JPL PUBLICATION 85-42

Some Properties of n-Dimensional Triangulations

C.L. Lawson

(NASA-CR-176535) SOME PROPERTIES OF n-DIMENSIONAL TRIANGULATIONS (Jet Propulsion Lab.) 35 p HC A03/MF A01 CSCL 09B

N86-19009

Unclas

G3/61 05451

June 15, 1985

National Aeronautics and
Space Administration

Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

TECHNICAL REPORT STANDARD TITLE PAGE

1. Report No. 85-42	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Some Properties of n-Dimensional Triangulations		5. Report Date June 15, 1985 6. Performing Organization Code
7. Author(s) C.L. Lawson		8. Performing Organization Report No.
9. Performing Organization Name and Address JET PROPULSION LABORATORY California Institute of Technology 4800 Oak Grove Drive Pasadena, California 91109		10. Work Unit No. 11. Contract or Grant No. NAS7-918 13. Type of Report and Period Covered External Report JPL Publication
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION Washington, D.C. 20546		14. Sponsoring Agency Code B-055-11-00-11-36

15. Supplementary Notes

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The principal results of the present report are

(a) A set of n+2 points in n-space may be triangulated in at most 2 different ways.

(b) The "sphere test" defined in this report selects a preferred one of these two triangulations.

(c) A set of parameters is defined that permits the characterization and enumeration of all sets of n+2 points in n-space that are significantly different from the point of view of their possible triangularizations.

(d) The local sphere test induces a global sphere test property for a triangulation.

(e) A triangulation satisfying the global sphere property is dual to the n-dimensional Dirichlet tessellation, i.e., it is a Delaunay triangulation.

17. Key Words (Selected by Author(s))	18. Distribution Statement
Computer Programming and Software; Numerical Analysis; Theoretical Mathematics	Unclassified; unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified		

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The research described in this publication was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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ACKNOWLEDGEMENTS

The author wishes to express his appreciation to C. deBoor, G. Farin, and W. V. Snyder for helpful discussions and comments on the drafts of this report.

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1. Introduction

Let P denote a set of m distinct points in n-space ($n \geq 2$, $m \geq n+1$.) Let C denote the convex hull of P. Assuming P does not lie entirely in some $(n-1)$-dimensional manifold, we are interested in the problem of constructing a simplicial tessellation, T, of C such that each simplex, $t \in T$, has $n+1$ points of P as its vertices, has non-null n-dimensional volume, and contains no other points of P. Such a tessellation will be called a **triangulation**.

Another type of tessellation with which we shall be concerned is the **Dirichlet tessellation**. This is the set of m cells, d_i, defined as

$$d_i = \{q : \|q-p_i\| \leq \|q-p_j\| \text{ for all } p_j \in P\}$$

Thus the interior points of the cell d_i are the points of n-space that are closer to the point p_i than to any other point of P. A triangulation of P can be defined that is, in a certain sense, dual to the Dirichlet tessellation. Such a triangulation is called a **Delaunay triangulation** and will be defined in Section 9.

Triangulations of point sets in 2-space are used in a variety of applications, particularly as an initial step in the analysis of data that is available at scattered points in the plane. After triangulation one may carry out interpolation, regridding to a rectangular grid, contour plotting, or other processes. The Dirichlet tessellation is used as a model in various scientific fields where it is appropriate to associate a unique region of space with each point in a finite point set.

Examples of applications of the Dirichlet tessellation are summarized and referenced in [Green '78].

Similar needs arise for the analysis of data defined at scattered points in higher dimensional spaces, but much less study has been devoted to triangulation algorithms and interpolation methods for these problems.

Some early algorithms proposed for triangulation in the plane required $O(m^2)$ time. Since 1972 a number of computer programs for this problem, or the closely related Dirichlet tessellation problem, have been reported with estimates of execution time in the range of $m^{1.3}$ to $m^{1.5}$. See [Lawson '72], [Lawson '77], [Akima '78], [Green '78], [Cline '84], and [Renka '84b]. Triangulation on the surface of a sphere is treated in [Lawson '84] and [Renka '84a]. The methods of various of these papers have been used in a number of proprietary graphics packages, and in the portable Fortran programs described in [Renka '84a] and [Renka '84b] that are available from the ACM software distribution service.

Bowyer [Bowyer '81] devised and implemented an algorithm for constructing the n-dimensional Dirichlet tessellation and the dual (Delaunay) triangulation. A very satisfactory execution time estimate of $O(a_k n^{(1+1/k)} + b_k n)$ was reported. The algorithm was implemented in ISO Fortran and performed well in a variety of test cases.

Barnhill and Little [Barnhill '84] presented ideas on a different approach to the n-dimensional triangulation problem and also gave interpolation methods for use with triangular grids in 3- and 4-dimensional space.

The present report establishes some mathematical properties of n-dimensional triangulations that provide additional understanding of the problem. These results are generalizations of properties of the 2-dimensional problem given in [Lawson '77].

The algorithm given in [Lawson '77] operates by successively triangulating various 4-point subsets of the given point set P. It is easily seen that a set of 4 points in the plane admits of at most 2 different triangulations. In [Lawson '77] a local "circle test" was introduced that selected one of these two possible triangulations. It was shown that the triangulation produced according to this criterion satisfied a global circle test property, a max-min angle property, and was dual to the Dirichlet tessellation. The equivalence of these latter two properties was established independently by Sibson in [Sibson '78].

The principal results of the present report are

(a) A set of n+2 points in n-space may be triangulated in at most 2 different ways.

(b) The "sphere test" defined in this report selects a preferred one of these two triangulations.

(c) A set of parameters is defined that permits the characterization and enumeration of all sets of n+2 points in n-space that are significantly different from the point of view of their possible triangularizations.

(d) The local sphere test induces a global sphere test property for a triangulation.

(e) A triangulation satisfying the global sphere property is dual to the n-dimensional Dirichlet tessellation, i.e., it is a Delaunay triangulation.

2. Barycentric coordinates and their geometric interpretation

The convex hull of $n+1$ points is called a **simplex**. The convex hull of a subset consisting of n of these points is called a **facet** of the simplex.

