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SECOND-ORDER CLOSURE MODELS FOR ROTATING TURBULENT FLOWS

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Abstract

The physical properties of the commonly used second-order closure models are examined theoretically for rotating turbulent flows. Comparisons are made with results which are a rigorous consequence of the Navier-Stokes equations for the problem of fully-developed turbulent channel flow in a rapidly rotating framework. It is demonstrated that all existing second-order closures yield spurious physical results for this test problem of rotating channel flow. In fact, the results obtained are shown to be substantially more unphysical than those obtained from the simpler K-ε and K-ℓ models. Modifications in the basic structure of these second-order closure models are proposed which can alleviate this problem.

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1. INTRODUCTION

The ability to accurately predict rotating turbulent flows could have
a wide variety of important applications ranging from the analysis of
turbomachinery to the description of turbulence in the atmosphere or in
oceans. However, with the exception of some recent work on the limiting
case of two-dimensional turbulence[1-3], there appears to have been little
work which accounts for the effects of rigid body rotations on turbulence
modeling by a direct analysis of the Navier-Stokes equations. In fact,
much of the work in modeling rotating flows has been conducted utilizing
the K-ε (or K-ℓ) model of turbulence which requires the effects of rotations
to be accounted for by the use of a variety of ad hoc empiricisms (c.f.,
Majumdar, Pratap, and Spalding[4], Howard, Patankar, and Bordynuik[5], and
Galmes and Lakshminarayana[6]). While such approaches can be useful in
correlating experimental results for a particular class of rotating turbulent
flows, they do not form the basis for a general theory which is needed if
models are to be developed that truly have a predictive value.

The purpose of the present paper is to examine the consistency of
various second-order closure models with results which are a rigorous
consequence of the Navier-Stokes equations in a rapidly rotating framework.
Although there have been a few previous studies of rotating turbulent flows
using second-order closure models (c.f., So[7], So and Peskin[8], and Mellor
and Yamada[9]), only a narrow range of flows have been considered so that
no definitive conclusions could be drawn about the correctness of the models.
In this paper the test problem of fully-developed turbulent channel flow
in a rapidly rotating framework will be considered. It will be proven,
as a rigorous consequence of the Navier-Stokes equations, that the Reynolds

stress tensor for this problem must be two-dimensional (as a direct conse-
quence of the Taylor-Proudman theorem[10]) and must have a non-zero Reynolds
shear stress in the plane of the flow. The commonly used second-order
closure models yield completely opposite results (i.e., a three-dimensional
Reynolds stress tensor with vanishing Reynolds shear stresses), and, hence,
are fundamentally inconsistent with the Navier-Stokes equations for turbulent
channel flow in a rapidly rotating framework. It is also demonstrated
that the results obtained from these second-order closures are substantially
more unphysical than those obtained from the simpler K-ε or K-ℓ models.
Modifications in existing second-order closure models which can alleviate
this problem will be explored. By the addition of one term to the second-
order closure obtained by Haworth and Pope[11] from a generalized Langevin
model, it will be shown that consistency with the Navier-Stokes equations
in a rapidly rotating frame can be achieved. Other possible modifications
will be discussed briefly in the last section along with the prospects for
future research.

2. Second-Order Closure Models and Rotating Channel Flow

The turbulent flow of a homogeneous and incompressible viscous fluid in a rotating frame of reference will be considered. The velocity field $\underset{\sim}{v}$ and pressure P will.be decomposed into ensemble mean and fluctuating parts, respectively, as follows

$$\underset{\sim}{v} = \overline{\underset{\sim}{v}} + \underset{\sim}{u}, \qquad P = \overline{P} + p \tag{1}$$

Of course, the decompositions in Eq. (1) are solutions of the mean and fluctuating parts of the Navier-Stokes equations and continuity equation which, in a steadily rotating frame of reference, take the form[12]

$$\frac{\partial \overline{\underset{\sim}{v}}}{\partial t} + \overline{\underset{\sim}{v}} \cdot \nabla \overline{\underset{\sim}{v}} = -\nabla \overline{P} + \nu \nabla^2 \overline{\underset{\sim}{v}} - \nabla \cdot \underset{\sim}{\tau} - 2\underset{\sim}{\Omega} \times \overline{\underset{\sim}{v}} \tag{2}$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \overline{\underset{\sim}{v}} \cdot \nabla \underset{\sim}{u} = -\underset{\sim}{u} \cdot \nabla \underset{\sim}{u} - \underset{\sim}{u} \cdot \nabla \overline{\underset{\sim}{v}} - \nabla p + \nu \nabla^2 \underset{\sim}{u} + \nabla \cdot \underset{\sim}{\tau} - 2\underset{\sim}{\Omega} \times \underset{\sim}{u} \tag{3}$$

$$\nabla \cdot \overline{\underset{\sim}{v}} = 0 \tag{4}$$

$$\nabla \cdot \underset{\sim}{u} = 0 \tag{5}$$

where ν is the kinematic viscosity of the fluid, $\underset{\sim}{\Omega}$ is the angular velocity of the reference frame relative to an inertial framing, and $\underset{\sim}{\tau}$ is the kinematic Reynolds stress tensor given by

$$\underset{\sim}{\tau} = \overline{\underset{\sim}{u}\underset{\sim}{u}} \tag{6}$$

It should be noted that $\overline{P}$ in Eq. (2) is the modified mean pressure which includes the centrifugal and gravitational body force potentials.

