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TAYLOR-GÖRTLER INSTABILITIES OF TOLLMIEN-SCHLICHTING WAVES
AND OTHER FLOWS GOVERNED BY
THE INTERACTIVE BOUNDARY LAYER EQUATIONS

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ABSTRACT

The Taylor-Görtler vortex instability equations are formulated for steady and unsteady interacting boundary layer flows of the type which arise in triple-deck theory. The effective Görtler number is shown to be a function of the wall shape in the boundary layer and the possibility of both steady and unsteady Taylor-Görtler modes exists. As an example the steady flow in a symmetrically constricted channel is considered and it is shown that unstable Görtler vortices exist before the boundary layers at the wall develop the Goldstein singularity discussed by Smith and Daniels (1981). As an example of an unsteady spatially varying basic state we also consider the instability of high frequency large amplitude Tollmien-Schlichting waves in a curved channel. It is shown that they are unstable in the first ´Stokes layer stage´ of the hierarchy of nonlinear states discussed by Smith and Burggraf (1985). The Tollmien-Schlichting waves are shown to be unstable in the presence of both convex and concave curvature.

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INTRODUCTION

Our concern is with the Taylor-Görtler instability of interactive boundary layer flows of the type which occur in triple-deck theory. Thus, we investigate the instability of the 'lower deck' boundary layer which is set up when a classical boundary layer with Reynolds number R_E encounters a hump of height and length order $R_E^{-5/8}$ and $R_E^{-3/8}$ respectively. We find that the form of the Taylor-Görtler instability equations in the lower deck are almost identical to those appropriate to a classical boundary layer. The main difference is that the wall shape function $f(X,T)$ enters the instability equations and, in fact, for steady flows f_{XX} plays the role of the Görtler number.

We shall see that for unsteady interactive boundary layer flows both steady and unsteady Taylor-Görtler vortices of the type discussed by Hall (1982, 1983) and Seminara and Hall (1976) respectively are possible. These flows are also potentially unstable to short wavelength Rayleigh modes and the reader is referred to the papers by Bodonyi and Smith (1985) and Tutty and Cowley (1985) for a discussion of that problem. In general the instability equations which we derive must be solved numerically because a parallel flow and/or a quasi-steady approximation cannot be justified. However, in order to demonstrate that some of these flows are unstable we shall here concentrate on two problems for which some asymptotic progress is possible.

Firstly, we look in detail at the steady flow in a symmetrically constricted channel. Smith and Daniels (1981) have shown that when h the scaled height of constriction becomes large a classical boundary layer of thickness $O(h^{-1/2})$ is set up within the wall boundary layer. This inner boundary layer develops a Goldstein singularity beyond the minimum channel

width position, but Smith and Daniels showed that the singularity could be removed without any upstream influence being set up. Here we consider the instability of the flow before the boundary layer develops the singularity. For large values of h we are able to solve the instability equations asymptotically and demonstrate the instability of the $h^{-1/2}$ layer. In general the instability occurs before the Goldstein singularity develops; however, it is possible, as in Smith and Daniels (1981), to choose humps with the required concave curvature only beyond the position where the singularity develops.

Secondly, we look at the unsteady interactive boundary layer which governs the growth of Tollmien-Schlichting waves in parallel or boundary layer flows. Here the unsteadiness is characterized by Ω the frequency of the Tollmien-Schlichting wave. For definiteness, and to avoid the complications of boundary layer growth, we look at the instability of the waves in a slightly curved channel. Recently Smith and Burggraf (1985) looked at the structure of high frequency large amplitude Tollmien-Schlichting waves in a variety of situations. Dependent on the size of the disturbance and the particular flow under investigation, they found a hierarchy of nonlinear partial differential systems to describe the disturbance. The first nonlinear stage discussed by Smith and Burggraf is such that the disturbance has the form of a Stokes layer near the wall. We investigate the instability of this flow and identify the critical disturbance size above which the Tollmien-Schlichting wave is unstable to the Stokes layer Taylor-Görtler mode identified by Seminara and Hall (1976). When the Tollmien-Schlichting wave has amplitude greater than this critical value, a three-dimensional flow containing streamwise vortices develops; the effect of this new flow on the

growth of the wave into the larger amplitude states of Smith and Burggraf is beyond the scope of this paper. The procedure adopted in the rest of this paper is as follows: in Section 2 we derive the equations governing the centrifugal instability of ´lower-deck´ boundary layer flows. In Section 3 we show how these equations can be solved for the steady symmetric channel flow problem. In Section 4 we carry out a similar analysis for Tollmien-Schlichting waves in curved channel flows whilst in Section 5 we draw some conclusions.

2. THE TAYLOR-GÖRTLER INSTABILITY EQUATIONS FOR TRIPLE-DECK FLOWS

It is useful at this stage to discuss briefly the Taylor-Görtler instability equations for a classical two-dimensional boundary layer flow over a curved wall. The reader is referred to the papers by, for example, Görtler (1940), Smith (1955), Floryan and Saric (1979), Hall (1982) for a discussion of the approximations required to obtain a self-consistent set of linear stability equations.