Let p_1, ..., p_{n+1}, be $n+1$ distinct points in n-space. Define the matrix

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_{n+1} \end{bmatrix}$$

Let t be the (possibly degenerate) simplex with vertices p_1, ..., p_{n+1}. The n-dimensional volume of t is given by

$$\mathrm{Vol}(t) = |\mathrm{Det}(B)|/n!$$

Let q be an arbitrary point in n-space. If t is nondegenerate, i.e., if $\mathrm{Vol}(t) \neq 0$, the numbers, b_1, ..., b_n, satisfying

$$(1) \qquad \begin{bmatrix} 1 & 1 & & 1 \\ p_1 & p_2 & \cdots & p_{n+1} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ q \end{bmatrix}$$

are called barycentric coordinates of q relative to the simplex, t.

For each s, the sign of b_s indicates the position of q relative to the hyperplane, H_s, containing the facet of t opposite vertex p_s. Thus $b_s = 0$ when q is in H_s, $b_s > 0$ when q is on the same side of H_s as p_s, and $b_s < 0$ when q is on the opposite side of H_s from p_s.

Some consequences of these facts are:

(a) The point, q, is in the simplex, t, if and only if all $b_s \geq 0$.

(b) If q is strictly outside the simplex, t, then one or more of the b_s's are negative. These negative b_s's identify the facets of t with whose vertices q can be connected to form nondegenerate simplices neighboring to t.

(c) At least one of the b_s's must be positive since $\Sigma b_s = 1$.

3. Triangulations of sets of n+2 points

In the 2D problem, study of all possible ways of triangulating the convex hull of four points provided the key ideas that led to a systematic procedure for improving the triangulation of the convex hull of larger sets of points as one point at a time was introduced. The study of sets of n+2 points provides similar insights in the n-dimensional problem.

In 2-space all of the different possibilities for configurations of four distinct points, not on a common line, are illustrated by the three cases in Figure 1.

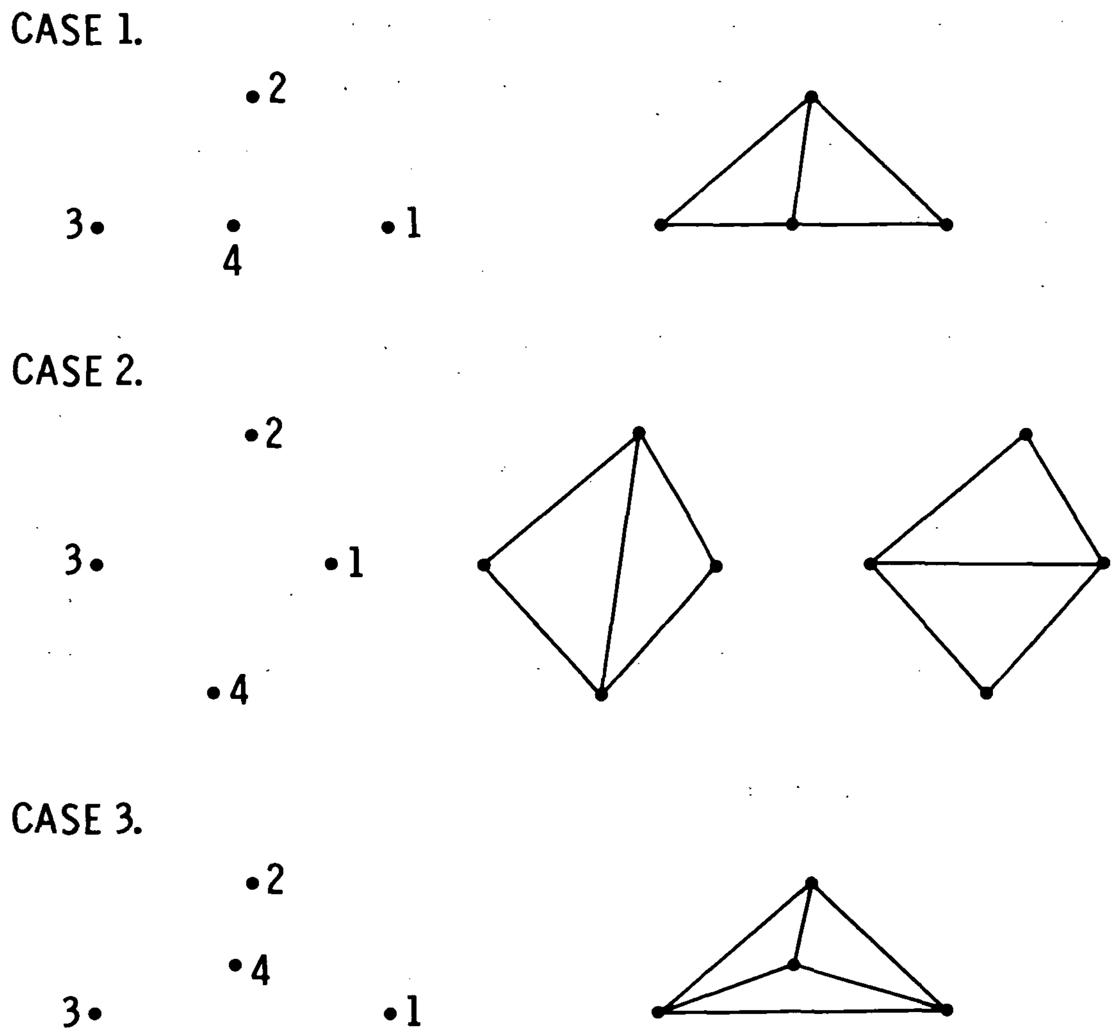


Figure 1. Configurations of four distinct non-colinear points in 2-space.

In Case 1 there is a unique triangulation using two triangles. In Case 2 there are two possible triangulations, each using two triangles. In Case 3 there is a unique triangulation using three triangles.

One might expect the general n-dimensional case to be much more complicated than the 2-D case. Although there is more complexity in the higher dimensional cases, there are, fortunately, two very useful properties that persist. It remains true that a given set of $n+2$ points in n-space admits of at most two different triangulations by n-dimensional simplices. Furthermore, if the $n+2$ points do not all lie on a common n-sphere, one of the possible triangulations is uniquely selected by the sphere test that will be described shortly.

These facts will be established in the following two sections.

