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VORTEX CONCEPTION OF ROTOR AND MUTUAL EFFECT OF SCREW/PROPELLERS

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16. Abstract

A vortex theory of screw/propellers with variable circulation according to the blade and its azimuth is proposed, the problem is formulated and circulation is expanded in a Fourier series. Equations are given for inductive velocities in space for screws, including those with an infinitely large number of blades and expansion of the inductive velocity by blade azimuth of a second screw. Multiparameter improper integrals are given as a combination of elliptical integrals and elementary functions, and it is shown how to reduce elliptical integrals of the third kind with a complex parameter to integrals with a real parameter.

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List of Symbols/Notations/Abbreviations

x, y, z -- coordinate system

V, V_x, V_y -- flight speed

R, r -- radii of the screw and blade cross section

ψ -- blade azimuth

k -- number of blades

a_0 -- angle of "blade cone"

v_d -- inductive velocity averaged over the disk of the screw

$\Gamma(r,\psi)$, $\Gamma_0(r)$, $\Gamma_{nc}(r)$, $\Gamma_{ns}(r)$ -- circulation of airspeed around blade cross section and its harmonics

v, $v_0(r)$, $v_{mc}(r)$, $v_{ms}(r)$ -- inductive velocity of screw $\Gamma(r,\psi)$ and its azimuth θ harmonics

Ω -- angular velocity of rotation of screw around its axis

w -- inductive velocity of abstract screw of radius ρ with constant circulation $\gamma(\psi)$ over the radius

$w_n, w_{n,0}(r)$, $w_{n,mc}(r)$, $w_{n,ms}(r)$ -- inductive velocity of abstract screw with circulation $\gamma_n = e^{in\psi}$ and its harmonics

x_2, y_2, z_2 -- coordinates of center of hub of second screw

a_{02} -- value of a_0 for second screw

sgnA -- unit with sign of value of A, sgn0=1 in this case

$$F_q = \partial F / \partial q$$

$$K = \partial K / \partial q$$

$$V_y^* = V_y + v_d, \quad \lambda = V_y^* / V_x, \quad y_e = y - a_0\rho - \lambda x$$

$$y_\nabla = y_2 + a_{02} r - a_0 \rho, \quad y_* = y_\nabla - \lambda x_2$$

VORTEX CONCEPTION OF ROTOR AND MUTUAL EFFECT OF SCREW/PROPELLERS

A.M. Lepilkin

In existing studies on the vortex conception of a screw, a screw system with an infinitely large number of blades is used (which permits avoidance of the extremely complicated accounting for the effect of nonperpendicular running off of the vortices from a rotor line and the time variable circulation of airspeed around the blade cross section), and only steady state conditions are considered. /77*

G.I. Maykapar in 1947 [1] and A.P. Proskuryakov in 1956 [2] used a system with constant circulation over time. G.I. Maykapar generalized the method of Zhukovskiy for the case of a slanting vortex cylinder, and he proposed breaking down the vortices into circular (parallel to the blade planes) and rectilinear (along the cylinder generatrices), and he gave formulas for inductive velocities only in the planes of the blades in the form of definite integrals. A.P. Proskuryakov considered only a flat vortex system and broke the vortices down into longitudinal and transverse, and he gave formulas for the axial inductive velocities involving elliptical integrals of the third kind, which changed in complex planes, the parameters of which are determined by the roots of an equation of the fourth degree.

A vortex theory of the screw is proposed below with a variable (according to the blade and by its azimuth) circulation. A three dimensional vortex system is used as the base, the form of which takes account of the conical nature of the relative locations of the blades and the first harmonic of their flywheel motion. The results obtained are suitable for all normal conditions (with the exception of the case of axial movement of the screw against the force of gravity).

In Section 1, formulation of the problem is given, and expansion of the circulation in a Fourier series by blade azimuth is proposed.

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*Numbers in the margin indicate pagination in the foreign text.

In Section 2 and Section 3, equations are given for the inductive velocities in space, in which in Section 3, it is for a screw with an infinitely large number of blades.

In Section 4, expansion of the inductive velocity of a screw by the second blade azimuth is given.

In Section 5, multiparameter, improper integrals are presented in the form of a combination of elliptical integrals and elementary functions.

In Section 6 (Appendix), a method is shown of reduction of elliptical integrals of the third kind with a complex parameter to integrals with a real parameter.

1. Formulation of Problem

In steady state motion of a screw, angle ε between the blade axis and the plane normal to the axis of rotation of the screw changes periodically /78

$$\varepsilon = a_0 + a_{1c} \cos \psi + a_{1s} \sin \psi + \ldots \quad (1)$$

The axes of the blades (if the short distance of the hinges from the screw axis and the higher harmonics of flywheel motion of the blades are disregarded) form a circular cone with angle of taper a_0, the axis of which is deflected from the screw axis by small angles a_{1c} (in the plane of movement of the screw axis) and a_{1s} (to the right or to the left).

We introduce a clockwise coordinate system with origin O in the center of the hub of the screw by drawing the y axis upward along the axis of the "blade cone" and by directing the x axis forward so that flight speed vector V is in the xy plane (Fig. 1). We define blade azimuth ψ as the angle between the projections of its axis on the xz plane and the x axis. In calculation of the forces and moments developed by the screw, it will be easy to change to a similar coordinate system with the Oy_1 axis along the axis of rotation of the screw.

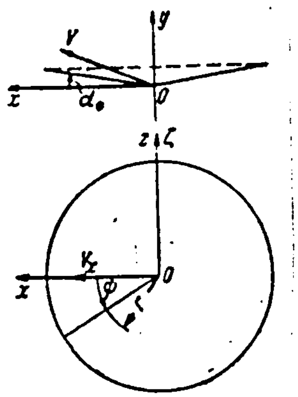

Fig. 1.

Angles a_0, a_{1c} and a_{1s} can be calculated approximately beforehand based on axial inductive velocity v_d averaged over the propeller "disk" (it can be determined from for example the theory of Glauert [3]).

Because of the smallness of angles a_{1c} and a_{1s}, the angular velocity of rotation of the "blade cone" can be considered uniform and equal to velocity Ω of rotation of the screw around its axis. In analysis of the mutual effect of the screws, the axes of the "blade cones" can be considered parallel. The blades of the screw are assumed to be uniformly arranged and identical in all respects.

In determination of the shape of the vortices flowing away, by disregarding the projections of inductive velocity v on the x and y axes, which are extremely small compared to the velocities of points of the blade V and Ωr, each element of a vortex can be considered to move parallel to the y axis from the point where it left the blade. In order for such a (linearized in shape) theory to retain meaning up to hovering ($V_x=0$) inclusive, velocity $V_y^*=V_y+v_d$ of departure of the vortices from the xz plane averaged over the "disk" of the screw (or in the xz plane) should be introduced.

Steady state conditions, circulation Γ of the airspeed around the blade cross section at radius r depends periodically on blade azimuth ψ.

We introduce the concept of an abstract screw of radius ρ with constant circulation $\gamma(\psi)$ along the blade, and we determine corresponding inductive velocity w.

Let t be the interval of time from a given moment to the time a vortex element flows away from the blade. The blade vortex connected with circulation $\gamma(\psi)$ and two "end" vortices with circulation $\gamma_\psi = \gamma(\psi - \Omega t)$, flowing away at radii ρ and 0, form a closed circuit filled (because of the law of conservation of circulation) with "transverse" rectilinear vortices with circulation $d\gamma_\phi = (d\gamma_\phi/d\phi)d\phi$, the ends of each of which rest on points of the end vortices which correspond to equal values of t. The coordinates of the vortex element in the x,y,z system will be (Fig. 2)

$$\xi = \rho' \cos(\psi - \varphi) - \varphi \frac{V_x}{\Omega}, \quad \eta = a_0 \rho' - \varphi \frac{V_{y*}}{\Omega}$$
$$\zeta = -\rho' \sin(\psi - \varphi) \quad (\varphi = \Omega t) \tag{1.1}$$

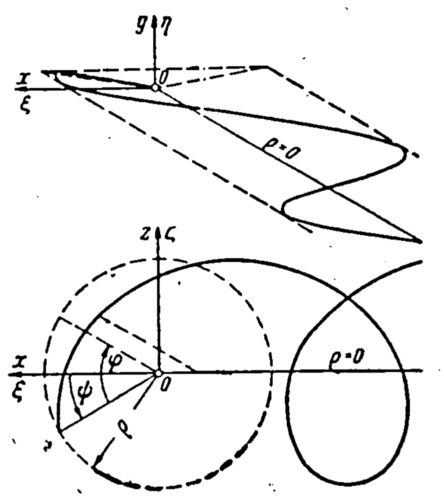

Fig. 2.

Only ϕ changes along the outside ($\rho' = \rho$) and inside ($\rho' = 0$) ends of the vortices ($0 \leqslant \phi < \infty$) and only ρ' changes along the transverse vortices ($0 \leqslant \rho' \leqslant \rho$). The equation of the connected vortex will be $\phi = 0$.

We call the $\rho' = \text{const}$ line (which is transformed at $V_x = 0$ into a screw with pitch V_{y*}/Ω and is projected on the xz plane in the form of a trochoid at $V_x \neq 0$) a helicotrochoid. The $\rho' = 0$ line will be rectilinear to the axis of the vortex system.

The inductive velocities which correspond to a given vortex line with circulation Γ are determined by the integrals along this line

$$v_x = \frac{1}{4\pi} \int \Gamma \left(\frac{\partial}{\partial y} \frac{1}{l} d\zeta - \frac{\partial}{\partial z} \frac{1}{l} d\eta \right)$$
$$v_y = \frac{1}{4\pi} \int \Gamma \left(\frac{\partial}{\partial z} \frac{1}{l} d\xi - \frac{\partial}{\partial x} \frac{1}{l} d\zeta \right) \tag{1.2}$$
$$v_z = \frac{1}{4\pi} \int \Gamma \left(\frac{\partial}{\partial x} \frac{1}{l} d\eta - \frac{\partial}{\partial y} \frac{1}{l} d\xi \right)$$
$$l = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

where ξ, η and ζ are the coordinates of element ds of the vortex line. These integrals can also be applied formally to vortices with circulation of variable length with such integrals considered to be branching "transverse" vortices. For construction of the velocity field in the xyz coordinate system semifixed to the propeller, Eq. (1.2) can be utilized, since their result depends only on the differences of the coordinates of the points of this system and the points where the vortex element is located at a given time.

At a fixed point of the xyz system, the inductive velocity will be a periodic function of blade azimuth ψ with period $\tau=2\pi/k$, where k is the number of blades.

A change to a multiblade screw can be accomplished by summation of the inductive velocities of a single blade screw shifted by ψ in τ, 2τ, . . . Therefore, only the cases k=1 and ∞ are considered further.

For a change to a screw of radius R with variable circulation $\Gamma(r,\psi)$ over radius r, the following must be used in equations for the inductive velocities of an abstract screw with circulations $\gamma(\psi)$,

$$\gamma(\psi) = -\left\{\frac{\partial\Gamma}{\partial r}\right\}_{r=\rho} d\rho \qquad (1.3)$$

and the integral must be taken along the blade axis ($0\leq\rho\leq R$).

In order to simplify practical application of the theory, the following expansion should be used

$$\Gamma(r,\psi) = \Gamma_0(r) + \sum_{n=1}^{\infty} [\Gamma_{nc}(r)\cos n\psi + \Gamma_{ns}(r)\sin n\psi] \qquad (1.4)$$

A screw with unit complex circulation

$$\gamma_n = e^{in\psi} = \cos n\psi + i \sin n\psi \qquad (n=0, 1, 2, \dots) \qquad (1.5)$$

will correspond to complex inductive velocity

$$w_n = w_{nc} + i w_{ns} \qquad (n=0,1,2,\ .\ .\ .). \tag{1.6}$$

The inductive velocities of circulation harmonics $\Gamma_0(r)$, $\Gamma_{nc}(r)$ and $\Gamma_{ns}(r)$ will be /80

$$v_{nc}(x,y,z) = -\int_0^R w_{nc}(\rho)\frac{d\Gamma_{nc}}{d\rho}d\rho, \quad v_{ns}(x,y,z) = -\int_0^R w_{ns}(\rho)\frac{d\Gamma_{ns}}{d\rho}d\rho \tag{1.7}$$

2. Inductive Velocities of $k\neq\infty$ Screw

By applying Eq. (1.2) and (1.1), passage around the edge of the vortex surface must be carried out so that the connected vortex passes from $\rho'=0$ to $\rho'=\rho$ and that the "transverse" vortices have to pass in the same direction. The following equations are obtained in this way:

$$4\pi w_x = \frac{V_y^*}{\Omega}\int_0^\infty \gamma_\varphi \frac{\partial}{\partial z}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\varphi + \rho\int_0^\infty \gamma_\varphi \cos(\psi-\varphi)\frac{\partial}{\partial y}\frac{1}{l}d\varphi - \int_0^\rho\int_0^\infty \frac{\partial\gamma_\varphi}{\partial\varphi}\times \tag{2.1}$$
$$\times\left[\sin(\psi-\varphi)\frac{\partial}{\partial y}\frac{1}{l} + a_0\frac{\partial}{\partial z}\frac{1}{l}\right]d\varphi\, d\rho' - \gamma(\psi)\int_0^\rho\left(\sin\psi\frac{\partial}{\partial y}\frac{1}{l_0'} + a_0\frac{\partial}{\partial z}\frac{1}{l_0'}\right)d\rho'$$

$$4\pi w_y = -\rho\int_0^\infty \gamma_\varphi\left[\cos(\psi-\varphi)\frac{\partial}{\partial x}\frac{1}{l} - \sin(\psi-\varphi)\frac{\partial}{\partial z}\frac{1}{l}\right]d\varphi + \tag{2.2}$$
$$+\int_0^\rho\int_0^\infty \frac{\partial\gamma_\varphi}{\partial\varphi}\left[\sin(\psi-\varphi)\frac{\partial}{\partial x}\frac{1}{l'} + \cos(\psi-\varphi)\frac{\partial}{\partial z}\frac{1}{l'}\right]d\varphi\, d\rho' -$$
$$-\frac{V_x}{\Omega}\int_0^\infty \gamma_\varphi\frac{\partial}{\partial z}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\varphi + \gamma(\psi)\int_0^\rho\left(\sin\psi\frac{\partial}{\partial x}\frac{1}{l_0'} + \cos\psi\frac{\partial}{\partial z}\frac{1}{l_0'}\right)d\rho'$$

$$4\pi w_z = -\rho\int_0^\infty \gamma_\varphi \sin(\psi-\varphi)\frac{\partial}{\partial y}\frac{1}{l}d\varphi + \frac{V_x}{\Omega}\int_0^\infty \gamma_\varphi\frac{\partial}{\partial y}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\varphi - \tag{2.3}$$
$$-\int_0^\rho\int_0^\infty \frac{\partial\gamma_\varphi}{\partial\varphi}\left[\cos(\psi-\varphi)\frac{\partial}{\partial y}\frac{1}{l'} - a_0\frac{\partial}{\partial x}\frac{1}{l'}\right]d\varphi d\rho' -$$
$$-\frac{V_y^*}{\Omega}\int_0^\infty \gamma_\varphi\frac{\partial}{\partial x}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\varphi - \gamma(\psi)\int_0^\rho\left(\cos\psi\frac{\partial}{\partial y}\frac{1}{l_0'} - a_0\frac{\partial}{\partial x}\frac{1}{l_0'}\right)d\rho'$$

$$l^2 = [x-\rho\cos(\psi-\varphi)+\varphi V_x/\Omega]^2 + [z+\rho\sin(\psi-\varphi)]^2 + \tag{2.4}$$
$$+(y-a_0\rho+\varphi V_y^*/\Omega)^2$$

Here, ℓ' and ℓ_a are the values of ℓ at $\rho=\rho'$ and $\rho=0$ respectively, and ℓ_0' is the value of ℓ' at $\phi=0$. Differentiation with respect to x, y and z is performed before integration with respect to ϕ (the condition of convergence of improper integrals). It is easy to see that

$$4\pi w_y = \rho\int_0^\infty \gamma_\varphi\left(\frac{\partial}{\partial\rho}\frac{1}{l} + a_0\frac{\partial}{\partial y}\frac{1}{l}\right)d\varphi - \frac{V_x}{\Omega}\int_0^\infty \gamma_\varphi\frac{\partial}{\partial z}\left(\frac{1}{l} - \frac{1}{l_a}\right)d\varphi + \\ + \int_0^\rho\int_0^\infty \frac{\partial\gamma_\varphi}{\partial\varphi}\frac{\partial}{\partial\psi}\frac{1}{l'}\,d\varphi\frac{d\rho'}{\rho'} + \gamma(\psi)\int_0^\rho \frac{\partial}{\partial\psi}\frac{1}{l_0'}\frac{d\rho'}{\rho'} \tag{2.5}$$

By substituting $\gamma=e^{in\psi}$ and $\gamma_\phi=e^{in(\psi-\phi)}$, complex functions of the type of equation (1.6) can be obtained for each projection of the inductive velocity.

