FUNDAMENTAL STUDIES ON A HEAT DRIVEN LAMP

FINAL REPORT FOR NASA GRANT NAG 3-437

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Submitted by the
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Abstract

A detailed theoretical study of a heat-driven lamp has been performed. This lamp uses a plasma produced in a thermionic diode. The light is produced by the resonance transition of cesium. An important result of this study is that up to 30% of the input heat is predicted to be converted to light in this device. This is a major improvement over ordinary thermionic energy converters in which only ~1% is converted to resonance radiation. Efficiencies and optimum inter-electrode spacings have been found as a function of cathode temperature and the radiative escape factor. The theory developed explains the operating limits of the device.

I. Introduction

The topic of this work is the direct conversion of heat to light. The method investigated is to use the cesium plasma of a thermionic discharge to produce resonance radiation. The motivation for this study is to determine the feasibility of the thermionic-photovoltaic energy converter proposed by D. L. Chubb. The reason cesium is used is that the frequency of its resonance radiation is closely matched to the band-gap of several semiconductors. This match permits efficient operation of photovoltaic cells in strong contrast to the inefficient operation of photovoltaics when powered by broadband solar radiation.

The major result of this work is that up to 30% of the input heat to the discharge may be converted to resonance radiation even with very low escape factors for the radiation. A brief description of the technical work is included in Appendix A. The details of the model, the calculations, and the results are given in Appendix B.

II. Papers and Presentations

- A paper on this work was presented at the 1985 IEEE International Conference on Plasma Science. The abstract of this paper is included as Appendix A.
- F. Stefani completed his Master's Thesis on this work. It is included as Appendix B.
- A paper by F. Stefani and J. L. Lawless will be submitted to *IEEE Journal of Plasma Science* (in progress).

III. Student Participation

Student: F. Stefani
Period: 1983-1985
Degree: Master of Science
Thesis: See Appendix B

APPENDIX A: IEEE Conference Abstract

Presented at the 1985 IEEE International
Conference on Plasma Science
Pittsburgh, Pennsylvania
June, 1985

Plasma Dynamics in a Thermionic-Photovoltaic Energy Converter[1]

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The resonance radiation from a thermionic discharge can be used to generate electricity by employing photovoltaic cells to collect this energy[2]. Since the Cs resonance line is well matched to the band-gap of many semiconductors, light-to-electricity conversion efficiencies of 50% or more are possible. This paper will investigate how efficiently a thermionic discharge may be made to convert heat to light.

Normally, an optimized thermionic energy converter produces very little resonance radiation. Our theoretical studies show that at higher currents and longer inter-electrode gaps, a significant portion of the input heat can be converted to line radiation even if the radiative escape factor is as low as 1%. Neglecting lead losses and optical collection difficulties, efficiencies of up to 30% for the conversion of heat to radiation in the discharge are predicted.

The theoretical studies model the electron transport, ion transport, and inelastic collisions leading to ionization, recombination, and emission of resonance radiation. The transport of resonance radiation is treated using an escape factor. The emission rate is coupled to the ionization and recombination rates through the Cs excited state kinetics. Diffusion and energy balances determine the electron temperature and density distribution. The model makes clear an unintuitive result that the maximum efficiency with which line radiation is emitted is only weakly dependent on the optical escape factor. High efficiencies are typically found at inter-electrode distances on the order of .5 cm

An experiment has been constructed to measure the efficiency for converting thermal energy to resonance radiation and also the maximum discharge length. Measurements are in progress. Available experimental results will be presented.

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[1]This work supported by NASA-Lewis Grant NAG 3-437

[2]D.L. Chubb, *Thermionic-Photovoltaic Energy converter*, Paper 5B6, IEEE International Conference on Plasma Science, San Diego, California (1983)

APPENDIX B: F. Stefani's Master's Thesis

LINE RADIATION EMISSION FROM A CESIUM THERMIONIC CONVERTER

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INTRODUCTION

It has been suggested [1] that if line radiation from a thermionic device could be emitted with sufficient intensity, then a device employing photo cells to collect this energy could be made to generate electricity efficiently. This follows from the observation that the band gap energies of several common photo-voltaics are well matched to the line radiation energies of cesium diode converters.

Resonance Radiation	
$6P_{1/2} - 6S_{1/2}$	1.39 eV
$6P_{3/2} - 6S_{1/2}$	1.46 ev
Excitation Band Gaps	
Silicon	1.12 eV
Indium Phosphide	1.27 eV
Gallium Aresenide	1.35 eV

Table 1 Radiation and band gap energies.

This study investigates the efficiency with which resonance radiation can be generated from a thermionic device and identifies the operating conditions for which radiation output and the radiation efficiency are maximized. It has been tentatively established that radiation efficiencies on the order of 30% are possible even for optically dense plasmas. This result, if confirmed by experiment, means that the prospects for the proposed device are indeed promising.

The first part of this report is a discussion of the model that was employed. This is followed by a brief outline of the solution procedure. The last part of this report is a presentation of the results and an attempt to relate them to the physical events occurring in the plasma. A listing of the computer progran developed for this study is included in the appendix.

AN OVERVIEW OF THE MODEL

The work presented herein is based on an analytical model of a cesium plasma diode. This model, described in detail in reference 2, is summarized here primarily to review the assumptions made in the context of the problem and to provide some insights that help make the results more understandable.

A control volume is taken as the region of the inter-electrode space extending from the top of the emitter sheath to the top of the collector sheath. Three governing equations are written for the plasma in the control volume. These are conservation equations for mass, momentum, and energy. A soluton procedure is employed that permits the optical thickness of the gas to be varied

as a parameter. Since the problem is 1-D in the direction of current flow it is possible to reduce the governing equations to algebraic expressions. When these are solved simultaneously for the electron temperature, the electron density, and the operating current, a solution is obtained.

From the onset, it was assumed that a thermionic device capable of generating high levels of radiation would be operating in the ignited mode. This means that the ions and electrons in the plasma are created predominantly by electron-atom collisions in the control volume rather than by contact ionization at the walls. In this case emission effects are of relatively little importance. Of greater importance are the assumptions made about the plasma in the inter-electrode space. This work has assumed the following:

1) The plasma is singly ionized and is composed of electrons, cesium atoms, and cesium ions.
2) The plasma exists at two temperatures, one for the heavier particles and one for the electrons.
3) Both temperatures are isothermally distributed in space.

The two temperature plasma assumption is based on the fact that most collisions are elastic, and very little energy is transferred from the fast moving electrons to the slower moving heavier particles. The isothermal assumption is in effect a statement that the thermal conductivity of the plasma is high. This assumption has been confirmed experimentally as well as through numerical tests [3].

AMBIPOLAR DIFFUSION EQUATION

The Ambipolar Diffusion Eqn. is a mass balance for the electrons and the ions in the control volume. It is derived by writing the conditions for mass conservation for the steady state plasma in terms of ionization and recombination rate constants S, and α.

$$\nabla\cdot\Gamma_e = \nabla\cdot\Gamma_i = S\,N_{os}\,n - \alpha n^3 \qquad (1)$$

To reflect conditions in the device transport equations for the electrons and ions must include the effects of electric fields as well as diffusion effects. In one dimension they are written as follows:

$$\Gamma_e = -\mu_e(n\, d\Psi/dx + (kT_e/e)\, dn/dx) \qquad (2)$$

$$\Gamma_i = -\mu_i(-n\, d\Psi/dx + (kT_i/e)\, dn/dx) \qquad (3)$$

where: Γ_e , Γ_i = electron and ion fluxes
μ_e , μ_i = electron and ion mobilities
e = elementary charge
n = electron density = ion density
Ψ = electron motive
k = Boltzmann's constant
T_e , T_i = electron and ion temperatures

Substituting the transport equation for the ions (eqn. 3) into the conservation equation (eqn. 1) yields a diffusion equation for the ions.

$$- d/dx((kT_e/e)\, dn/dx - n\, d\Psi/dx) = S\, N_{gs}\, n - \alpha n^3 \qquad (4)$$

Substuting the electron transport (eqn. 2) into this expression eliminates the electric field term and results in a conservation equation in terms of the coupled motion of electrons and ions:

$$-D_a\, d^2n/dx^2 + \Gamma_e d(\mu_i/\mu_e)/dx = (1 + \mu_i/\mu_e)[S\, N_{gs}\, n - \alpha n^3] \qquad (5)$$

$$\text{where } D_a = \mu_i\, (\, kT_i + kT_e\,) / e \qquad (6)$$

This equation is called the Ambipolar diffusion equation and D_a is the ambipolar diffusivity. The ambipolar diffusion equation can be simplified by neglecting the ratio of the ion and electron mobilities μ_i/μ_e (since it is much less than one) and by making the reasonable assumption that this ratio does not vary much across the inter-electrode distance. Assuming also that D_a is constant reduces the ambipolar diffusion equation to the following form:

$$-D_a\, d^2n/dx^2 = S\, N_{gs}\, n - \alpha n^3 \qquad (7)$$

Solving the ambipolar diffusion equation subject to the appropriate boundary conditions (Appendix B) yields the following solution and associated eigenvalue condition:

$$n(x) = n_{(max)}\, sn(2KX/d) \qquad (8)$$

$$SN_{gs} = (4K^2)D_a/d^2 + (1/2)\, \alpha\, n^2 \qquad (9)$$

where $n_{(max)}$ is the maximum electron density.

Both these equations are of interest and reveal different things about the plasma. The solution to the ambipolar diffusion equation gives the distribution of the electrons in the inter-electrode space. The eigenvalue condition gives the correspondence between electron temperature and the maximum density

for which ionization and recombination are such to balance the loss of particles to the walls by diffusion. This correspondence is termed the ignited condition.

The function sn that appears in the solution of the ambipolar diffusion equation is the Jacobi elliptic function [4]. It is characterized by the quarterperiod K, whose value can range from $\pi/2$ to infinity. Both limits represent important cases in the problem we are solving. In the limit where $K = \pi/2$, the sn function is exactly the sin function and the electron distribution is similar to that shown in fig. 1. This case corresponds to conditions where recombination is negligible, usually at lower values of the electron density. The other limiting case is that where the electron density is very close to the Saha value. This condition is referred to as the Saha limit. In this case the electron distribution departs significantly from the sin profile, being much steeper at the boundaries and flat at the center, as in fig. 2.