The Reynolds stress transport equation is obtained by taking the ensemble mean of the symmetric part of the outer product of Eq. (3) with $\underset{\sim}{u}$. This equation takes the form[12]

$$\frac{D\tau_{k\ell}}{Dt} + \tau_{km} \frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \bar{v}_k}{\partial x_m} = - \frac{\partial}{\partial x_m} (\overline{u_m u_k u_\ell})$$

$$- (\overline{u_k \frac{\partial p}{\partial x_\ell} + u_\ell \frac{\partial p}{\partial x_k}}) - 2\nu \overline{\frac{\partial u_k}{\partial x_m} \frac{\partial u_\ell}{\partial x_m}} + \nu \nabla^2 \tau_{k\ell}$$

$$+ 2 \varepsilon_{mkn} \Omega_m \tau_{n\ell} + 2\varepsilon_{m\ell n} \Omega_m \tau_{nk} \tag{7}$$

in a rotating frame of reference. In (7), Cartesian tensor notation is utilized where the Einstein summation convention applies to repeated indices and $\varepsilon_{k\ell m}$ represents the permutation tensor. Second-order closure models are obtained when closure relations are provided that tie the higher-order turbulence correlations in (7) to the Reynolds stresses (along with their spatial gradients), the mean velocity gradients, and the length scale of turbulence. The closure relations used for the higher-order correlations (i.e., the triple velocity correlation, pressure gradient-velocity correlation, and the dissipation rate correlation) in an inertial framing are of the general form

$$\overline{u_k u_\ell u_m} = C_{k\ell m}(\underset{\sim}{\tau}, \nabla\underset{\sim}{\tau}, \ell) \tag{8}$$

$$\overline{u_k \frac{\partial p}{\partial x_\ell} + u_\ell \frac{\partial p}{\partial x_k}} = - C_1(\tau_{km} \frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \bar{v}_k}{\partial x_m}) + A_{k\ell}(\underset{\sim}{\tau}, \nabla\underset{\sim}{\tau}, \bar{\underset{\sim}{D}}, \ell) \tag{9}$$

$$2\nu \overline{\frac{\partial u_k}{\partial x_m} \frac{\partial u_\ell}{\partial x_m}} = B_{k\ell}(\underset{\sim}{\tau}, \bar{\underset{\sim}{D}}, \ell) \tag{10}$$

respectively (see Mellor and Herring[13], Launder, Reece, and Rodi[14], and Lumley[15]). In (7)-(10), C_1 is a dimensionless constant, ℓ is the length scale of turbulence which must be obtained from a separate transport equation, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{v} \cdot \nabla \qquad (11)$$

$$\overline{D}_{k\ell} = \frac{1}{2}(\frac{\partial \overline{v}_k}{\partial x_\ell} + \frac{\partial \overline{v}_\ell}{\partial x_k}) \qquad (12)$$

are, respectively, the mean substantial derivative and rate of strain tensor. By extending (8)-(10) to rotating frames of reference (see Speziale[2]) and then substituting the results into (7), it is a simple matter to show that all existing second-order closure models are of the form

$$\frac{D\tau_{k\ell}}{Dt} + (1-C_1)(\tau_{km}\frac{\partial \overline{v}_\ell}{\partial x_m} + \tau_{\ell m}\frac{\partial \overline{v}_k}{\partial x_m}) = -\frac{\partial}{\partial x_m} C_{mk\ell}(\underset{\sim}{\tau},\underset{\sim}{\nabla\tau},\ell)$$

$$- A_{k\ell}(\underset{\sim}{\tau},\underset{\sim}{\nabla\tau},\overline{D},\ell) - B_{k\ell}(\underset{\sim}{\tau},\overline{D},\ell) + \nu\nabla^2\tau_{k\ell}$$

$$+ (2-C_1)(\epsilon_{mkn}\Omega_m\tau_{n\ell} + \epsilon_{m\ell n}\Omega_m\tau_{nk}) \qquad (13)$$

in a rotating frame of reference. It should be noted that in the Rotta-Kolmogorov model[13] the constant $C_1 = 0$ and in the Launder, Reece, and Rodi[14] and Lumley[15] models, C_1 is a non-zero constant which does not equal 2. Hence, the Coriolis term on the right-hand-side of (13) survives in all existing second-order closures where it constitutes the only non-inertial term.