Suppose then that ℓ and U_0 are typical length and velocity scales for the flow and that ν is the kinematic viscosity of the fluid. We define a Reynolds number R_E by

$$R_E = \frac{U_0 \ell}{\nu} ,$$

(2.1)

and take x and y to be dimensionless variables measuring distance along and normal to a surface with local curvature $a^{-1} K(x)$. The variable x is scaled on ℓ whilst y is a boundary layer variable scaled on $\ell R_E^{-1/2}$. The basic flow $\underset{\sim}{u}_B$ is of the form

$$\underset{\sim}{u}_B = U_0 \left(\overline{u}(x,y), R_E^{-1/2} \overline{v}(x,y), 0 \right) + \cdots,$$

and this flow is perturbed by writing

$$\underset{\sim}{u} = \underset{\sim}{u}_B + U_0 \left(U(x,y), R_E^{-1/2} V(x,y), R_E^{-1/2} W(x,y) \right) \exp \left\{ i R_E^{1/2} kz \right\}. \qquad (2.2)$$

Here k is a nondimensional wavenumber in the spanwise direction, and we have assumed that the instability occurs on the boundary layer thickness length scale. From the momentum and continuity equations we can show that in the limit $R_E \to \infty$ with the Görtler number $G = 2R_E^{1/2} \frac{\ell}{a}$ held fixed the linear stability equations are

$$U_x + V_y + ikW = 0, \qquad (2.3a)$$

$$\overline{u}U_x + U\overline{u}_x + \overline{v}U_y + V\overline{u}_y = \left\{ \partial_y^2 - k^2 \right\} U, \qquad (2.3b)$$

$$\overline{u}V_x + U\overline{v}_x + \overline{v}V_y + V\overline{v}_y + K(x)G\overline{u}U = -P_y + \left\{ \partial_y^2 - k^2 \right\} V, \qquad (2.3c)$$

$$\overline{u}W_x + \overline{v}W_y = -ikP + \left\{ \partial_y^2 - k^2 \right\} W. \qquad (2.3d)$$

Here P is the nondimensional pressure perturbation corresponding to (U, V, W) and we have assumed that the perturbation is steady. The generalization of (2.3) to a weakly three-dimensional boundary layer is given by Hall (1985).

The essential difficulty with (2.3) is that for G and k $0(1)$ there is no rational reason why a parallel flow approximation should be made and the partial differential system must be solved numerically as was done by Hall (1983). For $k \gg 1$, but $G \sim k^4$ an asymptotic solution to (2.3) was given

by Hall (1982) who showed that in this limit nonparallel effects can be taken care of in a self-consistent manner. For O(1) wavenumbers the numerical calculations of Hall (1983) showed that the position of neutral stability is a function of the initial disturbance. At higher Görtler numbers the local growth rate approaches the asymptotic result which, in this regime, is consistent with a parallel flow theory calculation. Unfortunateley, it has been assumed elsewhere that this latter result justifies the use of parallel flow theories. However, in the only regime where the parallel flow theories are valid, i.e., $k \gg 1$, an asymptotic result of at least the same accuracy as any parallel flow theory can be written down in closed form with little effort.

We now show how the equations corresponding to (2.3) can be derived for a basic flow governed by some interactive boundary layer structure. For definiteness we focus on a flow governed by triple-deck theory; the formulation for other structures is essentially identical. Consider then the flow over the wall

$$y = \epsilon^5 f(X,T),$$

where $X = \epsilon^{-3} x$, and $T = \epsilon^{-2} \dfrac{tU_0}{\ell}$. Here t denotes time whilst the small parameter $\epsilon = R_E^{-1/8}$. We define the lower deck variable Y by

$$Y = \epsilon^{-5} y,$$

and in the lower deck the basic state expands as

$$\frac{u_{BL}}{U_0} = \left(\epsilon \bar{u}(X,Y), \epsilon^3 \bar{v}(X,Y), 0\right) + \cdots,$$

whilst the pressure expands as

$$\frac{p\ell^2}{\mu U_0} = \varepsilon^2 \; \overline{p}(X) + \cdots.$$

The equations which determine the flow in the lower deck are

$$\overline{u}_T + \overline{u}\,\overline{u}_X + \overline{v}\,\overline{u}_Y = -\overline{P}_X + \overline{u}_{YY}, \qquad (2.4a)$$

$$\overline{u}_X + \overline{v}_Y = 0, \qquad (2.4b)$$

whilst the boundary conditions at the wall are

$$\overline{u} = 0, \quad \overline{v} = f_t \quad \text{on} \quad Y = f(X,T), \qquad (2.5a)$$

and at infinity we require

$$\overline{u} \rightarrow Y + A(X,T), \qquad (2.5b)$$

where A is the displacement function which must be related to $\overline{p}$ through a pressure-displacement law. If we make the unsteady Prandtl transform

$$Y \rightarrow Y + f(X,t), \quad \overline{v} \rightarrow \overline{v} + \overline{u}f_X + f_T, \qquad (2.6)$$

then (2.4a), (2.4b) are unchanged whilst (2.5) reduces to

$$\overline{u} = \overline{v} = 0, \quad Y = 0,$$

$$\overline{u} \rightarrow Y + A + f, \quad Y \rightarrow \infty. \qquad (2.7)$$

We now investigate the instabilty of this flow to Taylor-Görtler vortices which might be associated with either the steady or unsteady components of the basic state. We look for perturbations confined to the lower deck having spanwise wavelength comparable with the lower deck thickness. The possible source of instability is, of course, the curvature of the wall in the lower deck. We write