In non-Saha conditions the quarter period is calculated from the ratio of the electron density to the Saha density; first by calculating the modulus m, and then by taking the arithmetic geometric mean to find K [5].

$$m = 1 / (2 S N_{gs} / (\alpha n^2) - 1) \tag{10}$$

In the Saha limit, the modulus approaches an asymptotic value of unity and round-off error in the a.g.m. calculation of K becomes appreciable. In this case K is calculated from the following expression:

$$K = n d (\alpha / 8 m D_a)^{1/2}, \qquad m \Rightarrow 1 \tag{11}$$

This is derived by solving eqn. 10 for SN_{gs}, substituting this expression into the eigenvalue condition (eqn. 9) and solving for K.

TRANSPORT EQUATION

The second governing equation is a momentum balance for the electrons. It can be written as a generalized Ohm's law and is referred to in this work as the transport equation. The transport equation is obtained by solving the electron flux (eqn. 2) for the electric field and then integrating this over the length of the inter-electrode distance to obtain the voltage drop across the plasma.

$$\Gamma_e = - \mu_e (n \, d\Psi/dx + (kT_e/e) \, dn/dx) \tag{12}$$

$$d\Psi/dx = - \Gamma_e/\mu_e - (kT_e/e) (1/n) \, dn/dx \tag{13}$$

$$\Psi_0 - \Psi_1 = - \int \Gamma_e (1/\mu_e n) dx - (kT_e/e) \ln (n_d/n_0) \tag{14}$$

$$\Psi_0 - \Psi_1 = J R - (kT_e/e) \ln (n_d/n_0) \quad (15)$$

where n_0 and n_d are the electron densities at $x = 0$ and $x = d$.

The first term in the preceding expression is the voltage drop across the plasma due to ohmic heating. The second term represents the voltage drop due to concentration gradients (diffusion drop). To calculate the resistance in the plasma R, the first term in eqn. 14 must be integrated over the length of inter-electrode space. If the resistance is to be valid over a broad range of conditions then the electron mobility that appears in the integral must include the effects of both electron-atom collisions and electron-ion collisions.

In evaluating this integral it is convenient to make use of an approximate superposition rule for mobilities:

$$1/\mu_e = 1/\mu_{ea} + 1/\mu_{ei} \quad (16)$$

where μ_{ea} and μ_{ei} are the mobilities resulting from electron-atom collisions and from electron-ion collisions.

Thus: $R = R_{ea} + R_{ei}$

$$R = \int 1/(e\mu_e n)dx = \int 1/(e\mu_{ea} n)dx + \int 1/(e\mu_{ei} n)dx \quad (17)$$

Since conductivity due to electron-ion effects ($\sigma_{ei} = e\mu_{ei}n$) is a function of T_e and independent of the electron density it may be approximated as a constant and removed from the integration. For the conditions of interest to us the resistance from electron ion effects may be calculated from the following expression [6].

$$R_{ei} = d/\sigma_{ei}$$

$$\text{where } \sigma_{ei} = .15085\, T_e^{1.5} / \ln(\Lambda) \quad (18)$$

$$\text{and } \Lambda = 12389\, T_e^{1.5} (n/2)^{-1/2}$$

The electron-atom mobility is also a function of temperature and independent of the electron density so it too may be taken out of the integration. It is:

$$\mu_{ea} = v \lambda e/3KT \quad (19)$$

In the above expression the mean velocity of the atoms is taken from kinetic theory as $v=\sqrt{(8KT/\pi m)}$ and the mean free path is based on cross-sections found in the literature. This gives the following expression for the resistance:

$$R = i/\mu_{ea}\int(1/n)\,dx \quad + \quad d/\sigma_{ei} \tag{20}$$

If the appropriate substitutions are made the last remaining integral can be evaluated and the following expression is obtained.

$$R = d/(2\mu_{ea}n)\ \ln[4n^2 / n_0 n_d(1-m)] \quad + \quad d/\sigma_{ei} \tag{21}$$

The preceding discussion shows that the electron-atom resistance is inversely proportional to the electron density while the electron-ion resistance is practically independent of it. This being the case, at high electron densities resistance depends only on temperature. The general behavior of resistance versus electron density is plotted in figure 3. This behavior accounts for some important trends that appear in the results.

Often times the voltage across the electrodes is a more meaningful parameter than the voltage drop across the plasma. For this reason it is more useful to express the transport equation in terms of the former quantity. This requires that the voltage drops across the electrode sheaths be added to the voltage drop across the plasma. It is assumed that positive (electron retaining) sheaths are formed at the electrodes and that the flux at the sheaths is equal to the random thermal flux times a Boltzmann factor.

coll: $J = en_d v_e/4\ e^{(-eV_a/kT_e)} \quad \Rightarrow \quad V_a = kT_e/e\ \ln(en_d v_e/4J)$ (22)

emit: $J = J_E - en_d v_e/4\ e^{(-eV_a/kT_e)} \quad \Rightarrow \quad V_k = kT_e/e\ \ln[en_d v_e/4(J_E-J)]$ (23)

When the sheath voltages are added to the expression for the plasma drop derived above several of the terms combine and the result is the transport equation in its final form:

$$V_d = JR + (kT_e/e)\ \ln[J/(J_E-J)] \tag{24}$$

The first term still represents the ohmic drop while the second term now incorporates the combined effects of the sheath and diffusion drops.

ENERGY EQUATION

The third necessary equation is an energy balance for the system. In its final form this equation states that the electrical power loss in the plasma is caused by three separate processes. Energy is lost as thermionically emitted electrons deposit onto the collector, as electrons and ions that are created in the plasma diffuse to the walls and lastly as radiation out:

$$V_d\, j = 2KJ_E(T_e-T_E)/e + P_{diff} + P_{rad} \tag{25}$$

The fate of thermionically emitted electrons is illustrated in fig 4. These enter the control volume at the emitter temperature and at a rate corresponding to the saturation current given by the Richardson-Dushmann equation. Neglecting back-emission from the collector, the operating current is equal to the emitted current minus the backscattered current. Each of these currents has an energy flux associated with it consisting of potential and kinetic energy terms. The net sum of these fluxes is the first term in the energy equation. This term is only a function of the electron temperature and is referred to as the kinetic energy term.

The sum of diffusion and radiation losses is the power required to maintain the plasma ignited. This is calculated from the net rate of ionization established by the the ambipolar diffusion equation and can be expressed as the sum of the terms P_{diff} and P_{rad}.

$$P_{diff} = 4\, E_0\, K\, D_a\, N_e/d \qquad \text{where } E_0 \text{ is the ionization energy} \tag{26}$$

$$P_{rad} = E_{6p}\, N_{6p}\, gA \tag{27}$$

As one might expect the diffusion losses are inversely proportional to the inter-electrode distance. The power lost by radiation is equal to the product of the number of atoms in the first excited level, the frequency at which they emit and the energy of this radiation:
The escape factor g, is used to modify the Einstein A-coefficient so that (for example) if the gas is optically dense, g is close to zero and very little energy is lost in the form of radiation. To calculate the total number of atoms in the 6p state the expression for the local 6p population (eqn. 12) is integrated over the control volume. This integral cannot be evaluated analytically and so it is necessary to perform the integration numerically. In contrast to diffusion losses radiation losses are proportional to the inter-electrode distance. Also it should be understood that P_{rad} represents radiation emitted in all directions not just the radiation which is available for collection by photo-panels.

SOLUTION PROCEDURE

In summary of the preceeding sections, the model reduces to three nonlinear, algebraic equations in three unknowns. These are:

The ambipolar diffusion equation.

$$D_a\, d^2n/dx^2 = S\, N_{gs}\, n - \alpha n^3 \tag{7}$$

The transport equation.

$$V_d = JR + (kT_e/e)\, \ln [J/(J_E - J)] \tag{24}$$

And the energy equation.

$$V_d = (J/J_E)2K(T_e-T_E)/e + P_{diff}/J_E + P_{rad}/J_E \quad (25)$$

A solution of model requires that these be solved simultaneously. Although any method of solution will do, the method outlined below is the one that yields the most insight into the problem. This method takes advantage of the fact that the ambipolar diffusion equation is a function of only two of the three variables (T_e and n) and as such can be solved first for the ignited condition which the other two equations must satisfy (fig. 5). This reduces the problem to that of solving two equations in two unknowns. These can be solved simultaneously, however it is useful to solve them independently for the variable J and then to search for solution points (fig 6). This allows one to better understand why solutions exist where they do.

RESULTS

In the study the escape factor, the emitter temperature, and the inter-electrode distance were varied as parameters while other conditions were fixed at the following values.

Ion Temperature = 1500 K
Emitter work function = 2.4 V
Plasma drop = 1 V
Electrode area (planar) = 1 cm^2
Number density of plasma = $10^{16}/cm^3$

Briefly, the rationale that went into this selection was the following. 1) The escape factor is necessarily a parameter since the optical density is independent of the macroscopic quantities calculated in the model. For this reason it is also true that some of the curves, especially those for higher values of the escape factor, may not correspond to physically achievable solutions. 2) Varying the emitter temperature is equivalent to varying to power of the device. At the onset it was not clear what the relationship between power and efficiency would be. 3) Finally, the inter-electrode distance has a direct effect on the area available for radiation. Although distance does not appear explicitly in the plotted results it is perhaps the most important variable in determining the radiation output. One important result is that some optimal solutions involve inter-electrode spacings which are significantly greater than normally encountered.

Regarding the other conditions, the only one that warrants any mention is the plasma drop of one volt. It is this characteristic that sets the proposed device apart from conventional thermionic generators and is largely responsible for the higher radiaton eficiencies . This is discussed further in the next section.

The results of the study are summarized in figures 7,8, and 9. Three degrees of optical density were investigated corresponding to escape factor values of g=1, g=.1, g=.01. For all three cases the emitter temperature was varied from 1500K

to 2000K. And for each combination of g and T_E the inter-electrode distance was varied from the minimum value for which a solution existed to greatest possible value for which a solution existed. Since there is no simple criteria for determining beforehand where solutions exist the method used was in effect trial and error, letting the computer determine where solutions were impermissible. An important part of understanding the results involves understanding why under certain conditions solutions fail to exist.

In each case studied the three governing equations were solved for the electron temperature, electron density and operating current by the method outlined above. From this information the radiation output and the efficiencies were calculated. Radiation output is defined as P_{rad} and the overall efficiency is defined as the ratio of the radiation output to the total power consumed by the device.

$$\eta = P_{rad}/[P_{rad} + P_{diff} + J\,(\phi_A + 2kT_e/e)] \qquad (27)$$

where ϕ_A is the work functon of the anode.