Now, we will consider the test problem of fully-developed turbulent channel flow in a rapidly rotating framework (see Figure 1). This fully-developed channel flow is maintained by a constant axial mean pressure gradient

$$\frac{\partial \overline{P}}{\partial x} = - G \tag{14}$$

while the channel is subjected to a steady spanwise rotation with the angular velocity (see Figure 1)

$$\underset{\sim}{\Omega} = \Omega \underset{\sim}{k} \tag{15}$$

The mean velocity field is given by

$$\underset{\sim}{\overline{v}} = \overline{U}(y) \underset{\sim}{i} \tag{16}$$

and the Reynolds stress tensor is of the form

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}(y) \tag{17}$$

since the flow conditions are fully-developed. Hence, the axial component of the Reynolds equation (2) in a rotating channel flow is given by

$$\nu \frac{d^2\overline{U}}{dy^2} - \frac{d\tau_{xy}}{dy} + G = 0 \tag{18}$$

Since it is well known that the axial velocity profile in fully-developed turbulent channel flow is not parabolic like its laminar counterpart, it is clear that

$$\tau_{xy} \neq 0 \tag{19}$$

(also see the experimental results of Johnston, Halleen, and Lezius[16]).

The y and z- components of the Reynolds equation (2) take the form

$$- \frac{\partial \overline{P}}{\partial y} - \frac{d\tau_{yy}}{dy} - 2\Omega\overline{U} = 0 \qquad (20)$$

$$- \frac{\partial \overline{P}}{\partial z} - \frac{d\tau_{yz}}{dy} = 0 \qquad (21)$$

Since there is no forcing in the z-direction,

$$\frac{\partial \overline{P}}{\partial z} = 0$$

and (21) can be integrated to yield the result

$$\tau_{yz} = 0 \qquad (22)$$

since $\underset{\sim}{\tau}$ must satisfy the no slip condition at the channel walls. Similarly, it can be shown that

$$\tau_{xz} = 0 \qquad (23)$$

The vorticity transport equation in a steadily rotating framework (where $\underset{\sim}{\omega} = \underset{\sim}{\nabla} \times \underset{\sim}{v}$) takes the form [10]

$$\frac{D\underset{\sim}{\omega}}{Dt} = \underset{\sim}{\omega} \cdot \nabla \underset{\sim}{v} + \nu \nabla^2 \underset{\sim}{\omega} + 2\underset{\sim}{\Omega} \cdot \nabla \underset{\sim}{v} \qquad (24)$$

A rapidly rotating framework is defined as one in which

$$\Omega t_o \gg 1 \qquad (25)$$

where t_0 is the time scale of the turbulent fluctuations. For a rotating channel flow where $\underset{\sim}{\Omega} = \Omega \underset{\sim}{k}$, (24) can be written in the equivalent form

$$\frac{1}{\Omega} \frac{D\underset{\sim}{\omega}}{Dt} = \frac{1}{\Omega} \underset{\sim}{\omega} \cdot \nabla \underset{\sim}{v} + \frac{\nu}{\Omega} \nabla^2 \underset{\sim}{\omega} + 2\underset{\sim}{k} \cdot \nabla \underset{\sim}{v} \tag{26}$$

As a result of (25), equation (24) yields the approximate constraint (in a rapidly rotating framework) that

$$2\underset{\sim}{k} \cdot \nabla \underset{\sim}{v} = 0 \tag{27}$$

It, thus, follows that

$$\frac{\partial \underset{\sim}{v}}{\partial z} = 0 \tag{28}$$

Since the walls of the channel are parallel to the axis of rotation, (28) has the exact solution

$$\underset{\sim}{v} = \underset{\sim}{v}(x,y,t) \tag{29}$$

which is valid <u>throughout the channel</u>. Hence, the flow must become two-dimensional (in a strong approximate sense) as would be expected from the Taylor-Proudman theorem.[10] Since, $\overline{\underset{\sim}{v}} = \overline{U}(y)\underset{\sim}{i}$, it is clear that the fluctuating velocity is of the approximate form

$$\underset{\sim}{u} = \underset{\sim}{u}(x,y,t) \tag{30}$$

in a rapidly rotating channel flow. It will now be shown that since there is no mean forcing along the axis of rotation (i.e., since $\overline{v}_z = \partial \overline{P}/\partial z = 0$) it follows that

$$u_z = 0 \tag{31}$$

This result can be easily seen by examining the z-component of the fluctuating

momentum equation (3) which reduces to the form

$$\frac{\partial u_z}{\partial t} + \overline{U} \frac{\partial u_z}{\partial x} = - u_x \frac{\partial u_z}{\partial x} - u_y \frac{\partial u_z}{\partial y} + \nu \nabla^2 u_z \tag{32}$$

since $\partial p/\partial z = 0$ (the flow is unforced in the z-direction). Once $\overline{U}$, u_x, and

u_y are determined from the x and y-components of the momentum equations (2)

and (3), equation (32) constitutes a _linear_ partial differential equation

for the determination of u_z. This equation has the simple solution $u_z = 0$

when solved subject to the no slip condition at the channel walls. Hence,

the solution

$$\tau_{xz} = \overline{u_x u_z} = 0 \tag{33}$$

$$\tau_{zz} = \overline{u_z u_z} = 0 \tag{34}$$

is consistent with the Navier-Stokes equations. It is thus clear that,

as a rigorous consequence of the Navier-Stokes equations, the Reynolds

stress tensor in a rapidly rotating channel flow takes the two-dimensional form

$$\underset{\sim}{\tau} = \begin{bmatrix} \tau_{xx}(y) & \tau_{xy}(y) & 0 \\ \tau_{xy}(y) & \tau_{yy}(y) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

