optimum solution of the unconstrained minimzation, Eq. B.9, the iterations between Eqs. B.7-9 continue until both q_{n+1} and $\dot{q}_{n+1}$ reach the convergence criteria:

$$
\begin{aligned}
&|\underline{q}_{n+1}^{(j+1)} - \underline{q}_{n+1}^{(j)}| < e, \\
&|\dot{\underline{q}}_{n+1}^{(j+1)} - \dot{\underline{q}}_{n+1}^{(j)}| < e, \\
&|\underline{\lambda}_{n+1}^{(j+1)} - \underline{\lambda}_{n+1}^{(j)}| < e
\end{aligned}
\tag{B.12}
$$

where the notation $|\cdot|$ denotes the L^∞ norm and the e is a given small constant. Once the convergence is achieved, the computation moves to the next step and the iteration starts again.

The numerical algorithm is summarized as follows:

Step 1: Start with initial conditions and $\underline{\lambda}_0 = 0$.

Step 2: Select the initial values for $\underline{q}_{n+1}^{(0)}$, $\dot{\underline{q}}_{n+1}^{(0)}$ and $\underline{\lambda}_{n+1}^{(0)}$.

Step 3: Solve the matrix equation (B.7) for $\dot{\underline{q}}_{n+1}^{(j+1)}$ and $\underline{\lambda}_{n+1}^{(j+1)}$.

Step 4: Calculate the initial estimate of $\underline{q}_{n+1}^{(j+1)}$ by using equation B.8.

Step 5: Update the value of $\underline{q}_{n+1}^{(j+1)}$ by carrying out the unconstrained minimization, Eqs. B.9-11, in order to correct the constraint

deviation.

Step 6: Check the convergence criteria defined in equation B.12, if the convergence is achieved, move to Step 2 with n = n+1; otherwise, j = j+1, move to Step 3.

There are some remarks worthwhile mentioning here. First, if it is difficult to find the initial conditions for all of the dependent $\underline{q}$ and $\dot{\underline{q}}$ in Step 1, the unconstrainted minimization scheme given in Step 5 with independent variables fixed can be used to obtain accurate dependent $\underline{q}$ and $\dot{\underline{q}}$. Second, the generalized acceleration $\ddot{\underline{q}}$ can be calculated directly by rearranging Eq. B.3 as

$$\ddot{\underline{q}} = M^{-1} \left[\dot{M} \dot{\underline{q}} + \left(\frac{\partial \Phi}{\partial \underline{q}} \right)^T \underline{\lambda} + \underline{G} \left(\underline{q}, \dot{\underline{q}}, t \right) \right]$$

Compared to the method introduced in references 2 and 3, the proposed approach avoids the complicated and time-consuming process of deriving the second order derivative of equations of constraints, i.e., $\ddot{\underline{\Phi}}$.

B.III. DESIGN SENSITIVITY ANALYSIS

The design sensitivity analysis of a system with differential and algebraic equations associated with mechanical system dynamics has recently been a subject of study [10]. Two approaches have been discussed in the literature. One is the direct differentiation method [11]. The other is the adjoint variable technique [8, 12].

The differential/algebraic equations for the dynamics of a constrained mechanical system can be rewritten as follows:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\underline{q}}} \right) - \frac{\partial T}{\partial \underline{q}} = \underline{Q} + \underline{\Phi}_{\underline{q}}^T \underline{\lambda},$$

and

$$\underline{\Phi}\ (\underline{b},\ \underline{q},\ t) = 0$$

with the initial conditions given as

$$\begin{aligned} \underline{q} &= \underline{q}_0(\underline{b}), \qquad \text{at } t = 0 \\ \dot{\underline{q}} &= \dot{\underline{q}}_0(\underline{b}), \qquad \text{at } t = 0\ . \end{aligned} \tag{B.13}$$

The kinetic energy T is given as a quadratic function of $\dot{\underline{q}}$, i.e., $T = \frac{1}{2}\,\dot{\underline{q}}^T M\ (\underline{q},\ \underline{b},\ t)\ \dot{\underline{q}}$ for a symmetric matrix M.

It is very common in the optimal design formulation to have the cost or constraint written in a functional form as

$$\psi = \int_0^\tau F\ (\underline{b},\ \underline{q},\ \dot{\underline{q}},\ t)\ dt$$

The task for the design sensitivity analysis is to obtain the design derivative of ψ with respect to the design variable $\underline{b}$. The design variation of ψ, $\delta\psi$ is derived as

$$\delta\psi = \int_0^\tau \left(\frac{\partial F}{\partial \underline{b}}\,\delta\underline{b} + \frac{\partial F}{\partial \underline{q}}\,\underline{q}' + \frac{\partial F}{\partial \dot{\underline{q}}}\,\dot{\underline{q}}'\right) dt,$$

or after integrating by parts,

$$\delta\psi = \int_0^\tau \left\{\frac{\partial F}{\partial \underline{b}}\,\delta\underline{b} + \left[\frac{\partial F}{\partial \underline{q}} - \left(\frac{\partial F}{\partial \dot{\underline{q}}}\right)_t\right]\underline{q}'\right\} dt + \frac{\partial F}{\partial \dot{\underline{q}}}\,\underline{q}'\ \Big|_0^\tau \tag{B.14}$$

where the terms with apostrophes denote the variations due to the perturbations of design variables, $\delta\underline{b}$. It is revealed in Eqs. B.1 and B.2

that the relation between design variable $\underline{b}$ and state variable $\underline{q}$ is highly nonlinear. Thus, it is difficult to find $\underline{q}'$ explicitly in terms of $\underline{b}$. Nevertheless, the adjoint variable technique provides an alternative in which $\underline{q}'$ is not required to be defined analytically.

To begin the adjoint variable technique, one pre-multiplies two arbitrary vectors $\underline{\mu}(t)$ and $\underline{\nu}(t)$ to Eqs. B.1-2, and integrates the products over the time period $(0, \tau)$ to get the identities:

$$\int_0^\tau [-\dot{\underline{\mu}}^T M \dot{\underline{q}} - \underline{\mu}^T \frac{\partial T}{\partial \underline{q}} - \underline{\mu}^T \underline{Q} - \underline{\mu}^T (\underline{\Phi}_{,\underline{q}})^T \underline{\lambda}] \, dt + \underline{\mu}^T M \dot{\underline{q}} \,|_0^\tau = 0 \qquad (B.15)$$

and

$$\int_0^\tau \underline{\nu}^T \underline{\Phi} \, dt = 0 \qquad (B.16)$$

The design variation of Eq. B.15 is derived as

$$\int_0^\tau \{ - \dot{\underline{\mu}}^T (M \dot{\underline{q}})_{,\underline{q}} \underline{q}' - \dot{\underline{\mu}}^T (M \dot{\underline{q}})_{,\underline{b}} \delta \underline{b} - \dot{\underline{\mu}}^T M \dot{\underline{q}}'$$