The volumetric efficiency was also calculated. This is a figure of merit defined as the ratio of the radiation output to the power dissipated inside the plasma.

$$\eta_{vol} = P_{rad}/JV_d \qquad (28)$$

The data points that are plotted correspond to the solutions for which the efficiency was the greatest. In each case these coincide with the solutions for which the interelectrode spacing was also the greatest. That this is hardly a coincidence is taken up in the following section.

DISCUSSION

Two results of this study are of particular interest. These are that radiation efficiencies on the order of thirty percent are attainable and that this maximum is independent of the optical density of the gas. High radiation efficiencies result from the greater plasma drop that characterizes the proposed device. A greater plasma drop means that more energy is dissipated *inside* the plasma. additionally it happens that a larger fraction of this energy is dissipated as radiation. That this should be the case is not obvious and is explained in the following section.

One important consequence of a higher plasma drop is that solutions exist at greater inter-electrode spacings. This is because larger plasma drops can overcome a greater resistance. Ultimately it is the larger inter-electrode spacings that account for the higher radiation efficiencies. As mentioned earlier energy is lost from the plasma in three seperate processes. These are illustrated in figure 10 which shows the contributions of the various terms to the energy equation. The kinetic energy term describes the energy that is lost as

emitted electrons are deposited onto the collector. P_{diff} represents diffusion losses to the walls and P_{rad} is the energy lost as radiation. Since the electron temperature is determined by the ignited condition, the kinetic-energy term is independent of the plasma drop. Radiation and the diffusion losses on the other hand depend strongly on the inter-electrode spacing. At short distances diffusion losses are very high. At large distances radiation losses dominate. From this discussion one can understand why at higher voltage drops a greater percentage of the plasma losses will occur as radiation and consequently radiation efficiencies will be greater.

This does not yet explain why the maximum radiation efficiency is independent of the optical thickness of the gas, for Intuitively, one would expect optically dense gasses to be less efficient at producing radiation. The explanation lies two parts. First, optically dense solutions exist at greater distances then optically thin solutions. This can be understood by referring once again to figure 10 and noting that since Prad is proportional to the escape factor g, a smaller slope on the Prad line will cause the energy equation to intersect the tranport equation at a greater distance. The consequence of this is that *net* radiation output is not proportional to the escape factor. In some cases studied, reducing the escape factor by a factor of 100 had the effect of reducing the radiation output by only a factor of 5. In addition to this there is the fact that optically dense gasses produce their maximum radiation at lower value of the emitted current (fig 9) and hence at lower input powers.

It is interesting to compare the proposed device with a Carnot engine operating between the two electrode temperatures. If we take the collector temperature to be such that the emitted current from the collector is one percent of the emitted current from the emitter, then the device is operating between 1500K and 745K. The Carnot efficiency between these two temperatures is 50.3% and so our device is operating at 56% of Carnot efficiency.

The results then indicate that high radiation efficiencies can be obtained from cesium thermionic devices provided that there is an adequate votage drop across the plasma. The value of one volt that was used in the study is a reasonable value in that it can be generated without any electrical power input to the device. High efficiencies come by virtue of the higher inter-electrode distances which the solutions will accomodate. One particularly positive result is that efficiencies on the order of 30% can be obtained even with optically dense gasses. Since this is most likely to be the case under conditions of interest, this study lends some theoretical support to the proposed device.

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FIGURES

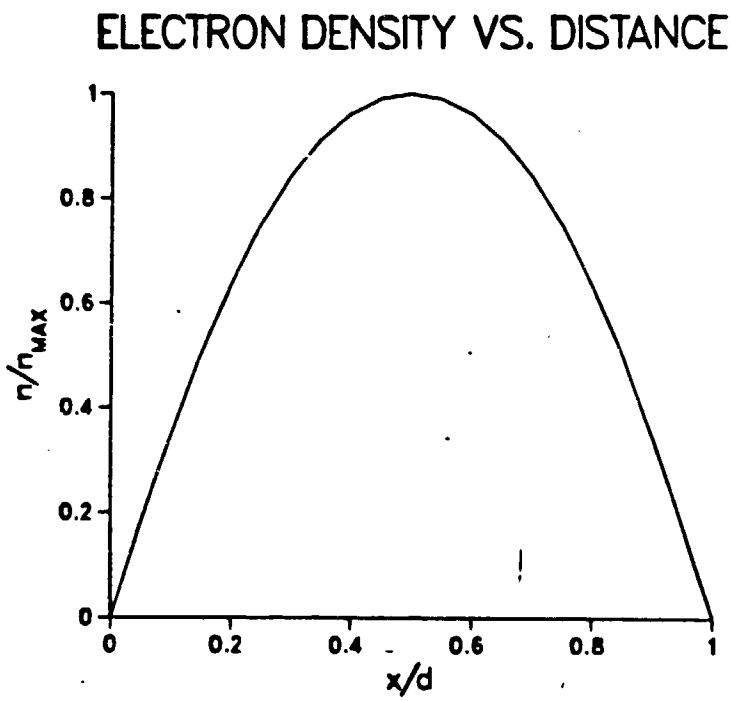

Figure 1. Electron distribution under non-Saha conditions.

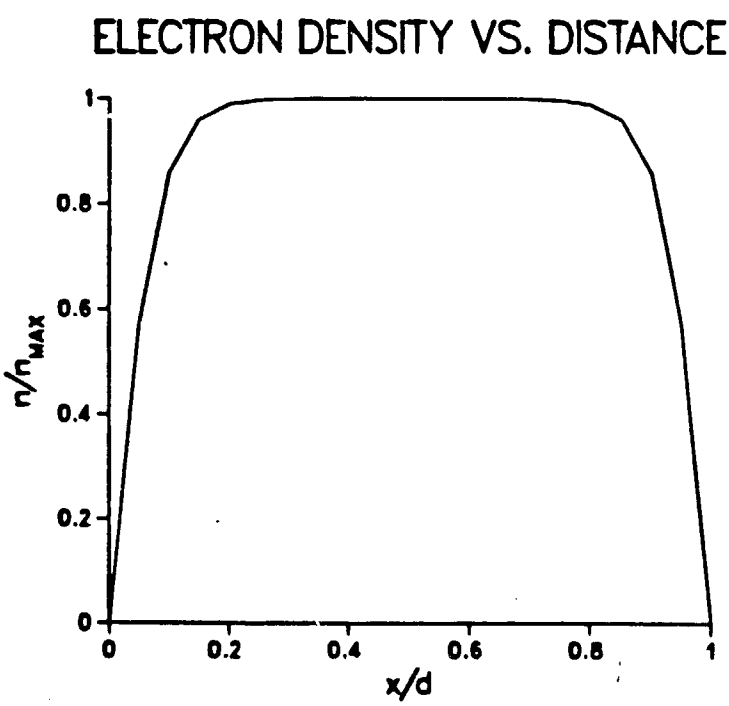

Figure 2. Electron distribution under Saha conditions.

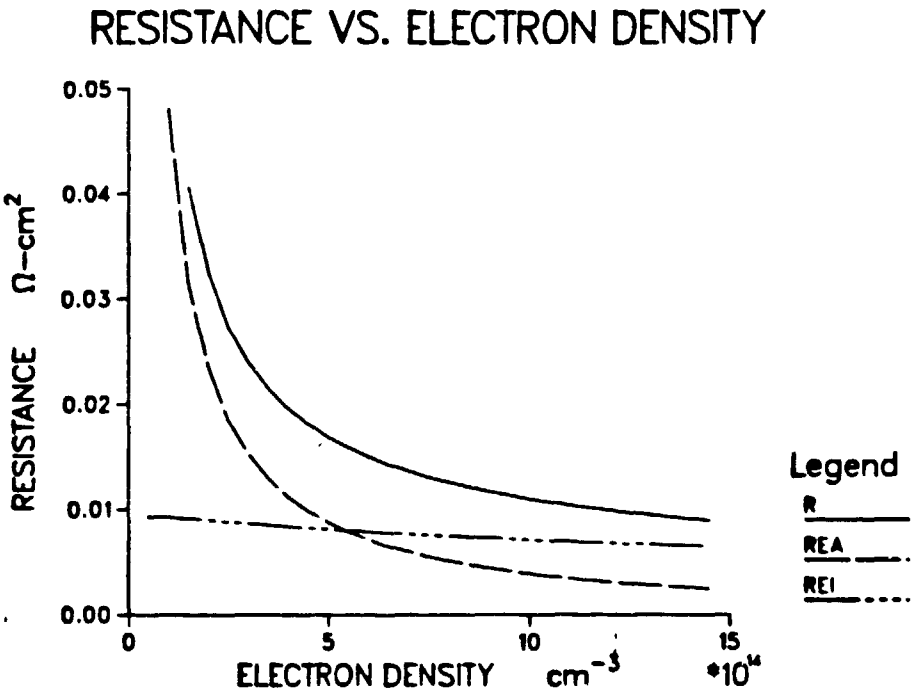

Figure 3. Plasma resistance vs. electron density.

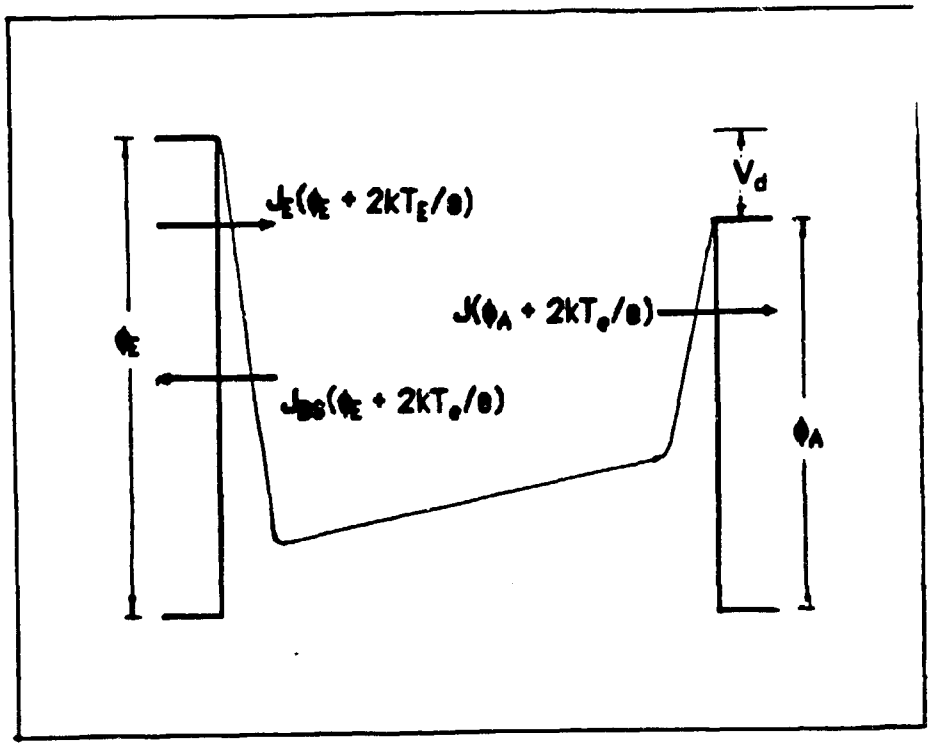

Figure 4. Energy fluxes for emitted electrons.