4. Signature sets

Theorem 1 *Let P be a set of $n+2$ points, $p_1, \ldots, p_{n+2}$, in n-space not lying entirely in any $(n-1)$-dimensional manifold. There is a partitioning of the index set $\{1, 2, \ldots, n+2\}$ into three sets, S_0, S_1, and S_2, and a set of numbers, c_i, satisfying*

$$(2) \qquad \sum_{i \in S_1} c_i p_i = \sum_{i \in S_2} c_i p_i$$

$$(3) \qquad \sum_{i \in S_1} c_i = \sum_{i \in S_2} c_i = 1$$

$$(4) \qquad c_i = 0, \quad i \in S_0$$

$$(5) \qquad c_i > 0, \quad i \in S_1 \cup S_2$$

The numbers c_i are uniquely determined by the set P. The sets S_0, S_1, and S_2, are also *unique, with the understanding that the labeling of S_1 and S_2 could be arbitrarily interchanged.*

Proof. The $(n+1)\times(n+2)$ matrix

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_{n+2} \end{bmatrix}$$

is of rank $n+1$ due to the hypothesis that the points of P do not all lie in any $(n-1)$-dimensional manifold. The system

$$(6) \qquad Ax = 0$$

has a 1-dimensional space of solution vectors. Let x^* be a nonzero vector satisfying Eq.(6) and normalized to satisfy $\Sigma|x_i| = 2$. Since the first row of Eq.(6) is $\Sigma x_i = 0$, the vector x^* must have both positive and negative components, and the sum of the positive components must equal the sum of the magnitudes of the negative components. In fact, with the specified normalization, these sums must each be 1.

If any components of x^* are zero, let S_0 be the index set for these components. Let S_1 be the index set for components of x^* of one sign and let S_2 be the index set for components of x^* of the other sign. Both S_1 and S_2 are non-null. Define

$$c_i = |x_i^*|, \quad i = 1, \ldots, n+2$$

Then $\Sigma_i c_i = 2$ and all of the Eqs.(2-5) are satisfied.

This specification of the sets, S_0, S_1, and S_2, and the numbers, c_i, is unique to within the possible interchange of S_1 and S_2, since if Eqs.(2-5) were satisfied by any other sets, S_0', S_1', and S_2', and numbers, c_i', it would permit the construction of a solution vector for Eq.(6) not in the same 1-dimensional subspace as x^*. $\square$

For any point set P satisfying the hypotheses of Theorem 1, the associated sets, S_0, S_1, and S_2, will be called the **signature sets** of P.

Let p^* be the point given by the equal left and right members of Eq.(2). Note that p^* is in the convex hull of $\{p_i : i \in S_1\}$ and also in the convex hull of $\{p_i : \text{d} \in S_2\}$. In fact the geometric interpretation of Theorem 1 is that the subsets of P indexed by S_1 and S_2 are the smallest two disjoint subsets of P whose convex hulls have a point in common, and the common point, p^*, is unique.

As examples, in Fig. 1, we can identify the signature sets and the common point p^* as follows:

Case I $\quad S_0 = \{2\}$, $S_1 = \{4\}$, $S_2 = \{1, 3\}$, $p^* = p_4$

Case II $\quad S_0 =$ Null, $S_1 = \{1, 3\}$, $S_2 = \{2, 4\}$, $p^* =$ the intersection of lines p_1p_3 and p_2p_4.

Case III $\quad S_0 =$ Null, $S_1 = \{4\}$, $S_2 = \{1, 2, 3\}$, $p^* = p_4$

To relate this to the possible triangulations of the convex hull of P, we next identify the possible nondegenerate simplices that can be formed from the points of P.

For each $i = 1, \ldots, n+2$, let t_i denote the (possibly degenerate) simplex formed using the points $P \backslash p_i$ as vertices. (The notation $P \backslash p_i$ denotes the subset of P remaining when point p_i is removed.) The n-dimensional volume of t_i is λc_i where the c_i's are given by Theorem 1, and λ is a positive constant independent of i. Thus the nondegenerate simplices are just the set $\{t_i : i \in S_1 \cup S_2\}$.

What are the possible groupings of these simplices to form a triangulation of the convex hull of P?

5. Admissible triangulations of $n+2$ points

Theorem 2 *Let P be a point set as in Theorem 1. There are at most two distinct triangulations of the convex hull of P, namely, $T_1 = \{t_i : i \in S_1\}$ and $T_2 = \{t_i : i \in S_2\}$, where the sets S_1 and*

S_2 *are as defined in Theorem* 1, *and* t_i *is the simplex with* vertex set $P-\{p_i\}$. *If one of the sets* S_1 *or* S_2 *is of cardinality* 1, *the corresponding set* T_1 *or* T_2 *is not an admissible triangulation.*

Proof. This theorem will be proved by establishing the following four assertions:

(a) Any pair of simplices, one indexed in S_1 and the other indexed in S_2, is mutually overlapping and thus cannot be used in the same triangulation.

(b) Any pair of simplices, both indexed in S_1 or both indexed in S_2, is nonoverlapping, and thus the sets $T_1 = \{t_i : i \in S_1\}$ and $T_2\{t_i : i \in S_2\}$ are each nonoverlapping sets of simplices.

(c) Each of the sets T_1 and T_2 covers the entire convex hull, C, of P, i.e., any point $q \in C$ is also contained in some $t_i \in T_1$ and in some $t_j \in T_2$.

(d) T_i is not an admissible triangulation of C if $|S_i| = 1$.

Note that Eq.(2) can be solved for any one of the p_i's, $i \in S_1 \cup S_2$, and the resulting equation gives the barycentric coordinates of one point of P with respect to the simplex formed by the others. For example, choose an index, $k \in S_1$, and solve Eq.(2) for p_k, obtaining

$$(7) \quad p_k = \sum_{i \in S_2} (c_i/c_k)p_i - \sum_{i \in S_1 \setminus k} (c_i/c_k)p_i + \sum_{i \in S_0} (c_i/c_k)p_i$$

from which we may write the barycentric coordinates of p_k relative to the simplex t_k as

$$(8) \quad b_i = c_i/c_k > 0 \quad \text{for } i \in S_2$$

$$(9) \quad b_i = -c_i/c_k < 0 \quad \text{for } i \in (S_1-\{k\})$$

$$(10) \quad b_i = 0 \quad \text{for } i \in S_0$$

Eq.(8) implies that for each $i \in S_2$, p_i and p_k are on the same side of the common facet shared by t_i and t_k. Thus t_i and t_k overlap, establishing assertion (a). Eq.(9) implies that for each $i \in S_1 \backslash k$, p_i and p_k are on opposite sides of the common facet shared by t_i and t_k. Thus t_i and t_k do not overlap, establishing assertion (b).