$$\frac{\underset{\sim}{u}}{U_0} = \underset{\sim}{u}_{BL} + \Delta(\varepsilon U(X,Y,T),\ \varepsilon^3\ V(X,Y,T),\ \varepsilon^3\ W(X,Y,T))E, \qquad (2.8)$$

where $\Delta \ll 1$ and $E = \exp\dfrac{ikZ}{\varepsilon^5}$. Here Z has been scaled on $\ell R_E^{-\frac{1}{2}}$ and we have assumed in (2.8) that the normal and spanwise velocity components are comparable. This is the usual case for Taylor-Görtler instabilities, and the relative scaling of the X and Y velocity components is again consistent with that appropriate to the small gap Taylor vortex problem (see, for example, Davey (1962)). The pressure perturbation p^+ in the lower deck expands as

$$\frac{p^+\ \ell}{\mu U_0} = \varepsilon^6\ P(X,Y,T)E, \qquad (2.9)$$

and the above relatively small scaling for P enables us to retain the convective and diffusive terms in the Y and Z momentum equations. It remains for us to substitute the perturbed flow into the Navier-Stokes equations for the lower deck and with $\Delta \ll 1$ linearize about the basic state. We note that at this stage it has not been necessary to define a Görtler number for the flow. After linearizing about the basic state and making the Prandtl transform (2.6) together with

$$V \to V + Uf_X,$$

we obtain

$$U_X + V_Y + ikW = 0,$$

$$U_T + \overline{u}U_X + U\overline{u}_X + \overline{v}U_Y + V\overline{u}_Y = \{\partial_Y^2 - k^2\}U,$$

$$V_T + \overline{u}V_X + U\overline{v}_X + \overline{v}V_Y + V\overline{v}_Y + 2U\{f_{XX}\overline{u} + f_{XT}\} = -P_Y + \{\partial_Y^2 - k^2\}V,$$

$$W_T + \overline{u}W_X + \overline{v}W_Y = -ikP + \{\partial_Y^2 - k^2\}W, \tag{2.10}$$

which must be solved subject to

$$U = V = W = 0, \quad Y = 0, \infty. \tag{2.11}$$

We see that the generalization of (2.3) to an unsteady triple-deck flow leads to almost the same equations but with $KG\overline{u}U$ replaced by $2\{f_{XX}\,\overline{u} + f_{XT}\}$. For steady triple-deck flows this means that f_{XX} plays the role of the Görtler number, whilst for unsteady flows an extra term proportional to Uf_{XT} arises due to the vertical motion of the boundary. For time-periodic basic states the system (2.10) contains the terms which lead to centrifugal instabilities in Stokes layers so in general we must be alert to the possibility of both types of Görtler instability.

The solution of (2.10) is in general a numerical problem which, for time-dependent flows, will be an order of magnitude more difficult than the steady state calculations of Hall (1983). In the next section we will look at a particular steady state which it is possible for us to solve (2.10)

asymptotically in a self-consistent manner. Finally we note that, with the appropriate scalings used, (2.10) apply to other interactive boundary layer flows governed by (2.4), (2.7).

3. SYMMETRIC CHANNEL FLOWS

In general it is not possible to solve (2.10) analytically; however, we now show how asymptotic methods can be used for a particular steady basic flow. We refer to the internal channel flows discussed by Smith and Daniels (1981). In that problem the wall boundary layer thickness is $\sim R^{-1/3}$ and the x variations are on an O(1) length scale. However, if the disturbance quantities are scaled appropriately then (2.10) still apply. The basic state satisfies (2.4) but with A = 0 in (2.7) since the channel is symmetric.

For steady flows we saw in Section 2 that f_{XX} plays the role of the Görtler number so that we expect the flow to become more unstable with increasing hump height. In view of the work of Hall (1982) we might then expect that an asymptotic solution of (2.10) should be possible. The Smith-Daniels problem is a suitable candidate for such an analysis because its structure for $|f| \gg 1$ is reasonably well understood. Suppose that we write

$$f(x) = hF(X), \tag{3.1}$$

with $h \gg 1$. In this situation a classical boundary layer extending to $X = -\infty$ and of thickness $\sim h^{-1/2}$ is attached to the hump. In this layer $\overline{u}, \overline{v}$, expand as

$$\overline{u} = h\overline{U} + \cdots, \tag{3.1a}$$

$$\overline{v} = h^{1/2} \overline{V} + \cdots, \tag{3.1b}$$

where $\overline{U}$ and $\overline{V}$ are functions of X and $\zeta = h^{1/2} Y$, and satisfy the classical boundary layer equations with pressure gradient $-FF_X$ and $\overline{U} \to F$ when $\zeta \to \infty$. The effective Görtler number for the inner $O(h^{-1/2})$ boundary layer then becomes $O(h^{7/2})$ so that, on the basis of the work of Hall (1983), we expect that neutral modes will have $k \sim O(h^{7/8})$. Moreover, these modes will be concentrated in an internal layer of depth $h^{-11/16}$ which is located so as to maximize the local spatial growth rate. At the location X we assume the layer is centered on $\zeta = \overline{\zeta}(X)$ and write