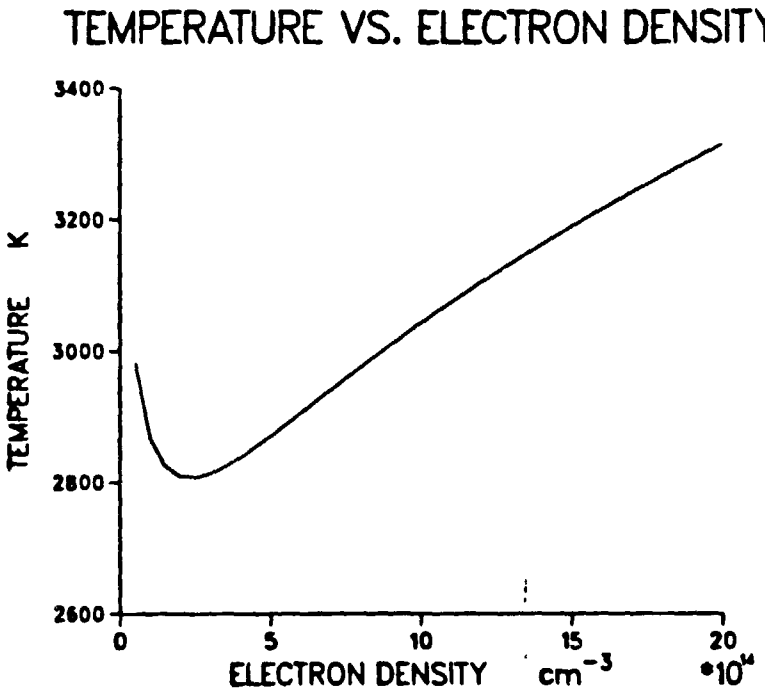

Figure 5. Solution to the ambipolar diffusion equation.

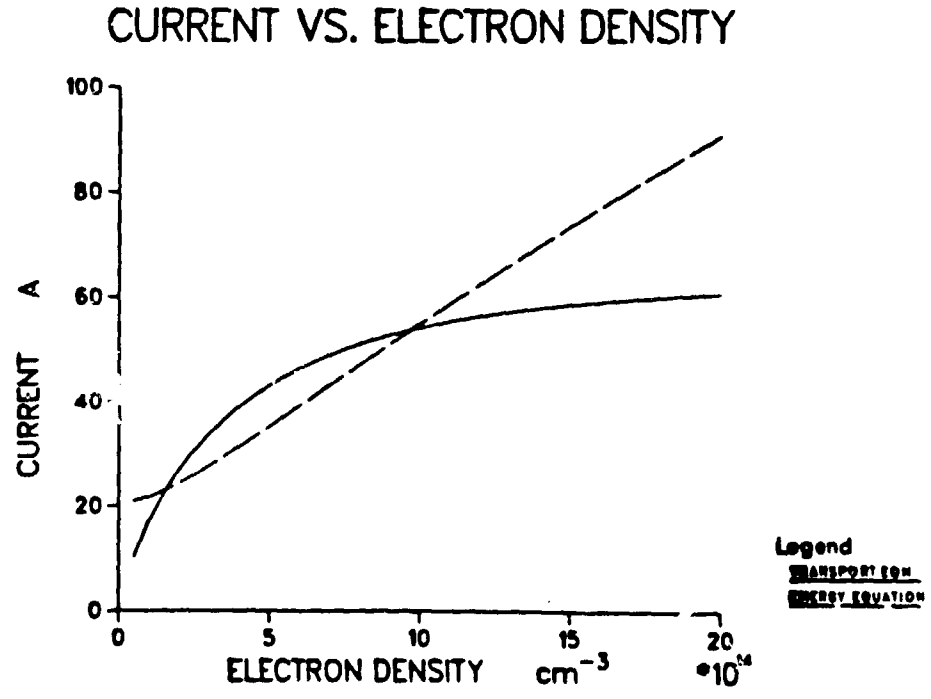

Figure 6. Solutions to the transport and energy equations.

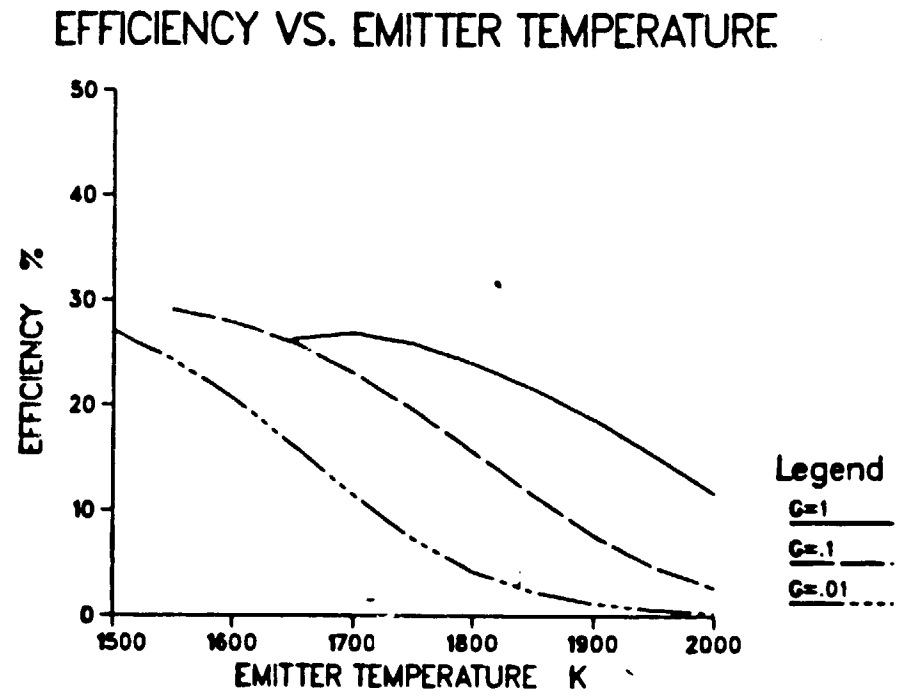

Figure 7. Efficiency vs. emitter temperature.

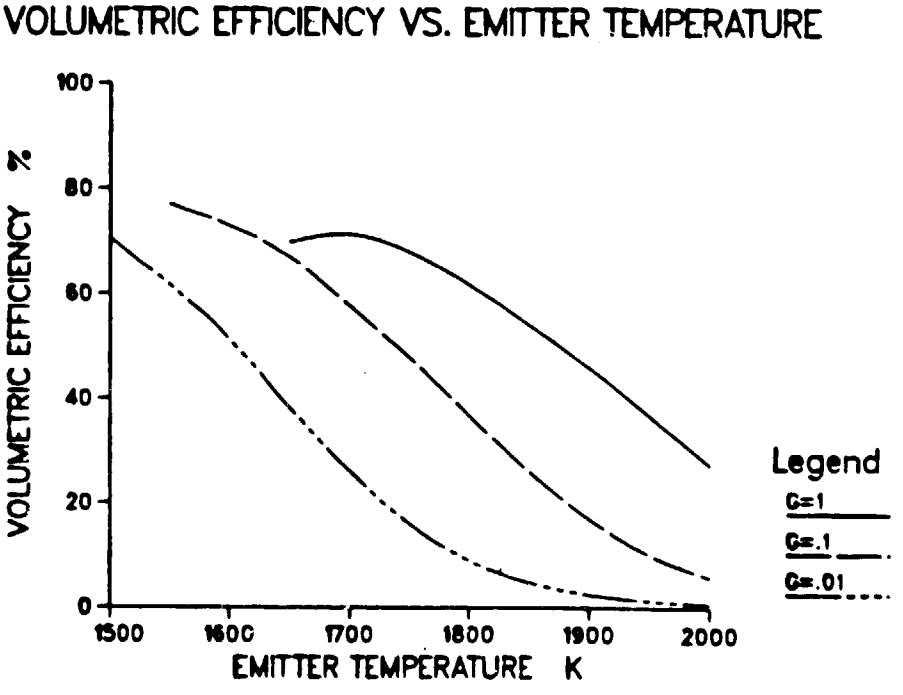

Figure 8. Volumetric efficiency vs. emitter temperature.

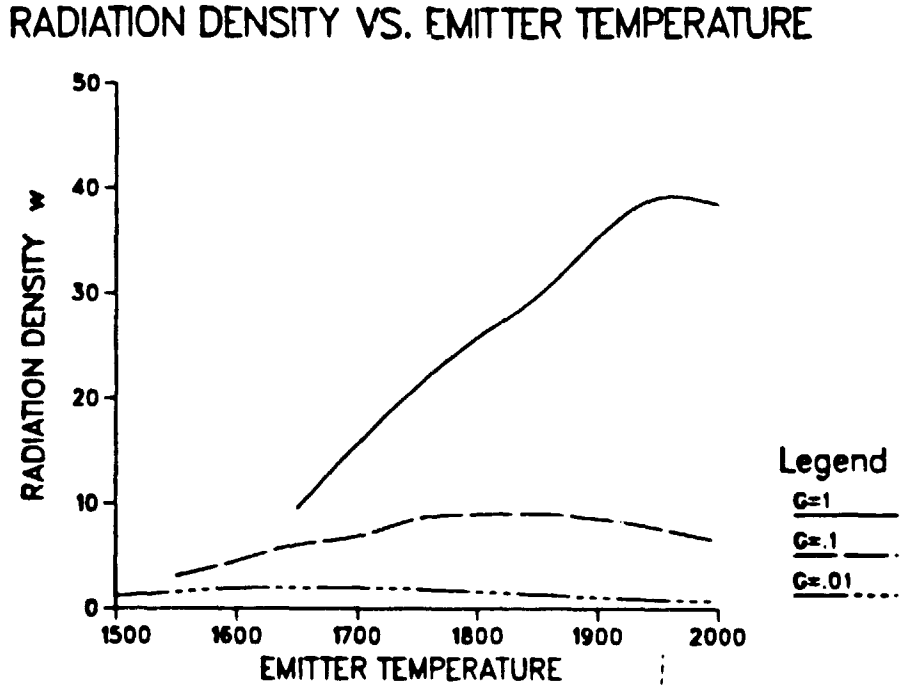

Figure 9. Radiation output vs. emitter temperature.