3. k=∞ Rotor

As $k\to\infty$ and with preservation of quantity $k\Gamma$, period $\tau=2\pi/k$ of inductive velocity $v(x,y,z,\psi)$ with respect to ψ tends toward zero. In the limit $k=\infty$, function v does not depend on blade azimuth ψ and evidently equals the average value of the inductive velocity of a single blade screw (with the same circulation) in one rotation of it /81

$$v^\circ(x, y, z) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x, y, z; \psi)\,d\psi\right\}_{k=1} \tag{3.1}$$

The use of function kv° for a $k\neq\infty$ screw designates approximate solution of the problem, the inaccuracy of which decreases with increase in number of blades. A k=∞ system, as numerous experiments have shown, gives satisfactory results with $k\geqslant 3$ and small values of $V/\Omega R$, when the coils of the vortex lines are located quite close to each other. Rotor parameters $V_x/\Omega R$ and $V_y^*/\Omega R$ also are small (usually less than 0.3).

By applying a k=∞ system in determination of circulation $\Gamma(r,\psi)$, a blade can replace a vortex line since, with k=∞, function w can be integrated everywhere with respect to ρ.

With $k=\infty$, the vortex system of an abstract screw consists of a vortex cone (connected vortices of the blades) and a vortex shell, each cross section of which is a circle in the y=const plane. The space inside the shell is filled with "transverse" vortices. In each infinitely thin parallel xz plane, the circulation distribution layer of the shell around the circumference is identical, if the calculation is carried out from the same helicotrochoid ψ=const.

In determination of an Eq. (3.1) type function for each projection of inductive velocity w, a detailed accounting will only be given for w_y. Transformation of the double integrals should be performed ahead of time for $\gamma(\psi)\neq$const. After integration over ϕ by parts, there will be

$$4\pi w_y{}^\circ = \rho\int\limits_0^\infty \gamma_\varphi\left(\frac{\partial}{\partial\rho}\frac{1}{l} + a_0\frac{\partial}{\partial y}\frac{1}{l}\right)d\varphi - \frac{V_x}{\Omega}\int\limits_0^\infty \gamma_\varphi\frac{\partial}{\partial z}\left(\frac{1}{l} - \frac{1}{l_a}\right)d\varphi - \\ - \int\limits_0^\rho\int\limits_0^\infty \gamma_\varphi\frac{\partial^2}{\partial\psi\,\partial\varphi}\frac{1}{l'}\,d\varphi\frac{d\rho'}{\rho'} \qquad (3.2)$$

In determination of the average value of Eq. (3.1) for a period, variable $\alpha=\psi-\phi$ can be introduced for each fixed value of ϕ. Then, $\alpha=\psi$ can be assumed. The hydrodynamic meaning of such a transformation is that summation of the action of the vortex elements in each parallel xz plane of the thin layer is carried out from the same helicotrochoidal surface ψ=const.

In this way, we obtain

$$4\pi w_y{}^\circ = \rho\int\limits_0^\infty \frac{\partial}{\partial\rho}\int\limits_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}d\psi\,d\varphi + \alpha_0\rho\int\limits_0^\infty\frac{\partial}{\partial y}\int\limits_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}d\psi\,d\varphi - \\ - \frac{V_x}{\Omega}\int\limits_0^\infty\frac{\partial}{\partial z}\int\limits_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi}\left(\frac{1}{l} - \frac{1}{l_a}\right)d\psi\,d\varphi - \int\limits_0^\rho\int\limits_0^\infty\int\limits_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi}\frac{\partial^2}{\partial\psi\,\partial\varphi}\frac{1}{l'}\,d\psi\,d\varphi\frac{d\rho'}{\rho'} \qquad (3.3)$$

$$l^2 = \left(x - \rho\cos\psi + \varphi\frac{V_x}{\Omega}\right)^2 + (z + \rho\sin\psi)^2 + \left(y - a_0\rho + \varphi\frac{V_y{}^*}{\Omega}\right)^2 \qquad (3.4)$$

Use of the equalities

$$\frac{\partial}{\partial x}\frac{1}{l}=\frac{\Omega}{V_x}\frac{\partial}{\partial\varphi}\frac{1}{l}-\frac{V_y^*}{V_x}\frac{\partial}{\partial y}\frac{1}{l},\qquad \int_0^\infty\frac{\partial}{\partial\varphi}\frac{1}{l}d\varphi=-\frac{1}{l_0}$$

(the first is necessary only for the w_z transformation) gives /82

$$4\pi w_x^\circ=\frac{V_y^*}{\Omega}\int_0^\infty\frac{\partial}{\partial z}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\psi\,d\varphi+\rho\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}\cos\psi\,d\psi\,d\varphi-$$
$$-\int_0^\rho\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l'}\cos\psi\,d\psi\,d\varphi\,d\rho'-\int_0^\rho\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}\sin\psi\,d\psi\,d\rho'-a_0\int_0^\rho\frac{\partial}{\partial z}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}d\psi\,d\rho' \quad (3.5)$$

$$4\pi w_y^\circ=\rho\int_0^\infty\frac{\partial}{\partial\rho}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}d\psi\,d\varphi+a_0\rho\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}d\psi\,d\varphi- \quad (3.6)$$
$$-\frac{V_x}{\Omega}\int_0^\infty\frac{\partial}{\partial z}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\psi\,d\varphi+\int_0^\rho\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi\rho'}\frac{\partial}{\partial\psi}\frac{1}{l_0'}d\psi\,d\rho'$$

$$4\pi w_z^\circ=\frac{V_x}{\Omega}\left(1+\frac{V_y^{*2}}{V_x^2}\right)\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi}\left(\frac{1}{l}-\frac{1}{l_a}\right)d\psi\,d\varphi+ \quad (3.7)$$
$$+\frac{V_y^*}{V_x}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi}\left(\frac{1}{l_0}-\frac{1}{l_*}\right)d\psi-\rho\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}\sin\psi\,d\psi\,d\varphi+$$
$$+\int_0^\rho\int_0^\infty\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l}\sin\psi\,d\psi\,d\varphi\,d\rho'-\int_0^\rho\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}\cos\psi\,d\psi\,d\rho'+a_0\int_0^\rho\frac{\partial}{\partial x}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}d\psi\,d\rho'$$
$$(l_*^2=x^2+y^2+z^2)$$

There now should be introduced variable $\xi=x+\phi V_x/\Omega$ and notations

$$\lambda=V_y^*/V_x,\qquad y_e=y-a_0\rho-\lambda x \quad (3.8)$$
$$l^2=\omega^2-2\omega\,(\xi\cos\psi-z\sin\psi)+\xi^2+z^2+(y_e+\lambda\xi)^2 \quad (3.9)$$

Then,

$$\frac{\partial}{\partial\rho}\frac{1}{l}+a_0\frac{\partial}{\partial y}\frac{1}{l}=\left(\frac{\partial}{\partial\omega}\frac{1}{l}\right)_{\omega=\rho} \quad (3.10)$$

$$4\pi w_x^\circ=-\lambda\,[K_z(\rho)-K_z(0)]+\frac{\Omega\rho}{V_x}K_y^{(1)}(\rho)-\frac{\Omega}{V_x}\int_0^\rho K_y^{(1)}(\rho')\,d\rho'-$$
$$-\int_0^\rho\frac{\partial}{\partial y}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}\sin\psi\,d\psi\,d\rho'-a_0\int_0^\rho\frac{\partial}{\partial z}\int_{-\pi}^{\pi}\frac{\gamma(\psi)}{2\pi l_0'}d\psi\,d\rho' \quad (3.11)$$

$$4\pi w_y^\circ = \frac{\Omega\rho}{V_x} K_\omega(\rho) - K_z(\rho) + K_z(0) + \int_0^\rho \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi\rho'} \frac{\partial}{\partial\psi} \frac{1}{l_0'}\, d\psi\, d\rho' \tag{3.12}$$

$$4\pi w_z^\circ = (1+\lambda^2)\,[K_y(\rho) - K_y(0)] + \lambda \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi}\left(\frac{1}{l} - \frac{1}{l_x}\right) d\psi - \tag{3.13}$$

$$-\frac{\Omega\rho}{V_x} K_y^{(2)}(\rho) + \frac{\Omega}{V_x}\int_0^\rho K_y^{(2)}(\rho')\,d\rho' - \int_0^\rho \frac{\partial}{\partial y} \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'} \cos\psi\, d\psi\, d\rho' + a_0 \int_0^\rho \frac{\partial}{\partial x} \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'}\, d\psi\, d\rho'$$

$$l_*^2 = x^2 + y^2 + z^2$$

Here, /83

$$K_q(\rho) = \left\{ \int_x^\infty \frac{\partial}{\partial q} \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l}\, d\psi\, d\xi \right\}_{\omega=\rho} \qquad (q = y, z, \omega) \tag{3.14}$$

$$K_y^{(1)} = \int_x^\infty \frac{\partial}{\partial y} \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l} \cos\psi\, d\psi\, d\xi, \qquad K_y^{(2)} = \int_x^\infty \frac{\partial}{\partial y} \int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l} \sin\psi\, d\psi\, d\xi \tag{3.15}$$

$$l_0^2 = \rho^2 - 2\rho\,(x\cos\psi - z\sin\psi) + x^2 + z^2 + (y - a_0\rho)^2 \tag{3.16}$$

Equations (3.14) can be presented as

$$K_y(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi} \int_x^\infty \frac{y_e + \lambda\xi}{l^3}\, d\xi\, d\psi, \quad K_z(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi} \int_x^\infty \frac{z + \rho\sin\psi}{l^3}\, d\xi\, d\psi \tag{3.17}$$

$$K_\omega(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi} \int_x^\infty \frac{\rho - \xi\cos\psi + z\sin\psi}{l^3}\, d\xi\, d\psi$$

Functions $K_y^{(1)}(\rho)$ and $K_y^{(2)}(\rho)$ are obtained from $K_y(\rho)$ by replacement of $\gamma(\psi)$ by $\gamma(\psi)\cos\psi$ and $\gamma(\psi)\sin\psi$. Since

$$l^2 = A + 2B\xi + C\xi^2$$

$$A = (z + \rho\sin\psi)^2 + \rho^2\cos^2\psi + y_e^2, \quad B = \lambda y_e - \rho\cos\psi, \quad C = 1 + \lambda^2 \tag{3.18}$$

then

$$\int_x^\infty \frac{d\xi}{l^3} = \frac{\sqrt{C}}{H} - \frac{B + xC}{HD}, \quad \int_x^\infty \frac{\xi\, d\xi}{l^3} = \frac{A + xB}{HD} - \frac{B}{H\sqrt{C}}, \quad H = AC - B^2 \tag{3.19}$$

$$H = (z + \rho\sin\psi)^2 (1 + \lambda^2) + (y_e + \lambda\rho\cos\psi)^2$$

$$D^2 = \rho^2 - 2\rho\,(x\cos\psi - z\sin\psi) + x^2 + z^2 + (y - a_0\rho)^2$$

Consequently,

$$K_y(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi H} \left\{ \left(\sqrt{C} - \frac{B + xC}{D}\right)(y - a_0\rho) + \left(\frac{A + xB}{D} - \frac{B}{\sqrt{C}}\right)\lambda \right\} d\psi \tag{3.20}$$

$$K_z(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi H}\left(\sqrt{C} - \frac{B+xC}{D}\right)(z+\rho\sin\psi)\,d\psi$$

$$K_\omega(\rho) = -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi H}\left\{\left(\sqrt{C} - \frac{B+xC}{D}\right)(\rho + z\sin\psi) - \left(\frac{A+xB}{D} - \frac{B}{\sqrt{C}}\right)\cos\psi\right\} d\psi \quad (3.20)$$

The remaining integrals with respect to ψ will be

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi}\frac{\partial}{\partial\psi}\frac{1}{l_0'}\,d\psi &= -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi}\frac{x\sin\psi + z\cos\psi}{l_0'^3}\,d\psi \\
\frac{\partial}{\partial x}\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'}\,d\psi &= -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi}\frac{x-\rho'\cos\psi}{l_0'^3}\,d\psi \\
\frac{\partial}{\partial z}\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'}\,d\psi &= -\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi}\frac{z+\rho'\sin\psi}{l_0'^3}\,d\psi \\
\frac{\partial}{\partial y}\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'}\begin{matrix}\cos\\ \sin\end{matrix}(\psi)\,d\psi &= -(y-a_0\rho')\int_{-\pi}^{\pi} \frac{\gamma(\psi)}{2\pi l_0'^3}\begin{matrix}\cos\\ \sin\end{matrix}(\psi)\,d\psi
\end{aligned} \quad (3.21)$$

All the integrals with fixed function $\gamma(\psi)$ can be found numerically. Calculation of integrals (3.20) is complicated however by the extreme peaks of the subintegral functions at small values of H. The case H=0 and D=0 is possible with $y - a_0\rho = \pm\lambda\sqrt{\rho^2 - z^2}$. /84

The calculations are substantially simplified if the circulation is expanded in a Fourier series with respect to ψ. For $\gamma_n = e^{in\psi}$ (n=0, 1, 2, . . .), integrals (3.14) are transformed into special functions

$$K_{nq}(\rho) = \left\{\int_x^{\infty} \frac{\partial}{\partial q}\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l}\,d\psi\,d\xi\right\}_{\omega=\rho} \qquad (q = y,\ z,\ \omega) \quad (3.22)$$

$$K_{nv}^{(1)}(\rho) = \frac{K_{n+1,\,v}(\rho) + K_{n-1,\,v}(\rho)}{2}, \qquad K_{nv}^{(2)}(\rho) = \frac{K_{n+1,\,v}(\rho) - K_{n-1,\,v}(\rho)}{2i} \quad (3.23)$$

The substitution $\psi = \alpha$ sgn z should be introduced here. There will then be

$$K_{nq}(\rho) = \left\{\int_x^{\infty} \frac{\partial}{\partial q}\int_{-\pi}^{\pi} \frac{\exp(in\alpha\,\mathrm{sgn}\,z)}{2\pi l}\,d\alpha\,d\xi\right\}_{\omega=\rho} \qquad (q = y,\ |z|,\ \omega) \quad (3.24)$$

$$l^2 = \omega^2 - 2\omega(\xi\cos\alpha - |z|\sin\alpha) + \xi^2 + z^2 + (y_e + \lambda\xi)^2$$

Functions of the type

$$F_\mu(x, g, h; \omega, \lambda) = \int_x^\infty \int_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi l}\, d\alpha\, d\xi \qquad (\omega \geqslant 0,\ g \geqslant 0, h \geqslant 0),\ (\mu = 1, 2, \ldots) \tag{3.25}$$

where

$$l^2 = \omega^2 - 2\omega(\xi\cos\alpha - g\sin\alpha) + \xi^2 + g^2 + (h + \lambda\xi)^2 \tag{3.26}$$

and its derivatives over g, h and ω (μ=0, 1, 2, . . .) are discussed below (in Section 5). These functions are presented in the form of a combination of elliptical integrals and elementary functions, in which

$$F_\mu = F_{\mu c} + iF_{\mu s}, \qquad F_{\mu q} = F_{\mu cq} + iF_{\mu sq} \tag{3.27}$$

It is now evident that, with $\lambda_e=|\lambda|\ \mathrm{sgn}\ (\lambda y_e)$,

$$K_{nq}(\omega) = F_{ncq}(x, |z|, |y_e|; \omega, \lambda_e) + i\,\mathrm{sgn}\,z\, F_{nsq}(x, |z|, |y_e|; \omega, \lambda_e) \tag{3.28}$$

Consequently, in the equations of Section 5, it is sufficient to assume $g=|z|$, $h=|y_e|$, where $|h+\lambda x|=|y-a_0\rho|$. It is then necessary to assume $\omega=\rho$.