$$\eta = h^{3/16}(\zeta - \overline{\zeta}(X)). \tag{3.2}$$

The wavenumber k is then expanded as

$$k = k_0 h^{7/8} + k_1 h^{1/2} + \cdots, \tag{3.3}$$

whilst we write

$$U = \{U_0(\eta,X) + h^{-3/16} U_1(\eta,X) + h^{-3/8} U_2(\eta,X) + \cdots\}E^*, \tag{3.4}$$

together with similar expansion for $\dfrac{V}{h^{1/4}}$, $\dfrac{W}{h^{1/16}}$ and $\dfrac{P}{h^{15/16}}$. Here the quantity E^* is defined by

$$E^* = \exp\left[h^{3/4} \int^X \{\beta_0(X) + h^{-3/8} \beta_1(X) + \cdots\}dX \right], \tag{3.5}$$

so that $\{\beta_i(X)\}$ determine the spatial growth of the disturbance. In fact, we will concentrate on the neutral case and to the order which we proceed here it is not necessary to distinguish between the growth rates for different flow quantities. Finally, near $\bar{\zeta}$ the basic state expands as

$$\bar{u} = h\{\bar{U}_0(X) + h^{-3/16} \eta\bar{U}_1(X) + h^{-3/8} \eta^2 \bar{U}_2(X) + \cdots\} \tag{3.6a}$$

$$\bar{v} = h^{1/2}\{\bar{V}_0(X) + h^{-3/16} \eta\bar{V}_1(X) + h^{-3/8} \eta^2 \bar{V}_2(X) + \cdots\} \tag{3.6b}$$

where

$$\bar{U}_i(X) = \frac{\bar{U}^{(i)}(X,\bar{\zeta})}{i!} , \quad \bar{V}_i(X) = \frac{\bar{V}^{(i)}(X,\bar{\zeta})}{i!} . \tag{3.6c}$$

It remains for us to substitute the above expansion into (2.10) and successively equate like powers of order $h^{-3/16}$. The zeroth order problem is found to be

$$(\beta_0 \bar{U}_0 + k_0^2)V_0 + 2F_{XX} \bar{U}_0 U_1 = 0, \tag{3.7a}$$

$$(\beta_0 \bar{U}_0 + k_0^2)U_0 + V_0 \bar{U}_1 = 0, \tag{3.7b}$$

$$iP_0 + k_0 W_0 = 0, \tag{3.7c}$$

$$V_0' + ik_0 W_0 = 0. \tag{3.7d}$$

The required consistency condition for (3.7a), (3.7b) yields the zeroth order eigenrelation

$$(\beta_0 \bar{U}_0 + k_0^2)^2 = \bar{U}_1 \bar{U}_0 F_{XX}, \tag{3.8}$$

and the potentially unstable root of this equation is

$$\overline{U}_0 \, \beta_0 = -k_0^2 - \sqrt{\overline{U}_1 \, \overline{U}_0 \, F_{XX}} \, ,$$

which exists only for $\overline{U}_1 \, \overline{U}_0 \, F_{XX} > 0$. At a position where $\overline{U}_1 \, \overline{U}_0 \, F_{XX}$ vanishes we have a coalescence of modes and a transition region is required. The most unstable position in the layer is such that

$$2\beta_0 \, \overline{U}_0 (\beta_0 \, \overline{U}_0 + k_0^2) = [\overline{U}_1^2 + 2\overline{U}_0 \, \overline{U}_2] F_{XX}, \tag{3.9}$$

and, with f, k_0 given, (3.8), (3.9) fix $\overline{\zeta}$ and β_0. From now on we restrict our attention to the neutral case and set $\beta_0 = \beta_1 = \beta_2 = 0$ so that the flow is neutral at X if

$$k_0^4 = \overline{U}_1 \, \overline{U}_0 \, F_{XX}, \tag{3.10}$$

and $\overline{\zeta}$ is determined by the condition

$$\overline{U}_1^2 + 2\overline{U}_0 \, \overline{U}_2 = 0,$$

which requires that $|\overline{U} \, \overline{U}''|$ has a maximum at $\zeta = \overline{\zeta}$. At this stage the vertical structure of the eigenfunction is not determined; at second-order the following equation for V_0 emerges as a solvability condition:

$$V_0'' - \frac{4}{3} k_0 \, k_1 \, V_0 + k_0^{-2} \, \eta^2 \, F_{XX} [\overline{U}_1 \, \overline{U}_2 + \overline{U}_3 \, \overline{U}_0] V_0 = 0. \tag{3.11}$$

The solutions of (3.11) which decay when $|\eta| \to \infty$ are

$$V_0 = V_{0n} = U(-n - 1/2 , \gamma n) \tag{3.12}$$

where $U(-n - 1/2 , \gamma n)$ is a parabolic cylinder function and

$$\gamma = \sqrt{2}\{-F_{xx}[\overline{U}_1 \, \overline{U}_2 + \overline{U}_3 \, \overline{U}_0]k_0^{-2}\}^{1/4} ,$$

the wavenumber k, must then satisfy

$$k_1 = k_{1n} = \frac{3\gamma^2}{4k_0} \{n + 1/2 \} . \tag{3.13}$$

The most unstable mode corresponds to $n = 0$ so that correct to order $h^{1/2}$ the neutral wavenumber is

$$k = \{\overline{U}_1 \, \overline{U}_0 \, F_{XX}\}^{1/4} h^{7/8} + \frac{3}{4} \left\{ - \frac{[\overline{U}_1 \, \overline{U}_2 + \overline{U}_3 \, \overline{U}_0]}{\overline{U}_0 \, \overline{U}_1} \right\}^{1/2} h^{1/2} + \dots . \tag{3.14}$$