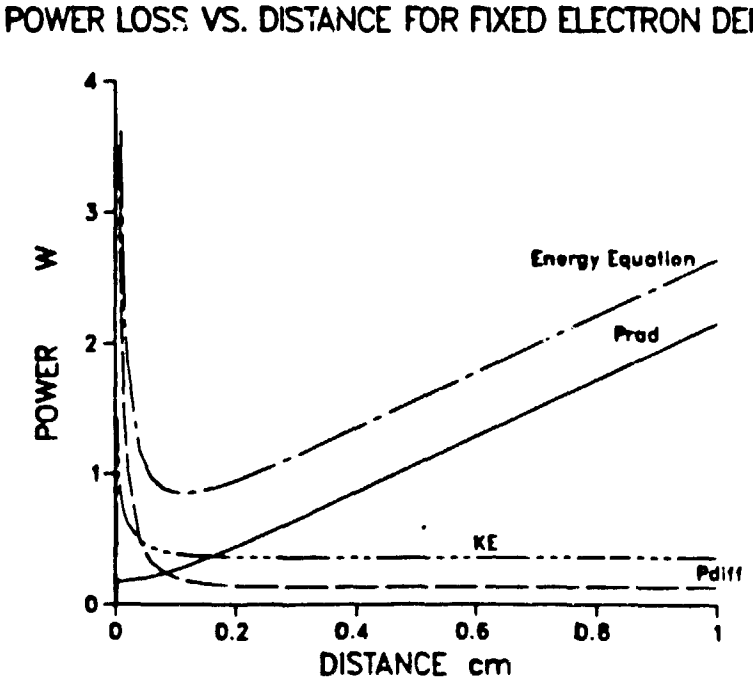

Figure 10. Energy losses as a function of inter-electrode spacing.

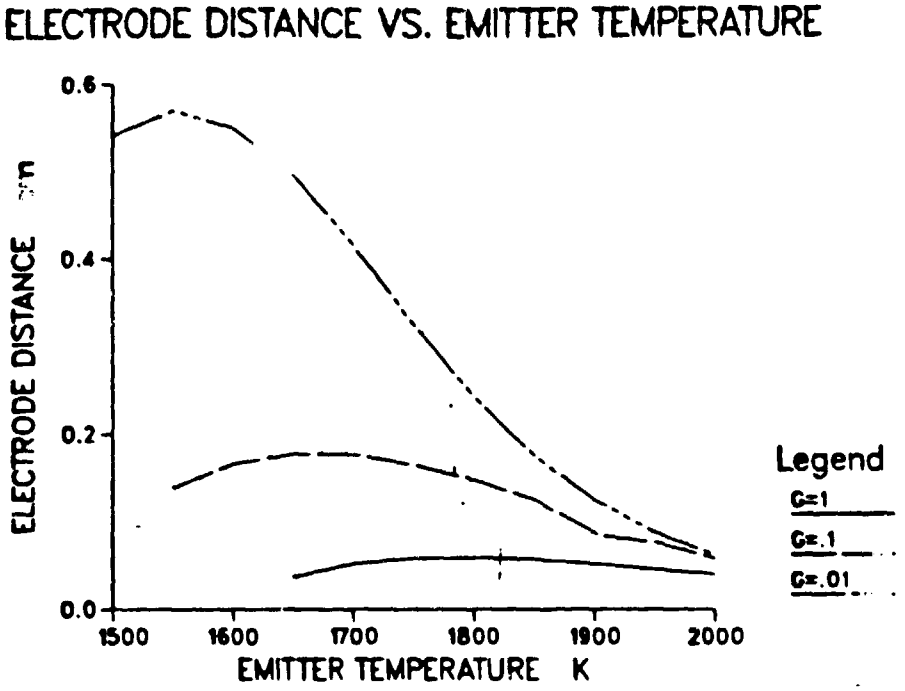

Figure 11. Maximum inter-electrode spacing vs. emitter temperature.

APPENDIX A: REACTION RATE CONSTANTS FOR EXCITED STATE KINETICS

To derive expressions for the ionizaton and recombination rate coefficients S and α, it is necessary to know the atomic states that are present in the plasma and the reactions that occur between these states. In this work three states and three reactions (shown below) were considered important in determining the ionization/recombination kinetics of the plasma.

$$Cs_{6s} + e^- \Leftrightarrow Cs_{6p} + e^-$$

$$Cs_{6p} \Rightarrow Cs_{6s} + h\nu$$

$$Cs_{6p} + e^- \Leftrightarrow Cs^+ + 2e^-$$

Based on these reactions the rate equations for the atomic states are written as follows:

$$dN_{6s}/dt = -N_{6s}nK_{6s\text{-}6p} + N_{6p}nK_{6p\text{-}6s} + N_{6p}gA$$

$$dN_{6p}/dt = -nN_{6p}K_{6p\text{-}6s} + nN_{6s}K_{6s\text{-}6p} - N_{6p}gA - nN_{6p}K_{6p\text{-}c} + n^3K_{c\text{-}6p}$$

where $K_{6s\text{-}6p}$ is the foreward rate constant for the reaction between the ground state and the 6p state, and $K_{6p\text{-}6s}$ the backwards rate constant. The subscript c refers to the continuum.

Under quasi-steady conditions the populations of the excited states change slowly in time. With this assumption the rate equation for the 6P state reduces to an algebraic expression that can be solved for the 6P population:

$$N_{6p} = R_o + R_1N_{6s}$$

where: $$R_0 = n^3K_{c\text{-}6p} / (nK_{6p\text{-}6s} + nK_{6p\text{-}c} + gA)$$

$$R_1 = nN_{6s}K_{6s\text{-}6p} / [nK_{6p\text{-}6s} + nK_{6p\text{-}c} + gA]$$

Substituting the expression for N_{6p} into the rate equation above then yields the desired form from which the coefficients S and α can be factored:

$$dN_{6s}/dt = -nSN_{6s} + \alpha n^3$$

where: $S = K_{6s\text{-}6p} - R_1 (K_{6p\text{-}6s} + gA / n)$

$\alpha = R_0 (nK_{6p\text{-}6s} + gA) / n^3$

To evaluate these expressions it is necessary to know the foreward and backward rate constants for the reactions between adjacent states.

$K_{6s\text{-}6p}$ is the rate constant for collisional excitation from the ground state to first excited level. It is found by integrating the product of the cross-section for the reaction [ref. 7] and the velocity distribution of the electrons (assumed Maxwellian) over the range of energies. Assuming that the cross-section increases linearly from the threshold the result is:

$$K_{6s\text{-}6p} = \exp(-CV_0^2) [B (V_0^4 + V_0^2/C + 1/C^2) + A (V_0^2 + 1/C)]/2C$$

where $A = (m_e/2\pi kT_e)^{3/2} 4\pi\sigma_0$

$B = (m_e/2\pi kT_e)^{3/2} 2\pi m_e \Delta\sigma/\Delta E$

$C = m_e/2kT_e$

and $V_0 = \sqrt{2m_e/E}$, $E = 1.33$ eV

$\sigma_0 = -1.02\times10^{-14}$ cm^2

$\Delta\sigma/\Delta E = 71\times10^{-16}$ cm^2/eV

From this expression it is possible to obtain an expression for the backward reaction, $K_{6p\text{-}6s}$. The quasi-equilibrium assumption states that:

$$N_{6s} K_{6s\text{-}6p} = N_{6p} K_{6p\text{-}6s}.$$

If equilibrium with the continuum is also assumed then the ratio of the populations can be gotten from the Saha equation:

$$N_{6S}/N_{6P} = (1/3) \exp(E_{6s\text{-}6p}/kT_e)$$

This gives the following expression for $K_{6p\text{-}6s}$:

$$K_{6p\text{-}6s} = K_{6s\text{-}6p} (1/3) \exp(E_{6s\text{-}6p}/kT_e).$$

$K_{c\text{-}6p}$ is an effective rate constant for collisional de-excitation from the continuum to the 6P state. This is given in reference 8 as:

$$K_{c\text{-}6p} = .46\ (e^2/(4\pi\varepsilon_0 kT_e))^5\ (8kT_e/\pi m_e)^{1/2}$$

Invoking once again quasi-equilibrium and the Saha equation yields:

$$K_{6p\text{-}c} = K_{c\text{-}6p}\ (2\pi h^2/mkT_e)^{3/2}\ \exp\ (E_{6p}/kT_e)/(2L_{6p} + 1)$$

APPENDIX B: BOUNDARY CONDITIONS

In this work two different sets of boundary conditions are used for the electron density at the electrodes. In the lowest order approximation, the walls are considered absorbing so that:

$$n_0 = 0 \qquad n_d = 0$$

where the subsctipts 0 and d are used to denote the emitter are collector sheaths.

This assumption works well in finding the solution to the ambipolar diffusion equation. It is not adequate however for expressions in which terms on the order of λ/d are important. This is the case when evaluating the resistance of the plasma. For here the assumption of zero electron density at the walls will lead to an infinite resistance.