$$- \underline{\mu}^T (\frac{\partial T}{\partial \underline{q}})_{,\underline{q}} \underline{q}' - \underline{\mu}^T (\frac{\partial T}{\partial \underline{q}})_{,\underline{b}} \delta \underline{b} - \underline{\mu}^T (\frac{\partial T}{\partial \underline{q}})_{,\dot{\underline{q}}} \dot{\underline{q}}'$$

$$- \underline{\mu}^T \underline{Q}_{,\underline{q}} \underline{q}' - \underline{\mu}^T \underline{Q}_{,\underline{b}} \delta \underline{b} - \underline{\mu}^T \underline{Q}_{,\dot{\underline{q}}} \dot{\underline{q}}'$$

$$- \underline{\mu}^T [(\underline{\Phi}_{,\underline{q}})^T \underline{\lambda}]_{,\underline{q}} \underline{q}' - \underline{\mu}^T [(\underline{\Phi}_{,\underline{q}})^T \underline{\lambda}]_{,\underline{b}}$$

$$- \underline{\mu}^T (\underline{\Phi}_{,\underline{q}})^T \underline{\lambda}'\} \, dt$$

$$+ \underline{\mu}^T (M\dot{\underline{q}})_{,\underline{q}} \underline{q}' \Big|_0^\tau + \underline{\mu}^T (M\dot{\underline{q}})_{,\dot{\underline{q}}} \dot{\underline{q}}' \Big|_0^\tau$$

$$+ \underline{\mu}^T (M \bar{\dot{\underline{q}}})_{,\underline{b}} \delta\underline{b} \Big|_0^\tau = 0$$

where the subscripts denote derivatives and where the terms with a bar on top are not subject to the differentiation. Integrating the terms with $\dot{\underline{q}}'$ in the preceding equation by parts, one obtains

$$\int_0^\tau \{\underline{q}'^T \{\frac{d}{dt} (M \dot{\underline{\mu}}) + [(\frac{\partial T}{\partial \underline{q}})^T_{,\underline{q}} \underline{\mu}]_{,t} + [\underline{Q}^T_{,\underline{q}} \underline{\mu}]_{,t}$$

$$- (M \dot{\underline{q}})^T_{,\underline{q}} \dot{\underline{\mu}} - (\frac{\partial T}{\partial \underline{q}})_{,\underline{q}} \underline{\mu} - \underline{Q}^T_{,\underline{q}} \underline{\mu} - [\bar{\underline{\lambda}}^T \underline{\Phi}_{,\underline{q}}]_{,\underline{q}} \underline{\mu}\}$$

$$- \delta\underline{b}^T \{(M \bar{\dot{\underline{q}}})_{,\underline{b}}^T \dot{\underline{\mu}} + (\frac{\partial T}{\partial \underline{q}})_{,\underline{b}}^T \mu + \underline{Q}_{,\underline{b}}^T \mu - [\bar{\underline{\lambda}}^T \underline{\Phi}_{,\underline{q}}]_{,\underline{b}} \underline{\mu}\}$$

$$- \underline{\lambda}'^T (\underline{\Phi}_{,\underline{q}} \underline{\mu})\} \, dt$$

$$- \dot{\underline{\mu}}^T M \underline{q}' \Big|_0^\tau - \underline{\mu}^T (\frac{\partial T}{\partial \underline{q}})_{,\dot{\underline{q}}} \underline{q}' \Big|_0^\tau - \underline{\mu}^T \underline{Q}_{,\dot{\underline{q}}} \underline{q}' \Big|_0^\tau$$

$$+ \underline{\mu}^T (M \dot{\underline{q}})_{,\underline{q}} \underline{q}' \Big|_0^\tau + \underline{\mu}^T M \dot{\underline{q}}' \Big|_0^\tau - \underline{\mu}^T (M \bar{\dot{\underline{q}}})_{,\underline{t}} \cdot \delta\underline{b} \Big|_0^\tau = 0$$

Furthermore, the design variation of Eq. B.16 provides

$$\int_0^\tau (\underline{\nu}^T \underline{\Phi}_{,\underline{q}}\, \underline{q}' + \underline{\nu}^T \underline{\Phi}_{,\underline{b}}\, \delta\underline{b})\, dt = 0.$$

Adding the last two equations to Eq. B.14 and grouping the corresponding terms together, one is able to express the design variation of the given functional ϕ symbolically as

$$\delta\phi = \int_0^\tau (\underline{\Lambda}_{\underline{b}}^T \delta\underline{b} - \underline{\Lambda}_{\underline{q}}^T \underline{q}' + \underline{\mu}^T \underline{\Phi}^T_{,\underline{q}}\, \underline{\lambda}')dt + \text{boundary terms} \qquad (B.17)$$

where $$\underline{\Lambda}_{\underline{b}} \equiv (M\, \dot{\bar{\underline{q}}})_{,\underline{b}}^T \dot{\underline{\mu}} + (\frac{\partial T}{\partial \underline{q}})_{,\underline{b}}^T \underline{\mu} + \underline{Q}_{,\underline{b}}^T \underline{\mu}$$

$$+ (\bar{\underline{\lambda}}\, \underline{\Phi}_{,\underline{q}})_{,\underline{b}}\, \underline{\mu} + \underline{\Phi}_{,\underline{b}}^T \underline{\nu} + F_{,\underline{b}}^T$$

and

$$\underline{\Lambda}_{\underline{q}} \equiv \frac{d}{dt}(M\, \dot{\underline{u}}) + [(\frac{\partial T}{\partial \underline{q}})_{,\underline{q}}^T \underline{\mu}]_{,t} + [\underline{Q}_{,\underline{q}}^T \underline{\mu}]_{,t}$$

$$- (M\, \dot{\underline{q}})_{,\underline{q}}^T \dot{\underline{\mu}} - (\frac{\partial T}{\partial \underline{q}})_{,\underline{q}}^T \underline{\mu} - \underline{Q}_{,\underline{q}}^T \underline{\mu}$$

$$- [\bar{\underline{\lambda}}^T \underline{\Phi}_{,\underline{q}}]_{,\underline{q}}\, \underline{\mu} + \underline{\Phi}_{,\underline{q}}^T \underline{\nu} + F_{,\underline{q}}^T - (F_{,\dot{\underline{q}}})_{,t}^T$$