For $\gamma=e^{in\psi}$, the remaining integrals are reduced to elliptical integrals. Actually,

$$\begin{aligned}
&\frac{\partial}{\partial y}\int_{-\pi}^{\pi} \begin{matrix}\cos\\ \sin\end{matrix}(\psi)\, \frac{\gamma_n(\psi)}{2\pi l_0'}\, d\psi = -(y - a_0\rho')\int_{-\pi}^{\pi} \frac{e^{i(n+1)\psi} \pm e^{i(n-1)\psi}}{\sqrt{\pm 1}\, 4\pi l_0'^3}\, d\psi \\
&\frac{\partial}{\partial z}\int_{-\pi}^{\pi} \frac{\gamma_n(\psi)}{2\pi l_0'}\, d\psi = -z\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l_0'^3}\, d\psi + i\rho'\int_{-\pi}^{\pi} \frac{e^{i(n+1)\psi} - e^{i(n-1)\psi}}{4\pi l_0'^3}\, d\psi \\
&\frac{\partial}{\partial x}\int_{-\pi}^{\pi} \frac{\gamma_n(\psi)}{2\pi l_0'}\, d\psi = -x\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l_0'^3}\, d\psi + \rho'\int_{-\pi}^{\pi} \frac{e^{i(n+1)\psi} + e^{i(n-1)\psi}}{4\pi l_0'^3}\, d\psi \\
&\int_{-\pi}^{\pi} \frac{\gamma_n(\psi)}{2\pi}\, \frac{\partial}{\partial\psi}\, \frac{1}{l_0'}\, d\psi = -\int_{-\pi}^{\pi} \frac{d\gamma_n}{d\psi}\, \frac{d\psi}{2\pi l_0'} = -in\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l_0'}\, d\psi
\end{aligned} \tag{3.29}$$

With the structure of expression (3.16) taken into account, it should be assumed that /85

$$\cos\theta = x / r, \qquad \sin\theta = -z / r, \qquad r = \sqrt{x^2 + z^2} \tag{3.30}$$

After this, use of the substitution $\psi-\theta=\pi-2\phi$ gives

$$\int_0^{\pi}\frac{\cos n\psi}{\pi l_0'^{m}}\,d\psi=\frac{2}{\pi}\frac{(-1)^n E_n^{(-m/2)}(k)}{[(\rho+r)^2+(y-a_0\rho)^2]^{m/2}},\quad k^2=\frac{4\rho r}{(\rho+r)^2+(y-a_0\rho)^2} \tag{3.31}$$

$$E_n^{(-m/2)}(k)=\int_0^{\pi/2}\frac{\cos 2n\varphi\,d\varphi}{(1-k^2\sin^2\varphi)^{m/2}}\qquad \begin{pmatrix} m=1\\ m=3\end{pmatrix} \tag{3.32}$$

For an abstract screw, integration must also be performed with respect to ρ' from 0 to ρ. The way of calculation of the corresponding double integrals can be indicated. For a real screw with variable circulation however, $\Gamma(0,\psi)=\Gamma(R,\psi)=0$. Therefore, integration over ρ' proves to be superfluous. Actually, for a $g(\rho)$ type function with $\Gamma(0)=\Gamma(R)=0$,

$$\int_0^R\left(\int_0^{\rho} g(\rho')\,d\rho'\right)\frac{\partial\Gamma}{\partial\rho}\,d\rho=-\int_0^R g(\rho)\,\Gamma(\rho)\,d\rho$$

Function

$$E_n^{(\nu)}(k,\alpha)=\int_0^{\alpha}(1-k^2\sin^2\varphi)^{\nu}\cos 2n\varphi\,d\varphi \tag{3.33}$$

can be called a generalized elliptical integral, since

$$E_0^{(-1/2)}(k,\alpha)=F(k,\alpha),\qquad E_0^{(1/2)}(k,\alpha)=E(k,\alpha) \tag{3.34}$$

It is easy to verify by differentiation with respect to α that

$$k^2[(n-\nu-1)E_{n-1}^{(\nu)}(k,\alpha)+(n+\nu+1)E_{n+1}^{(\nu)}(k,\alpha)]+ \\ +2n(2-k^2)E_n^{(\nu)}(k,\alpha)=2\sin 2n\alpha\,(1-k^2\sin^2\alpha)^{\nu+1} \tag{3.35}$$

The base here will be functions $E_0^{(\nu)}(k,\alpha)$ and

$$E_1^{(\nu)}(k,\alpha)=-\frac{2-k^2}{k^2}E_0^{(\nu)}(k,\alpha)+\frac{2}{k^2}E_0^{(\nu+1)}(k,\alpha) \tag{3.36}$$

It also is easy to determine that

$$E_0^{(\nu+1)}(k,\alpha)-\frac{2\nu+1}{2(\nu+1)}(2-k^2)E_0^{(\nu)}(k,\alpha)+ \\ +\frac{\nu}{\nu+1}(1-k^2)E_0^{(\nu-1)}(k,\alpha)=\frac{k^2}{2(\nu+1)}(1-k^2\sin^2\alpha)^{\nu}\sin\alpha\cos\alpha \tag{3.37}$$

where elliptical integrals of the first and second kinds (3.34) will be the base.

With small k^2, when recurrent Eq. (3.35) is inconvenient because of the effect of small differences (increasing with increase of n), it is better to use the expansion

$$E_n^{(\nu)}(k,\alpha) = \Phi_{n0} - \binom{\nu}{1} k^2 \Phi_{n,1} + \binom{\nu}{2} k^4 \Phi_{n,2} - \dots \tag{3.38}$$

$$\Phi_{n,s} = \int_0^\alpha (\sin^2\varphi)^s \cos 2n\varphi \, d\varphi \tag{3.39}$$

By using the corresponding indeterminate integral [4, p. 155], we have

$$\int_0^\alpha (\sin^2\varphi)^s \cos 2n\varphi \, d\varphi = \frac{(\sin^2\alpha)^s \sin 2n\alpha}{2(n+s)} - \frac{n}{n+s}\int_0^\alpha (\sin^2\varphi)^s \frac{\sin(2n-1)\varphi}{\sin\varphi} d\varphi$$

But

$$\frac{\sin k\varphi}{\sin\varphi} = \begin{cases} 1 + 2[\cos 2\varphi + \cos 4\varphi + \dots + \cos(k-1)\varphi] & (k = 3, 5, \dots) \\ 2[\cos\varphi + \cos 3\varphi + \dots + \cos(k-1)\varphi] & (k = 2, 4 \dots) \end{cases}$$

On this basis, the recurrent formula /86

$$\Phi_{n,s} = \frac{(\sin^2\alpha)^s \sin 2n\alpha}{2(n+s)} - \frac{n}{n+s} f_s(\alpha) - \frac{2n}{n+s}(\Phi_{1,s} + \dots + \Phi_{n-1,s}) \tag{3.40}$$

$$f_s(\alpha) = \int_0^\alpha (\sin^2\varphi)^s d\varphi$$

is obtained, where $f_s(\alpha)$ is a known function which can be determined by the recurrent formula

$$2sf_s = (2s-1) f_{s-1} - (\sin\alpha)^{2s-1}\cos\alpha, \quad f_0 = \alpha, \quad 2f_1 = \alpha - \sin\alpha\cos\alpha$$

The substitution of $\sqrt{1-k^2}$ tan ϕ tan θ=1 gives

$$d\varphi = -\frac{\sqrt{1-k^2}\, d\vartheta}{1-k^2\sin^2\vartheta}, \quad 1 - k^2\sin^2\varphi = \frac{1-k^2}{1-k^2\sin^2\vartheta}$$
$$\int_\alpha^{\pi/2} (1-k^2\sin^2\varphi)^\nu d\varphi = (1-k^2)^{\nu+1/2}\int_0^\beta \frac{d\vartheta}{(1-k^2\sin^2\vartheta)^{\nu+1}} \tag{3.41}$$

where α and β are connected by the relationship tan α tan $\beta\sqrt{1-k^2}$=1.

For $\alpha=\pi/2$, integrals (3.33) will be "full,"

$$E_n^{(\nu)}(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^\nu \cos 2n\varphi \, d\varphi \tag{3.42}$$

According to the identity

$$(\sin^2 \varphi)^s = \left[2 \sum_{q=0}^{s-1} (-1)^{s-q} \cos 2(s-q) + \binom{2s}{s}\right] \frac{1}{2^{2s}}$$

$$\Phi_{n,s} = \int_0^{\pi/2} (\sin^2 \varphi)^s \cos 2n\varphi \, d\varphi = (-1)^n \frac{\pi}{2} \binom{2s}{s-n} \frac{1}{2^{2s}}, \quad \Phi_{n,s} = 0 \quad (s < n)$$

Consequently, for n=1, 2, 3, . . .,

$$E_n^{(\nu)}(k) = \frac{\pi}{2}\left(\frac{k^2}{4}\right)^n \left\{\binom{\nu}{n} - \binom{\nu}{n+1}\binom{2n+2}{1}\frac{k^2}{4} + \binom{\nu}{n+2}\binom{2n+4}{2}\left(\frac{k^2}{4}\right)^2 - \dots\right\} \tag{3.43}$$

For $\nu<0$, all the coefficients of this series have the same sign $(-1)^n$, so functions $E_n^{(\nu)}(k)$ do not have roots.

For $\alpha=0$ when $\beta=\pi/2$, Eq. (3.41) gives

$$\int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^\nu \, d\varphi = (1 - k^2)^{\nu+1/2} \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2 \sin^2 \varphi)^{\nu+1}} \tag{3.44}$$

For $\nu=\mu-1/2$, it follows from this that

$$\begin{gathered} E_0^{(-\mu-1/2)}(k) = (1 - k^2)^{-\mu} E_0^{(\mu-1/2)}(k) \qquad (\mu = 1, 2, 3, \dots) \\ E_0^{(-3/2)}(k) = \frac{E(k)}{1 - k^2}, \qquad E_0^{(-5/2)}(k) = \frac{E_0^{(3/2)}(k)}{(1 - k^2)^2}, \dots \end{gathered} \tag{3.45}$$

It follows from Eq. (3.35) that

$$E_{n+1}^{(-1/2)}(k) = -\frac{4n}{2n+1}\,\frac{2 - k^2}{k^2}\, E_n^{(-1/2)}(k) - \frac{2n-1}{2n+1}\, E_{n-1}^{(-1/2)}(k) \tag{3.46}$$

$$E_0^{(-1/2)}(k) = F(k), \quad k^2 E_1^{(-1/2)}(k) = 2E(k) - (2 - k^2) F(k)$$

$$E_{n+1}^{(-3/2)}(k) = -\frac{4n}{2n-1}\,\frac{2 - k^2}{k^2}\, E_n^{(-3/2)}(k) - \frac{2n+1}{2n-1}\, E_{n-1}^{(-3/2)}(k) \tag{3.47}$$

$$(1 - k^2) E_0^{(-3/2)}(k) = E(k), \quad k^2 (1 - k^2) E_1^{(-3/2)}(k) = 2(1 - k^2) F(k) - (2 - k^2) E(k)$$

Equations (3.46) and (3.47) are necessary for calculation of /87
integrals (3.32).

Point k=1 is singular. As is known,

$$F(k) = \ln\frac{4}{\sqrt{1-k^2}} + \frac{1}{4}\left(\ln\frac{4}{\sqrt{1-k^2}} - 1\right)(1-k^2) + \dots$$
$$E(k) = 1 + \frac{1}{2}\left(\ln\frac{4}{\sqrt{1-k^2}} - \frac{1}{2}\right)(1-k^2) + \dots \qquad (3.48)$$

It is therefore convenient to introduce functions limited to point k=1

$$A_n(k) = (-1)^n (1-k^2) E_n^{(-3/2)}(k), \qquad A_n(1) = 1 \qquad (3.49)$$

which are determined according to Eq. (3.47) by the recurrent formula

$$A_{n+1} = \frac{4n}{2n-1}\frac{2-k^2}{k^2}A_n - \frac{2n+1}{2n-1}A_{n-1}, \qquad (A_{-n} = A_n)$$
$$A_0(k) = E(k), \qquad k^2A_1(k) = (2-k^2)E(k) - 2(1-k^2)F(k) \qquad (3.50)$$

4. Inductive Velocity Harmonics and Mutual Effect of Screws

We discuss a pair of screws with parallel "blade cone" axes, with origin O of the xyz coordinate system placed in the center of the hub of the screw, the effect of which is under consideration. We place origin O_2 of the identically oriented coordinate system in the center of the hub of the second (to be calculated) screw. As before, let ψ be the blade azimuth of the first screw and θ similarly be the azimuth to be calculated of the blade of the second screw (Fig. 3). Actually, in a two rotor helicopter system, a pair of counterrotating screws is used. This is then easy to take into account, by substituting the sign of azimuth θ in the final equations. Let L be the distance between the screw axes, β be the angle between normal L which connects the axis and the xy plane (Fig. 3), the "angle of slide" of the pair. The coordinates of center O_2 of the hub of the second (to be calculated) screw will then be

$$x_2 = L\cos\beta, \qquad y_2, \qquad z_2 = -L\sin\beta \qquad (4.1)$$

The coordinates of the point on the axis of the blade of the second screw at distance r from the axis of this screw at angle "of taper"

a_{02} of the location of this blade will be

$$x = x_2 + r\cos\theta, \quad y = y_2 + a_{02}r, \quad z = z_2 - r\sin\theta \qquad (4.2)$$

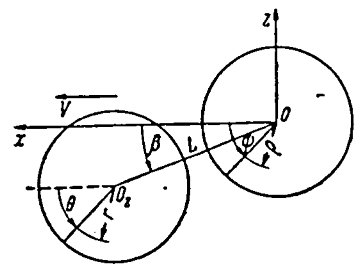

Fig. 3.