Appropriate nonzero boundary conditions are derived in reference 2 by developing expressions for electron and ion fluxes at the walls. These are the following:

$$n_0 = 4\hat{n}/v_i \sqrt{T_i/T_e} \; (D_a \, dn/dx + (\mu_i/\mu_e)\Gamma_e)$$

$$n_d = 4\hat{n}/v_i \sqrt{T_i/T_e} \; (-D_a \, dn/dx - (\mu_i/\mu_e)\Gamma_e)$$

Where: $\hat{n}$ is the unit vector normal to the electrodes. On the cathode surface $\hat{n}$ is directed into the plasma. On the anode it is directed out of the plasma. Γ_e is the electron flux emitted thermionically from the cathode, given by the Richardson-Dushmann equation

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                       C
C   PROGRAM PL1                                                         C
C                                                                       C
C   WRITTEN BY F. STEFANI, 1984.                                        C
C                                                                       C
C   THIS PROGRAM CALCULATES THE OPERATING CONDITIONS FOR A THERMIONIC    C
C   DEVICE BASED ON AN ANALYTICAL MODEL DEVELOPED BY J.L. LAWLESS.      C
C   SOLUTIONS ARE FOUND BY FIRST CALCULATING THE TEMPERATURE THAT       C
C   SATISFIES THE IGNITED CONDITION AT A GIVEN ELECTRON DENSITY, AND    C
C   THEN SOLVING THE TRANSPORT AND ENERGY EQUATIONS FOR THE OPERATING   C
C   CURRENT BASED ON THAT TEMPERATURE.  THIS IS DONE FOR A RANGE OF     C
C   ELECTRON DENSITIES AND THE POINT WHERE THE THREE EQUATIONS ARE      C
C   SATISFIED SIMULTANEOUSLY CAN BE FOUND BY INSPECTION OF THE OUTPUT.  C
C   THE PROGRAM ALSO CALCULATES THE RADIATION EFICIENCY OF THE DEVICE.  C
C                                                                       C
C   [REF:   AN ANALYTICAL MODEL OF THERMIONIC DISCHARGES, J.L. LAWLESS, C
C           CARNEGIE-MELLON UNIVERSITY]                                 C
C                                                                       C
C   SOME ADDITIONAL COMMENTS:                                           C
C                                                                       C
C   * THE FOLLOWING QUANTITIES ARE ASSIGNED FIXED VALUES IN THE PROGRAM: C
C       PLASMA VOLTAGE DROP          VD = 1.0      [V]                  C
C       ION TEMPERATURE              TI = 1500.    [K]                  C
C       # OF NUCLEI                NNUC = 1.0E16  [CM**-3]             C
C       ELECTRODE AREA            AELEC = 1.0      [CM**2]             C
C                                                                       C
C   * ANODE WORK FUNCTION: WFA = WFC - VD      (WFC; CATHODE WK. FTN.)  C
C                                                                       C
C   * RECOMBINATION IS ASSUMED THROUGHOUT.                              C
C                                                                       C
C   * N6P IS COMPUTED BY NUMERICALLY INTEGRATING THE LOCAL 6P POPULATION C
C     OVER THE INTER-ELECTRODE SPACE.                                   C
C                                                                       C
C   * RADIATION EFFICIENCY IS CALCULATED IN TWO WAYS:                   C
C                              EFF1=PRAD/(PRAD+PDIFRJ(VA+2KTE/CHG))     C
C                              EFF2=PRAD/(RJ*VD)                        C
C                                                                       C
C   * SIMPSONS RULE HAS BEEN MODIFIED TO TAKE ADVANTAGE OF SYMMETRY.    C
C                                                                       C
C   * OUTPUT IS WRITTEN TO FILE PL1.DAT                                 C
C                                                                       C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   LIST OF VARIABLES
C
C   INPUTS:
C
C   DIST      INTER-ELECTRODE GAP                          [CM]
C   TEMIT     EMITTER TEMPERATURE                          [K]
C   WFC       EMITTER WORK FUNCTION                        [V]
C   G         RADIATION ESCAPE FACTOR
C
C   OUTPUTS:
C
C   NE        ELECTRON DENSITY                             [CM**-3]
C   XN1       ELECTRON DENSITY AT THE ANODE                [CM**-3]
C   TE        ELECTRON TEMPERATURE                         [K]
C   SM1       1 - MODULUS
C   XJEN      OPERATING CURRENT BASED ON ENERGY EQN.       [A/CM**2]
```

```
C   RJ        OPERATING CURENT BASED ON TRANSPORT EQN.     [A/CM**2]
C   R         PLASMA RESISTANCE                            [OHMS]
C   REA       RESISTANCE DUE TO ELECTRON-ATOM COLLISIONS   [OHMS]
C   REI       RESISTANCE DUE TO ELECTRON-ION COLLISIONS    [OHMS]
C   VK        CATHODE SHEATH VOLTAGE                       [V]
C   VA        ANODE SHEATH VOLTAGE                         [V]
C   EFF1      RADIATION EFFICIENCY (SEE HEADER)
C   EFF2      VOLUMETRIC EFFICIENCY  "    "
C   SAHA      FLAG FOR SAHA CONDITION:
C                   SAHA=0 => PLASMA IS RECOMBINING
C                   SAHA=1 => RECOMBINATION NEGLIGIBLE
C                   SAHA=2 =>    "     "      "   "   ,  SM1 = 0
C
C   PHYSICAL CONSTANTS:
C
C   ME        ELECTRON MASS                                [KG/1.602E-15]
C   MI        ION MASS                                     [KG/1.602E-15]
C   CHG       FUNDAMENTAL CHARGE                           [C/1.602E-19]
C   K         BOTZMANNS CONST                              [eV/K]
C   E6P       ENERGY Cs6p-Cs6s                             [eV]
C   ESP       ENERGY Cs6s-Cs6p                             [eV]
C   ACOEFF    EINSTEIN A COEFFICIENT                       [1/S]
C   LS        DEGENERACY OF THE GROUND STATE
C   LP        DEGENERACY OF THE FIRST EXCITED LEVEL
C   SIGIA     CROSS-SECTION FOR ION-ATOM COLLISIONS        [CM**2]
C   SIGEA     CROSS-SECTION FOR ELECTRON-ATOM COLLISIONS   [CM**2]
C   SIG0      Y-INTERCEPT OF CROSS-SECTION VS. ENERGY CRV. [CM**2]
C   DLTXCT    SLOPE OF CROSS-SECTION VS. ENERGY CURVE.     [CM**2/EV]
C
      EXTERNAL F
      EXTERNAL SIMPER
      EXTERNAL TMIN
      EXTERNAL XJT
      EXTERNAL TDIFF
      REAL KSP,KPS,KSPA,KSPB,KCP,KPC,NGS,NE,N6P,MUI,MUE,MFPI,MFPE
      REAL MI,KSPCON,ME,HBAR,PI,LS,LP,K,NNUC,JE,JNET,QP,MOD,JION
      DIMENSION DDIST(15),DTEMIT(15),DATWFC(15),DATG(15)
      COMMON/C1/ME,MI,CHG,EPS0,K,HBAR,E6P,ESP,ACOEFF,PI,LS,
     *LP,SIGIA,SIGEA,DLTXCT,VD,JE,TI,TEMIT,AELEC,DIST,G,NE,
     *SIGSP,SIG0,NNUC
      COMMON/C3/KPS,KPC,KSP,KCP,NGS
      COMMON/C4/MOD,QP,SM1
      COMMON/C2/TE,DA,MUE
      COMMON/C5/REA,REI,R
      COMMON/C6/SAHA
      OPEN(UNIT=7,DEVICE='DSK',FILE='PL1')
C
C  PHYSICAL CONSTANTS
C
      ME=5.686E-16
      MI=1.377E-10
      CHG=1.0
      EPS0=5.527E5
      K=8.6176E-5
      HBAR=6.5784E-16
      E6P=2.4625
      ESP=1.4319
      E0=E6P+ESP
      ACOEFF=3.57E7
      PI=3.14159265
```

```
      LS=0.0
      LP=1.0
      SIGIA=1.2E-13
      SIGEA=4.E-14
      SIGSP=50.E-16
      SIGO=-1.01E-14
C
C   SLOPE OF CROSS-SECTION FOR 6S-6P TRANSITION IN CESIUM (DLTXCT)
C   OBTAINED FROM NORCROSS & STONE: PP. 667, FIGURE 1, CURVE #4; REF. 29;
C   NOLAN AND PHELPS, 1965.
C
      DLTXCT=71E-16
C
C  OTHER INPUTS
C
      VD=1.0
      TI=1500.
      AELEC=1.
      NNUC=1.E16
C
C  READ AND STORE RUN PARAMATERS
C
      ICOUNT=0
100   ICOUNT=ICOUNT+1
      WRITE(5,475)ICOUNT
475   FORMAT(2X,'INPUT DATA:DIST,TEMIT,WFC,G',12X,'CASE',2I)
      READ(5,476)DDIST(ICOUNT),DTEMIT(ICOUNT),DATWFC(ICOUNT),
     $DATG(ICOUNT)
476   FORMAT(4G)
      WRITE(5,700)DDIST(ICOUNT),DTEMIT(ICOUNT),DATWFC(ICOUNT),
     $DATG(ICOUNT)
700   FORMAT(2X,4(2X,E12.5))
      IF(DTEMIT(ICOUNT).NE.0.0)GO TO 100
C
C  MAIN PROGRAM DO LOOP.
C
      ICOUNT=ICOUNT-1
      DO 10 NX=1,ICOUNT
      DIST=DDIST(NX)
      TEMIT=DTEMIT(NX)
      WFC=DATWFC(NX)
      G=DATG(NX)
      JE=120.*TEMIT**2*EXP(-WFC/(K*TEMIT))
C
C  WRITE RELEVANT INPUT DATA AND HEADER TO OUTPUT FILE.
C
      WRITE(7,480)NX
      WRITE(7,481)DIST
      WRITE(7,482)TEMIT
      WRITE(7,483)G
      WRITE(7,484)WFC
      WRITE(7,485)JE
      WRITE(7,486)VD
      WRITE(7,487)
480   FORMAT(1H1,'   INPUTS PL1      CASE',I2,/)
481   FORMAT(2X,'DIST ='F5.3)
482   FORMAT(2X,'TEMIT='F8.2)
483   FORMAT(2X,'G    ='F5.3)
484   FORMAT(2X,'WFC  ='F5.2)
485   FORMAT(2X,'JE   ='F6.2)
```