It is noted that the design variation $\delta\phi$ is a linear functional of variations, $\delta\underline{b}$, $\underline{q}'$ and $\underline{\lambda}'$, and that the vectors $\underline{\mu}$ and $\underline{\nu}$ still remain unspecified. One may then assign values for $\underline{\mu}$ and $\underline{\nu}$ so that $\underline{\Lambda}_{\underline{q}} = 0$ and $\underline{\Phi}_{,\underline{q}}\, \underline{\mu} = 0$. In other words, the adjoint variables $\underline{\mu}$ and $\underline{\nu}$ are defined so as to eliminate the influence of unknowns $\underline{q}'$ and $\underline{\lambda}'$ in the formulation of $\delta\phi$. The condition $\underline{\Lambda}_{\underline{q}} = 0$, along with $\underline{\Phi}_{,\underline{q}}\, \underline{\mu} = 0$, can be arranged and written as follows:

$$\frac{d}{dt}(M\dot{\underline{\mu}}) + \underline{\Phi}_{,\underline{q}}^{T}\underline{\nu} = \underline{R}(\underline{b}, \underline{q}, \dot{\underline{q}}, \underline{\mu}, \dot{\underline{\mu}}, \underline{\nu}, t) \tag{B.18}$$

and

$$\underline{\Phi}_{,\underline{q}}\underline{\mu} = 0 \tag{B.19}$$

where $\underline{R}$ symbolically denotes the rest of the terms defined in $\underline{A}_q$. It is evident that the adjoint equations B.18 and B.19 are linear functions in terms of adjoint variables $\underline{\mu}$ and $\underline{\nu}$ and they represent a mixed system of differential/algebraic equations. More specifically, Eq. B.19 provides a set of linear constraints on $\underline{\mu}$, and $\underline{\nu}$ serves as a vector of lagrangian multipliers.

Following a similar procedure one can investigate the boundary terms shown in Eq. B.17 and determine the terminal conditions of the adjoint variable $\underline{\mu}$ in such a way that the influence of design variations $\underline{q}'$ and $\dot{\underline{q}}'$ is eliminated from the boundary terms of $\delta\phi$. Note that the boundary terms of $\delta\phi$ have the design variations $\underline{q}'$ and $\dot{\underline{q}}'$ defined at both $t = 0$ and $t = \tau$. If the initial conditions of all the independent and dependent coordinates, as well as velocities, are given explicitly, for instance:

$$\underline{q} = \underline{q}_o(\underline{b}) \quad , \text{at } t = 0$$

$$\dot{\underline{q}} = \dot{\underline{q}}_o(\underline{b}) \quad , \text{at } t = 0,$$

then the design variations $\underline{q}'$ and $\dot{\underline{q}}'$ at $t = 0$ are found without difficulty as

$$\underline{q}' = \left(\frac{\partial \underline{q}_o}{\partial \underline{b}}\right)\delta\underline{b}, \qquad \text{at } t = 0$$

$$\dot{\underline{q}}' = \left(\frac{\partial \dot{\underline{q}}_o}{\partial \underline{b}}\right) \delta \underline{b}, \qquad \text{at } t = 0.$$

Hence, the boundary terms of $\delta\psi$ can be rearranged to obtain:

Boundary terms =

$$- \{[\dot{\underline{\mu}}^T M + \underline{\mu}^T \left(\frac{\partial \underline{T}}{\partial \underline{q}}\right)_{,\dot{\underline{q}}} + \underline{\mu}^T \underline{Q}_{,\dot{\underline{q}}} - \underline{\mu}^T (M \dot{\underline{q}})_{,\underline{q}}] \left(\frac{\partial \underline{q}_o}{\partial \underline{b}}\right)$$

$$- \underline{\mu}^T M \left(\frac{\partial \dot{\underline{q}}_o}{\partial \underline{b}}\right) - \underline{\mu}^T (M \bar{\dot{\underline{q}}})_{,\underline{b}}\} \cdot \delta \underline{b}|_{t=0}$$

$$- \{[\dot{\underline{\mu}}^T M + \underline{\mu}^T \left(\frac{\partial \underline{T}}{\partial \underline{q}}\right)_{,\dot{\underline{q}}} + \underline{\mu}^T \underline{Q}_{,\dot{\underline{q}}} - \underline{\mu}^T (M \dot{\underline{q}})_{,\underline{q}}] \underline{q}' + \underline{\mu}^T M \dot{\underline{q}}'\}_\tau$$

$$+ \underline{\mu}^T (M \bar{\dot{\underline{q}}})_{,\underline{b}} \delta \underline{b}|_o^\tau \qquad \text{(B.20)}$$

On the other hand, while the boundary conditions are known only for the independent coordinates $\underline{q}_I$ and velocities $\dot{\underline{q}}_I$, i.e.,

$$\underline{q}_I = \underline{q}_o (\underline{b}), \qquad \text{at } t=o$$

$$\dot{\underline{q}}_I = \dot{\underline{q}}_o (\underline{b}), \qquad \text{at } t=o, \qquad \text{(B.21)}$$

the equations of constraints defined at t=o should be used in order to find the design variations of the dependent quantities. The design variations of Eq. B.21 and Eq. B.2 at t=o provide the following matrix equation

$$\frac{\partial \underline{\Phi}}{\partial \underline{q}_D}\,\underline{q}'_D = -\left(\frac{\partial \underline{\Phi}}{\partial \underline{b}} + \frac{\partial \underline{\Phi}}{\partial \underline{q}_I}\,\frac{\partial \underline{q}_D}{\partial \underline{b}}\right)\delta \underline{b} \qquad \text{(B.22)}$$

Which can be used to determine the design variation $\underline{q}'_D$ explicitly in terms of $\delta \underline{b}$ at t=0 Provided that the matrix $\partial \underline{\Phi}/\partial \underline{q}_D$ is not singular. Furthermore, the design variations of $\dot{\underline{q}}'_I$ and $\dot{\underline{q}}'_D$ can be obtained in a similar way by constructing Eq. B.22 based on the design variation of $\dot{\underline{\Phi}}$ at t = 0. Therefore, the boundary terms of $\delta\phi$ are still able to be written in the form of equation B.20, although only the independent coordinates and velocities are given at t=0.

Next the boundary terms of $\delta\phi$ defined at $t=\tau$ are investigated. The best way to avoid calculating the unknown variations $\underline{q}'$ and $\dot{\underline{q}}'$ at the terminal time τ is to specify the terminal conditions of adjoint variable μ at $t = \tau$ so that the terms associated with $\underline{q}'$ and $\dot{\underline{q}}'$ can be dropped. To achieve this, it is sufficient to obtain the following identifies from Eq. B.20,

$$M\,\dot{\underline{\mu}} + \left[\left(\frac{\partial T}{\partial \underline{q}}\right)_{,\dot{\underline{q}}}^{T} + \underline{Q}^{T}_{,\dot{\underline{q}}} - (M\,\dot{\underline{q}})_{,\underline{q}}\right]^{T} \underline{\mu} = 0, \quad \text{at } t=\tau$$

and

$$M\,\underline{\mu} = 0 \qquad \text{, at } t=\tau$$

Because of the positiveness of the mass matrix M, it is simply concluded from the above conditions that $\underline{\mu}(\tau) = \dot{\underline{\mu}}(\tau) = 0$. Based on these two terminal conditions, along with the adjoint equations, Eqs. B.18-19, the adjoint variables $\underline{\mu}(t)$ and $\underline{\nu}(t)$ can then be determined uniquely in the entire period of time $(0, \tau)$. Note that the same numerical scheme used for dynamic analysis, Eqs. B.1-2, can be applied here to solve the adjoint equation

numerically, though a better scheme could be implemented to take advantage of the linearity of the adjoint equations.