The problem is formulated: to find for an abstract screw with circulation $\gamma_n = e^{in\psi}$ the expansion

$$w_n(r,\theta) = w_{n,0}(r) + \sum_{m=1}^{\infty} [w_{n,mc}(r)\cos m\theta + w_{n,ms}(r)\sin m\theta] \qquad (4.3)$$

With complex harmonics

$$w_{n,mc} = w_{nc,mc} + iw_{ns,mc}, \quad w_{n,ms} = w_{nc,ms} + iw_{ns,ms} \qquad (4.4)$$

for unit circulation $\gamma_n = e^{in\psi}$ and using expansion (1.4) of circulation $\Gamma(\rho,\psi)$, the Fourier series /88

$$v_n(r,\theta) = v_{n,0}(r) + \sum_{m=1}^{\infty} [v_{n,mc}(r)\cos m\theta + v_{n,ms}(r)\sin m\theta] \qquad (4.5)$$

can be plotted for the inductive velocity of a real screw, by calculating the required number of harmonics by the equations

$$\begin{gathered} v_{nc,0} = -\int_0^R w_{nc,0}(r,\rho)\frac{d\Gamma_{nc}}{d\rho}d\rho, \quad v_{ns,0} = -\int_0^R w_{ns,0}(r,\rho)\frac{d\Gamma_{ns}}{d\rho}d\rho \\ \begin{matrix} v_{nc,mc} \\ v_{nc,ms} \end{matrix} = -\int_0^R \begin{matrix} w_{nc,mc} \\ w_{nc,ms} \end{matrix} \frac{d\Gamma_{nc}}{d\rho}d\rho, \quad \begin{matrix} v_{ns,mc} \\ v_{ns,ms} \end{matrix} = -\int_0^R \begin{matrix} w_{ns,mc} \\ w_{ns,ms} \end{matrix} \frac{d\Gamma_{ns}}{d\rho}d\rho \end{gathered} \qquad (4.6)$$

for $0<r<R_2$. Usually, $R_2=R$ (the screw radii are the same).

In the case $x_2=y_2=z_2=0$, the harmonics of the natural inductive velocity of the screw are obtained. Subsequently, only an analysis of axial inductive velocity $w_y{}^\circ$ of a $k=\infty$ screw will be given ($w_x{}^\circ$ and $w_z{}^\circ$ are unimportant for the operation of rotors). Exponent $^\circ$ is dropped here.

Equation (3.6) for $\gamma=e^{in\psi}$ can be presented as

$$4\pi w_n = -in\int_0^{\rho}\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l_0'}\frac{d\rho'}{\rho'} +$$
$$+\int_0^{\infty}\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi}\left\{\rho\left(\frac{\partial}{\partial\rho}\frac{1}{l} + a_0\frac{\partial}{\partial y}\frac{1}{l}\right) - \frac{V_x}{\Omega}\frac{\partial}{\partial z}\left(\frac{1}{l} - \frac{1}{l_a}\right)\right\} d\psi\, d\varphi \qquad (4.7)$$

where ℓ^2 function (3.4), in accordance with Eq. (4.2), will have this form:

$$l^2 = (x_2 - \rho\cos\psi + r\cos\theta + \varphi V_x / \Omega)^2 + (y_\nabla + \varphi V_y^* / \Omega)^2 +$$
$$+ (z_2 + \rho\sin\psi - r\sin\theta)^2, \qquad y_\nabla = y_2 + a_{02}r - a_0\rho \qquad (4.8)$$

Differentiation is carried out in Eq. (4.7) only up to the integration operator; if n=1, 2, 3, . . ., the integrals for $\rho=0$ are absent (this subsequently becomes less evident). On the assumption that

$$\omega^2 = \rho^2 - 2\rho r\cos(\psi - \theta) + r^2, \qquad e^{i\chi} = \frac{\rho - re^{i(\theta-\psi)}}{\omega}$$

equality (4.8) can be transformed thus:

$$l^2 = \omega^2 - 2\omega D\cos(\psi + \chi - \tau) + D^2 + (y_\nabla + \varphi V_y^* / \Omega)^2$$

where

$$D^2 = (x_2 + \varphi V_x / \Omega)^2 + z_2^2, \qquad e^{i\tau} = (x_2 - iz_2 + \varphi V_x / \Omega) / D \qquad (4.9)$$

It is significant that ω and $e^{i\chi}$ are functions of $\psi-\theta$. In Eq. (4.7), the average value of the subintegral function for the period is used. It can therefore be written that

$$4\pi w_n = -ine^{in\theta}\int_0^{\rho}\int_{-\pi-\theta}^{\pi-\theta} \frac{e^{in(\psi-\theta)}}{2\pi l_0'} d\psi \frac{d\rho'}{\rho'} +$$
$$+ e^{in\theta}\int_0^{\infty}\int_{-\pi-\theta}^{\pi-\theta} \frac{e^{in(\psi-\theta)}}{2\pi}\left\{\rho\left(\frac{\partial}{\partial\rho}\frac{1}{l} + a_0\frac{\partial}{\partial y}\frac{1}{l}\right) - \frac{V_x}{\Omega}\frac{\partial}{\partial z}\left(\frac{1}{l} - \frac{1}{l_a}\right)\right\} d\psi\, d\varphi$$

and $\psi-\varepsilon$ can be replaced by ψ; the result will be /89

$$4\pi w_n = -ine^{in\theta}\int_0^{\rho}\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi l_0'} d\psi \frac{d\rho'}{\rho'} +$$
$$+ e^{in\theta}\int_0^{\infty}\int_{-\pi}^{\pi} \frac{e^{in\psi}}{2\pi}\left\{\rho\left(\frac{\partial}{\partial\rho}\frac{1}{l} + a_0\frac{\partial}{\partial y}\frac{1}{l}\right) - \frac{V_x}{\Omega}\frac{\partial}{\partial z}\left(\frac{1}{l} - \frac{1}{l_a}\right)\right\} d\psi\, d\varphi \qquad (4.10)$$

$$l^2 = \omega^2 - 2\omega D\cos(\psi + \theta + \chi - \tau) + D^2 + (y_\nabla + \varphi V_y^* / \Omega)^2 \qquad (4.11)$$

$$\omega = \sqrt{\rho^2 - 2\rho r\cos\psi + r^2}, \qquad e^{i\chi} = \frac{\rho - re^{-i\psi}}{\omega} \qquad (4.12)$$

We call this transformation the ω transformation. We introduce the series

$$\frac{e^{in\theta}}{l} = a_0 + \sum_{m=1}^{\infty} (a_{mc} \cos m\theta + a_{ms} \sin m\theta)$$

$$a_0 = \int_{-\pi}^{\pi} \frac{e^{in\theta}}{2\pi l} d\theta, \qquad \begin{matrix} a_{mc} \\ a_{ms} \end{matrix} = \int_{-\pi}^{\pi} \frac{e^{i(n+m)\theta} \pm e^{i(n-m)\theta}}{2\pi l \sqrt{\pm 1}} d\theta \tag{4.13}$$

By introducing variable θ=θ-σ, where σ=τ-χ-ψ does not depend on θ, it is easy to obtain

$$l^2 = \omega^2 - 2\omega D \cos\vartheta + D^2 + (y_\nabla + \varphi V_y^* / \Omega)^2 \tag{4.14}$$

$$a_0 = e^{in\sigma} \int_{-\pi}^{\pi} \frac{e^{in\vartheta}}{2\pi l} d\vartheta, \qquad \begin{matrix} a_{mc} \\ a_{ms} \end{matrix} = \frac{e^{i(n+m)\sigma}}{\sqrt{\pm 1}} \int_{-\pi}^{\pi} \frac{e^{i(n+m)\vartheta}}{2\pi l} d\vartheta \pm \frac{e^{i(n-m)\sigma}}{\sqrt{\pm 1}} \int_{-\pi}^{\pi} \frac{e^{i(n-m)\vartheta}}{2\pi l} d\vartheta$$

or

$$a_0 = A_0 e^{-in\psi}, \qquad \begin{matrix} a_{mc} \\ a_{ms} \end{matrix} = \frac{A_m \pm A_{-m}}{\sqrt{\pm 1}} e^{-in\psi}$$

where, if the integration variables are selected from the left,

$$A_{\pm m}(\rho, \varphi; \psi) = e^{i(n\pm m)(\tau-\chi)\mp im\psi} \int_{-\pi}^{\pi} \frac{e^{i(n\pm m)\vartheta}}{2\pi l} d\vartheta \tag{4.15}$$

Now, based on Eq. (4.10), it follows that

$$4\pi w_{n0} = -in \int_0^{\rho}\int_{-\pi}^{\pi} A_0(\rho', 0)\, d\psi \frac{d\rho'}{\rho'} - \frac{V_x}{\Omega} \int_0^{\infty}\int_{-\pi}^{\pi} \frac{\partial}{\partial z} [A_0(\rho, \varphi) - A_0(0, \varphi)] \times$$
$$\times\, d\psi d\varphi + \rho \int_0^{\infty}\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial \rho} A_0(\rho, \varphi) + a_0 \frac{\partial}{\partial y} A_0(\rho, \varphi)\right] d\psi d\varphi \quad (m=0)$$

$$4\pi \begin{matrix} w_{n,mc} \\ w_{n,ms} \end{matrix} = -in \int_0^{\rho}\int_{-\pi}^{\pi} \frac{A_m(\rho', 0) \pm A_{-m}(\rho', 0)}{\rho' \sqrt{\pm 1}} d\psi\, d\rho' +$$
$$+ \rho \int_0^{\infty}\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial \rho} \frac{A_m(\rho, \varphi) \pm A_{-m}(\rho, \varphi)}{\sqrt{\pm 1}} + a_0 \frac{\partial}{\partial y} \frac{A_m(\rho, \varphi) \pm A_{-m}(\rho, \varphi)}{\sqrt{\pm 1}}\right] d\psi\, d\varphi -$$
$$- \frac{V_x}{\Omega} \int_0^{\infty}\int_{-\pi}^{\pi} \left[\frac{\partial}{\partial z} \frac{A_m(\rho, \varphi) \pm A_{-m}(\rho, \varphi)}{\sqrt{\pm 1}} - \frac{\partial}{\partial z} \frac{A_m(0, \varphi) \pm A_{-m}(0, \varphi)}{\sqrt{\pm 1}}\right] d\psi\, d\varphi$$

After substitution of the order of integration over ψ and φ by introducing the notations /90

$$
\left.\begin{aligned}
K_{\mu}(\omega) &= \frac{V_x}{\Omega}\int_{0}^{\infty} e^{i\mu\tau}\int_{-\pi}^{\pi}\frac{e^{i\mu\vartheta}}{2\pi l}\,d\vartheta\,d\varphi \quad (\pm\mu = 1, 2, 3, \ldots)\\
K_{\mu q}(\omega) &= \frac{V_x}{\Omega}\int_{0}^{\infty}\frac{\partial}{\partial q}\left\{e^{i\mu\tau}\int_{-\pi}^{\pi}\frac{e^{i\mu\vartheta}}{2\pi l}\,d\vartheta\right\}d\varphi \quad (\pm\mu = 0, 1, 2, \ldots;\ q = y, z, \omega)
\end{aligned}\right\} \qquad (4.16)
$$

since D=L and $\tau=\beta$ for $\phi=0$ we obtain:

for m=0,

$$
\left.\begin{aligned}
4\pi w_{n,0} = &-ine^{in\beta}\int_{0}^{\rho}\int_{-\pi}^{\pi}\frac{e^{-in\chi'}}{2\pi\rho'}\int_{-\pi}^{\pi}\frac{e^{in\vartheta}}{2\pi l_0'}\,d\vartheta\,d\psi\,d\rho' +\\
&+\frac{\Omega\rho}{V_x}\left\{\int_{-\pi}^{\pi}\frac{e^{-in\chi}}{2\pi}\left(K_{n\omega}\frac{\partial\omega}{\partial\rho} - inK_n\frac{\partial\chi}{\partial\rho}\right)_{\omega}d\psi + a_0\int_{-\pi}^{\pi}\frac{e^{-in\chi}}{2\pi}K_{ny}(\omega)\,d\psi\right\} -\\
&-\int_{-\pi}^{\pi}\frac{e^{-in\chi}}{2\pi}K_{nz}(\omega)\,d\psi + \int_{-\pi}^{\pi}\frac{e^{-in\chi_a}}{2\pi}K_{nz}(r)\,d\psi
\end{aligned}\right\} \qquad (4.17)
$$

for m=1, 2, 3, . . .,

$$
\left.\begin{aligned}
4\pi\,\begin{matrix}w_{n,mc}\\ w_{n,ms}\end{matrix} = &-in\left\{\frac{e^{i(n+m)\beta}}{\sqrt{\pm 1}_i}\int_{0}^{\rho}\int_{-\pi}^{\pi}\frac{e^{-i(n+m)\chi'-im\psi}}{2\pi\rho'}\int_{-\pi}^{\pi}\frac{e^{i(n+m)\vartheta}}{2\pi l_0'}\,d\vartheta\,d\psi\,d\rho' \pm\right.\\
&\left.\pm\frac{e^{i(n-m)\beta}}{\sqrt{_i\pm 1}}\int_{0}^{\rho}\int_{-\pi}^{\pi}\frac{e^{-i(n-m)\chi'+im\psi}}{2\pi\rho'}\int_{-\pi}^{\pi}\frac{e^{i(n-m)\vartheta}}{2\pi l_0'}\,d\vartheta\,d\psi\,d\rho'\right\} +\\
&+\frac{\Omega\rho}{V_x}\left\{\int_{-\pi}^{\pi}\frac{e^{-i(n+m)\chi-im\psi}}{2\pi\sqrt{\pm 1}}\left[K_{n+m,\omega}(\omega)\frac{\partial\omega}{\partial\rho} - i(n+m)K_{n+m}(\omega)\frac{\partial\chi}{\partial\rho}\right]d\psi \pm\right.\\
&\left.\pm\int_{-\pi}^{\pi}\frac{e^{-i(n-m)\chi+im\psi}}{2\pi\sqrt{\pm 1}}\left[K_{n-m,\omega}(\omega)\frac{\partial\omega}{\partial\rho} - i(n-m)K_{n-m}(\omega)\frac{\partial\chi}{\partial\rho}\right]d\psi\right\} +\\
&+a_0\frac{\Omega\rho}{V_x}\left\{\int_{-\pi}^{\pi}\frac{e^{-i(n+m)\chi-im\psi}}{2\pi\sqrt{\pm 1}}K_{n+m,y}(\omega)\,d\psi \pm \int_{-\pi}^{\pi}\frac{e^{-i(n-m)\chi+im\psi}}{2\pi\sqrt{\pm 1}}K_{n-m,y}(\omega)\,d\psi\right\} -\\
&-\left\{\int_{-\pi}^{\pi}\frac{e^{-i(n+m)\chi-im\psi}}{2\pi\sqrt{\pm 1}}K_{n+m,z}(\omega)\,d\psi \pm \int_{-\pi}^{\pi}\frac{e^{-i(n-m)\chi+im\psi}}{2\pi\sqrt{\pm 1}}K_{n-m,z}(\omega)\,d\psi\right\} +\\
&+\left\{\int_{-\pi}^{\pi}\frac{e^{-i(n+m)\chi_a-im\psi}}{2\pi\sqrt{\pm 1}}K_{n+m,z}(r)\,d\psi \pm \int_{-\pi}^{\pi}\frac{e^{-i(n-m)\chi_a+im\psi}}{2\pi\sqrt{\pm 1}}K_{n-m,z}(r)\,d\psi\right\}
\end{aligned}\right\} \qquad (4.18)
$$