Let q be an arbitrary point in C. Then there are non-negative coefficients, d_i, such that

$$(11) \qquad q = \sum_{i=1}^{n+2} d_i p_i$$

$$(12) \qquad 1 = \sum_{i=1}^{n+2} d_i$$

Rewrite Eqs.(2-3) as

$$(13) \qquad 0 = \sum_{i \in S_1} c_i p_i - \sum_{i \in S_2} c_i p_i$$

$$(14) \qquad 0 = \sum_{i \in S_1} c_i - \sum_{i \in S_2} c_i$$

Using an indeterminate, λ, form λ times Eq.(13) plus Eq.(11), and λ times Eq.(14) plus Eq.(12):

$$(15) \qquad q = \sum_{i \in S_1} (d_i + \lambda c_i) p_i + \sum_{i \in S_2} (d_i - \lambda c_i) p_i + \sum_{i \in S_0} d_i p_i$$

$$(16) \qquad 1 = \sum_{i \in S_1} (d_i + \lambda c_i) + \sum_{i \in S_2} (d_i - \lambda c_i) + \sum_{i \in S_0} d_i$$

There is a range of values of λ, $\lambda_{min} \le \lambda \le \lambda_{max}$, for which all of the coefficients, $(d_i + \lambda c_i)$, $i \in S_1$, and $(d_i - \lambda c_i)$, $i \in S_2$, appearing in Eqs.(15-16) are nonnegative. At the low end of this range, at least one of the coefficients indexed in S_1, say $d_j + \lambda c_j$, is zero, showing that $q \in t_j$. Similarly at the high end of the λ range at least one of the coefficients indexed in S_2, say

$d_\ell - \lambda c_\ell$, is zero, showing that $q \in t_\ell$. These limiting values of λ are given by

$$\lambda_{min} = \max\{-d_i/c_i : i \in S_1\}$$

$$\lambda_{max} = \min\{d_i/c_i : i \in S_2\}$$

This establishes assertion (c).

Consider now the case in which one of the sets S_1 or S_2 has cardinality 1, e.g., suppose $|S_1| = 1$. Then the triangulation, T_1, consists of a single simplex, say t_j. The point p_j that is not a vertex of t_j is therefore not a vertex in the triangulation, T_1. Thus the triangulation T_1 is not admissible because it does not include all points of P as vertices. □

In a different context, namely in developing a stable method for evaluation of multivariate splines, Grandine [Grandine '84] proved a theorem encompassing a subset of the above Theorem 2. His theorem states that an arbitrary point in the convex hull of a set of n+2 points in n-space can be in the interior of at most two of the n+2 simplices that can be formed using these points.

As examples of Theorem 2, consider again Figure 1. In Case 1, with $S_1 = \{4\}$ and $S_2 = \{1, 3\}$, the triangulation $T_2 = \{t_1, t_3\}$ is admissible while $T_1 = \{t_4\}$ is not, again because p_4 is not used as a vertex. In Case 2, with $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$, there is a choice of two admissible triangulations, $T_1 = \{t_1, t_3\}$ and $T_2 = \{t_2, t_4\}$. In Case 3, with $S_1 = \{4\}$ and $S_2 = \{1, 2, 3\}$, the triangulation $T_2 = \{t_1, t_2, t_3\}$ is admissible while $T_1 = \{t_4\}$ is not, because p_4 is not used as a vertex.

Corollary 1 *In the context of building triangulations, an enumeration of all possible significantly different* *configurations of $n+2$ distinct points in n-space, not lying in any $(n-1)$-dimensional manifold, is given by all of the possible ways of assigning values to $|S_0|$, $|S_1|$, and $|S_2|$ satisfying*

(17) $|S_2| \geq |S_1| \geq 1$

(18) $|S_2| \geq 2$

(19) $|S_0| \geq 0$

(20) $|S_0| + |S_1| + |S_2| = n+2$

Proof. The sets S_1 and S_2 must each be nonempty to satisfy Eq.(3). Since the sets S_1 and S_2 are not mutually distinguished we may arbitrarily use S_1 as the label of the smaller of the two sets when they are not of the same size. These considerations establish Eq.(17).

The sets S_1 and S_2 cannot both be singletons for then they would have to index the same single point in order for Eq.(2) to hold. This is ruled out by the hypothesis that all points of P are distinct. Along with the convention of Eq.(17) this gives Eq.(18).

Eqs.(19-20) follow from previous discussions. □

Using Corollary 1 one may list the values of $(|S_0|, |S_1|, |S_2|)$ for all significantly different configurations of $n+2$ points in n-space. This is done for dimensions 1, 2, 3, and 4 in Table 1.

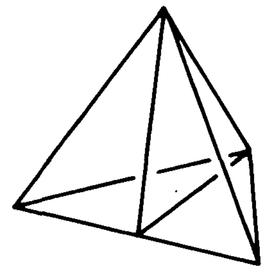
Table 1. Characterizing parameters for all of the significantly different configurations of $n+2$ points in n-space for n = 1, 2, 3, and 4.

n = 1			n = 2			n = 3			n = 4		
$\lvert S_0\rvert$	$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_0\rvert$	$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_0\rvert$	$\lvert S_1\rvert$	$\lvert S_2\rvert$	$\lvert S_0\rvert$	$\lvert S_1\rvert$	$\lvert S_2\rvert$
0	1	2	1	1	2	2	1	2	3	1	2
			0	2	2	1	2	2	2	2	2
			0	1	3	1	1	3	2	1	3
						0	2	3	1	2	3
						0	1	4	1	1	4
									0	3	3
									0	2	4
									0	1	5

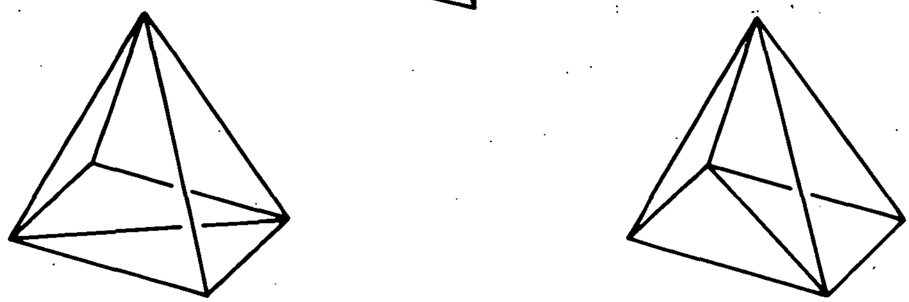
For each dimension in Table 1 the triples are listed in reverse lexicographic order. With this ordering cases in the same row have the same pair of values of $|S_1|$ and $|S_2|$, differing only in $|S_0|$. Since the possible triangulations are determined by $|S_1|$ and $|S_2|$, there is a significant geometric relationship between the possible triangulations for cases appearing in the same row. This will be explained further after introducing Figure 2.