This differs from $\underset{\sim}{\tau}$ for turbulent channel flow in an inertial framework only

in that τ_{zz} is zero -- a result which is a direct consequence of the Taylor-

Proudman reorganization that occurs in a rapidly rotating framework.

It will now be demonstrated that all existing second-order closure models yield results that are fundamentally inconsistent with (35) and, hence are in serious violation of the Navier-Stokes equations for rotating channel flow. By establishing a one-to-one correspondence between (x,y,z) and (x_1, x_2, x_3), the general form of second-order closures given by (13) reduces to the equation

$$\frac{1}{\Omega} \left[\frac{D\tau_{k\ell}}{Dt} + (1-C_1)(\tau_{km} \frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \bar{v}_k}{\partial x_m}) \right] = -\frac{1}{\Omega} \frac{\partial C_{mk\ell}(\underset{\sim}{\tau}, \nabla\underset{\sim}{\tau}, \ell)}{\partial x_m}$$

$$-\frac{1}{\Omega} A_{k\ell} (\underset{\sim}{\tau}, \nabla\underset{\sim}{\tau}, \bar{\underset{\sim}{D}}, \ell) - \frac{1}{\Omega} B_{k\ell}(\underset{\sim}{\tau}, \bar{\underset{\sim}{D}}, \ell) + \frac{\nu}{\Omega} \nabla^2 \tau_{k\ell}$$

$$+ (2-C_1)(\varepsilon_{3kn}\tau_{n\ell} + \varepsilon_{3\ell n}\tau_{nk}) \tag{36}$$

where we have made use of the fact that $\underset{\sim}{\Omega} = \Omega\underset{\sim}{k}$ and we have divided through by Ω. Taking the limit of (36) as $\Omega \rightarrow \infty$, we conclude that all existing second-order closure models yield the constraint

$$(2-C_1)(\varepsilon_{3kn}\tau_{n\ell} + \varepsilon_{3\ell n}\tau_{nk}) = 0 \tag{37}$$

for a rapidly rotating channel flow. As noted earlier,

$$2 - C_1 \neq 0 \tag{38}$$

for <u>all</u> of the existing second-order closures (in fact, $C_1 = 0$ for the Rotta-Komogorov model and $C_1 < 1$ in the Launder models[14]). It is then a simple matter to show that (37) has the general solution

$$\tau_{xx} = \tau_{yy} \tag{39}$$

$$\tau_{xy} = 0 \tag{40}$$

Hence, all existing second-order closures predict a state of transverse isotropy

for turbulent channel flow in a rapidly rotating framework -- a result which

is in conflict with the Navier-Stokes equations as demonstrated earlier.

In fact, this constitutes a completely spurious physical result since it

is well known that a turbulent shear flow must be accompanied by a non-zero

turbulent shear stress. Furthermore, since the Coriolis term vanishes in

the zz-component of all existing second-order closures (see Eq. (13)), it

is clear that these models predict that

$$\tau_{zz} \neq 0 \qquad\qquad\qquad (41)$$

as they do for a turbulent channel flow in an inertial framing. This result

is inconsistent with the Taylor-Proudman theorem for rotating channel flow

as discussed earlier. These inconsistencies arise because all existing

second-order closure models violate the principle of material frame-indifference

in the limit of two-dimensional turbulence -- a result which is a rigorous

consequence of the Navier-Stokes equations as proven by Speziale.[1,17] To

be more specific, the inertial terms in (13) do not vanish in a two-

dimensional turbulence unless it is isotropic. In the next section, it will

be shown how these deficiencies can be remedied.