Here and subsequently, χ' and ω' designate quantities $\chi(\rho)$ and $\omega(\rho)$ of Eq. (4.12) with $\rho=\rho'$, and χ_a is the value of χ with $\rho=0$. It follows from Eq. (4.12) that

$$\cos\chi = \frac{\rho - r\cos\psi}{\omega}, \quad \sin\chi = \frac{r\sin\psi}{\omega}, \quad \frac{\partial\omega}{\partial\rho} = \cos\chi, \quad \frac{\partial\chi}{\partial\rho} = -\frac{\sin\chi}{\omega} \quad (4.19)$$

where $\chi(-\psi)=-\chi(\psi)$. Therefore, in the notations

$$\begin{matrix} K_\mu^+ \\ K_\mu^- \end{matrix} = \frac{\partial K_\mu}{\partial\omega} \pm \frac{\mu}{\omega} K_\mu, \qquad K_0^+ = \frac{\partial K_0}{\partial\omega} = K_0^- \quad (4.20)$$

the following relationships are obtained: /91

for m=0

$$K_{\mu\omega}\frac{\partial\omega}{\partial\rho} - i\mu K_\mu \frac{\partial\chi}{\partial\rho} = \frac{e^{i\chi}}{2}K_\mu^+ + \frac{e^{-i\chi}}{2}K_\mu^-$$

$$4\pi^2 w_{n0} = -\, i n e^{in\beta} \int_0^\rho\int_0^\pi \frac{\cos n\chi'}{\rho'} \int_{-\pi}^{\pi} \frac{e^{in\vartheta}}{2\pi l_0'}\, d\vartheta\, d\psi\, d\rho' + \quad (4.21)$$

$$+ \frac{\Omega\rho}{V_x}\left\{\frac{1}{2}\int_0^\pi [\cos(n-1)\chi\, K_n^+(\omega) + \cos(n+1)\chi\, K_n^-(\omega)]\, d\psi + \right.$$

$$\left. + a_0 \int_0^\pi \cos n\chi K_{ny}(\omega)\, d\psi\right\} - \int_0^\pi \cos n\chi K_{nz}(\omega)\, d\psi + \int_0^\pi \cos n\chi_a K_{nz}(r)\, d\psi$$

for m=1, 2, 3, . . .,

$$4\pi^2 \begin{matrix} w_{n,mc} \\ w_{n,ms} \end{matrix} = -in\left\{\frac{e^{i(n+m)\beta}}{\sqrt{\pm 1}} \int_0^\rho\int_0^\pi \frac{\cos[(n+m)\chi' + m\psi]}{\rho'} \int_{-\pi}^{\pi} \frac{e^{i(n+m)\vartheta}}{2\pi l_0'}\, d\vartheta\, d\psi\, d\rho' \pm \right.$$

$$\left. \pm \frac{e^{i(n-m)\beta}}{\sqrt{\pm 1}} \int_0^\rho\int_0^\pi \frac{\cos[(n-m)\chi' - m\psi]}{\rho'} \int_{-\pi}^{\pi} \frac{e^{i(n-m)\vartheta}}{2\pi l_0'}\, d\vartheta\, d\psi\, d\rho'\right\} +$$

$$+ \frac{\Omega\rho}{2V_x}\left\{\int_0^\pi \cos[(n+m-1)\chi + m\psi]\, \frac{K_{n+m}^+(\omega)}{\sqrt{\pm 1}}\, d\psi + \right. \quad (4.22)$$

$$+ \int_0^\pi \cos[(n+m+1)\chi + m\psi]\, \frac{K_{n+m}^-(\omega)}{\sqrt{\pm 1}}\, d\psi \pm$$

$$\pm \int_0^\pi \cos[(n-m-1)\chi - m\psi]\, \frac{K_{n-m}^+(\omega)}{\sqrt{\pm 1}}\, d\psi \pm$$

$$\left. \pm \int_0^\pi \cos[(n-m+1)\chi - m\psi]\, \frac{K_{n-m}^-(\omega)}{\sqrt{\pm 1}}\, d\psi\right\} +$$

$$+ a_0 \frac{\Omega\rho}{V_x}\left\{\int_0^\pi \cos[(n+m)\chi + m\psi]\, \frac{K_{n+m,y}(\omega)}{\sqrt{\pm 1}}\, d\psi \pm \right.$$

$$\pm\int_0^\pi \cos[(n-m)\chi - m\psi]\frac{K_{n-m,y}(\omega)}{\sqrt{\pm 1}}d\psi\Big\} -$$

$$\Big\{\int_0^\pi \cos[(n+m)\chi + m\psi]\frac{K_{n+m,z}(\omega)}{\sqrt{\pm 1}}d\psi \pm$$

$$\pm\int_0^\pi \cos[(n-m)\chi - m\psi]\frac{K_{n-m,z}(\omega)}{\sqrt{\pm 1}}d\psi\Big\} +$$

$$\Big\{\int_0^\pi \cos[(n+m)\chi_a + m\psi]\frac{K_{n+m,z}(r)}{\sqrt{\pm 1}}d\psi \pm$$

$$\pm\int_0^\pi \cos[(n-m)\chi_a - m\psi]\frac{K_{n-m,z}(r)}{\sqrt{\pm 1}}d\psi\Big\}$$

It should be remembered that, for n=1, 2, 3, . . ., the integrals for $\rho=0$ ($\omega=r$) are absent. A change in Eq. (4.16) to the integration variable $\xi=x_2+\phi V_x/\Omega$ with notations /92

$$\lambda = V_y^* / V_x^{\cdot}, \qquad y_* = y_\nabla - \lambda x_2 \tag{4.23}$$

in accordance with Eq. (4.14) and (4.9), gives

$$K_\mu(\omega) = \int_{x_2}^{\infty}\frac{e^{i\mu\tau}}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\mu\vartheta}\,d\vartheta}{\sqrt{\omega^2 - 2\omega D\cos\vartheta + D^2 + (y_* + \lambda\xi)^2}}d\xi \qquad (\mu \neq 0)$$

$$D = \sqrt{\xi^2 + z_2^2}, \qquad e^{i\tau} = (\xi - iz_2)/D \tag{4.24}$$

By introducing the substitution θ=a sgn z_2, it can be determined that

$$K_\mu(\omega) = \int_{x_2}^{\infty}\int_{-\pi}^{\pi}\frac{\exp(i\mu\alpha\,\mathrm{sgn}\,z)}{2\pi l}\,d\alpha\,d\xi$$

$$l^2 = \omega^2 - 2\omega\,(\xi\cos\alpha - |z_2|\sin\alpha) + \xi^2 + z_2^2 + (y_* + \lambda\xi)^2 \tag{4.25}$$

Thus, Eq. (3.25) introduced above and functions $F_\mu(x,g,h;\,\omega,\lambda)$, ($\mu\geqslant 0$) described in Section 5 arise. With $\lambda_*=|\lambda|$ sgn (λy_*),

$$K_\mu(\omega) = F_{|\mu|c}(x_2, |z_2|, |y_*|;\,\omega, \lambda_*) + i\,\mathrm{sgn}\,(\mu\,\mathrm{sgn}\,z_2)\,F_{|\mu|s}(x_2, |z_2|, |y_*|;\,\omega, \lambda_*)$$
$$(\pm\mu = 1, 2, \ldots)$$
$$K_{\mu q}(\omega) = F_{|\mu|cq}(x_2, |z_2|, |y_*|;\,\omega, \lambda_*) + i\,\mathrm{sgn}\,(\mu\,\mathrm{sgn}\,z_2)F_{|\mu|sq}(x_2, |z_2|, |y_*|;\,\omega, \lambda_*) \tag{4.26}$$
$$(\pm\mu = 0, 1, \ldots)$$

where q=$|z_2|$, y_*, ω. It is therefore sufficient to assume in the equations of Section 5 that

$$x = x_2,\ g = |z_2|,\ h = |y_*|,\ |h + \lambda x| = |y_\nabla|$$

Equations (4.21-4.22) also include the integrals

$$\int_{-\pi}^{\pi} \frac{e^{i\mu\vartheta}}{2\pi l_0} d\vartheta = \int_{-\pi}^{\pi} \frac{e^{i\mu\vartheta} d\vartheta}{2\pi \sqrt{\omega^2 - 2\omega D_0 \cos\vartheta + D_0^2 + y_\nabla^2}}, \qquad D_0 = L \qquad (\mu = 0, 1, \ldots) \quad (4.27)$$

By operating as specified above in Section 3, the following can be obtained:

$$\int_{-\pi}^{\pi} \frac{e^{i\mu\vartheta}}{2\pi l_0} d\vartheta = \frac{2}{\pi} \frac{(-1)^\mu}{\sqrt{(\omega + L)^2 + y_\nabla^2}} E_\mu^{(-1/2)}(k), \quad k^2 = \frac{4\omega L}{(\omega + L)^2 + y_\nabla^2} \qquad (4.28)$$

$$E_\mu^{(-1/2)}(k) = \int_0^{\pi/2} \frac{\cos 2\mu\varphi \, d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \qquad (4.29)$$

Functions $E_\mu^{(-1/2)}(k)$ were introduced and studied in Section 3. We study functions ω and χ. According to Eq. (4.12) and (4.19),

$$\omega = \sqrt{(\rho - r)^2 + 4\rho r \sin^2 \frac{\psi}{2}}, \qquad \chi = \operatorname{arc\,tg} \frac{r \sin\psi}{\rho - r\cos\psi} \qquad (4.30)$$

$$\omega = \rho, \quad \chi = 0 \text{ with } r = 0 \qquad (4.31)$$
$$\omega = 2r \sin {}^1\!/_2\psi, \quad \chi = {}^1\!/_2(\pi - \psi) \text{ with } \rho = r$$
$$\omega = r, \quad \chi = \pi - \psi \text{ with } \rho = 0$$

$$\omega = |\rho - r|, \qquad \chi = \begin{cases} 0 & (\rho > r) \\ \pi/2 & (\rho = r) \text{ with } \psi = 0 \\ \pi & (\rho < r) \end{cases}$$

It is easy to compile tables of functions $\omega(\psi)$ and $\chi(\psi)$ for a /93 number of calculated radii, by taking the value of ψ every 10 or 15° but a smaller step with small values of ψ. It is evident that $0 \leqslant \omega \leqslant R + R_2$; usually, $R_2 = R$. $R=1$ can be adopted in the calculations in compiling tables for $0 \leqslant r \leq 1$ and $0 \leqslant \rho \leq 1$. The author has tables of $\omega(\psi)$, $\chi(\psi)$ and $\cos[(n+m)\chi + m\psi]$ for $\pm m = 0, 1, 2$ and $n=0$, $n=1$.

With $r=0$ or $\rho=0$, ω does not depend on ψ. Functions $K_\mu(\omega)$ and $K_{\mu q}(\omega)$ can therefore be taken out of the integral sign. With $r=0$, we have $\chi=0$, $\omega=\rho$, and it turns that $w_{n,m} \equiv 0$, but

$$4\pi w_{n,0} = -ine^{in\beta}\int_0^{\rho}\int_{-\pi}^{\pi}\frac{e^{in\vartheta}}{2\pi l_0'}d\vartheta\frac{d\rho'}{\rho'}+ \qquad (4.32)$$

$$+\frac{\Omega\rho}{V_x}\{K_{n\omega}(\rho)+a_0K_{nv}(\rho)\}-K_{nz}(\rho)+K_{nz}(0) \quad (r=0)$$

In the case of $\rho=0$, all $w_{n,m}\equiv 0$.

The practical value of the ω transformation is that, with $a_0=0$, parameters r and ρ are combined in one parameter ω (4.12) which, for a given relative location of screw pairs (x_2, y_2, z_2), with a given orientation of it relative to the speed of movement (angles β and arctan λ), is the unique argument of functions $F_\mu(x, g, h; \omega, \lambda)$. Even with $a_0\neq 0$, the first three parameters of functions $F_\mu(\omega)$ and $F_{\mu q}(\omega)$ provide complete solution of the problem, since it is sufficient to assume $h=|y_*|=|y_\nabla-\lambda x_2|$.

After calculating the required number of functions of $\omega(\psi)$ for the lower harmonics, integration with respect to ψ can be performed by numerical methods.

The following individual cases are of great importance:

1. $z_2=0$ (flight of a "longitudinal" pair without slipping);

2. $x_2=0=z_2$ (pair of coaxial screws or screw with $a_0\neq 0$);

3. $x_2=y_*=z_2=0$ (isolated screw with $a_0=0$);

4. $x_2=0=y_*$ (flight of "transverse" pair without slipping).

In the latter case, the functions have two branches each:

$$(4a)\ L^2(1+\lambda^2)-\omega^2>0,\ (4\ \ )\ L^2(1+\lambda^2)-\omega^2<0$$

Case (4a) occurs in the absence of "overlap" of the screws ($L>R+R_2$).

In Section 5, functions $F_\mu(x, g, h; \omega, \lambda)$ are written out for

$$(1)\ g = 0, \quad (2)\ g = 0 = x, \quad (3)\ g = x = h = 0, \quad (4)\ x = 0 = h$$

In the first and fourth cases, the parameters of integrals of the third kind are real. In the second and third cases, functions $F_\mu(\omega)$ and their derivatives are simple algebraic functions.

In the case g=0, the algebraic functions in $F_\mu(\omega)$ and $F_{\mu q}(\omega)$, with $\lambda h<0(\lambda y_*<0)$, have two branches each with branching point $h=\omega|\lambda|$. This must be taken into account in integration with respect to ψ in Eq. (4.21), (4.22). Transition point $\psi_\wedge$ is defined according to Eq. (4.12) as

$$\omega|\lambda| = |y_*|, \qquad 2\rho r \cos\psi_\wedge = \rho^2 + r^2 - (y_* / \lambda)^2 \tag{4.33}$$

in which, in segment $[0, \psi_\wedge)$, we have $|\rho-r|\leqslant\omega<|y_*/\lambda|$. It is evident that

$$\begin{aligned} &\omega \geqslant |y_* / \lambda|, \quad \psi_\wedge = 0 \text{ with } (\rho - r)^2 \geqslant (y_* / \lambda)^2 \\ &\omega \leqslant |y_* / \lambda|, \quad \psi_\wedge = \pi \text{ with } (\rho + r)^2 \leqslant (y_* / \lambda)^2 \end{aligned} \tag{4.34}$$

Transition point $\psi_\wedge$, in case (4), will be at $\omega>L$ ("transverse" pair of screws with "overlap"). It is determined by the equality $L^2(1+\lambda^2)=\omega^2$. It is sufficient therefore in Eq. (4.33) and (4.34) to replace $(y_*/\lambda)^2$ by $L^2(1+\lambda^2)$.

In the second and third cases, it proved to be possible to calculate all the integrals of ψ by presenting them in the form of a combination of elliptical integrals and elementary functions. These results will be the subject of a separate article. In the fourth case, half of functions $F_\mu(\omega)$ and $F_{\mu q}(\omega)$ are simple algebraic functions, namely:

$$\begin{aligned} &F_{\mu s}, \partial F_{\mu c} / \partial h \text{ and } \partial F_{\mu s} / \partial g \text{ with } \mu = |n \pm m| = 1, 3, 5, \ldots, \\ &F_{\mu c}, \partial F_{\mu s} / \partial h \text{ and } \partial F_{\mu c} / \partial g \text{ with } \mu = |n \pm m| = 2, 4, \ldots, \partial F_0/\partial g \end{aligned}$$

Integration over ψ also is successfully performed here in a similar way.