```
486   FORMAT(2X,'VD   ='F5.2)
487   FORMAT(' ',2X,2X,'NE',8X,' XN1'
     *,9X,'TE',12X,'SM1 ',
     *8X,'XJEN',8X,'RJ',6X,'R',8X,'REI',8X,'VK',8X,'VA',6X,
     *'EFF',2X,'PRAD/PIN',1X,'SAHA')
C
C   MAIN DO LOOP INDEXING NE
C   OPTION 1:  SCALING ORDERS OF MAGNITUDE
C
C      DO 11 J=1,3
C      DO 12 I=1,9
C      NE=.1E12*FLOAT(I)*10.**FLOAT(J)
C+
C   OPTION 2:  FIXED INCREMENT
      DO 11 I=1,30
      NE=.05E15*FLOAT(I)
C
C          ***SOLVE THE AMBIPOLAR DIFFUSION EQUATION***
C
C
C   FIRST FIND 'TMIN'. WHICH IS THE SOLUTION TO THE AMBIPOLAR DIFFUSION
C   EQUATION NEGLECTING RECOMBINATION.
C   THIS TEMPERATURE IS THE LOWER BOUND IN THE ROOTFINDING
C   FOR THE AMBIPOLAR DIFFUSION EQUATION WITH RECOMBINATION
C   (WHICH IS EVALUATED BY THE FUNCTION TDIFF).
C
      A=1500.
      B=5500.
      SAHA=0.
      CALL BISECT(TMIN,A,B,.5,IFLAG)
      IF(IFLAG.LT.2)GO TO 46
      WRITE(7,333)IFLAG
333   FORMAT(2X,'ROOTFINDING FOR TMIN BOMBED',2X,2I)
      GO TO 9
46    CONTINUE
      TSAHA=(A+B)/2.
C
C   SOLVE THE AMBIPOLAR DIFFUSION EQUATION AND CALCULATE THE FOLLOWING
C   QUANTITIES IN THE PROCESS.
C
C
C   IF ROOTFINDING BOMBS THEN THE SOLUTION IS TOO CLOSE TO THE
C   LOWER LIMIT CALCULATED BY TMIN. CONDITIONS ARE CALCULATED
C   AT THE SAHA LIMIT. SAHA FLAG IS SET TO 1.
C
      A=B
      B=5500.
      CALL BISECT(TDIFF,A,B,.01,IFLAG)
      IF(IFLAG.LT.2)GO TO 47
C      WRITE(7,334)IFLAG
C334   FORMAT(2X,'ROOTFINDING FOR DIFF. EQN. BOMBED:
C     *DEFAULT TO SAHA VALUES',2X,2I)
      SAHA=1.
      TE=TSAHA
      GO TO 212
47    CONTINUE
      TE=(A+B)/2.
      ERROR=ABS(A-B)/2.
C
C   CALCULATION OF THE RATE CONSTANTS.
```

```
C
212   CC=ME/(2.*K*TE)
      DD=ESP/(K*TE)
      VMSQ=2.*ESP/ME
C
C     FORMULATION FOR KSP BASED ON VARIABLE CROSS-SECTION.
      AA=(CC/PI)**(3./2.)*4.*PI*SIG0
      KSPA=(AA/(2.*CC))*EXP(-CC*VMSQ)*(VMSQ+1./CC)
      BB=(CC/PI)**1.5*2.*PI*(ME*1.E20)*DLTXCT
      KSPB=BB/(2.*CC)*EXP(-CC*VMSQ)*((VMSQ/1.E20)*VMSQ+2.*VMSQ
     */(CC*1.E20)+2./(CC*1.E20*CC))
      KSP=KSPA+KSPB
      KPS=KSP*EXP(DD)*(2.*LS+1.)/(2.*LP+1.)
      KCP=.46*EXP(.5*(ALOG(8.*TE/PI)+ALOG(K)-ALOG(ME))-5.*(ALOG(4.*PI*TE
     *)+ALOG(EPS0*K)))
      KPC=KCP/((2.*LP+1.)*(2.*PI*(HBAR/ME)*(HBAR/K)/TE)**(3./2.)
     **EXP(E6P/(K*TE)))
      R1=KSP/(KPC+KPS+G*ACOEFF/NE)
      S=KSP-R1*(KPS+G*ACOEFF/NE)
      R0=NE**2.*KCP/(KPS+KPC+G*ACOEFF/NE)
      ALPHA=R0/NE*(NE*KPS+G*ACOEFF)/NE**2.
      NGS=NNUC-NE
      MFPI=1./(NNUC*SIGIA)
      MFPE=1./(NNUC*SIGEA)
      MUI=2.*CHG*MFPI/(3.*(K*TI*PI*MI/2.)**.5)
      MUE=2.*CHG*MFPE/(3.*(K*TE*PI*ME/2.)**.5)
      DA=MUI*K*(TE+TI)/CHG
      IF(SAHA.EQ.0.0)GO TO 214
      QP=SQRT(ALPHA/(8.*DA))*NE*DIST
      IF(QP.LE.20.)GO TO 213
C
C  IN THE CASE WHERE SM1 (SM1=1-MODULUS) IS TOO LOW TO BE CALCULATED BY
C  THE A.G.M. LADDER (QP>20) THEN SM1 IS SET TO ZERO AND SAHA FLAG IS 2.
C
      SAHA=2.
      SM1=0.
      MOD=1.
      N6P=(R0+R1*NGS)*AELEC*DIST
      GO TO 215
C
C       ***CALCUALTE OUTPUT CURRENT BY THE ENERGY EQN.***
C
C
C  OUTPUT CURRENT CALCULATION (VARIABLE XJEN):
C  N6P CALCULATED BY INTEGRATION OVER THE LENGTH.
C  THE INTEGRAND, N6P(X), IS CALCULATED BY THE EXTERNAL FUNCTION F.
C
213   SM1=16.*EXP(-2.*QP)
      MOD=1.-SM1
214   CALL SIMPER(F,AREA,DIST)
      N6P=AREA
215   PDIFF=4.*E0*AELEC*DA*QP*NE/DIST*1.602E-19
      PRAD=G*ACOEFF*ESP*N6P*1.602E-19
      Q=PDIFF+PRAD
C
      XJEN1=2.*JE*K*(TE-TEMIT)/(VD*CHG)
      XJEN2=Q/VD
      XJEN=XJEN1+XJEN2
C
C         ***SOLVE THE TRANSPORT EQUATION***
```

```
C
C
C  CALCULATE RJCRIT, THE UPPER LIMIT FOR ROOTFINDING, BASED ON THE CONDITION
C  THAT THE ELECTRON DENSITY AT THE ANODE XN1, MUST BE POSITIVE.
C
      RJCRIT=(SIGIA/SIGEA)*SQRT((TI/TE)*(MI/ME))
     **(CHG*AELEC)*DA*NE*(PI/DIST)*1.602E-19
      RJ=RJCRIT-RJCRIT*(.00001)
      IF(RJ.GE.JE)RJ=JE-JE*(.001)
C
C   ROOTFINDING FOR OUTPUT CURRENT BASED ON THE TRANSPORT EQN.
C
C   TRANSPORT EQUATION IS SOLVED BY EXTERNAL FUNCTION XJT.
C   THE VARABLE USED FOR THE OUTPUT CURRENT IS RJ.
C
      A=RJ/100.
      B=RJ
      CALL BISECT(XJT,A,B,.001,IFLAG)
      IF(IFLAG.LT.2)GO TO 99
      WRITE(7,717)NE,IFLAG
717   FORMAT(' ',X,E7.1,' ROOTFINDING XJT BOMBED',2X,'IFLAG=',2I)
      GO TO 9
99    CONTINUE
      RJ=(A+B)/2.
      ERROR=ABS(A-B)/2.
C
C  CALCULATION OF SHEATH VOLTAGES
C
      VEBAR=SQRT(8.*K*TE/(PI*ME))
      VIBAR=SQRT(8.*K*TI/(PI*MI))
      ALFA=SQRT(TE*2.*PI/TI)
      XNA=(4./(ALFA*VIBAR))*DA*2.*QP*NE/DIST
      XNB=(SIGEA/SIGIA)*SQRT((TE/TI)*(ME/MI))*RJ/(1.602E-19*AELEC)
     **(4./(ALFA*VIBAR))
      XN1=XNA-XNB
      XN0=XNA+XNB
      ARGVK=XN0*VEBAR*CHG*1.602E-19/(4.*(JE-RJ))
      VK=K*TE/CHG*ALOG(ARGVK)
      ARGVA=XN1*VEBAR*CHG*1.602E-19/(4.*RJ)
      VA=K*TE/CHG*ALOG(ARGVA)
C
C WFC IS THE CATHODE WORK FUNCTION
C WFA IS THE ANODE WORK FUNCTION.
C
      WFA=WFC-VD
C
C CALCULATE SYSTEM EFFICIENCIES
C
      EFF1=PRAD/(PRAD+RJ*(WFA+2.*K*TE/CHG))
      EFF2=PRAD/(RJ*VD)
C
C ESTABLISH ION CURRENT
C
      JION=PDIFF/E0
      RATIO=JION/RJ
C
C WRITE CALCULATED QUANTITIES
C
      WRITE(7,761)NE,XN1,TE,SM1,XJEN,RJ,R,REI,VK,VA,EFF1,EFF2,SAHA
761   FORMAT(' ',X,E8.3,2X,E9.3,3X,F10.4,4X,E9.3,4X,F7.2,4X
```

```
     *,F7.2,3X,F5.3,3X,F7.4,3X,F7.3,3X,F7.3,3X,F5.3,3X,F6.3,3X,F2.0)
9     CONTINUE
12    CONTINUE
11    CONTINUE
10    CONTINUE
      WRITE(5,752)
752   FORMAT(2X,'  OUTPUT WRITTEN TO FILE: PL1.DAT ')
      CLOSE(UNIT=5)
      STOP
      END