Thus for a given hump size h and shape f the flow is neutrally stable at X to a vortex flow with wavenumber k if (3.14) is satisfied. Clearly instability occurs only for $\overline{U}_1 \, \overline{U}_0 \, F_{XX} > 0$, and if we restrict our attention to humps with $f > 0$ then upstream separation does not occur and it can be shown that until the Goldstein singularity develops $\overline{U}_1 \, \overline{U}_0 > 0$ so that only a locally concave wall can lead to instability.

If we assume that $F(X)$ and $k \gg 1$ are given, then an alternative interpretation of (3.14) is necessary. In this situation we can think of (3.14) as an implicit equation for the hump height $h \gg 1$ which makes the Görtler vortex flow with wavenumber $k \gg 1$ neutral at some position in the flow field. In the following discussion we assume that $F \to 0$, $X \to \pm\infty$ and

that F_{XX} is positive in $(-\infty, -C)$ and negative in $(-\infty, D)$ where C and D are positive constants. Since we also know that

$$\overline{U} = 0, \; \zeta = 0, \; \overline{U}_\zeta \to 0, \; \zeta \to \infty$$

and

$$\overline{U} \sim O(f), \; X \to -\infty$$

it follows that $\overline{U}_0 \, \overline{U}_1 \, F_{XX}$ has at least one maximum in $-\infty < X < -C$, $0 < \zeta < \infty$. Suppose that the largest maximum occurs at

$$X = X_C, \; \overline{\zeta} = \zeta_C;$$

then ignoring the order $h^{1/2}$ term in (3.14) we see that the minimum hump height h_C which leads to a neutral vortex anywhere (in fact at (X_C, ζ_C)) is given by

$$h_C = \{\overline{U}_1(X_C)\overline{U}_0(X_C)F_{XX}\}^{-2/7} k^{8/7}. \tag{3.15}$$

If h is increased beyond h_C there will be two neutral locations at $-C - \alpha, \; -C + \beta$ with $\alpha, \beta > 0$ each corresponding to the wavenumber k. Between these positions the flow is formally unstable; the instability will amplify by an amount $O(\exp[h^{3/4} I])$ for some $I > 0$ in this interval and will become nonlinear if its initial size is sufficiently large. For humps with $h \sim O(1)$ we expect that a similar situation arises for $k \sim O(1)$ with at least one finite interval where instability occurs. Beyond the position of the maximum constriction a further region where instability can occur will exist so long as the Goldstein singularity is not encountered.

4. THE INSTABILITY OF TOLLMIEN-SCHLICHTING WAVES IN CURVED CHANNEL FLOWS

In the previous section we described how a particular steady solution of (2.4), (2.7) becomes unstable to steady Taylor-Görtler vortices. We now show how a time-periodic solution of that system can also become unstable to a Stokes layer Taylor-Görtler vortex. The particular type of time-periodic basic state which we consider corresponds to a large amplitude high frequency Tollmien-Schlichting wave propagating in a curved channel. Here the curvature which causes the instability is not on the triple deck length scale, so it is necessary to say a few words about the derivation of the appropriate form of the equations corresponding to (2.10).

We consider the flow driven by an azimuthal pressure gradient between concentric cylinders of radii a, a + d. The maximum flow velocity is taken to be $\frac{U_0}{2}$ and if we define dimensionless variables x and y by

$$x = \frac{a\theta}{d} = \delta^{-1} \theta, \quad y = \frac{r - a - d}{d}$$

then in the absence of either Tollmien-Schlichting waves or Taylor-Görtler vortices the basic state for $\delta \ll 1$ is

$$\underline{u}_B = U_0(u_0 + 0(\delta), \ 0, \ 0) \tag{4.1}$$

with

$$u_0 = y(1 - y). \tag{4.2}$$

The Reynolds numer R_E is defined by

$$R_E = \frac{U_0 \ d}{\nu}$$

and Dean (1928) showed that (4.1) is centrifugally unstable for $R_E^2 \delta \sim O(1)$. The flow is also unstable to Tollmien-Schlichting waves for $\frac{1}{4} R_E > 5774$. We ignore the steady Taylor-Görtler mode and examine the instability of finite amplitude Tollmien-Schlichting waves to Stokes layer Taylor-Görtler vortex modes. These occur for $R_E^{4/7} \delta \sim O(1)$ and so are apparently less important than the steady vortex mode. However, we shall see that they occur both near the inner and outer cylinders so that in external flows over convex walls we can expect this mode to be the only centrifugal one available. Our choice of the curved channel flow problem rather than an external boundary layer enables us to investigate the instability mechanism without the possibly insurmountable difficulties of handling the effect of boundary layer growth.