5. Special Functions of Vortex Theory

We consider function (3.25) of five real variables /94

$$F_\mu(x, g, h; \omega, \lambda) = \int\limits_x^\infty \int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi l}\, d\alpha\, d\xi \qquad (\mu = 0, 1, 2, \ldots) \tag{5.1}$$

$$l^2 = \omega^2 - 2\omega(\xi\cos\alpha - g\sin\alpha) + \xi^2 + g^2 + (h + \lambda\xi)^2, \quad \omega \geqslant 0,\ g \geqslant 0,\ h \geqslant 0 \tag{5.2}$$

After integration by parts over α, there will be

$$i\frac{\mu}{\omega} F_\mu = \int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi} \int\limits_x^\infty \frac{\xi\sin\alpha + g\cos\alpha}{l^3}\, d\xi\, d\alpha, \qquad l^2 = A + 2B\xi + C\xi^2 \tag{5.3}$$

$$A = \omega^2 + 2\omega g\sin\alpha + g^2 + h^2, \quad B = \lambda h - \omega\cos\alpha, \quad C = 1 + \lambda^2 \tag{5.4}$$

Consequently, $F_0(\omega)$ does not exist. However, derivatives

$$\frac{\partial F_\mu}{\partial h} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi} \int\limits_x^\infty \frac{h + \lambda\xi}{l^3}\, d\xi\, d\alpha, \qquad \frac{\partial F_\mu}{\partial g} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi} \int\limits_x^\infty \frac{g + \omega\sin\alpha}{l^3}\, d\xi\, d\alpha$$
$$\frac{\partial F_\mu}{\partial \omega} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi} \int\limits_x^\infty \frac{\omega - \xi\cos\alpha + g\sin\alpha}{l^3}\, d\xi\, d\alpha \tag{5.5}$$

exist for μ=0. Since

$$\int\limits_x^\infty \frac{d\xi}{l^3} = \frac{\sqrt{C}}{H} - \frac{B + xC}{HD}, \quad \int\limits_x^\infty \frac{\xi\, d\xi}{l^3} = \frac{A + xB}{HD} - \frac{B}{H\sqrt{C}}, \qquad H = AC - B^2 \tag{5.6}$$
$$D^2 = \omega^2 - 2\omega(x\cos\alpha - g\sin\alpha) + x^2 + g^2 + (h + \lambda x)^2$$

then

$$i\frac{\mu}{\omega} F_\mu = \int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi H}\left\{\left(\sqrt{C} - \frac{B + xC}{D}\right) g\cos\alpha + \left(\frac{A + xB}{D} - \frac{B}{\sqrt{C}}\right)\sin\alpha\right\} d\alpha$$
$$\frac{\partial F_\mu}{\partial h} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi H}\left\{\left(\sqrt{C} - \frac{B + xC}{D}\right) h + \left(\frac{A + xB}{D} - \frac{B}{\sqrt{C}}\right)\lambda\right\} d\alpha$$
$$\frac{\partial F_\mu}{\partial g} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi H}\left(\sqrt{C} - \frac{B + xC}{D}\right)(g + \omega\sin\alpha)\, d\alpha \tag{5.7}$$
$$\frac{\partial F_\mu}{\partial \omega} = -\int\limits_{-\pi}^{\pi} \frac{e^{i\mu\alpha}}{2\pi H}\left\{\left(\sqrt{C} - \frac{B + xC}{D}\right)(\omega + g\sin\alpha) - \left(\frac{A + xB}{D} - \frac{B}{\sqrt{C}}\right)\cos\alpha\right\} d\alpha$$

$$H = (g + \omega\sin\alpha)^2(1 + \lambda^2) + (h + \lambda\omega\cos\alpha)^2 \geqslant 0 \tag{5.8}$$

With ω=0, only these functions are different from zero:

$$\frac{\partial F_0}{\partial g} = -\frac{g}{H_0}\left(\sqrt{C} - \frac{\lambda h + xC}{D_0}\right), \quad H_0 = g^2(1+\lambda^2) + h^2, \quad D_0^2 = x^2 + g^2 + (h+\lambda x)^2$$

$$\frac{\partial F_1}{\partial h} = -\frac{1}{H_0}\left\{\left(\sqrt{C} - \frac{\lambda h + xC}{D_0}\right)h + \left(\frac{g^2+h^2+x\lambda h}{D_0} - \frac{\lambda h}{\sqrt{C}}\right)\right\} \tag{5.9}$$

$$\frac{\partial F_1}{\partial \omega} = -\frac{1}{2H_0}\left\{ig\left(\sqrt{C} - \frac{\lambda h + xC}{D_0}\right) - \left(\frac{g^2+h^2+x\lambda h}{D_0} - \frac{\lambda h}{\sqrt{C}}\right)\right\}$$

For $\omega \neq 0$, function (5.8) should be presented as

$$H = \frac{P(\omega\cos\alpha)}{(g-\omega\sin\alpha)^2(1+\lambda^2) + (h+\lambda\omega\cos\alpha)^2} \tag{5.10}$$

In Eq. (5.10), /95

$$P(\varkappa) = \{\varkappa^2 - 2\lambda(h + ig\sqrt{1+\lambda^2})\varkappa - (h + ig\sqrt{1+\lambda^2})^2 - \omega^2(1+\lambda^2)\} \times$$
$$\times \{\varkappa^2 - 2\lambda(h - ig\sqrt{1+\lambda^2})\varkappa - (h - ig\sqrt{1+\lambda^2})^2 - \omega^2(1+\lambda^2)\}, \quad \varkappa = \omega\cos\alpha \tag{5.11}$$

Equation $P(\kappa)=0$ is broken down into two. The first of them

$$\varkappa^2 - 2\lambda(h + ig\sqrt{1+\lambda^2})\varkappa - (h + ig\sqrt{1+\lambda^2})^2 - \omega^2(1+\lambda^2) = 0 \tag{5.12}$$

has the roots

$$\left.\begin{matrix}\varkappa_1\\ \varkappa_2\end{matrix}\right. = \lambda(h + ig\sqrt{1+\lambda^2}) \pm \operatorname{sgn}\lambda\sqrt{1+\lambda^2}\sqrt{\omega^2 + (h + ig\sqrt{1+\lambda^2})^2} \tag{5.13}$$

and the second has roots κ_3 and κ_4, which are complexly conjugated with roots κ_1 and κ_2. It is easy to determine that

$$\frac{1}{H} = \frac{R(\omega\cos\alpha) - 2g(1+\lambda^2)\omega\sin\alpha}{P(\omega\cos\alpha)} \tag{5.14}$$

$$R(\varkappa) = h^2 + 2\lambda h\varkappa - \varkappa^2 + (g^2+\omega^2)(1+\lambda^2) \tag{5.15}$$

$$\frac{1}{H} = \sum_{\nu=1}^{4}\frac{R(\varkappa_\nu) - 2g(1+\lambda^2)\omega\sin\alpha}{(\omega\cos\alpha - \varkappa_\nu)P'(\varkappa_\nu)}, \quad \frac{\omega\cos\alpha}{H} = \sum_{\nu=1}^{4}\frac{R(\varkappa_\nu) - 2g(1+\lambda^2)\omega\sin\alpha}{(\omega\cos\alpha - \varkappa_\nu)P'(\varkappa_\nu)}\varkappa_\nu$$

$$\frac{\omega\sin\alpha}{H} = \sum_{\nu=1}^{4}\frac{R(\varkappa_\nu)\omega\sin\alpha - 2g(1+\lambda^2)(\omega^2 - \varkappa_\nu^2)}{(\omega\cos\alpha - \varkappa_\nu)P'(\varkappa_\nu)}$$

For κ_1 and κ_2, with the use of equalities (5.12) and (5.13),

$$\left.\begin{matrix}\varkappa_1^2 - \omega^2\\ \varkappa_2^2 - \omega^2\end{matrix}\right. = \left\{(h + ig\sqrt{1+\lambda^2})\sqrt{1+\lambda^2} \pm |\lambda|\sqrt{\omega^2 + (h + ig\sqrt{1+\lambda^2})^2}\right\}^2$$

is obtained and if, for these roots, there are introduced the radicals

$$\frac{\sqrt{\varkappa_1^2-\omega^2}}{\sqrt{\varkappa_2^2-\omega^2}}=(h+ig\sqrt{1+\lambda^2})\sqrt{1+\lambda^2}\pm|\lambda|\sqrt{\omega^2+(h+ig\sqrt{1+\lambda^2})^2} \quad (5.16)$$

$R(\kappa)=-i2g(1+\lambda^2)\sqrt{\kappa^2-\omega^2}$. Therefore, by assuming that

$$\sqrt{\omega^2-\varkappa_\nu^2}=(\sqrt{-1})_\nu\sqrt{\varkappa_\nu^2-\omega^2}, \qquad (\sqrt{-1})_\nu=\begin{cases} i \text{ for } \kappa_1 \text{ and } \kappa_2 \\ -i \text{ for } \kappa_3 \text{ and } \kappa_4 \end{cases} \quad (5.17)$$

it can result that $R(\kappa)=-2g(1+\lambda^2)\sqrt{\omega^2-\kappa^2}$ and consequently,

$$\frac{1}{H}=-\sum_{\nu=1}^{4}T_\nu\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{(\omega\cos\alpha\mp\varkappa_\nu)P'(\varkappa_\nu)}, \qquad T_\nu=\frac{2g(1+\lambda^2)}{P'(\varkappa_\nu)}$$

$$\frac{\omega}{H}\begin{matrix}\cos\\ \sin\end{matrix}(\alpha)=-\sum_{\nu=1}^{4}T_\nu\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{(\omega\cos\alpha-\varkappa_\nu)P'(\varkappa_\nu)}\quad\frac{\varkappa_\nu}{\sqrt{\omega^2-\varkappa_\nu^2}} \quad (5.18)$$

Paired equalities are used here and subsequently. By differentiating first polynomial (5.11) over κ and substituting the values of κ_1 and κ_2 in $P'(\kappa)$ and utilizing equalities (5.12) and (5.13), it can be determined that

$$P'\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix}=\pm i\,\mathrm{sgn}\,\lambda\,8g(1+\lambda^2)\sqrt{\omega^2+(h+ig\sqrt{1+\lambda^2})^2}\left(h+\lambda\begin{matrix}\varkappa_1\\ \varkappa_2\end{matrix}\right)$$

$$T_\nu=\frac{(-1)^\nu i\,\mathrm{sgn}\,\lambda}{4(h+\lambda\varkappa_\nu)\sqrt{\omega^2+(h+ig\sqrt{1+\lambda^2})^2}} \qquad (\nu=1,\nu=2) \quad (5.19)$$

and, for $\nu=3$ and $\nu=4$, it is sufficient to change the sign in front of i.

Roots κ_ν can be complex. We will therefore consider separately the real and imaginary parts of integrals (5.7), based on the notations

$$F_\mu=F_{\mu c}+iF_{\mu s} \quad (5.20)$$

In using identities (5.18) in Eq. (5.7), in place of A and B, it is sufficient to substitute the numbers /96

$$A_\nu=g^2+2g\sqrt{\omega^2-\varkappa_\nu^2}+\omega^2+h^2,\ B_\nu=\lambda h-\varkappa_\nu \qquad (A_\nu C-B_\nu^2=0) \quad (5.21)$$

for which $C(A_\nu+xB_\nu)=B_\nu(B_\nu+xC)$. After noting the identities

$$\frac{\sqrt{\omega^2-\varkappa^2}+\omega\sin\alpha}{\omega\cos\alpha-\varkappa}\,\omega\cos\alpha=\varkappa\,\frac{\sqrt{\omega^2-\varkappa^2}+\omega\sin\alpha}{\omega\cos\alpha-\varkappa}+\sqrt{\omega^2-\varkappa^2}+\omega\sin\alpha$$

$$\frac{\sqrt{\omega^2-\varkappa^2}+\omega\sin\alpha}{\omega\cos\alpha-\varkappa}\,\omega\sin\alpha=\sqrt{\omega^2-\varkappa^2}\,\frac{\sqrt{\omega^2-\varkappa^2}+\omega\sin\alpha}{\omega\cos\alpha-\varkappa}-\omega\cos\alpha-\varkappa$$

and introducing the notation

$$\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{\omega\cos\alpha-\varkappa_\nu}\left(1-\frac{B_\nu+xC}{D\sqrt{C}}\right)\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha \tag{5.22}$$

from Eq. (5.7), it is easy to obtain the formulas

$$\frac{\mu}{\omega}\begin{matrix}F_{\mu c}\\ F_{\mu s}\end{matrix}=\mp\sum_{\nu=1}^{4}\frac{gC\varkappa_\nu-B_\nu\sqrt{\omega^2-\varkappa_\nu^2}}{\omega\sqrt{C}}T_\nu\begin{matrix}H_{\mu s}\\ H_{\mu c}\end{matrix}(\varkappa_\nu)\mp\sum_{\nu=1}^{4}\frac{T_\nu}{2\pi\sqrt{C}}\times$$
$$\times\int\limits_{-\pi}^{\pi}\left(gC\,\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{\omega}+B_\nu\frac{\varkappa_\nu+\omega\cos\alpha}{\omega}\right)\left(1-\frac{B_\nu+xC}{D\sqrt{C}}\right)\begin{matrix}\sin\\ \cos\end{matrix}(\mu\alpha)\,d\alpha \tag{5.23}$$

$$\begin{matrix}\partial F_{\mu c}/\partial g\\ \partial F_{\mu s}/\partial g\end{matrix}=\sqrt{C}\sum_{\nu=1}^{4}\left(g+\sqrt{\omega^2-\varkappa_\nu^2}\right)T_\nu\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu)-\frac{\sqrt{C}}{2\pi}\sum_{\nu=1}^{4}T_\nu\times$$
$$\times\int\limits_{-\pi}^{\pi}(\varkappa_\nu+\omega\cos\alpha)\left(1-\frac{B_\nu+xC}{D\sqrt{C}}\right)\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha \tag{5.24}$$

$$\begin{matrix}\partial F_{\mu c}/\partial h\\ \partial F_{\mu s}/\partial h\end{matrix}=\sum_{\nu=1}^{4}\frac{hC-\lambda B_\nu}{\sqrt{C}}T_\nu\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu) \tag{5.25}$$

$$\begin{matrix}\partial F_{\mu c}/\partial\omega\\ \partial F_{\mu s}/\partial\omega\end{matrix}=\sum_{\nu=1}^{4}\frac{C\left(\omega^2+g\sqrt{\omega^2-\varkappa_\nu^2}\right)+B_\nu\varkappa_\nu}{\omega\sqrt{C}}T_\nu\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu)-\sum_{\nu=1}^{4}\frac{T_\nu}{2\pi\sqrt{C}}\times$$
$$\times\int\limits_{-\pi}^{\pi}\left(gC\,\frac{\varkappa_\nu+\omega\cos\alpha}{\omega}-B_\nu\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{\omega}\right)\left(1-\frac{B_\nu+xC}{D\sqrt{C}}\right)\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha \tag{5.26}$$

We transform the algebraic factors in these equations. Equality $A_\nu C-B_\nu^2=0$, with Eq. (5.21) taken into consideration, can be presented as

$$\left(g+\sqrt{\omega^2-\varkappa_\nu^2}\right)^2(1+\lambda^2)+(h+\lambda\varkappa_\nu)^2=0 \tag{5.27}$$

According to Eq. (5.21) and (5.27),

$$hC-\lambda B_\nu=h+\lambda\varkappa_\nu,\quad g+\sqrt{\omega^2-\varkappa_\nu^2}=(\sqrt{-1})_\nu\frac{h+\lambda\varkappa_\nu}{\sqrt{1+\lambda^2}} \tag{5.28}$$