Finally, with the knowledge of $\underline{\mu}$, $\underline{\nu}$, $\underline{q}$ and $\underline{\lambda}$, the design variation of the cost functional $\delta\phi$, Eq. B.14, can be expressed as a linear functional of the perturbation of the design variable, $\delta\underline{b}$,

$$\delta\phi = \int_0^\tau \underline{A}_b^T \delta\underline{b}\, dt + \underline{B}_b\, \delta\underline{b}\Big|_0 + \underline{\mu}^T (M\, \dot{\underline{q}})_{,\underline{b}}\, \delta\underline{b}\Big|_0^\tau$$

where $\underline{A}_b$ is defined in Eq. B.17. While the initial condition of all the independent and dependent coordinates, as well as velocities, are given explicitly, the term $\underline{B}_b$ is defined as

$$\underline{B}_b \equiv [\dot{\underline{\mu}}^T M + \underline{\mu}^T (\frac{\partial T}{\partial \underline{q}})_{,\underline{q}} + \underline{\mu}^T \underline{Q}_{,\underline{q}} - \underline{\mu}^T (M\, \dot{\underline{q}})_{,\underline{q}}] (\frac{\partial \underline{q}_0}{\partial \underline{b}})$$

$$- \underline{\mu}^T M (\frac{\partial \dot{\underline{q}}_0}{\partial \underline{b}}) - \underline{\mu}^T (M\, \dot{\underline{q}})_{,\underline{b}} .$$

Otherwise, a similar form can still be obtained based on Eq. B.22.

B.IV. NUMERICAL EXAMPLE

A modified slider-crank mechanism with one degree of freedom is studied here as an example to validate the numerical algorithm presented previously. This mechanism is composed of two linkages, the crank AB and the connecting rod BC, as shown in Fig. B.1. While each of the hinge joints A and B entertains two constraints, the joint point C is forced to slide along the x-axis. With a torque H applied at the joint A, the system is subjected to a planar motion. The definitions of body-fixed coordinates, as well as the

vectors $\underline{\ell}_i$ and $\underline{d}_i$, are indicated in Fig. B.2. The total kinetic energy of the system is obtained as

$$T = \sum_{1}^{2} \frac{1}{2} (M_i \underline{R}_i^T \underline{R}_i + J_i \omega_i^2)$$

The kinematic constraints for joints A, B and C are given as,

$$\left.\begin{aligned} \underline{\phi}_1 &\equiv \underline{\ell}_1 + \underline{R}_1 = 0 \\ \underline{\phi}_2 &\equiv \underline{R}_1 + \underline{d}_1 - \underline{\ell}_2 - \underline{R}_2 = 0 \\ \phi_3 &\equiv \underline{J}^T (\underline{R}_2 + \underline{d}_2) = 0 \end{aligned}\right\} \qquad (B.23)$$

where the unit vector $\underline{J}$ is parallel to the Y-direction of the inertia frame. The constraint ϕ_3 means that there is no Y-component of joint C's movement at any time.

Analysis

Based on Hamilton's principle and theory of lagrangian mulipliers, a system of eleven equations can be set up in the form of Eq. B.7 for the slider-crank mechanism. The unknowns to be solved are the six degrees of freedom, $\underline{R}_1$, $\underline{R}_2$, w_1 and w_2, as well as the five lagrangian multipliers, $\underline{\lambda}_1$, $\underline{\lambda}_2$ and λ_3 associated with the equations of constraints, Eq. B.23. The diagonal components of the 6 x 6 mass matrix M are M_1, M_1, M_2, M_2, J_1 and J_2. The fifth component, equal to the given torque H, is the only non-zero element in the forcing term $\underline{Q}$. The detailed formulation of the Jacobian, $\frac{\partial \underline{\Phi}}{\partial \underline{q}}$, is given in the Appendix.

With the non-dimensional data: M_1=M_2=1, H=2, J_1 = 0.08333, J_2 = 0.33333, $|\underline{\ell}_1| = |\underline{d}_1| = 0.5$, and $|\underline{\ell}_2| = |\underline{d}_2| = 1$, the slider-crank mechanism of

concern is analyzed on the DEC-10 system with double precision. The time step Δt is set as 0.005 seconds. It takes 25.5 CPU seconds to simulate the motion for the time period of one second. The convergence criteria, Eq. B.12, for the coordinates and velocities are set as 10^{-5}. It takes at most three iteratives at each time grid to achieve the given convergence requirement. As for the unconstrained minimization scheme for the coordinate correction, it also takes at most three iterations at each time grid to achieve the convergence given as

$$|| \underline{\Phi} || < 10^{-6}$$

Some of the numerical results are listed in Table B.1. The last column contains the deviation of the constraint, ϕ_3, regarding the sliding joint C. Furthermore, the results, obtained by using a commerically available program, DADS [13], are also listed in Table B.1 for the purpose of comparison. A good agreement between the purposed scheme and DADS is observed.

Design Sensitivity Analysis and Optimization

The optimization of slider-crank mechanism studied here is to find the control torque H(t) so that the motion of the sliding joint C can follow a desired path $\underline{\eta}(t)$. This example falls into the category of inverse dynamics. Nevertheless, an optimization formulation is set up to approximate the best torque profile H(t) in terms of given functions. That is

$$\underset{H(t)}{\text{Min}}\ \phi_0 = \int_0^\tau [\underline{R}_2 + \underline{\ell}_2 - \underline{\eta}(t)]^2\, dt$$

where the control torque is expressed in terms of given functions $N_i(t)$ and design variables b_i as

$$H(t) = \Sigma\, b_i\, N_i(t). \tag{B.24}$$

The function $N_i(t)$ can be a polynominal or trigonometric function. The desired path, $\underline{\eta} = [\eta_x, \eta_y]^T$, is given as

$$\eta_x(t) = \alpha + 10\,(\beta-\alpha)\,t^3 - 15\,(\beta-\alpha)t^4 + 6\,(\beta-\alpha)\,t^5,\ 0 \le t \le 1,$$
$$\eta_y(t) = 0$$