We first describe the large amplitude high frequency Tollmien-Schlichting disturbances to (4.1) which exist for $R_E \gg 1$. The discussion we give is taken from the recent work of Smith and Burggraf (1985) who investigated a range of flow regimes where such disturbances can exist. We take z and t to be dimensionless axial and time variables scaled on d and U_0/d. The appropriate length scale for a Tollmien-Schlichting wave in a channel is $O(R_E^{1/7})$ so that if we define

$$\varepsilon = R_E^{-1/7} \tag{4.3}$$

and

$$X = \varepsilon x \tag{4.4}$$

then the wall layers at $y = 0, 1$ are of thickness $O(\varepsilon^2)$ so that near $y = 0$ we write

$$Y = y/\varepsilon^2. \tag{4.5}$$

Following Smith and Burggraf we seek a flow for $Y \sim O(1)$ of the form

$$\underline{u} = U_0\left(\epsilon^2 \, \bar{u}(X,Y,T), \; \epsilon^5 \, \bar{v}(X,Y,T), 0\right) + \cdots \qquad (4.6)$$

where $T = \epsilon^3 t$. The corresponding pressure perturbation is $\dfrac{\mu U_0}{d} \epsilon^4 \, p(X,T)$ and $\bar{u}, \bar{v}, \bar{p}$ are determined by

$$\bar{u}_T + \bar{u}\,\bar{u}_X + \bar{v}\,\bar{u}_Y = -\bar{P}_X + \bar{u}_{YY}$$

$$\bar{u}_X + \bar{v}_Y = 0$$

$$\bar{u} = \bar{v} = 0, \quad Y = 0$$

$$\bar{u} \to Y + A(X,T), \quad Y \to \infty \qquad (4.7)$$

Near $y = 1$ a similar boundary layer exists whilst in the core we have

$$\bar{u} = U_0(u_0,0,0) + U_0\left(\epsilon^2 \, \tilde{u}(X,y,T), \; \epsilon^3 \, \tilde{v}(X,y,T), 0\right) + \cdots$$

$$p = \frac{\mu U_0}{d} \epsilon^4 \, \tilde{p}(X,y,T) + \cdots. \qquad (4.8)$$

Here $\tilde{u}, \tilde{v}, \tilde{p}$ satisfy

$$u_0 \, \tilde{u}_X + \tilde{v}u_{0y} = 0$$

$$\tilde{u}_X + \tilde{v}_y = 0$$

$$u_0 \, \tilde{v}_X = -\tilde{p}_0' \qquad (4.9)$$

and the appropriate solution is

$$(\tilde{u}, \tilde{v}) = \left(A(X,T)u_{0y}, \ -A_X(X,T)u_0\right), \tag{4.10}$$

$$P_0 = -A_{XX} \int^y u_0^2(s)\,ds \tag{4.11}$$

so that the required pressure-displacement relationship is

$$P_0\bigg|_{y=1} - P_0\bigg|_{y=-0} = -\frac{1}{30} A_{XX}. \tag{4.12}$$

We are interested in the large amplitude high frequency solutions of (4.7) discussed by Smith and Burggraf (1985). The latter authors investigated a hierarchy of high frequency large amplitude states beginning with the case $\frac{\partial}{\partial T} \sim O(\Omega) \gg 1$, $\frac{\partial}{\partial X} \sim O(\Omega^{1/2})$, $\bar{u} \sim O(1)$. We shall concern ourselves here only with the latter state and the reader is referred to the Smith-Burggraf paper for a discussion of the remarkable range of more nonlinear states which occur for $\bar{u} \gg 1$.

For $\Omega \gg 1$ we write

$$\eta = \Omega^{1/2} Y,$$

$$\partial_T \to \Omega \partial_{\bar{T}} + \Omega^{-1/2} \partial_{\tilde{T}} + \cdots$$

$$\partial_X \to \Omega^{1/2} \partial_{\bar{X}}$$

and note that linear stability would then lead to instability on the $\tilde{T}$ scale. We note that Smith and Burggraf looked for spatial rather than

temporal growth. Here it is convenient to look for growth in time because it leads to an eigenvlaue problem already solved in a different context. The spatially growing case, except in the locally neutral state, leads to a different eigenvalue problem. The work of Smith and Burggraf shows directly that in this regime

$$\bar{u} = u_{01} + O(\Omega^{-1/2}) \qquad (4.13a)$$

$$\bar{v} = v_{01} + O(\Omega^{-1/2}) \qquad (4.13b)$$

where

$$u_{01} = \{p_{01}(1 - e^{m\eta})E + C.C.\}, \qquad (4.13c)$$

$$v_{01} = \{i\bar{\alpha}p_{01}(\eta - \frac{e^{m\eta}}{m} + \frac{1}{m})E + C.C.\}. \qquad (4.13d)$$

Here C.C. deontes ´complex conjugate´ and E denotes $\exp i\{\bar{\alpha}\,\bar{X} - \bar{T}\}$ with $\bar{\alpha}$ given by the eigenrelation

$$\bar{\alpha} = \sqrt{30},$$

whilst $m = e^{\frac{3}{4}\pi i}$.

Thus the Tollmien-Schlichting wave is bigger than the steady basic state for $\eta \sim O(1)$ and is just a Stokes layer flow. At higher order the Smith-Burggraf approach shows that p_{01} grows exponentially with $\tilde{T}$ and does not equilibrate. We will look at the instability of the Tollmien-Schlichting wave during this stage of its growth.