By applying Eq. (5.21) and (5.28),

$$gC\varkappa_\nu - B_\nu \sqrt{\omega^2 - \varkappa_\nu^2} = (h + \lambda\varkappa_\nu)\left\{\lambda g - (\sqrt{-1})_\nu \frac{\lambda h - \varkappa_\nu}{\sqrt{1+\lambda^2}}\right\} \tag{5.29}$$

can be obtained and, according to equations of the (5.12) and (5.28) type,

$$C(\omega^2 + g\sqrt{\omega^2 - \varkappa_\nu^2}) + B_\nu\varkappa_\nu = -(h + \lambda\varkappa_\nu)\{h + (\sqrt{-1})_\nu g\sqrt{1+\lambda^2}\} \tag{5.30}$$

Further, for $P(\kappa)=\kappa^n+a_1\kappa^{n-1}+\ldots+a_n$, if all roots of κ_ν are different,

$$\sum_{\nu=1}^{n} \frac{\varkappa_\nu^{n-k}}{P'(\varkappa_\nu)} = \begin{cases} 1 & (k=1) \\ 0 & (k=2,3,\ldots,n) \end{cases}$$

The use of this theorem for n=4 with account taken of determina- /97
tion of T_ν (5.18) simplifies the second sums of Eq. (5.23)-(5.26); only terms with factor (5.30) remain in Eq. (5.23), and only terms with factor (5.29) remain in Eq. (5.26). Therefore, with account taken of Eq. (5.28) with the notations

$$N_\nu = B_\nu / \sqrt{1+\lambda^2}, \qquad G_\nu = (h + \lambda\varkappa_\nu)\, T_\nu \tag{5.31}$$

the formulas

$$\frac{\mu}{\omega} \begin{matrix} F_{\mu c} \\ F_{\mu s} \end{matrix} = \mp \sum_{\nu=1}^{4} \frac{G_\nu}{\omega\sqrt{C}} \left[\lambda g - (\sqrt{-1})_\nu \frac{\lambda h - \varkappa_\nu}{\sqrt{C}}\right] \begin{matrix} H_{\mu s} \\ H_{\mu c} \end{matrix} (\varkappa_\nu) \pm$$

$$\pm \sum_{\nu=1}^{4} \frac{G_\nu}{\omega}\left[\frac{h}{\sqrt{C}} + (\sqrt{-1})_\nu g\right] \int_{-\pi}^{\pi} \left(1 - \frac{B_\nu + xC}{D\sqrt{C}}\right] \begin{matrix} \sin \\ \cos \end{matrix} (\mu\alpha) \frac{d\alpha}{2\pi}$$

$$\begin{matrix} \partial F_{\mu c} / \partial h \\ \partial F_{\mu s} / \partial h \end{matrix} = \sum_{\nu=1}^{4} \frac{G_\nu}{\sqrt{C}} \begin{matrix} H_{\mu c} \\ H_{\mu s} \end{matrix} (\varkappa_\nu), \qquad \begin{matrix} \partial F_{\mu c} / \partial g \\ \partial F_{\mu s} / \partial g \end{matrix} = \sum_{\nu=1}^{4} (\sqrt{-1})_\nu G_\nu \begin{matrix} H_{\mu c} \\ H_{\mu s} \end{matrix} (\varkappa_\nu)$$

$$\begin{matrix} \partial F_{\mu c} / \partial \omega \\ \partial F_{\mu s} / \partial \omega \end{matrix} = -\sum_{\nu=1}^{4} \frac{G_\nu}{\omega}\left[\frac{h}{\sqrt{C}} + (\sqrt{-1})_\nu g\right] \begin{matrix} H_{\mu c} \\ H_{\mu s} \end{matrix} (\varkappa_\nu) -$$

$$-\sum_{\nu=1}^{4} \frac{G_\nu}{\omega\sqrt{C}}\left[\lambda g - (\sqrt{-1})_\nu \frac{\lambda h - \varkappa_\nu}{\sqrt{C}}\right] \int_{-\pi}^{\pi} \left(1 - \frac{B_\nu + xC}{D\sqrt{C}}\right) \begin{matrix} \cos \\ \sin \end{matrix} (\mu\alpha) \frac{d\alpha}{2\pi}$$

are obtained and then, with account taken of the complex conjugate nature of the roots,

$$\mu \begin{matrix} F_{\mu c} \\ F_{\mu s} \end{matrix} = \mp \lambda g \begin{matrix} \partial F_{\mu s} / \partial h \\ \partial F_{\mu c} / \partial h \end{matrix} \mp \frac{2}{\sqrt{1+\lambda^2}} \sum_{\nu=1}^{2} \operatorname{Im} \left\{ G_\nu N_\nu \begin{matrix} H_{\mu s} \\ H_{\mu c} \end{matrix} (\varkappa_\nu) \right\} \pm$$

$$\pm 2g \int_{-\pi}^{\pi} \begin{matrix} \sin \\ \cos \end{matrix} (\mu\alpha) \frac{d\alpha}{2\pi D} \sum_{\nu=1}^{2} \operatorname{Im} \{ G_\nu (N_\nu + \varkappa \sqrt{1+\lambda^2}) \} \quad (\mu \neq 0) \tag{5.32}$$

$$\begin{matrix} \partial F_{\mu c} / \partial h \\ \partial F_{\mu s} / \partial h \end{matrix} = \frac{2}{\sqrt{1+\lambda^2}} \sum_{\nu=1}^{2} \operatorname{Re} \left\{ G_\nu \begin{matrix} H_{\mu c} \\ H_{\mu s} \end{matrix} (\varkappa_\nu) \right\} \tag{5.33}$$

$$\begin{matrix} \partial F_{\mu c} / \partial g \\ \partial F_{\mu s} / \partial g \end{matrix} = -2 \sum_{\nu=1}^{2} \operatorname{Im} \left\{ G_\nu \begin{matrix} H_{\mu c} \\ H_{\mu s} \end{matrix} (\varkappa_\nu) \right\} \tag{5.34}$$

$$\begin{matrix} \partial F_{\mu c} / \partial \omega \\ \partial F_{\mu s} / \partial \omega \end{matrix} = -\frac{h}{\omega} \begin{matrix} \partial F_{\mu c} / \partial h \\ \partial F_{\mu s} / \partial h \end{matrix} - \frac{g}{\omega} \begin{matrix} \partial F_{\mu c} / \partial g \\ \partial F_{\mu s} / \partial g \end{matrix} +$$

$$+ \frac{2}{\omega \sqrt{1+\lambda^2}} \int_{-\pi}^{\pi} \begin{matrix} \cos \\ \sin \end{matrix} (\mu\alpha) \frac{d\alpha}{2\pi D} \sum_{\nu=1}^{2} \operatorname{Im} \{ G_\nu N_\nu (N_\nu + \varkappa \sqrt{1+\lambda^2}) \} \quad (\mu \neq 0) \tag{5.35}$$

$$\frac{\partial F_0}{\partial \omega} = -\frac{h}{\omega} \frac{\partial F_0}{\partial h} - \frac{g}{\omega} \frac{\partial F_0}{\partial g} - \frac{2}{\omega \sqrt{1+\lambda^2}} \sum_{\nu=1}^{2} \operatorname{Im} \{ G_\nu N_\nu \} +$$

$$+ \frac{2}{\omega \sqrt{1+\lambda^2}} \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi D} \sum_{\nu=1}^{2} \operatorname{Im} \{ G_\nu N_\nu (N_\nu + \varkappa \sqrt{1+\lambda^2}) \} \tag{5.36}$$

We will therefore study roots κ_ν and functions of κ_ν only for $\nu=1$ and $\nu=2$.

Let a and b be real numbers. Then,

$$\sqrt{a+ib} = \frac{1}{\sqrt{2}} \left(\sqrt{\sqrt{a^2+b^2}+a} + i \sqrt{\sqrt{a^2+b^2}-a} \right) \qquad (b > 0)$$

Based on this, with the notation /98

$$\begin{matrix} M_1 \\ M_2 \end{matrix} = \frac{1}{\sqrt{2}} \{ \sqrt{[\omega^2 + h^2 - g^2(1+\lambda^2)]^2 + 4h^2 g^2 (1+\lambda^2)} \pm [\omega^2 + h^2 - g^2(1+\lambda^2)] \}^{1/2} \tag{5.37}$$

$$\sqrt{\omega^2 + (h + ig\sqrt{1+\lambda^2})^2} = M_1 + iM_2, \qquad M_1 M_2 = gh\sqrt{1+\lambda^2} \tag{5.38}$$

Equations (5.13) and (5.16) can now be presented as

$$\begin{matrix} \varkappa_1 \\ \varkappa_2 \end{matrix} = \lambda (h + ig\sqrt{1+\lambda^2}) \pm \operatorname{sgn} \lambda \sqrt{1+\lambda^2} (M_1 + iM_2) \tag{5.39}$$

$$\begin{matrix} \sqrt{\varkappa_1^2 - \omega^2} \\ \sqrt{\varkappa_2^2 - \omega^2} \end{matrix} = \sqrt{1+\lambda^2} (h + ig\sqrt{1+\lambda^2}) \pm |\lambda| (M_1 + iM_2) \tag{5.40}$$

and Eq. (5.31), with Eq. (5.21) and (5.19) taken into account, take the form

$$\begin{matrix} N_1 \\ N_2 \end{matrix} = -i\lambda g \mp \operatorname{sgn}\lambda\,(M_1 + iM_2) \tag{5.41}$$

$$G_\nu = \frac{(-1)^\nu i \operatorname{sgn}\lambda}{4\sqrt{\omega^2 + (h + ig\sqrt{1+\lambda^2})^2}} = \frac{(-1)^\nu \operatorname{sgn}\lambda\,(M_2 + iM_1)}{4\sqrt{[\omega^2 + h^2 - g^2(1+\lambda^2)]^2 + 4h^2g^2(1+\lambda^2)}} \tag{5.42}$$

By assuming $e^{i\alpha}=\zeta$ and applying the method of subtractions,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\mu\alpha\,d\alpha}{\cos\alpha - y} = \frac{1}{\pi i}\oint\frac{\zeta^\mu\,d\zeta}{\zeta^2 - 2y\zeta + 1} = \frac{(\varepsilon\sqrt{y^2-1}+y)^\mu}{\varepsilon\sqrt{y^2-1}} \quad (\mu = 0, 1, 2, \ldots) \tag{5.43}$$

can be found, where $\varepsilon=\pm 1$, in which the sign is selected from the condition $|\varepsilon\sqrt{y^2-1}+y|<1$.

Since $2\sin\mu\alpha\sin\alpha = \cos(\mu-1)\alpha - \cos(\mu+1)\alpha$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\mu\alpha\sin\alpha}{\cos\alpha - y}\,d\alpha = \begin{cases} 0 & (\mu = 0) \\ -(\varepsilon\sqrt{y^2-1}+y)^\mu & (\mu = 1, 2, \ldots) \end{cases} \tag{5.44}$$

Consequently, the first part of integral (5.22) will be

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{i\sqrt{\varkappa_\nu^2-\omega^2}+\omega\sin\alpha}{\omega\cos\alpha - \varkappa_\nu}\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha = \begin{cases} \begin{matrix} i\varepsilon \\ 0 \end{matrix} & (\mu = 0) \\ \pm\begin{matrix} i\varepsilon \\ 1 \end{matrix}\,\Phi^\mu(\varkappa_\nu) & (\mu = 1, 2, \ldots) \end{cases} \tag{5.45}$$

where

$$\Phi(\varkappa_\nu) = \frac{\varepsilon\sqrt{\varkappa_\nu^2-\omega^2}+\varkappa_\nu}{\omega} = \Phi_\nu \tag{5.46}$$

and ε is selected from the condition $|\Phi(\kappa_\nu)|<1$. According to Eq. (5.39) and (5.40),

$$\omega\Phi\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix} = (\varepsilon\sqrt{1+\lambda^2}+\lambda)\,[h + ig\sqrt{1+\lambda^2} \pm \varepsilon\operatorname{sgn}\lambda\,(M_1 + iM_2)] \tag{5.47}$$

Further, expression (5.6) can be presented as

$$D^2 = \omega^2 - 2\omega L\cos(\alpha - \sigma) + L^2 + (h + \lambda x)^2 \tag{5.48}$$

$$L = \sqrt{x^2 + g^2}, \quad \cos\sigma = x/L, \quad \sin\sigma = -g/L$$

Let $\theta=\alpha-\sigma$. Then,

$$D^2=\omega^2-2\omega L\cos\vartheta+L^2+(h+\lambda x)^2 \tag{5.49}$$

$$\int_{-\pi}^{\pi}\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{(\omega\cos\alpha-\varkappa_\nu)D}\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha=\int_{-\pi}^{\pi}\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin(\vartheta+\sigma)}{[\omega\cos(\vartheta+\sigma)-\varkappa_\nu]D}\begin{matrix}\cos\\ \sin\end{matrix}[\mu(\vartheta+\sigma)]\,d\vartheta$$

It can be proved in succession that /99

$$\frac{1}{\omega\cos(\vartheta+\sigma)-\varkappa}=\frac{\omega\cos\sigma\cos\vartheta-\varkappa+\omega\sin\sigma\sin\vartheta}{(\omega\cos\vartheta)^2-2\varkappa\cos\sigma(\omega\cos\vartheta)+\varkappa^2-\omega^2\sin^2\sigma}$$

$$\frac{\sqrt{\omega^2-\varkappa^2}+\omega\sin(\vartheta+\sigma)}{\omega\cos(\vartheta+\sigma)-\varkappa}=\frac{\sqrt{\omega^2-\varkappa^2}\cos\sigma-\varkappa\sin\sigma+\omega\sin\vartheta}{\omega\cos\vartheta-\varkappa\cos\sigma-\sqrt{\omega^2-\varkappa^2}\sin\sigma}.$$

On this basis, the second part of expression (5.22) will be

$$\int_{-\pi}^{\pi}\frac{\sqrt{\omega^2-\varkappa_\nu^2}+\omega\sin\alpha}{2\pi(\omega\cos\alpha-\varkappa_\nu)D}\begin{matrix}\cos\\ \sin\end{matrix}(\mu\alpha)\,d\alpha=$$

$$=\mp\begin{matrix}\sin\\ \cos\end{matrix}(\mu\sigma)\frac{\omega}{\pi}\int_{0}^{\pi}\frac{\sin\mu\vartheta\sin\vartheta\,d\vartheta}{(\omega\cos\vartheta-\varkappa_\nu\cos\sigma-\sqrt{\omega^2-\varkappa_\nu^2}\sin\sigma)D}+$$

$$+\begin{matrix}\cos\\ \sin\end{matrix}(\mu\sigma)\frac{\sqrt{\omega^2-\varkappa_\nu^2}\cos\sigma-\varkappa_\nu\sin\sigma}{\omega}\int_{0}^{\pi}\frac{\cos\mu\vartheta\,d\vartheta}{(\omega\cos\vartheta-\varkappa_\nu\cos\sigma-\sqrt{\omega^2-\varkappa_\nu^2}\sin\sigma)D}$$

As a result, after substitution of $\theta=\pi-2\phi$,

$$H_{0s,\nu}\equiv 0,\quad \begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu)=\pm\begin{matrix}ie\\ 1\end{matrix}\Phi^{\mu}(\varkappa_\nu)-(-1)^{\mu}\frac{p_\nu(N_\nu+x\sqrt{1+\lambda^2})}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}}\times$$
$$\times\left\{\begin{matrix}\cos\\ \sin\end{matrix}(\mu\sigma)\frac{\sqrt{\omega^2-\varkappa_\nu^2}\cos\sigma-\varkappa_\nu\sin\sigma}{\omega}\Pi_\mu(k,p_\nu)\pm\begin{matrix}\sin\\ \cos\end{matrix}(\mu\sigma)\Pi_\mu^{\Delta}(k,p_\nu)\right\} \tag{5.50}$$

is obtained, where

$$k^2=\frac{4\omega L}{(\omega+L)^2+(h+\lambda x)^2},\quad p_\nu=-\frac{2\omega}{\omega+\varkappa_\nu\cos\sigma+\sqrt{\omega^2-\varkappa_\nu^2}\sin\sigma} \tag{5.51}$$

$$\Pi_\mu(k,p)=\int_{0}^{\pi/2}\frac{\cos 2\mu\varphi\,d\varphi}{(1+p\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}},\quad \Pi_\mu^{\Delta}(k,p)=\frac{\Pi_{\mu-1}(k,p)-\Pi_{\mu+1}(k,p)}{2} \tag{5.52}$$

Function $\Pi_\mu(k, p)$ is a particular case of the generalized elliptical integral introduced and studied by the author. In the Appendix, recurrent formula

$$\tfrac{1}{2}\{\Pi_{\mu+1}(k,p)+\Pi_{\mu-1}(k,p)\}=(1+2/p)\,\Pi_\mu(k,p)-(2/p)\,E_\mu^{(-1/2)}(k) \tag{5.53}$$

was obtained and convenient formulas for calculation of $\Pi_0(k, p)$ are given for any (real and complex) value of parameter p.