3. ALTERNATIVE TURBULENCE MODELS FOR ROTATING FLOWS

As a prelude to examining alterations in existing second-order closure

models which can remedy the deficiencies discussed in the previous section,

we will consider the consistency of the simpler $K\text{-}\varepsilon$ or $K\text{-}\ell$ models of

turbulence with the Navier-Stokes equations for rotating channel flow. In

the $K\text{-}\varepsilon$ or $K\text{-}\ell$ models of turbulence, the Reynolds stress tensor takes the

form

$$\tau_{k\ell} = \frac{2}{3} K \delta_{k\ell} - \alpha K^{\frac{1}{2}} \ell (\frac{\partial \overline{v}_k}{\partial x_\ell} + \frac{\partial \overline{v}_\ell}{\partial x_k}) \tag{42}$$

where

$$K = \frac{1}{2} \tau_{mm} \tag{43}$$

is the turbulent kinetic energy per unit mass which is obtained from a

modeled version of its transport equation, ℓ is the length scale of

turbulence, and α is a dimensionless constant. For the $K\text{-}\ell$ model of turbulence,

ℓ is either specified algebraically or is determined from a modeled

transport equation obtained from the contracted form of the evolution

equation for the two-point velocity correlations.[13] In the $K\text{-}\varepsilon$ model,

the length scale is given by

$$\ell = \frac{K^{3/2}}{\varepsilon} \tag{44}$$

where ε is the dissipation rate of the turbulence which is obtained from a

modeled version of its transport equation.[14] Since the tensorial dependence

of (42) is only on the Kronecker delta and the symmetric part of the mean

velocity gradients which are frame-indifferent,[20] these traditional $K\text{-}\varepsilon$

or $K\text{-}\ell$ models are of the same form in a rotating frame as in an inertial

framing. The only way that the effects of rotation are accounted for is
through a change in the length scale which is usually done with ad hoc
empiricisms (see Howard, Patankar, and Bordynuik[5]). This result (along
with the fact that the mean flow is unidirectional so that the Coriolis
acceleration in (2) affects only the mean pressure[17]) guarantees that
the basic structure of the Reynolds stress tensor predicted by the K-ε or
K-ℓ models will be the same in an inertial or a rotating frame of reference.
To be more specific, the Reynolds stress tensor obtained from K-ε or K-ℓ
models of turbulence will be of the form

$$
\underset{\sim}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{bmatrix}
\tag{45}
$$

(where $\tau_{xx} = \tau_{yy} = \tau_{zz}$) for rotating channel flow or for channel flow in
an inertial framing. Of course, the specific values of the non-vanishing
components of $\underset{\sim}{\tau}$ can be different in the rotating frame as a result of changes
in the length scale. While (45) is not correct for a turbulent channel flow
in a rapidly rotating framework (since $\tau_{zz} \neq 0$), it is substantially less
unphysical than results obtained from existing second-order closures where
τ_{xy} is zero.

More recently, Galmes and Lakshminarayana[6] developed an algebraic
model for rotating flows which is given by

$$
\tau_{k\ell} = \frac{2}{3} K\delta_{k\ell} - \frac{2K}{C_1\varepsilon} \left(1 - \frac{1}{2} C_2\right)\left(\varepsilon_{kmn}\Omega_m\tau_{n\ell} + \varepsilon_{\ell mn}\Omega_m\tau_{nk}\right)
$$

$$
+ \frac{K(1-C_2)}{C_1\varepsilon} \left[\frac{2}{3} \tau_{mn} \frac{\partial \bar{v}_m}{\partial x_n} \delta_{k\ell} - \left(\tau_{km} \frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \bar{v}_k}{\partial x_m}\right)\right]
\tag{46}
$$

where C_1 and C_2 are dimensionless constants that take on the approximate

values of 1.5 and 0.6, respectively. For rotating channel flow where

$$\underset{\sim}{\Omega} = \Omega \underset{\sim}{k}$$

Eq. (46) can be written in the equivalent form

$$\frac{1}{\Omega} \tau_{k\ell} = \frac{2}{3} \frac{K}{\Omega} \delta_{k\ell} + \frac{2K}{C_1 \varepsilon} (1 - \frac{1}{2} C_2)(\varepsilon_{3kn} \tau_{n\ell} + \varepsilon_{3\ell n} \tau_{nk})$$

$$+ \frac{K(1-C_2)}{\Omega C_1 \varepsilon} [\frac{2}{3} \tau_{mn} \frac{\partial \overline{v}_m}{\partial x_n} \delta_{k\ell} - (\tau_{km} \frac{\partial \overline{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \overline{v}_k}{\partial x_m})] \qquad (47)$$

which, in a rapidly rotating framework (i.e., for $\Omega \to \infty$), yields the approximate

constraint

$$\varepsilon_{3kn} \tau_{n\ell} + \varepsilon_{3\ell n} \tau_{nk} = 0 \qquad (48)$$

Eq. (48) requires that $\tau_{xx} = \tau_{yy}$ and $\tau_{xy} = 0$. Hence, the algebraic Reynolds

stress model for rotating flows proposed by Galmes and Lakshminarayana[6]

yields the same spurious result for a rapidly rotating channel flow (i.e.,

a vanishing Reynolds shear stress in a turbulent shear flow) as that predicted

by existing second-order closure models. This problem arises since the

inertial term in (47) does not vanish for a two-dimensional turbulence

and, hence, (47) is in violation of the principle of material frame-indifference

which is a rigorous consequence of the Navier-Stokes equations in the two-

dimensional limit.