We shall firstly consider the instability of (4.6) to a Taylor-Görtler vortex perturbation with axial wavelength $O(\epsilon^2)$. We thus write

$$u = U_0(\epsilon^2\,\bar{u} + \epsilon^2[U(X,Y,T)\exp ikz\epsilon^{-2} + C.C.] + \cdots), \qquad (4.14a)$$

$$v = U_0\left(\epsilon^5\ \bar{v} + \epsilon^5\left[V(X,Y,T)\exp\ ikz\epsilon^{-2} + C.C.\right] + \cdots\right), \qquad (4.14b)$$

$$w = U_0\left(\epsilon^5\left[W(X,Y,T)\exp\ ikz\epsilon^{-2} + C.C.\right] + \cdots\right), \qquad (4.14c)$$

$$p = \frac{\mu U_0}{a}\left(\epsilon^4\ \bar{p} + \epsilon^{10}\ P(X,Y,T)\exp\ ikz\epsilon^{-2} + C.C.\right] + \cdots\right), \qquad (4.14d)$$

and then

$$2\delta = D\epsilon^4$$

where D can be interpreted as a Taylor-Görtler number for the Tollmien-Schlichting wave. The equations (4.14a), (4.14b), (4.14c), (4.14d) are then substituted into the momentum and continuity equations, and after linearizing about the basic state we find that (U, V, W, P) satisfies (2.10) but with the term $2U\{f_{XX}\ \bar{u} + f_{XT}\}$ replaced by $D\bar{u}U$. At this stage the eigenvalue problem D = D(k) could in principle be solved numerically for any given basic state. We shall proceed by looking at the high frequency limit of the Tollmien-Schlichting waves in order to make some analytical progress. We stress that we expect instability to occur for O(1) frequencies but do not pursue the necessary large scale computational task required to verify this speculative remark.

Let us now look at the high frequency limit of (2.10) and show that the eigenvalue problem governing the instability of the growing Tollmien-Schlichting waves can be reduced to one already studied in the context of Stokes layer instabilities.

Thus, in the Stokes layer we expand U, V, W, P in the form

$$U = [U_0 \ (\overline{X},\eta,\widetilde{T}) + 0(\Omega^{-\frac{1}{2}})]\exp \Omega^{3/2} \int^{\widetilde{T}} \sigma(\widetilde{T})d\widetilde{T}, \qquad (4.15a)$$

$$V = [\Omega^{\frac{1}{2}} V_0(\overline{X},\eta,T) + 0(\Omega^0)]\exp \Omega^{3/2} \int^{\widetilde{T}} \sigma(\widetilde{T})d\widetilde{T}, \qquad (4.15b)$$

$$W = [\Omega^{\frac{1}{2}} W_0(\overline{X},\eta,T) + 0(\Omega^0)]\exp \Omega^{3/2} \int^{\widetilde{T}} \sigma(\widetilde{T})d\widetilde{T}, \qquad (4.15c)$$

$$P = [\Omega P_0(\overline{X},\eta,T) + 0(\Omega^{\frac{1}{2}})]\exp \Omega^{3/2} \int^{\widetilde{T}} \sigma(\widetilde{T})d\widetilde{T} \qquad (4.15d)$$

whilst k and D expand as

$$k = k_0 \ \Omega^{\frac{1}{2}} + 0(\Omega^0), \qquad (4.15e)$$

$$D = D_0 \ \Omega^{3/2} + 0(\Omega). \qquad (4.15f)$$

The $\overline{X}$ variation enters the zeroth order problem only through the $\overline{X}$
dependence of u_{01}. The growth rate $\sigma(\widetilde{T})$ depends on $\widetilde{T}$ through the $\widetilde{T}$
dependence of P_{01}.

If the expansions (4.14), (4.15) are substituted into (2.10), the zeroth
order problem is found to be

$$V_{0\eta} + ik_0 \ W_0 = 0, \qquad (4.16a)$$

$$\sigma U_0 + U_{0\overline{T}} + V_0 \ u_{01\eta} = \{\partial_\eta^2 - k_0^2\}U_0, \qquad (4.16b)$$

$$\sigma V_0 + V_{0\overline{T}} - D_0 \, u_{01} \, U_{01} = -P_{0\eta} + \{\partial_\eta^2 - k_0^2\}V_0, \qquad (4.16c)$$

$$\sigma W_0 + W_{0\overline{T}} = -ik_0 \, P_0 + \{\partial_\eta^2 - k_0^2\}W_0, \qquad (4.16d)$$

which must be solved subject to

$$U_0 = V_0 = W_0 = 0, \qquad (4.17a)$$

$$U_0, \ V_0, \ W_0 \to 0, \quad \eta \to \infty \qquad (4.17b)$$

so that the vortex structure is confined to the Stokes layer. The eigenvalue problem specified (4.16), (4.17) is identical to that studied by Hall (1984) in the context of Schlichting's (1932) transversely oscillating cylinder problem. In fact, we identify (k,T) of (2.12) of Hall (1984) with $\left(\sqrt{2}k_0, \sqrt{2}D_0 \, |P_{01}|^2\right)$.