Note that the path $\eta_x(t)$, starting from the initial position α to the final position β, entertains zero velocities and accelerations at both $t=0$ and $t=\tau$. The design variation of the cost function is derived as

$$\delta\phi_0 = \int_0^\tau \left(\frac{\partial \underline{Q}}{\partial H}\right)^T \underline{\mu} \cdot \delta H\, dt$$

where the forcing term $\underline{Q}$ is given as $(0,\ 0,\ 0,\ 0,\ H,\ 0)^T$, and where $\delta H = \sum_1^1 N_i(t) \cdot \delta q_i$ because of equation B.24. Thus,

$$\delta\phi_0 = \Sigma\left[\int_0^\tau \mu_5 \cdot N_i(t)\, dt\right] \cdot \delta q_i$$

$$\equiv \underline{\ell}^T\, \delta\underline{q} \tag{B.25}$$

where $\underline{\ell}$ is defined as a sensitivity vector. The adjoint variable $\underline{\mu}$ is the solution of the following adjoint equations:

$$\underline{\underline{M}}\,\dot{\underline{u}} + \left(\frac{\partial \underline{\underline{\Phi}}}{\partial \underline{q}}\right)^T \underline{\nu} = -\left(\underline{\lambda}^T \frac{\partial \underline{\underline{\Phi}}}{\partial \underline{q}}\,\underline{\mu}\right)_{,\underline{q}} + 2\,(\underline{R}_2 + \underline{d}_2)^T{}_{,\underline{q}} \cdot (\underline{R}_2 + \underline{d}_2 - \underline{\eta}) \tag{B.26}$$

and

$$\frac{\partial \Phi}{\partial \underline{q}} \underline{\mu} = 0 \tag{B.27}$$

with terminal conditions; $\underline{\mu}(\tau) = \dot{\underline{\mu}}(\tau) = 0$. The detailed formulation of the terms in Eq. B.26 can be found in the Appendix.

The accuracy of the design sensitivity analysis is investigated next. First, the control torque is fixed as H=2. The design sensitivity vector calculated by the analytical equation is compared with the one calculated by the direct finite difference method. The result plotted in Fig. B.4 shows a very good agreement between the aforementioned two methods when the perturbation of the design variable is up to 10%. Second, the control torque is described by a quadratic function $H(t) = a_0 + a_1 t + a_2 t^2$ with three coefficents as design variables. The components of the design sensitivity vector provided by Eq. B.25, as well as those calculated by the finite difference method are listed in Table B.2. The results are obtained based on $a_0 = a_1 = a_2 = 1$ and 2% perturbation for each design variable. It is interesting to observe from the design sensitivity vector that to increase the design coefficient a_0 is more beneficial in terms of the reduction of error in the path generation than to increase a_1 and a_2.

Once the accurate design sensitivity is produced, any gradient-based mathematical programming can be used to generate the optimum solution iteratively. The following numerical results are obtained by using a recursive quadratic program called the linearization method [9]. The control torque is assumed to have one of the following forms:

$$H(t) = a_0,$$

$$H(t) = a_0 \text{ t } a_1 t,$$

$$H(t) = a_0 + a_1 t + a_2 t^2,$$

$$H(t) = b_1 \sin\pi t + b_2 \cos\pi t$$

where the coefficients, a_0, a_1, a_2, b_1 and b_2 are treated as design variables. Corresponding to differently prescribed torque functions, the optimum solutions and their associated data are listed in Table B.3, and plotted in Fig. B.5 as well. Note that none of the trial torque functions are able to establish a path pattern so as to make the terminal velocity and acceleration approach zero. Finally, to demonstrate the stable performance of the optimum algorithm, the convergence progressions of the cost function and the convergence criteria, i.e., L^2 norm of the design gradient, are also plotted in Fig. B.6 only for the quadratic control torque.

Table B.1. Numerical Results of Analysis

Time	Prop. Algorithm		DADS		ϕ_3
	θ_1(Rad)	w_1	θ_1(Rad)	w_1	
0.1	0.01499	0.29993	0.01697	0.29987	0.32×10^{-19}
0.2	0.05992	0.59759	0.06346	0.59698	0.14×10^{-17}
0.3	0.13410	0.88239	0.13822	0.88085	0.13×10^{-16}
0.4	0.23524	1.1324	0.23958	1.1294	0.45×10^{-16}
0.5	0.35865	1.3254	0.36317	1.3201	0.38×10^{-16}
0.6	0.49826	1.4578	0.50234	1.4522	0.29×10^{-16}
0.7	0.64877	1.5473	0.65248	1.5415	0.14×10^{-17}
0.8	0.80714	1.6189	0.81052	1.6136	0.35×10^{-16}
0.9	0.97270	1.6946	0.97595	1.6906	0.17×10^{-15}
1.0	1.1468	1.7927	1.1501	1.7908	0.34×10^{-15}

Table B.2 Design Sensitivity Analysis of the Slider-Cranker Mechanism

With Quadratic Control Torque:
$H(t) = a_0 + a_1 t + a_2 t^2$

Design Gradients	(1) Adjoint Variable Technique eq.	(2) Finite Difference Method $[\phi(a_1+\Delta a_1) - \phi(a_1)]/\Delta a_1$	$\frac{(1)-(2)}{(1)}$ %
$\frac{\delta \psi_0}{\delta a_0}$	-0.22357	-0.219	2.04%
$\frac{\delta \psi_0}{\delta a_1}$	-0.05845	-0.059	0.94%
$\frac{\delta \psi_0}{\delta a_2}$	-0.02389	-0.0245	2.93%

Table B.3 Optimum Control Torques of the Slider-Crank Mechanism

Type of Torque	Initial Design	Optimum Design	Initial Cost	Final Cost	No. of Iterations*
Constant	$H = 2.$	$H = 4.1915$	0.25049	0.044542	38
Linear	$H = 2.$	$H = 4.0919+0.39877t$	0.25049	0.045865	37
Quadratic	$H = 2.$	$H = 4.0912+0.39464t + 0.11552t^2$	0.25049	0.04614	38
Periodic	$H = 4 \sin\pi t + 4 \cos\pi t$	$H = 3.691 \sin\pi t + 3.872 \cos\pi t$	0.03585	0.03031	24

*The convergence criterion is that the L^2 norm of the design gradient is less than 0.01.

ADDENDUM TO APPENDIX B

1. The Jacobian, $\frac{\partial \underline{\Phi}}{\partial \underline{q}}$:

$$\frac{\partial \underline{\Phi}}{\partial \underline{q}} = \begin{bmatrix} -1 & 0 & 0 & 0 & \ell_{1y} & 0 \\ 0 & -1 & 0 & 0 & -\ell_{1x} & 0 \\ -1 & 0 & 1 & 0 & d_{1y} & -\ell_{2y} \\ 0 & -1 & 0 & 1 & -d_{1x} & \ell_{2x} \\ 0 & 0 & 0 & -1 & 0 & -d_{2x} \end{bmatrix}$$

where the first subscript denotes the number of the body and the second subscript represents the component along the specified direction.