It will now be demonstrated that these problems can be overcome by making a small modification in the second-order closure recently obtained by Haworth and Pope[11] from a generalized Langevin model. In this Langevin model, a Reynolds stress closure is obtained which, in an inertial framing, takes the form

$$\frac{D\tau_{k\ell}}{Dt} + \tau_{km} \frac{\partial \overline{v}_\ell}{\partial x_m} + \tau_{\ell m} \frac{\partial \overline{v}_k}{\partial x_m} = - \frac{\partial}{\partial x_m} (\overline{u_m u_k u_\ell})$$

$$+ G_{km} \tau_{m\ell} + G_{\ell m} \tau_{mk} + C_o \, \varepsilon \delta_{k\ell} \tag{49}$$

where C_o is a dimensionless constant and $\underset{\sim}{G}$ is a second rank tensor which is assumed to be of the general functional form

$$\underset{\sim}{G} = G(\underset{\sim}{\tau}, \, \overline{\nabla v}, \varepsilon) \tag{50}$$

It should be noted that in this Langevin model it is not necessary to model the triple velocity correlation $\overline{uuu}$; this term is closed since it arises from the convection in physical space of the joint probability density function which is obtained from a separate transport equation. Haworth and Pope[11] arrived at a model for $\underset{\sim}{G}$ of the form

$$G_{k\ell} = (2+\beta_1) \frac{\partial \overline{v}_k}{\partial x_\ell} - \frac{1}{2K} \beta_1 \frac{\partial \overline{v}_k}{\partial x_m} \tau_{m\ell} + H_{k\ell}(\underset{\sim}{\tau}, \overline{\underset{\sim}{D}}, \varepsilon) \tag{51}$$

where β_1 is a dimensionless constant and $\underset{\sim}{H}$ is a frame-indifferent function since it depends only on the frame-indifferent quantities $\underset{\sim}{\tau}$, $\overline{\underset{\sim}{D}}$, and ε. Hence the Langevin model (49) is, in an inertial framing, given by the equation

$$\frac{D\tau_{k\ell}}{Dt} - \tau_{km}\frac{\partial \bar{v}_\ell}{\partial x_m} - \tau_{\ell m}\frac{\partial \bar{v}_k}{\partial x_m} = -\frac{\partial}{\partial x_m}(\overline{u_m u_k u_\ell})$$

$$+ \beta_1(\tau_{km}\frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m}\frac{\partial \bar{v}_k}{\partial x_m}) - \frac{\beta_1}{2K}(\frac{\partial \bar{v}_k}{\partial x_m}\tau_{mn}\tau_{n\ell}$$

$$+ \frac{\partial \bar{v}_\ell}{\partial x_m}\tau_{mn}\tau_{nk}) + H_{km}\tau_{m\ell} + H_{\ell m}\tau_{mk} + C_o \varepsilon \delta_{k\ell} \tag{52}$$

where the reader is referred to Haworth and Pope[11] for the precise form of H. It is a simple matter to show that in a rotating framework (with angular velocity $\underset{\sim}{\Omega} = \Omega \underset{\sim}{k}$), Eq. (52) is given by

$$\frac{D\tau_{k\ell}}{Dt} - \tau_{km}\frac{\partial \bar{v}_\ell}{\partial x_m} - \tau_{\ell m}\frac{\partial \bar{v}_k}{\partial x_m} = -\frac{\partial}{\partial x_m}(\overline{u_m u_k u_\ell})$$

$$+ \beta_1(\tau_{km}\frac{\partial \bar{v}_\ell}{\partial x_m} + \tau_{\ell m}\frac{\partial \bar{v}_k}{\partial x_m}) - \frac{\beta_1}{2K}(\frac{\partial \bar{v}_k}{\partial x_m}\tau_{mn}\tau_{n\ell}$$

$$+ \frac{\partial \bar{v}_\ell}{\partial x_m}\tau_{mn}\tau_{nk}) + H_{km}\tau_{m\ell} + H_{\ell m}\tau_{mk} + C_o \varepsilon \delta_{k\ell}$$

$$- \beta_1 \Omega[\varepsilon_{3\ell m}\tau_{km} + \varepsilon_{3km}\tau_{\ell m} - \frac{1}{2K}(\varepsilon_{3km}\tau_{mn}\tau_{n\ell} + \varepsilon_{3\ell m}\tau_{mn}\tau_{nk})] \tag{53}$$

Haworth and Pope[11] proved that the inertial term in (53) vanishes for a two-dimensional turbulence and, hence, this Langevin model satisfies the principle of material frame-indifference in the two-dimensional limit unlike all other existing second-order closures. However, these authors also

showed that this model does not give rise to a Taylor-Proudman reorganization in a rapidly rotating framework. In fact, it can be shown that for a turbulent channel flow in a rapidly rotating framework, (53) yields a stress tensor of the form (45). While this is a substantial improvement over what is predicted by all other existing second-order closures, it is still not completely correct in that τ_{zz} should vanish in the limit as $\Omega \to \infty$.