It follows from the numerical calculations of Hall (1984) for the neutral case $\sigma = 0$ that the Tollmien-Schlichting wave is formallly unstable for

$$D_0 \, |P_{01}|^2 > 8.48 \qquad (4.18)$$

which, for a given value of D_0, determines $|P_{01}|$ the critical Tollmien-Schlichting wave amplitude.

Not surprisingly a similar analysis governs the instability problem for the Tollmien-Schlichting wall layer at $y = 1$. The only change is that, because the layer is now on a concave wall, the sign of D in (4.16) must be switched. Papageorgiu (1985) has investigated that eigenvalue problem, and the critical state is then determined by

$$D \ |p_{01}|^2 > 5.02. \tag{4.19}$$

Thus the layer at $y = 1$ becomes unstable first, and presumably the flow becomes three-dimensional before the larger amplitude two-dimensonal states of Smith and Burggraf develop. It is known from the experimental work of Seminara and Hall (1976) and Park, Barenghi, and Donnelly (1980) that the initial Stokes layer instability is followed by a secondary mode of instability at about 30% above the first critical Taylor number. In this regime the vortices interact and the disturbance persists beyond the Stokes layer. No adequate theoretical description of this nonequilibrium state is yet available, but the consequences for the problem discussed here are important. We refer to the fact that if the Tollmien-Schlichting wave also undergoes this secondary mode of instability then there will be a mechanism for disturbances inside the Stokes layer to penetrate outside the boundary layer.

We further note that if the mechanism described here does indeed apply to external flows then Tollmien-Schlichting waves will generate Taylor-Görtler vortices if convex or concave regions exist. In the concave regions the Taylor-Görtler mechanism associated with the main deck basic state will be more unstable so that the Tollmien-Schlichting breakdown into the Stokes layer mode is probably only of practical importance in convex regions.

Finally, we note from the calculations of Hall (1984) that when instability occurs V_0, W_0, P_0 are of the form

$$\begin{pmatrix} V_0 \\ W_0 \\ P_0 \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} V_{0n} \\ W_{0n} \\ P_{0n} \end{pmatrix} E^{2n+1}$$

whilst

$$U_0 = \sum_{-\infty}^{\infty} U_{0n} E^{2n}.$$

Some of the functions $U_{0n}(\eta)$, $V_{0n}(\eta)$ can be found in the paper by Hall (1984).

5. CONCLUSION

We have shown that interactive boundary layer flows of the type which arise in triple-deck theory can support Taylor-Görtler vortices. The form of the equations for steady flows is identical to that found for classcial boundary layers if we interpret the wall curvature as the Görtler number. For unsteady boundary layers, an extra term proportional to the streamwise gradient of the wall velocity is introduced into the equations.

We have seen that both steady and time-periodic boundary layer Görtler vortices can be described within the above framework. In particular, we showed that a large amplitude high frequency Tollmien-Schlichting wave can interact with a curved wall to give a Stokes layer Görtler vortex. If such a result also holds for external flows, it means that wall curvture, either concave or convex, can lead to the breakdown of two-dimensional Tollmien-Schlichting waves into a three-dimensional flow having streamwise vortices in a sublayer.

In general, the partial differential equations describing the evolution of Taylor-Görtler vortices in interactive boundary layer flows will have to be solved numerically. The results we found for the Smith-Daniels flow in a symmetric channel suggest that for $O(1)$ values of h instability will

occur. Thus we expect that a numerical investigation of the instability equations for $h \sim 0(1)$ will show the presence of growing Görtler vortices for some range of values of X whilst for large h we would expect to recover the asymptotic results of Section 3.

It is not yet known what effect the above type of instability will have on separation; certainly linear stability theory can lead to no predictions on this matter. However, in situations where the small wavelength analysis of Hall (1982) can be used we know that the instability will be concentrated away from the wall. Thus for the Smith-Daniel problem we can show that when the Goldstein singularity develops the instability still persists since the new structure required to remove the singularity is confined to a region near the wall.

We should not overlook the fact that the flows which we have investigated are also susceptible to Tollmien-Schlichting waves. In a situation where the basic flow can support both modes of instability we expect that, in view of the much larger spatial growth rates, the Görtler mode will dominate. An adequate description of even a weakly nonlinear interaction between Tollmien-Schlichting waves and Görtler vortices has not, to the author's knowledge, been given for any flow.

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16. Abstract

The Taylor-Görtler vortex instability equations are formulated for steady and unsteady interacting boundary layer flows of the type which arise in triple-deck theory. The effective Görtler number is shown to be a function of the wall shape in the boundary layer and the possibility of both steady and unsteady Taylor-Görtler modes exists. As an example the steady flow in a symmetrically constricted channel is considered and it is shown that unstable Görtler vortices exist before the boundary layers at the wall develop the Goldstein singularity discussed by Smith and Daniels (1981). As an example of an unsteady spatially varying basic state we also consider the instability of high frequency large amplitude Tollmien-Schlichting waves in a curved channel. It is shown that they are unstable in the first 'Stokes layer stage' of the hierarchy of nonlinear states discussed by Smith and Burggraf (1985). The Tollmien-Schlichting waves are shown to be unstable in the presence of both convex and concave curvature.

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