The three cases shown for n = 2 are those previously illustrated in Figure 1, and in the same order. Diagrams illustrating the five cases for n = 3 are given in Figure 2. Recall that cases with $|S_1|$ = 1 admit only one triangulation each while cases with $|S_1|$ > 1 admit two distinct triangulations each.

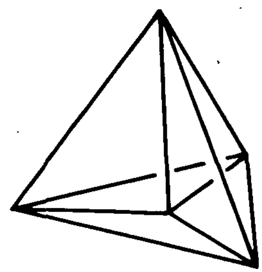
CASE 1.



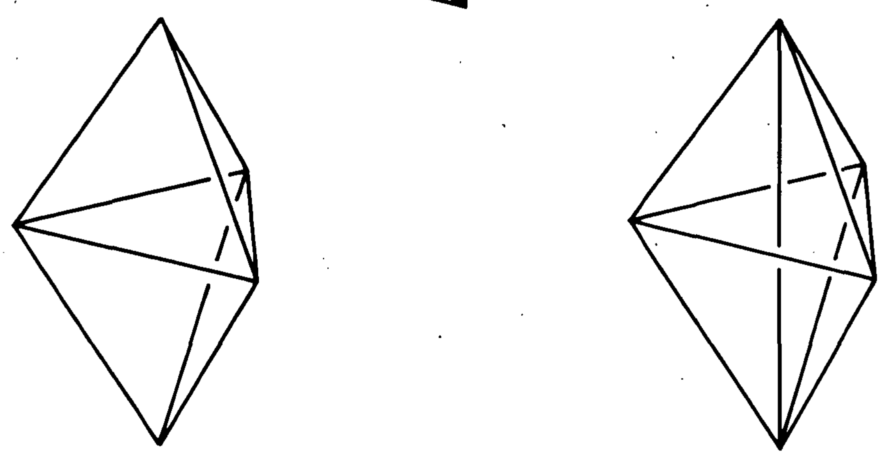
CASE 2.



CASE 3.



CASE 4.



CASE 5.

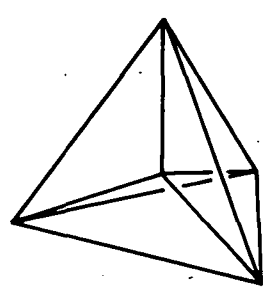


Figure 2. Configurations of five distinct non-coplanar points in 3-space.

Note that in the first three cases of Figure 2 the base facet of each diagram has the same configuration as the corresponding case in Figure 1. Case 1 admits of a further stage of reduction since the base edge in Figure 1 is the diagram for the case of 3 points in 1-space, i.e., the case corresponding to the first and only entry for $n = 1$ in Table 2.

These are examples of the following general principle: If a point set P has parameters $(\sigma_0, \sigma_1, \sigma_2)$ with $\sigma_0 \neq 0$, the subset P^* consisting of the points indexed in S_1 and S_2 will lie in an $(n-\sigma_0)$-dimensional manifold and have a $(0, \sigma_1, \sigma_2)$ configuration. (Note that these two related configurations are in the same row in Table 2.) Any triangulation of P is related to a tesselation of P^* by the fact that any simplex in a triangulation of P is the convex hull of the union of a simplex in a triangulation of P^* with the points of $P \backslash P^*$.

6. Configurations having all pairs of points connected

Note that in Case 3 and in the second triangulation shown for Case 4 in Figure 2, every pair of points is connected by an edge used in the triangulation. This is true of any triangulation of $n+2$ points in n-space if the triangulation consists of three or more simplices. This follows from the observation that if any three different subsets of size $n+1$ are selected from a set of $n+2$ points, every possible pairing of points must appear in one or more of the selected subsets.

This observation has particular significance in dimensions 4 and higher, since then there are configurations of $n+2$ points having 3 or more points in both S_1 and S_2; e.g., see Row 6 for

$n = 4$ in Table 1. In such cases both of the possible triangulations will have all points connected by edges of the triangulations. Therefore the two possible simplicial triangulations are not distinguished from each other by information about connectivity of pairs of points.

We shall return to this point in Section 9 in connection with the Dirichlet tessellation.

7. The sphere test for a set of $n+2$ points in n-space

Let P continue to denote a set of $n+2$ points in n-space, not lying entirely in any $n-1$ dimensional manifold. In the preceding section it was seen that there may be either one or two ways to triangulate the convex hull C of such a set P. For the cases in which two triangulations are possible we introduce the **sphere test** as a way of choosing one of the triangulations.

Suppose P is a configuration that admits two possible triangulations. Using the notation and results of Theorems 1 and 2, it follows that $|S_1| \geq 2$, $|S_2| \geq 2$, and $n \geq 2$. The two possible triangulations are $T_1 = \{t_i : i \in S_1\}$ and $T_2 = \{t_i : i \in S_2\}$, where t_i is the nondegenerate simplex with vertex set $P \backslash p_i$.

For each $i \in S_1 \cup S_2$, let E_i be the unique n-sphere circumscribing the simplex t_i and let B_i be the open n-ball whose boundary is E_i.

The Sphere Test

If all points of P lie on the same sphere, i.e., all of the spheres E_i are coincident, then the sphere test does not distinguish between T_1 and T_2. Otherwise choose $j \in S_1 \cup S_2$. If $p_j \notin B_j$ select the triangulation T_1 or T_2 that includes t_j, while if $p_j \in B_j$ select the triangulation T_1 or T_2 that does not include t_j.