It will now be demonstrated that by making a small modification in the model for the tensor $\underset{\sim}{G}$ proposed by Haworth and Pope[11], a second-order closure can be obtained that gives rise to a Taylor-Proudman reorganization in rapidly rotating turbulent flows, thus, alleviating the problems discussed above. This proposed modification to (51) is as follows:

$$
G_{k\ell} = \frac{\beta_2}{K^{1/2}} \left| \overline{W}_{mn}\overline{W}_{np}\tau_{pm} - K\,\overline{W}_{mn}\overline{W}_{nm} \right|^{1/2} \delta_{k\ell}
$$
$$
+ \left(2+\beta_1\right)\frac{\partial\overline{v}_k}{\partial x_\ell} - \frac{1}{2K}\beta_1\frac{\partial\overline{v}_k}{\partial x_m}\;\tau_{m\ell} + H_{k\ell}(\underset{\sim}{\tau},\underset{\sim}{\overline{D}},\varepsilon) \tag{54}
$$

where

$$
\overline{W}_{k\ell} = \frac{1}{2}\left(\frac{\partial\overline{v}_k}{\partial x_\ell} - \frac{\partial\overline{v}_\ell}{\partial x_k}\right) \tag{55}
$$

and β_2 is a dimensionless constant (for $\beta_2 = 0$, this model reduces to that of Haworth and Pope). Since the new term containing the coefficient β_2 vanishes in a two-dimensional turbulence, it follows that this model also satisfies the principle of material frame-indifference in the limit of two-dimensional turbulence. In a rotating framework with an angular velocity $\underset{\sim}{\Omega} = \Omega\underset{\sim}{k}$, this modified expression for $\underset{\sim}{G}$ gives rise to a Langevin model of the form

$$\frac{D\tau_{k\ell}}{Dt} - \tau_{km}\frac{\partial\overline{v}_\ell}{\partial x_m} - \tau_{\ell m}\frac{\partial\overline{v}_k}{\partial x_m} = -\frac{\partial}{\partial x_m}(\overline{u_m u_k u_\ell})$$

$$+ \beta_1(\tau_{km}\frac{\partial\overline{v}_\ell}{\partial x_m} + \tau_{\ell m}\frac{\partial\overline{v}_k}{\partial x_m}) - \frac{\beta_1}{2K}(\frac{\partial\overline{v}_k}{\partial x_m}\tau_{mn}\tau_{n\ell}$$

$$+ \frac{\partial\overline{v}_\ell}{\partial x_m}\tau_{mn}\tau_{nk}) + H_{km}\tau_{m\ell} + H_{\ell m}\tau_{mk} + C_o\varepsilon\delta_{k\ell}$$

$$- \beta_1\Omega[\varepsilon_{3\ell m}\tau_{km} + \varepsilon_{3km}\tau_{m\ell} - \frac{1}{2K}(\varepsilon_{3km}\tau_{mn}\tau_{n\ell}$$

$$+ \varepsilon_{3\ell m}\tau_{mn}\tau_{nk})] + \frac{2\beta_2}{K^{1/2}} |(\overline{W}_{mn} - \varepsilon_{3mn}\Omega)(\overline{W}_{np}$$

$$- \varepsilon_{3np}\Omega)\tau_{pm} - K(\overline{W}_{mn} - \varepsilon_{3mn}\Omega)(\overline{W}_{nm} - \varepsilon_{3nm}\Omega)|^{1/2}\tau_{k\ell} \qquad (56)$$

If we consider the case of a rapidly rotating framework (i.e., the limit as $\Omega \to \infty$), equation (56) yields the constraint

$$- \beta_1[\varepsilon_{3\ell m}\tau_{km} + \varepsilon_{3km}\tau_{\ell m} - \frac{1}{2K}(\varepsilon_{3km}\tau_{mn}\tau_{n\ell} + \varepsilon_{3\ell m}\tau_{mn}\tau_{nk})]$$

$$+ \frac{2\beta_2}{K^{1/2}} |\varepsilon_{3mn}\varepsilon_{3np}\tau_{pm} - K \varepsilon_{3mn}\varepsilon_{3nm}|^{1/2} \tau_{k\ell} = 0 \qquad (57)$$

A straightforward, although somewhat tedious calculation shows that (57) is satisfied if and only if the Reynolds stress tensor takes the two-dimensional form

$$\underset{\sim}{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \tau_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{58}$$

The key result that τ_{zz} must be zero can easily be seen by examining the zz-component of (57) which reduces to[†]

$$\frac{2\beta_2}{K^{1/2}} \, \tau_{zz}^{3/2} = 0 \tag{59}$$

It is thus clear that with just one minor modification, the Langevin second-order closure of Haworth and Pope yields a turbulence model that is consistent with the Taylor-Proudman theorem and is, thus, suitable for the description of rapidly rotating turbulent flows unlike all other existing turbulence models.