      FUNCTION TMIN(TE)
C
C CALCULATES IGNITED CONDITION BASED ON NO RECOMBINATION
C
      REAL KSP,KPS,KSPA,KSPB,KCP,KPC,NGS,NE,MUI,MUE,MFPI,MFPE
      REAL MI,ME,HBAR,PI,LS,LP,K,NNUC,JE,JNET,MOD
      COMMON/C1/ME,MI,CHG,EPS0,K,HBAR,E6P,ESP,ACOEFF,PI,LS,
     *LP,SIGIA,SIGEA,DLTXCT,VD,JE,TI,TEMIT,AELEC,DIST,G,NE,
     *SIGSP,SIG0,NNUC
      CC=ME/(2.*K*TE)
      DD=ESP/(K*TE)
      VMSQ=2.*ESP/ME
      AA=(CC/PI)**(3./2.)*4.*PI*SIG0
      KSPA=(AA/(2.*CC))*EXP(-CC*VMSQ)*(VMSQ+1./CC)
      BB=(CC/PI)**1.5*2.*PI*(ME*1.E20)*DLTXCT
      KSPB=BB/(2.*CC)*EXP(-CC*VMSQ)*((VMSQ/1.E20)*VMSQ+2.*VMSQ
     */(CC*1.E20)+2./(CC*1.E20*CC))
      KSP=KSPA+KSPB
      KPS=KSP*EXP(DD)*(2.*LS+1.)/(2.*LP+1.)
      KCP=.46*EXP(.5*(ALOG(8.*TE/PI)+ALOG(K)-ALOG(ME))-5.*(ALOG(4.*PI*TE
     *)+ALOG(EPS0*K)))
      KPC=KCP/((2.*LP+1.)*(2.*PI*(HBAR/ME)*(HBAR/K)/TE)**(3./2.)
     **EXP(E6P/(K*TE)))
      R1=KSP/(KPC+KPS+G*ACOEFF/NE)
      S=KSP-R1*(KPS+G*ACOEFF/NE)
      R0=NE**2.*KCP/(KPS+KPC+G*ACOEFF/NE)
      ALPHA=R0/NE*(NE*KPS+G*ACOEFF)/NE**2.
      NGS=NNUC-NE
      TMIN=S*NGS-NE*ALPHA*NE
      RETURN
      END
C
      FUNCTION TDIFF(TE)
C
C CALCULATES IGNITED CONDITION W/ RECOMBINATION ASSUMED.
C
      REAL KSP,KPS,KSPA,KSPB,KCP,KPC,NGS,NE,MUI,MUE,MFPI,MFPE
      REAL MI,ME,HBAR,PI,LS,LP,K,NNUC,JE,JNET,MOD
      COMMON/C1/ME,MI,CHG,EPS0,K,HBAR,E6P,ESP,ACOEFF,PI,LS,
     *LP,SIGIA,SIGEA,DLTXCT,VD,JE,TI,TEMIT,AELEC,DIST,G,NE,
     *SIGSP,SIG0,NNUC
      COMMON/C4/MOD,QP,SM1
      CC=ME/(2.*K*TE)
      DD=ESP/(K*TE)
      VMSQ=2.*ESP/ME
      AA=(CC/PI)**(3./2.)*4.*PI*SIG0
      KSPA=(AA/(2.*CC))*EXP(-CC*VMSQ)*(VMSQ+1./CC)
      BB=(CC/PI)**1.5*2.*PI*(ME*1.E20)*DLTXCT
      KSPB=BB/(2.*CC)*EXP(-CC*VMSQ)*((VMSQ/1.E20)*VMSQ+2.*VMSQ
```

```
     *)/(CC*1.E20)+2./(CC*1.E20*CC))
      KSP=KSPA+KSPB
      KPS=KSP*EXP(DD)*(2.*LS+1.)/(2.*LP+1.)
      KCP=.46*EXP(.5*(ALOG(8.*TE/PI)+ALOG(K)-ALOG(ME))-5.*(ALOG(4.*PI*TE
     *)+ALOG(EPS0*K)))
      KPC=KCP/((2.*LP+1.)*(2.*PI*(HBAR/ME)*(HBAR/K)/TE)**(3./2.)
     **EXP(E6P/(K*TE)))
      R1=KSP/(KPC+KPS+G*ACOEFF/NE)
      S=KSP-R1*(KPS+G*ACOEFF/NE)
      RO=NE**2.*KCP/(KPS+KPC+G*ACOEFF/NE)
      ALPHA=RO/NE*(NE*KPS+G*ACOEFF)/NE**2.
      NGS=NNUC-NE
      MFPI=1./(NNUC*SIGIA)
      MUI=2.*CHG*MFPI/(3.*(K*TI*PI*MI/2.)**.5)
      DA=MUI*K*(TE+TI)/CHG
C
C A.G.M. CALCULATION OF QUARTER PERIOD; QP
C FOR MORE INFO SEE SECTION 16.4 IN THE  HANDBOOK OF MATH. FUNCTIONS.
C MODULUS (MOD) BASED ON RATIO OF NE(MAX) TO N(SAHA).
C
      MOD=1./(2.*S*NGS/(NE*ALPHA*NE)-1.)
      SM1=1.-MOD
      AO=1.
      BO=SQRT(SM1)
      CO=SQRT(MOD)
15    ANEW=(AO+BO)/2.
      BNEW=SQRT(AO*BO)
      CNEW=(AO-BO)/2.
      ADIF=AO-ANEW
      IF(ADIF.LT..00001)GO TO 16
      AO=ANEW
      BO=BNEW
      CO=CNEW
      GO TO 15
16    CONTINUE
      QP=PI/(2.*ANEW)
      X1=S*NGS
      X2=4.*QP**2.*DA/DIST**2.
      X3=.5*NE*ALPHA*NE
      TDIFF=X1-X2-X3
28    RETURN
      END
C
      FUNCTION XJT(RJ)
C
C CALCULATES THE TRANSPORT EQUATION
C
      REAL KSP,KPS,KCP,KPC,NGS,NE,MUI,MUE,MFPI,MFPE
      REAL MI,ME,HBAR,PI,LS,LP,K,NNUC,JE,JNET,MOD
      COMMON/C2/TE,DA,MUE
      COMMON/C1/ME,MI,CHG,EPS0,K,HBAR,E6P,ESP,ACOEFF,PI,LS,
     *LP,SIGIA,SIGEA,DLTXCT,VD,JE,TI,TEMIT,AELEC,DIST,G,NE,
     *SIGSP,SIG0,NNUC
      COMMON/C4/MOD,QP,SM1
      COMMON/C5/REA,REI,R
      COMMON/C6/SAHA
      VIBAR=SQRT(8.*K*TI/(PI*MI))
      ALFA=SQRT(TE*2.*PI/TI)
      XNA=(4./(ALFA*VIBAR))*DA*2.*QP*NE/DIST
      XNB=(SIGEA/SIGIA)*SQRT((TE/TI)*(ME/MI))*RJ/(1.602E-19*AELEC)
```

```
     **(4./(ALFA*VIBAR))
      XN1=XNA-XNB
      XNO=XNA+XNB
      ARG1=NE**2./(XNO*XN1*4.)
      TERM1=1./QP*ALOG(ARG1)
      IF(SAHA.EQ.0.0)GO TO 100
      TERM2=2
      GO TO 101
100   TERM2=ALOG(16./SM1)/QP
101   REA=DIST/(2.*NE*MUE*1.602E-19)*(TERM1+TERM2)
      XLAM=12389.*TE**1.5/SQRT(NE/2.)
      REI=DIST*100/AELEC*ALOG(XLAM)/(.013085*TE**1.5)
      R=REA+REI
      C1=RJ*R
      C2=RJ/(JE-RJ)
      C3=ALOG(C2)
      C4=C3*K*TE/CHG
      XJT=(C1+C4)-VD
      RETURN
      END
C
      SUBROUTINE BISECT(F,A,B,XTOL,IFLAG)
C
C NO-FRILLS BISECTION ROOT-FINDING SUBROUTINE
C
      IFLAG=0
      N=-1
      FA=F(A)
      FB=F(B)
C      WRITE(7,790)FA,FB
C790   FORMAT(2X,'ENDPOINTS:',3X,'FA=',E15.7,7X,'FB=',E15.7)
      IF (FA*F(B).LE.0.) GO TO 5
      IFLAG=2
C
C      WRITE(7,601)A,B
C601   FORMAT(43H F(X) IS OF SAME SIGN AT THE TWO ENDPOINTS
C     *      2F15.7)
      RETURN
    5 ERROR=ABS(B-A)
    6 ERROR=ERROR/2.
C
C CHECK FOR SUFFICIENTLY SMALL INTERVAL
C
      IF(ERROR.LE.XTOL)  RETURN
      XM=(A+B)/2.
C
C CHECK FOR UNREASONABLE ERROR REQUIREMENTS
C
      IF(XM+ERROR.EQ.XM)  GO TO 20
      FM=F(XM)
C
C TEMPORARY PRINTOUT
C
      N=N+1
C
C CHANGE TO NEW INTERVAL
C
      IF(FA*FM.LE.0.)  GO TO 9
      A=XM
      FA=FM
```

```
      GO TO 6
    9 B=XM
      GO TO 6
   20 IFLAG=1
      RETURN
      END
C
      FUNCTION F(X)
C
C CALCULATES N6p(X) = R0 + R1*NGS SO THAT IT CAN BE INTEGRATED BY
C THE SUBROUTINE SIMPER.
C
      COMMON/C1/ME,MI,CHG,EPS0,K,HBAR,E6P,ESP,ACOEFF,PI,LS,
     *LP,SIGIA,SIGEA,DLTXCT,VD,JE,TI,TEMIT,AELEC,DIST,G,NE,
     *SIGSP,SIG0,NNUC
      COMMON/C3/KPS,KPC,KSP,KCP,NGS
      COMMON/C4/MOD,QP,SM1
      DIMENSION A(20),B(20),C(20),PSI(20)
      REAL KPS,KPC,KSP,KCP,NGS,NE,MOD
C
C CALCULATE AND STORE A.G.M. SCALE
C
      A(1)=1.
      B(1)=SQRT(SM1)
      C(1)=SQRT(MOD)
      N=2
20    A(N)=(A(N-1)+B(N-1))/2.
      B(N)=SQRT(A(N-1)*B(N-1))
      C(N)=(A(N-1)-B(N-1))/2.
      ADIF=A(N-1)-A(N)
      IF(ADIF.LT..0001)GO TO 26
      N=N+1
      GO TO 20
26    PSI(N)=2**(N-1)*A(N)*2.*QP*X/DIST
C
C RECURSION CALCULATION
C
      DO 13 I=1,N
      NN=N-I+1
      TARG=C(NN)/A(NN)*SIN(PSI(NN))
      IF(TARG.LT.1.0)GO TO 29
C      WRITE(5,68)
C      WRITE(5,69)C(N),A(N),PSI(NN),TARG
68    FORMAT(2X,'SUB:F(SAHA) BOMB: C(N),A(N),PSI(NN),TARG')
69    FORMAT(2X,E15.9,4X,E15.9,4X,E15.9,4X,E15.9)
      TARG=1.
29    PSI(NN-1)=(PSI(NN)+ASIN(TARG))/2.
13    CONTINUE
      SN=SIN(PSI(1))
      F=(SN**3.*NE*KCP*NE**2.+NE*SN*KSP*NGS)
     */(SN*NE*(KPS+KPC)+G*ACOEFF)
      RETURN
      END
C
      SUBROUTINE SIMPER(FX,AREA,DIST)
C
C     SIMPSONS RULE: NSEG MUST BE EVEN, FX MUST BE SYMMETRICAL ABOUT MIDPOINT.
C
      REAL LSEG
      NSEG=20
```

```
      LSEG=DIST/FLOAT(NSEG)
      SUM=0
      X3=0.
      NSTEP=NSEG/4
      DO 39 I=1,NSTEP
      X1=X3
      FX1=FX(X3)
      X2=X1+LSEG
      X3=X2+LSEG
      SUM=SUM+FX1+4.*FX(X2)+FX(X3)
39    CONTINUE
      AREA=2.*LSEG/3.*SUM
      RETURN
      END
```