2. The terms on the right side of Eq. B.26,

(a) $(\underline{\lambda}^T \frac{\partial \underline{\Phi}}{\partial \underline{q}} \underline{\mu})_{,\underline{q}}$

$$= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \ell_{1x}\,\mu_5\,\lambda_1 + \ell_{1y}\,\lambda_2 + d_{1x}\,\mu_5\,\lambda_3 + d_{1y}\,\mu_5\,\lambda_4 \\ -\ell_{2x}\,\mu_6\,\lambda_3 - \ell_{2y}\,\mu_6\,\lambda_4 + d_{2y}\,\mu_6\,\lambda_5 \end{Bmatrix}$$

(b) $2\,(\underline{R}_2 + \underline{d}_2)^T_{,\underline{q}} \cdot (\underline{R}_2 + \underline{d}_2 - \underline{\eta})$

$$= \begin{Bmatrix} 0 \\ 0 \\ 2(R_{2x} + d_{2x} - \eta_x) \\ 2(R_{2y} + d_{2y}) \\ 0 \\ -2\,d_{2y}\,(R_{2x} + d_{2x} - \eta_x) \\ 2\,d_{2x}\,(R_{2y} + d_{2y}) \end{Bmatrix}$$

REFERENCES OF APPENDIX B

1. Haug, E. J., Nikravesh, P. E., Sohoni, V. N., and Wehage, R. A., "Computer Aided Analysis of Large Scale, Constrained, Mechanical Systems," 4th International Symposium on Large Engineering Systems, Calgary Alberta, Canada, June 1982.

2. Chace, M. A., "Methods and Experience in Computer Aided Design of Large - Displacement Mechanical Systems," Computer Aided Analysis and Optimization of Mechanical System Dynamics, (ed. E. J. Haug), Springer-Verlag, Heidelberg, 1984, pp. 232-259.

3. Wehage, R. A., and Haug, E. J., "Generalized Coordinate Partitioning for Dimension Reduction in Analysis of Constrained Dynamics Systems," Journal of Mechanical Design, Vol. 104, January 1982, pp. 247-255.

4. Singh, R. P., and Likins, P. W., "Singular Value Decomposition for Constrained Dynamical Systems," ASME Journal of Applied Mechanics, to appear.

5. Mani, N. K., Haug, E. J., and Atkinson, K. E., "Application of Singular Value Decomposition for Analysis of Mechanical System Dynamics," ASME Journal of Mechanisms, Transmissions, and Automation in Design, to appear.

6. Vance, J. M., and Sitchin, A., "Derivation of First-Order Difference Equations for Dynamical Systems by Direct Application of Hamilton's Principle," ASME Journal of Applied Mechanics, Vol. 37, 1970, pp. 276-278.

7. Vance, J. M., "Numerical Solution of Dynamical Systems by Direct Application of Hamilton's Principle," International Journal for Numerical Methods in Engineering, Vol. 4, 1972, pp. 207-216.

8. Haug, E. J., Choi, K. K. and Komkov, V., Design Sensitivity Analysis of Structural System, Academic Press, December 1984.

9. Choi, K. K., Haug, E. J., Hou, J. W., and Sohoni, V. N., "Pshenichny's Linearization Method for Mechanical System Optimization," ASME Journal of Mechanisms, Transmissions, and Automation in Design, Vol. 105, No. 1, March 1983, pp. 97-103.

10. Haug, E. J., Mani, N. K., and Krishnaswami, P., "Design Sensitivity Analysis and Optimization of Dynamically Driven System," Computer Aided Analysis and Optimization of Mechanical System Dynamics, (ed. E. J. Haug), Springer-Verlag, Heidelberg, 1984, pp. 232-259.

11. Krishnaswami, P., Wehage, R. A., and Haug, E. J., Design Sensitivity Analysis of Constrained Dynamic Systems by Direct Differentiation, Technical Report No. 83-5, Center for Computer Aided Design, The University of Iowa, 1983.

12. Haug, E. J., and Ehle, P. E., "Second Order Design Sensitivity Analysis of Mechanical System Dynamics," International Journal for Numerical Methods in Engineering, Vol. 18, 1982, pp. 1699-1717.

13. DADS (Dynamic Analysis and Design System), commerically available from Computer Aided Design Software Incorporation, P.O. Box 203, Oakdale, Iowa 52319.

14. Gabriele, G. A., and Ragsdell, K. M., "Large Scale Nonlinear Programming Using the Generalized Reduced Gradient Method," ASME Journal of Mechanical Design, Vol. 102, No. 3, 1980, pp. 566-573.