As an example consider Case 2 of Figure 1 with $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$. To apply the sphere test we may choose any one of the four points and ask whether it is inside or outside the open ball circumscribing the other three points. Figure 3 illustrates the four possible ways of applying the test for this example. We find $p_1 \in B_1$, $p_3 \in B_3$, $p_2 \notin B_2$, and $p_4 \notin B_4$. Thus any one of these tests results in the selection of $T_2 = \{t_2, t_4\}$ as the preferred triangulation.

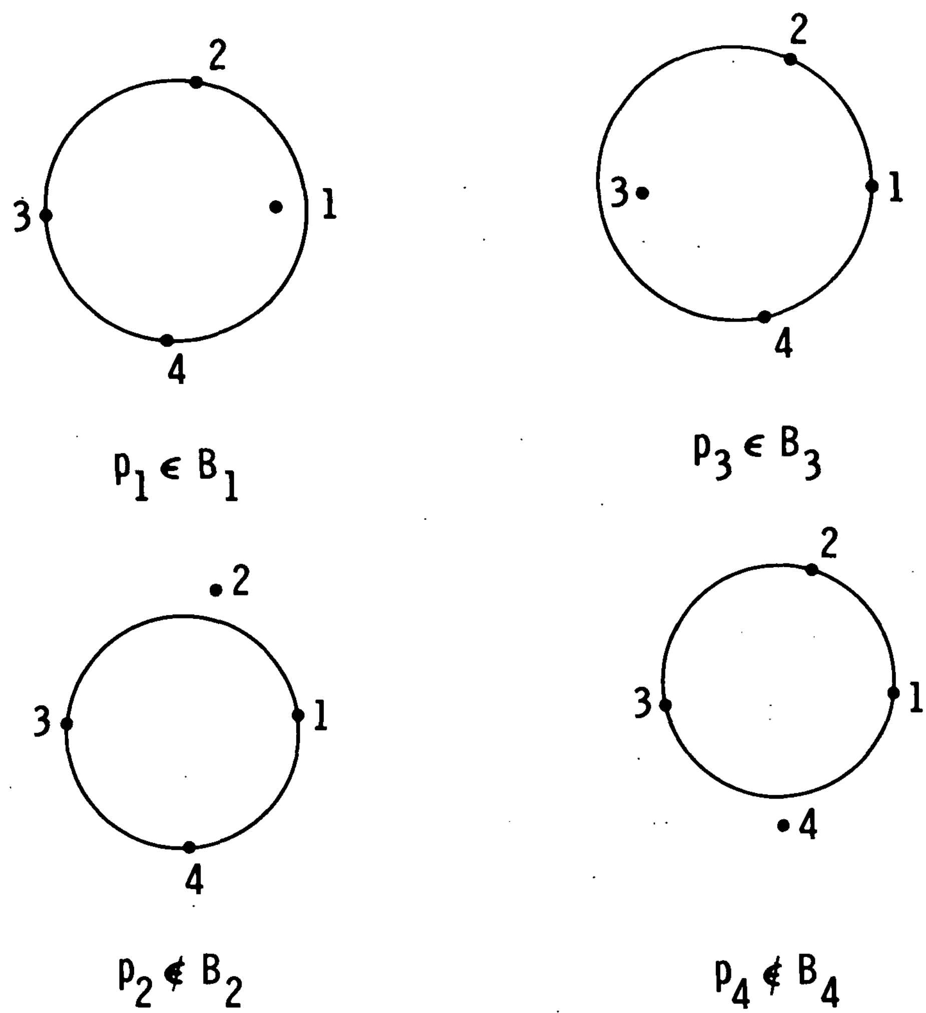


Figure 3. Example of application of the sphere test.

The crucial fact that the result of the sphere test is unique independent of the choice of the test point is established by the following theorem.

Theorem 3 *Let P be a point set satisfying the hypotheses of Theorem 1 with signature sets satisfying $|S_1| \geq 2$ and $|S_2| \geq 2$. If the points of P do not all lie on the same n-sphere then either*

$$p_i \notin B_i \text{ for all } i \in S_1 \text{ and } p_i \in B_i \text{ for all } i \in S_2$$

or

$$p_i \notin B_i \text{ for all } i \in S_2 \text{ and } p_i \in B_i \text{ for all } i \in S_1.$$

For the proof of this theorem it is useful to have the following two lemmas on intersecting spheres.

Lemma 1 *Let E_1 and E_2 be two distinct n-spheres in n-space, $(n \geq 2)$, intersecting in a set E_{12} consisting of more than a single point. Then E_{12} is an $(n-1)$-sphere contained in a uniquely determined hyperplane h_{12}.*

Lemma 2 *Let B_1 and B_2 denote the* open *balls bounded by E_1 and E_2, respectively. On one side of the hyperplane, h_{12}, B_1 will contain B_2, while the reverse inclusion will prevail on the other side of h_{12}. Let H_1 denote the open halfspace on the side of h_{12} in which B_1 contains B_2 and let H_2 denote the other open halfspace. Then*

(21) $\quad H_1 \cap B_2 \subset H_1 \cap B_1$

(22) $\quad (H_1 \cap B_2) \cap (H_1 \cap E_1) = \text{Null}$

(23) $\quad B_2 \cap (H_1 \cap E_1) = \text{Null}$

(24) $\quad H_2 \cap B_1 \subset H_2 \cap B_2$

(25) $\quad (H_2 \cap B_1) \cap (H_2 \cap E_2) = \text{Null}$

(26) $\quad B_1 \cap (H_2 \cap E_2) = \text{Null}$

Proof of Lemmas. The validity of these lemmas for n=2 is clear from consideration of Figure 4.

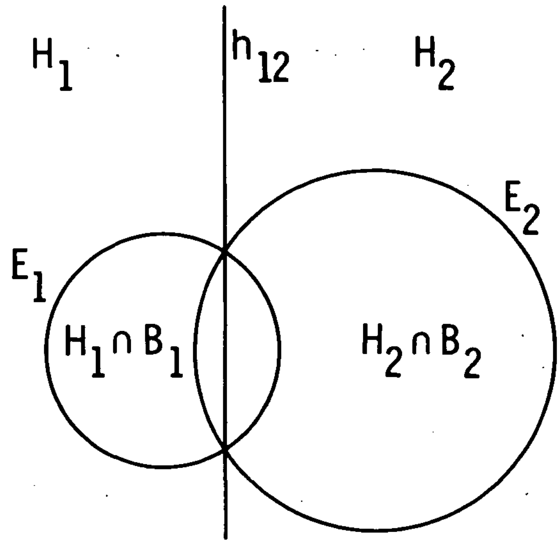


Figure 4. Configuration of Lemmas 1 and 2 in 2-space.