[†] No constraint is placed on τ_{xx}, τ_{xy}, and τ_{yy} since Eq. (57) vanishes for a two-dimensional turbulence.

4. CONCLUSION

It has been proven that all existing second-order closure models are fundamentally inconsistent with the Navier-Stokes equations in a rapidly rotating framework. In particular, it was demonstrated that, for the test problem of fully-developed turbulent channel flow subjected to a rapid spanwise rotation, all existing second-order closures yield the spurious physical result of a vanishing Reynolds shear stress and are inconsistent with the Taylor-Proudman theorem. The type of results predicted for this problem by these second-order closures are, in fact, substantially more unphysical than those obtained from the simpler K-ε or K-ℓ models of turbulence as indicated in Section 3. By adding one term to the second-order closure of Haworth and Pope obtained from a Langevin model, it was demonstrated that these problems can be overcome. This modified Langevin model appears to yield the first Reynolds stress closure that is generally consistent with the Navier-Stokes equations in a rapidly rotating framework.

Future calculations should be conducted on rotating turbulent flows with this modified Langevin model. This is quite important since the closure model for $\underset{\sim}{G}$ given in (54) is not the most general and, hence, there may be other terms which could have been overlooked. Unfortunately, such a computational study is quite involved and is beyond the scope of the present paper. Future research is also needed on the effect of rotations on the evolution of the joint probability density function since such evolution equations play an important role in all Langevin models. Few, if any, studies along these lines appear to have been reported.

With some additional refinements, turbulent closure models can be developed that apply to rotating flows for a wide range of rotation rates without the need for ad hoc empiricisms. Such a development could have a profound effect on the analysis of a wide variety of turbulent flows that are of interest to engineers and geophysicists.

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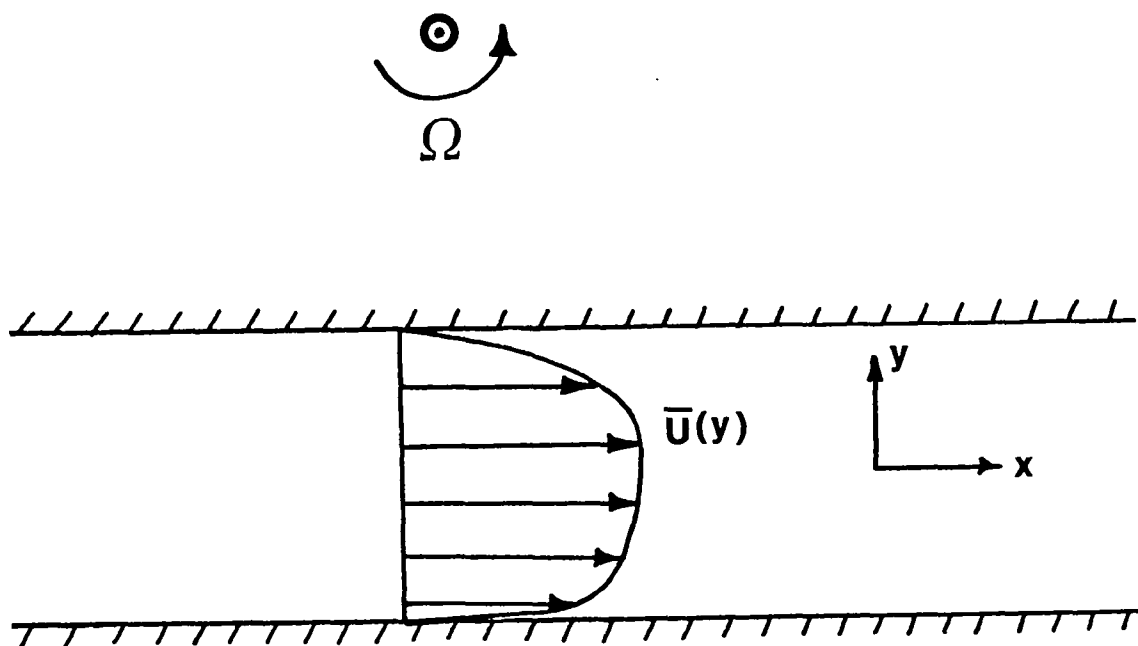[17] C. Speziale, ASME J. Appl. Mech. 51, 942 (1984).

FIGURE 1.   Fully-Developed Turbulent Channel Flow Subjected to a
Spanwise Rotation after Johnston, Halleen, and Lezius (1972).

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16. Abstract

    The physical properties of the commonly used second-order closure models are examined theoretically for rotating turbulent flows. Comparisons are made with results which are a rigorous consequence of the Navier-Stokes equations for the problem of fully-developed turbulent channel flow in a rapidly rotating framework. It is demonstrated that all existing second-order closures yield spurious physical results for this test problem of rotating channel flow. In fact, the results obtained are shown to be substantially more unphysical than those obtained from the simpler K-ε and K-ℓ models. Modifications in the basic structure of these second-order closure models are proposed which can alleviate this problem.

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