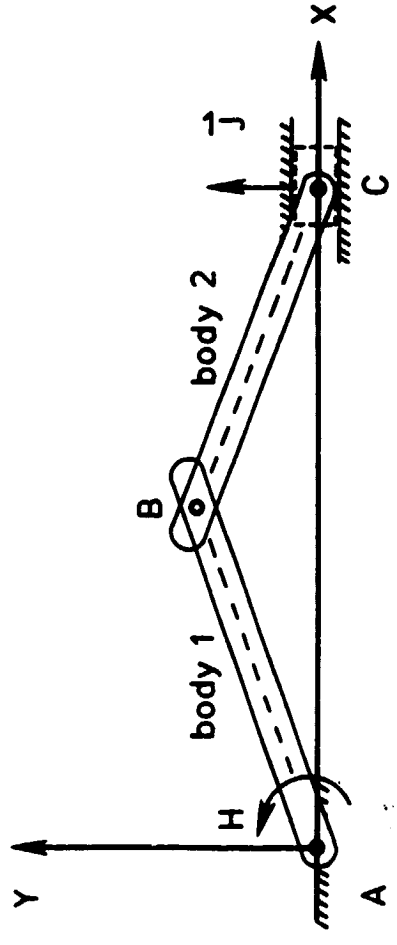


Fig. B.1. A Modified Slider-Crank Mechanism

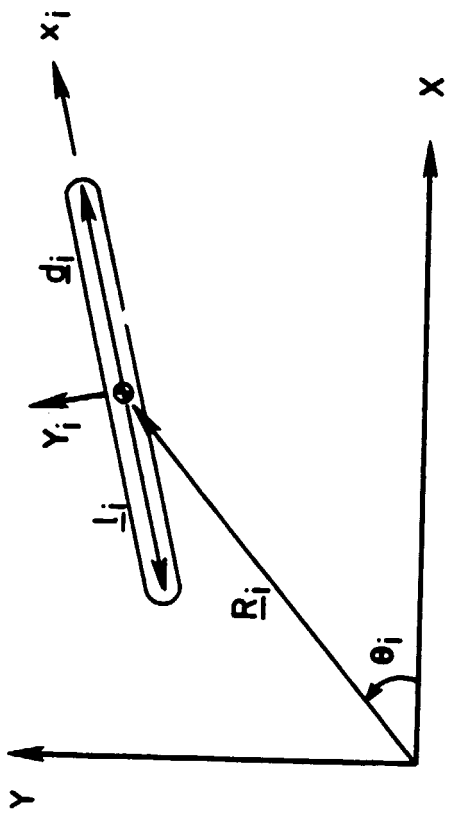
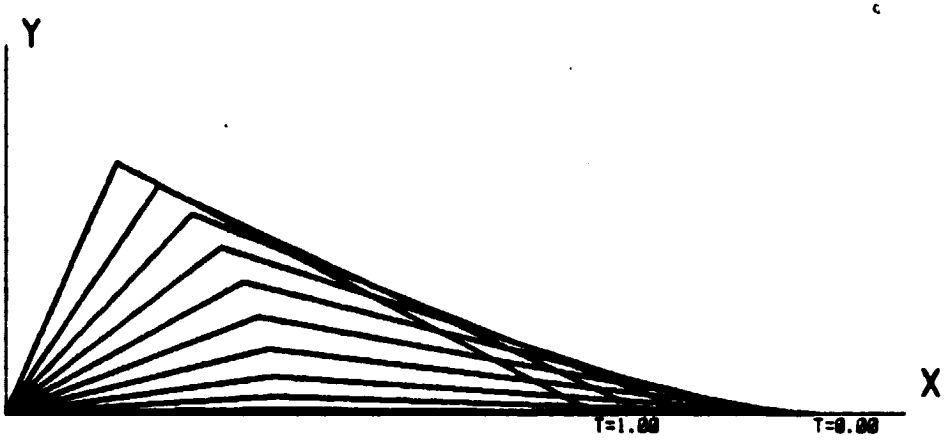
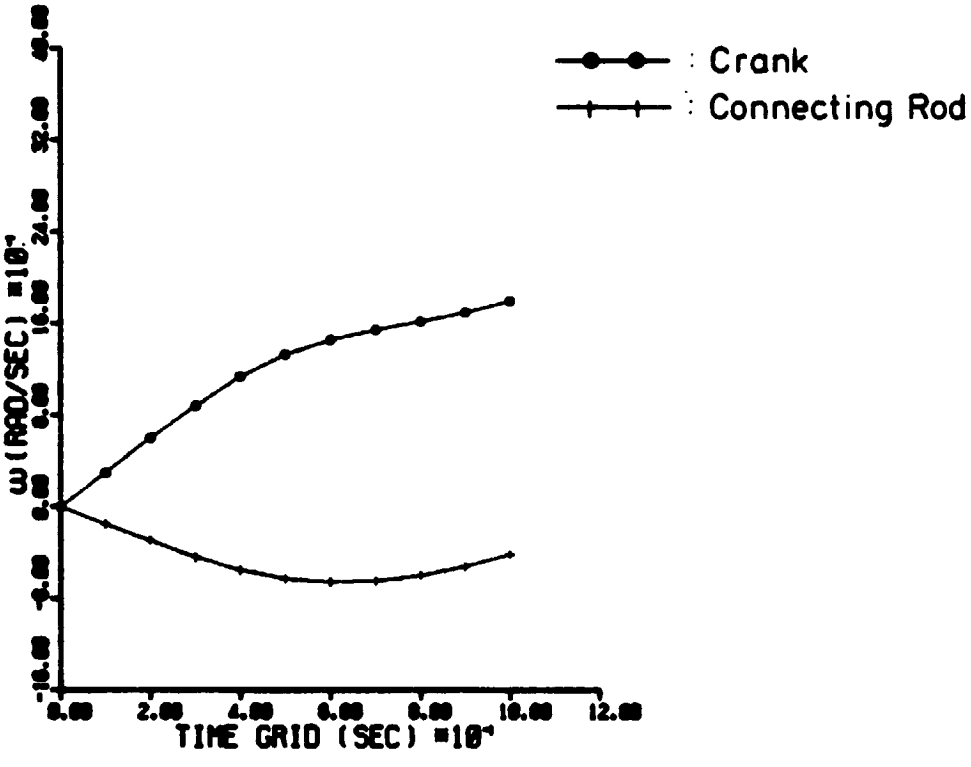


Fig. B.2. Definition of the Body-Fixed Coordinate



(a) The Motion of the Slider-Crank Mechanism



(b) Angular Velocity of the Crank and Connecting Rod

Fig. B.3. The Motion of the Slider-Crank Mechanism

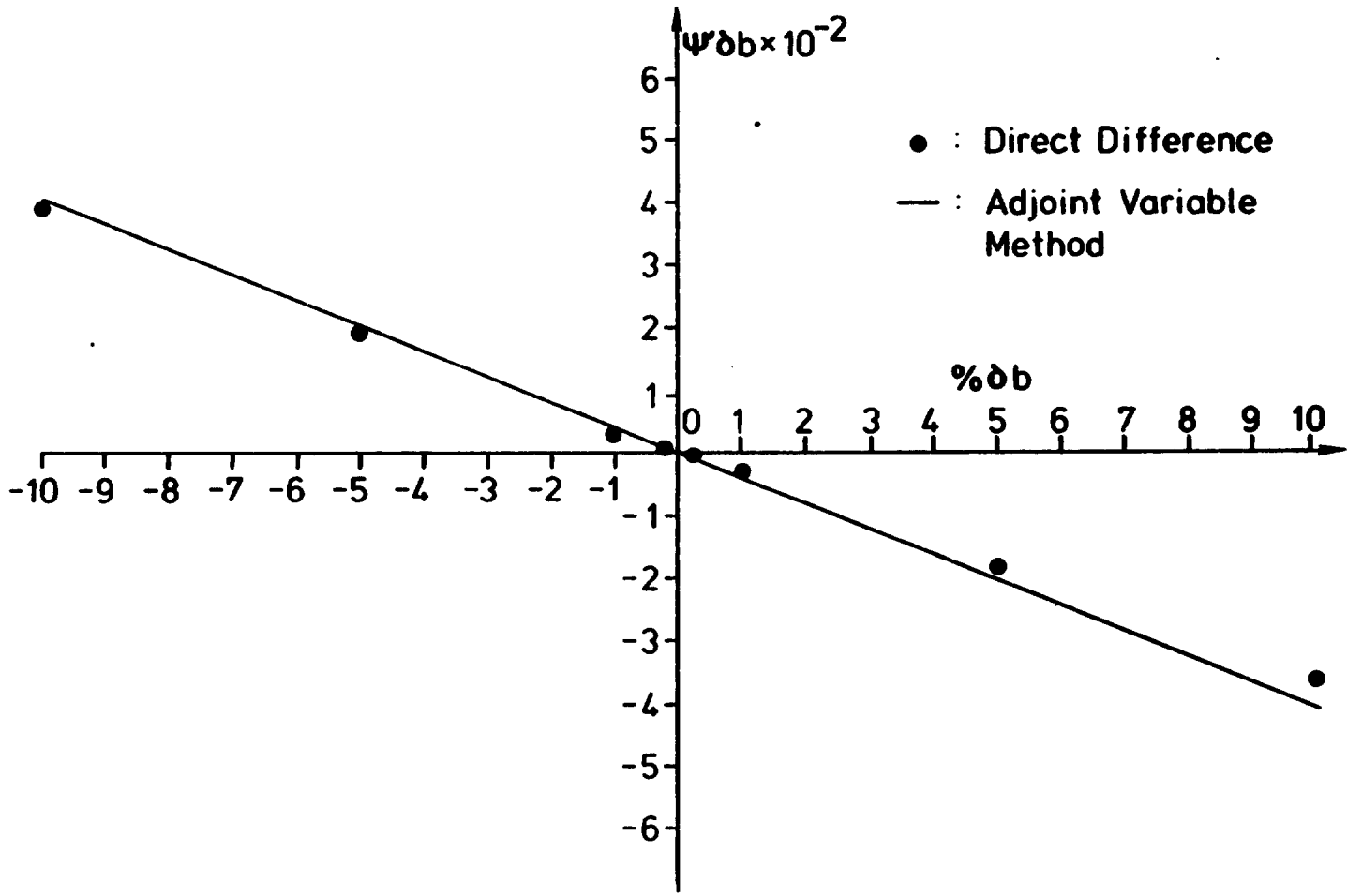


Fig. B.4. Design Sensitivity Analysis of the Slider-Crank Mechanism with Constant Control Torque: H = 2.