In higher dimensional spaces the configuration of the two intersecting balls is symmetric about the line, ℓ, connecting the centers of the two balls. Thus the intersection of the objects E_1, E_2, B_1, B_2, h_{12}, H_1, and H_2 with any 2-dimensional manifold containing ℓ again gives the configuration of Figure 4. We omit further details of the proof. □

The principal conclusions to be used subsequently from Lemma 2 are Eqs.(23) and (26). For example from Eq.(23) we know that if a point, q, is in $H_1 \cap E_1$, then $q \notin B_2$.

Proof of Theorem 3. Without loss of generality let $j \in S_1$ and suppose $p_j \notin B_j$ so that T_1 is selected. It suffices to show that $p_i \notin B_i$ for every $i \in S_1$ and $p_i \in B_i$ for every $i \in S_2$.

Let $i \in S_1 \cup S_2$. Let t_{ij} denote the $(n-1)$-simplex forming the common facet of the simplices t_i and t_j. Thus t_{ij} is the $(n-1)$-simplex with vertex set $P \backslash \{p_i, p_j\}$.

Let h_{ij} denote the unique hyperplane containing t_{ij}. The intersection of the n-spheres E_i and E_j is the $(n-1)$-sphere E_{ij} that circumscribes t_{ij}. E_{ij} lies in h_{ij}. On one side of h_{ij}, B_i is contained in B_j, while on the other side of h_{ij}, B_j is contained in B_i. From information about one point of E_i, namely p_j, that is not in h_{ij}, the relative containment relations between B_i and B_j can be determined.

Let H_j denote the open halfspace on the same side of h_{ij} as p_j and let H'_j denote the open halfspace on the other side of h_{ij}. See Figure 5.

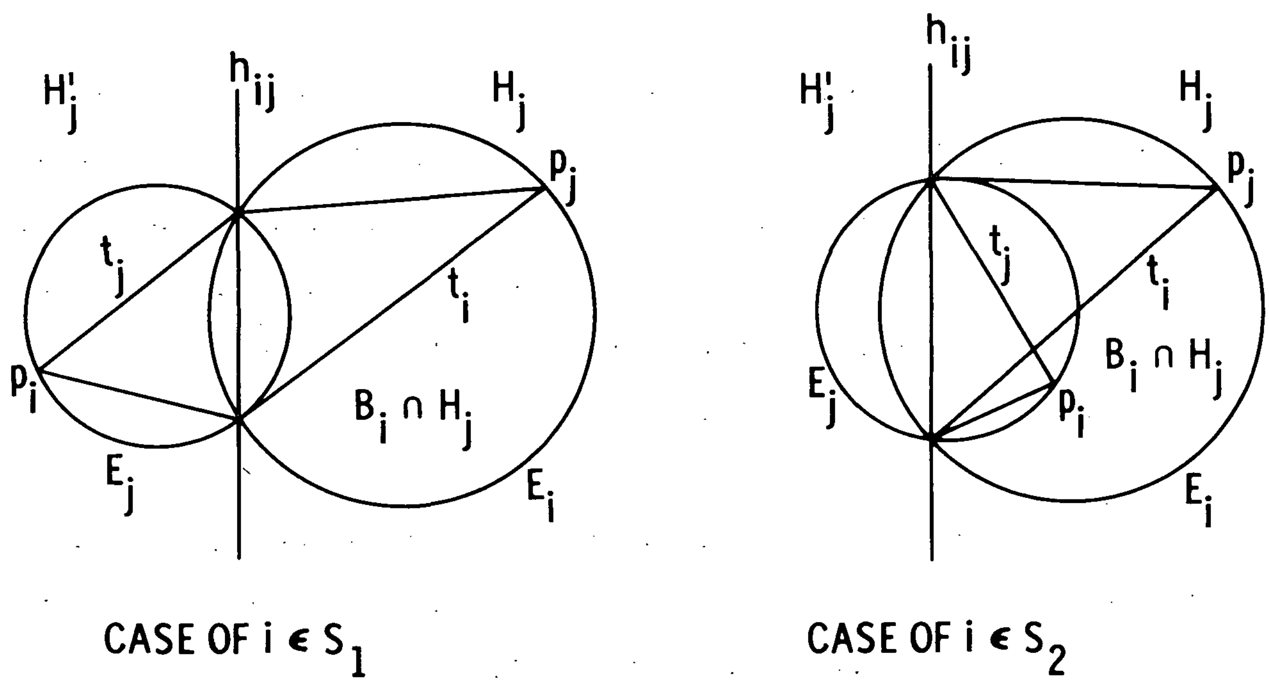


Figure 5. Configuration of Theorem 3 in 2-space.

Since $p_j \notin B_j$ but $p_j \in E_i$, it follows that

(27) $$B_j \cap H_j \subset B_i \cap H_j$$

Then, as in Lemma 1, in the other halfspace, H'_j, we have

(28) $$B_i \cap H'_j \subset B_j \cap H'_j$$

We now consider the two possible cases of $i \in S_1$ or $i \in S_2$.

If $i \in S_1$ then p_i and p_j are on opposite sides of h_{ij} and thus

$p_i \in H'_j$. Also $p_i \in E_j$ and thus $p_i \in E_j \cap H'_j$. From Eq.(28) and Lemma 2 it follows that $p_i \notin B_i$.

If $i \in S_2$ then p_i and p_j are on the same side of h_{ij} and thus $p_i \in H_j$. Also $p_i \in E_j$ and thus $p_i \in E_j \cap H_j$. From Eq.(27) it follows that $p_i \in B_i$. □

8. The local and global sphere tests

Let T be a triangulation of a point set P in n-space. The triangulation T satisfies the **global sphere test** if for each simplex $t_i \in T$ the open n-ball B_i circumscribing t_i contains no points of P.

A pair of n-simplices, t_i and t_j, sharing a common $(n-1)$-dimensional facet t_{ij} will be said to satisfy the **local sphere test** if the vertex of t_i not in t_{ij} is outside the open ball B_j circumscribing t_j. By Theorem 3, this is equivalent to the requirement that the vertex of t_j not in t_{ij} is outside the open ball B_i circumscribing t_i.