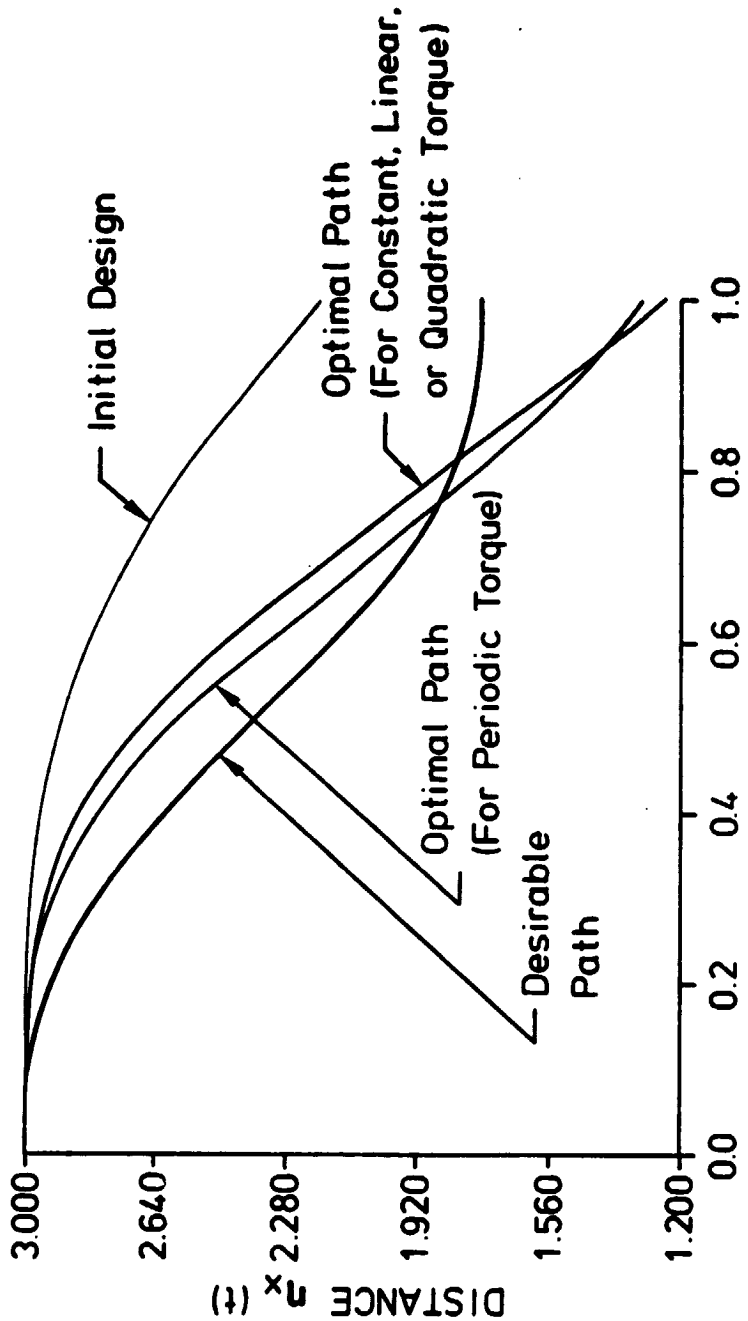


Fig. B.5. The Optimal Paths for Different Torque Functions

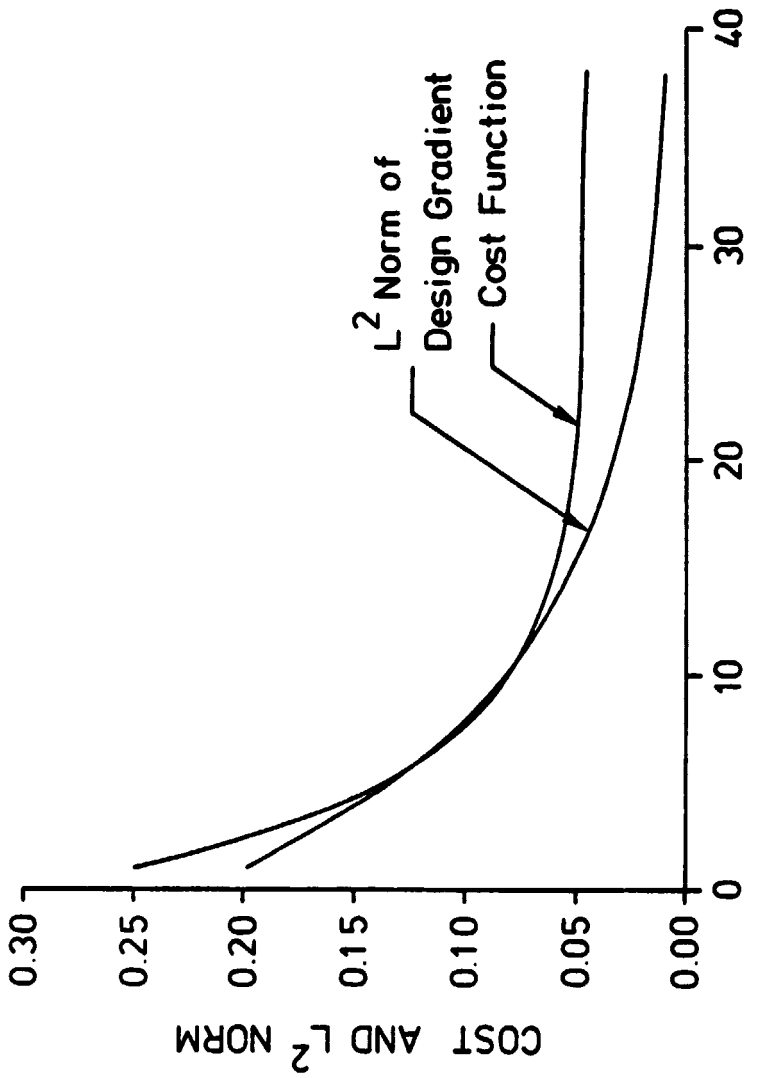


Fig. B.6. Numerical Convergence History of the Slider-Crank Mechanism (with quadratic torque)