Theorem 4 *If a triangulation T of a point set P has the property that every pair of simplices sharing a common facet satisfies the local sphere test, then T satisfies the global sphere test.*

Proof. The proof will be by contradiction. Suppose the hypothesis is satisfied, but there is some point $p^* \in P$ and some simplex $t' \in T$ such that the open ball B' circumscribing t' contains p^*.

Let ℓ be a line segment from p^* to some point q interior to t'. By a small perturbation of the position of the end point q, if necessary, we may assume that wherever ℓ passes from one simplex to another it intersects the relative interior of a common facet.

Relabel the simplices intersected by ℓ so they are denoted by $t_1, t_2, \ldots, t_k$, ordered along the line ℓ from t_1, which was previously called t', to t_k, which has p^* as one of its vertices.

For $i = 2, \ldots, k$, let p_i be the vertex of t_i that is not a vertex of t_{i-1}. For $i = 1, \ldots, k$, let E_i be the sphere circumscribing t_i, and let B_i be the open ball whose boundary is E_i. Note that $p_k = p^*$ and $B_1 = B'$.

By hypothesis, $p_i \notin B_{i-1}$, $i = 2, \ldots, k$, but we are assuming $p_k \in B_1$. Since $p_k \in B_1$ and $p_k \notin B_{k-1}$, there exists a smallest index, j, such that $p_k \in B_j$ and $p_k \notin B_{j+1}$. Let h denote the unique hyperplane containing the facet common to t_j and t_{j+1}. Figure 6 illustrates t_j, t_{j+1}, and related objects for the case of $n = 2$.

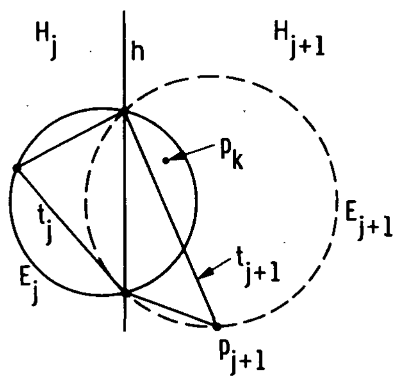


Figure 6. Configuration of Theorem 4 in 2-space.

Let H_{j+1} be the open halfspace on the same side of h as t_{j+1}. Let H_j be the opposite open halfspace, i.e., the halfspace on the same side of h as t_j. Then

$$p_{j+1} \in E_{j+1} \cap H_{j+1}$$

and

$$p_{j+1} \notin B_j$$

thus

$$B_j \cap H_{j+1} \subset B_{j+1} \cap H_{j+1}$$

$$p_k \in B_j \cap H_{j+1}$$

$$p_k \in B_{j+1} \cap H_{j+1}$$

contradicting the assumption that $p_j \notin B_{j+1}$. □

9. <u>The global sphere test and the Dirichlet tessellation</u>

Let P again denote a finite set of distinct points in n-space, not lying entirely in any $(n-1)$-dimensional manifold. With each point, $p_i \in P$, we associate the cell,

$$d_i = \{q : \|q-p_i\| \leq \|q-p_j\|, \text{ for all } p_j \in P\}$$

The cell d_i is called the **Dirichlet cell** associated with p_i relative to the point set P. The set of all d_i's is called the **Dirichlet tessellation** of n-space associated with the point set P. Clearly the cells d_i are disjoint except for common boundaries, and the union of all of the d_i's covers all of n-space.

Each cell, d_i, is the intersection of a finite number of halfspaces. Each such halfspace is bounded by the hyperplane that perpendicularly bisects the line segment connecting the point p_i to another point of P.

Let Q denote the set of points, q_j, that occur as vertices of the d_i's. Each q_j is the unique intersection point of at least n facets of some cell, d_i, and thus is the unique intersection point of at least n of the bisection hyperplanes, with some n of them being linearly independent.

Let $r_j = \|q_j - p_i\|$. Since $q_j \in d_i$, there are no points of P whose distance from q_j is less than r_j. For each bisecting hyperplane on which q_j lies, there is another point, $p \in P$, which is the reflection of p_i relative to this hyperplane and whose distance from q_j is also r_j. Thus there are $n+1$ or more points of P at the distance r_j from q_j, and this set of points does not lie in any $(n-1)$-dimensional manifold.

Conversely it can be verified that any point in n-space that attains its minimum distance from points of P at $n+1$ or more points of P that do not lie in an $(n-1)$-dimensional manifold must be a vertex of one or more of the d_i's, i.e., must belong to the set Q.

With each point $q_j \in Q$, associate the convex hull, s_j, of the points of P that are at the minimal distance, r_j, from q_j. The set, s_j, is called a **Delaunay cell**. The set of all s_j's constitutes the **Delaunay tessellation** of the convex hull of P. In particular it can be verified that the s_j's are mutually disjoint except for common boundaries, and the union of all of the s_j's coincides with the convex hull of P.

The Delaunay tessellation and the Dirichlet tessellation are dual to each other in the sense that each cell, d_i, of the Dirichlet tessellation is associated with a vertex, p_i, of the Delaunay tessellation, and each cell, s_j, of the Delaunay tessellation is associated with a vertex, q_j, of the Dirichlet tessellation.

If the points of P are in "general" position, each cell, s_j, will have just n+1 vertices, and thus will be a simplex. In this case, the Delaunay tessellation may be called a triangulation. In practical applications, e.g., [Bowyer '81], where one uses the Delaunay tessellation as a means toward producing a triangulation, one can replace any cell, s_j, that has more than n+1 vertices by an arbitrary triangulation of that cell, thus producing an overall triangulation of P. It appears common to extend the name Delaunay tessellation to such a triangulation.

We may now observe that a triangulation satisfying the global sphere test of Section 8 is, in fact, a Delaunay tessellation, possibly in the extended sense just mentioned. Let T be a triangulation of P satisfying the global sphere test. With each simplex, $t \in T$, associate the point, q, at the center of the circumsphere of t. If two of more simplices have the same circumcenter, replace these simplices by their union, s. Note that all vertices of such a cell, s, lie on a common sphere.

The circumcenter points, q, associated with the cells of this tessellation satisfy the properties of the set, Q, noted previously. Thus these cells are all Delaunay cells. Since their union covers the convex hull of P, no Delaunay cells are missing, so this is a Delaunay tessellation for P.

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