Further, based on Eq. (5.48) and with the use of the substitution $\alpha-\sigma=\pi-2\phi$, we obtain with module k^2 (5.51)

$$\int_{-\pi}^{\pi} \frac{\cos}{\sin}(\mu\alpha)\frac{d\alpha}{2\pi D} = \frac{2}{\pi}\frac{(-1)^{\mu}E_{\mu}^{(-1/2)}(k)}{\sqrt{(\omega+L)^2+(h+\lambda x)^2}}\frac{\cos}{\sin}(\mu\sigma) \tag{5.54}$$

In the case of g=0, when $M_1=\sqrt{\omega^2+h^2}$ and $M_2\equiv 0$

$$\Phi_1 = \operatorname{sgn}\lambda\frac{\sqrt{\omega^2+h^2}-h}{\omega}\left(\sqrt{1+\lambda^2}-|\lambda|\right), \quad \varepsilon=-\operatorname{sgn}\lambda \tag{5.55}$$

$$\Phi_2 = \begin{cases} -\operatorname{sgn}\lambda\dfrac{\sqrt{\omega^2+h^2}+h}{\omega}\left(\sqrt{1+\lambda^2}-|\lambda|\right), & \varepsilon=-\operatorname{sgn}\lambda, \ (h^2\leqslant\omega^2\lambda^2) \\ -\operatorname{sgn}\lambda\dfrac{\sqrt{\omega^2+h^2}-h}{\omega}\left(\sqrt{1+\lambda^2}+|\lambda|\right), & \varepsilon=\operatorname{sgn}\lambda, \quad (h^2\geqslant\omega^2\lambda^2) \end{cases} \tag{5.56}$$

$$H_{\mu c}(\varkappa_\nu) = i\varepsilon\Phi_\nu^{\mu} + i\frac{\sqrt{\varkappa_\nu^2-\omega^2}}{\omega}\frac{(-\operatorname{sgn}x)^{\mu+1}p_\nu\left(N_\nu+x\sqrt{1+\lambda^2}\right)}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}}\Pi_\mu(k, p_\nu)$$

$$H_{\mu s}(\varkappa_\nu) = -\Phi_\nu^{\mu} + \frac{(-\operatorname{sgn}x)^{\mu}p_\nu\left(N_\nu+x\sqrt{1+\lambda^2}\right)}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}}\Pi_\mu^{\Delta}(k, p_\nu),\ H_{0s,\nu}\equiv 0 \tag{5.57}$$

$$k^2=\frac{4\omega L}{(\omega+L)^2+(h+\lambda x)^2}, \quad \frac{p_1}{p_2} = \mp\frac{2\omega\operatorname{sgn}\lambda\operatorname{sgn}x}{\sqrt{\omega^2+h^2}\sqrt{1+\lambda^2}\pm\left[|\lambda h|+\omega\operatorname{sgn}\lambda\operatorname{sgn}x\right]}$$

and with the notations /100

$$\frac{\xi_1}{\xi_2} = \frac{x\sqrt{1+\lambda^2}\mp\operatorname{sgn}\lambda\sqrt{\omega^2+h^2}}{\sqrt{\omega^2+h^2}\sqrt{1+\lambda^2}\pm\left[|\lambda h|+\omega\operatorname{sgn}\lambda\operatorname{sgn}x\right]} \tag{5.58}$$

$$\frac{Z_1}{Z_2} = h\sqrt{1+\lambda^2}\pm|\lambda|\sqrt{\omega^2+h^2}$$

the following final formulas are obtained from Eq. (5.32)-(5.36):

$$\mu F_{\mu c} = \frac{\Phi_1^{\mu}+\Phi_2^{\mu}}{2\sqrt{1+\lambda^2}} - \frac{(-\operatorname{sgn}x)^{\mu+1}\operatorname{sgn}\lambda}{\sqrt{1+\lambda^2}}\omega\frac{\xi_1\Pi_\mu^{\Delta}(k, p_1)-\xi_2\Pi_\mu^{\Delta}(k, p_2)}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}}$$

$$\frac{\partial F_{\mu c}}{\partial h} = -\frac{\Phi_1^{\mu}+\operatorname{sgn}(h^2-\omega^2\lambda^2)\Phi_2^{\mu}}{2\sqrt{1+\lambda^2}\sqrt{\omega^2+h^2}} + \frac{(-\operatorname{sgn}x)^{\mu}}{\sqrt{1+\lambda^2}\sqrt{\omega^2+h^2}}\frac{Z_1\xi_1\Pi_\mu(k, p_1)+Z_2\xi_2\Pi_\mu(k, p_2)}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}} \tag{5.59}$$

$$\frac{\partial F_{\mu s}}{\partial g} = -\operatorname{sgn}\lambda\frac{\Phi_1^{\mu}-\Phi_2^{\mu}}{2\sqrt{\omega^2+h^2}} + \frac{(-\operatorname{sgn}x)^{\mu+1}}{\sqrt{\omega^2+h^2}}\omega\frac{\xi_1\Pi_\mu^{\Delta}(k, p_1)+\xi_2\Pi_\mu^{\Delta}(k, p_2)}{\pi\sqrt{(\omega+L)^2+(h+\lambda x)^2}}$$

$$\frac{\partial F_{\mu c}}{\partial \omega} = -\frac{h}{\omega}\frac{\partial F_{\mu c}}{\partial h} + \frac{2}{\pi}\frac{(-\operatorname{sgn} x)^{\mu} x/\omega}{\sqrt{\omega + L)^2 + (h + \lambda x)^2}} E_{\mu}^{(-1/2)}(k)$$

$$\frac{\partial F_0}{\partial \omega} = -\frac{h}{\omega}\frac{\partial F_0}{\partial h} + \frac{2}{\pi}\frac{x/\omega}{\sqrt{(\omega + L)^2 + (h + \lambda x)^2}} E_0^{(-1/2)}(k) - \frac{1}{\omega\sqrt{1+\lambda^2}}$$

The remaining functions are absent. Since parameters p_ν(5.57) satisfy the condition $1+p_\nu>0$, the corresponding integrals of the third kind of $\Pi(k, p_\nu)$ can be expressed through integrals of $F(k, \alpha)$ and $E(k, \alpha)$ by Eq. (6.1)-(6.4) specified below.

If $x=0$, $D=\sqrt{\omega^2+h^2}$ and, in Eq. (5.22),

$$1 - \frac{B_\nu + xC}{D\sqrt{C}} = 1 - \frac{N_\nu + x\sqrt{1+\lambda^2}}{D} = 1 - (-1)^\nu \operatorname{sgn}\lambda$$

There are obtained as a result the simple formulas

$$\begin{aligned}
&2\sqrt{1+\lambda^2}\,\mu F_{\mu c} = (1 + \operatorname{sgn}\lambda)\,\Phi_1^{\mu} + (1 - \operatorname{sgn}\lambda)\Phi_2^{\mu} \\
&2\sqrt{1+\lambda^2}\sqrt{\omega^2+h^2}\,\frac{\partial F_{\mu c}}{\partial h} = -(1+\operatorname{sgn}\lambda)\,\Phi_1^{\mu} - \operatorname{sgn}(h^2 - \omega^2\lambda^2)\,(1-\operatorname{sgn}\lambda)\Phi_2^{\mu} \\
&\operatorname{sgn}\lambda\, 2\sqrt{\omega^2+h^2}\,\frac{\partial F_{\mu s}}{\partial g} = -(1+\operatorname{sgn}\lambda)\Phi_1^{\mu} + (1-\operatorname{sgn}\lambda)\Phi_2^{\mu} \\
&\frac{\partial F_{\mu c}}{\partial \omega} = -\frac{h}{\omega}\frac{\partial F_{\mu c}}{\partial h}, \qquad \frac{\partial F_0}{\partial \omega} = -\frac{h}{\omega}\frac{\partial F_0}{\partial h} - \frac{1}{\omega\sqrt{1+\lambda^2}}
\end{aligned} \tag{5.60}$$

The remaining functions are absent.

Equations (5.60) are simplified in the important case of $x=0=h$ with $\lambda \geqslant 0$, /101

$$\begin{aligned}
&\mu F_{\mu c} = \frac{\Phi_1^{\mu}}{\sqrt{1+\lambda^2}}, \qquad \frac{\partial F_{\mu s}}{\partial g} = -\frac{\Phi_1^{\mu}}{\omega}, \qquad \Phi_1 = \sqrt{1+\lambda^2} - \lambda \\
&\frac{\partial F_{\mu c}}{\partial \omega} = \begin{cases} -1/\omega\sqrt{1+\lambda^2} & (\mu = 0) \\ 0 & (\mu = 1, 2, \ldots) \end{cases}
\end{aligned} \tag{5.61}$$

In considering the case of $x=0=h$, it must be noted that it is evident from Eq. (5.2) that, for $h=0$, functions F_μ, $\partial F_\mu/\partial g$, $\partial F_\mu/\partial\omega$ and $\lambda\partial F_\mu/\partial h$ do not depend on the sign of λ. Functions only of $\lambda>0$ will therefore be considered subsequently.

According to Eq. (5.48), for x=0, L=g, cos σ=0, sin σ=-1; therefore,

$$e^{i\mu\sigma} = (-i)^{\mu}, \qquad \begin{matrix}\cos\\ \sin\end{matrix}(\mu\sigma) = (-1)^{\mu}\begin{matrix}\cos\\ \sin\end{matrix}\left(\mu\frac{\pi}{2}\right) \tag{5.62}$$

$$\begin{gathered} H_{0s}(\varkappa_\nu) \equiv 0, \qquad \begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_\nu) = \pm \begin{matrix}i\varepsilon\\ 1\end{matrix}\,\Phi^{\mu}(\varkappa_\nu) - \\ - \frac{p_\nu N_\nu}{\pi(\omega+L)}\left\{\begin{matrix}\cos\\ \sin\end{matrix}\left(\mu\frac{\pi}{2}\right)\frac{\varkappa_\nu}{\omega}\Pi_\mu(k, p_\nu) \pm \begin{matrix}\sin\\ \cos\end{matrix}\left(\mu\frac{\pi}{2}\right)\Pi_\mu{}^{\Delta}(k, p_\nu)\right\} \\ k^2 = \frac{4\omega L}{(\omega+L)^2}, \qquad p_\nu = -\frac{2\omega}{\omega - \sqrt{\omega^2 - \varkappa_\nu^2}} = -\frac{2\omega}{\omega - i\sqrt{\varkappa_\nu^2 - \omega^2}} \end{gathered} \tag{5.63}$$

It is useful here to use the transformation established by the author (see Appendix)

$$\begin{gathered} \Pi_0(k, p_\nu) = -\frac{1+k_*}{2}\,\frac{p_\nu}{p_\nu+1}\left\{\left(\frac{k_*}{\ } - 1 - \frac{2}{p_\nu}\right)\Pi_0(k_*, p_*) - \frac{k_*}{p_*}F(k_*)\right\} \\ k_* = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{k^2}{(1+\sqrt{1-k^2})^2}, \quad p_* = \frac{1-2k_*\gamma+k_*^2}{\gamma^2-1}, \quad \gamma = -\left(1+\frac{2}{p_\nu}\right) \end{gathered} \tag{5.64}$$

According to Eq. (5.63),

$$\gamma = -\frac{\sqrt{\omega^2-\varkappa_\nu^2}}{\omega}, \qquad \gamma^2 - 1 = -\frac{\varkappa_\nu^2}{\omega^2}, \qquad k_* = \begin{cases}\omega/L \leqslant 1\\ L/\omega \leqslant 1\end{cases} \tag{5.65}$$

and, for g=L and h=0, with Eq. (5.27) taken into account, the result is

$$p_* = \begin{cases} -\dfrac{\omega^2 + 2L\sqrt{\omega^2-\varkappa_\nu^2} + L^2}{\varkappa_\nu^2 L^2/\omega^2} = -\dfrac{\omega^2/L^2}{1+\lambda^2} & (\omega \leqslant L) \\ -\dfrac{\omega^2 + 2L\sqrt{\omega^2-\varkappa_\nu^2} + L^2}{\varkappa_\nu^2} = -\dfrac{1}{1+\lambda^2} & (\omega \geqslant L) \end{cases} \tag{5.66}$$

Parameter p_* turns out to be real and the same for all κ_ν. Moreover, for x=0=h, function F_μ and its derivatives have two branches each

$$\text{(a) } L^2(1+\lambda^2) - \omega^2 > 0, \qquad \text{(b) } L^2(1+\lambda^2) - \omega^2 < 0 \tag{5.67}$$

where quantities κ_ν and p_ν are complex in case (b). An extremely simple case of change to integrals of the third kind with a real parameter thus occurs here (see Appendix).

In case (a)

$$M_1 = 0, \qquad M_2 = \sqrt{L^2(1+\lambda^2) - \omega^2}, \qquad \genfrac{}{}{0pt}{}{p_1}{p_2} = -\frac{2\omega}{\omega + L(1+\lambda^2) \pm \lambda M_2}$$
$$\omega\Phi\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix} = i\left(\varepsilon\sqrt{1+\lambda^2} + \lambda\right)\left(L\sqrt{1+\lambda^2} \pm \varepsilon M_2\right) \tag{5.68}$$

Since $L\sqrt{1+\lambda^2}=\sqrt{M_2{}^2+\omega^2}$, it follows from condition $|\Phi(\kappa_\nu)|<1$ that

$$\Phi(\varkappa_1) = -i\,\frac{L\sqrt{1+\lambda^2} - M_2}{\omega}\left(\sqrt{1+\lambda^2} - \lambda\right) \qquad \varepsilon = -1$$
$$\Phi(\varkappa_2) = \begin{cases} -i\,\dfrac{L\sqrt{1+\lambda^2} + M_2}{\omega}\left(\sqrt{1+\lambda^2} - \lambda\right), & \varepsilon = -1 \quad (\omega\lambda > M_2) \\ i\,\dfrac{L\sqrt{1+\lambda^2} - M_2}{\omega}\left(\sqrt{1+\lambda^2} + \lambda\right), & \varepsilon = 1 \quad (\omega\lambda < M_2) \end{cases} \tag{5.69}$$

However, $M_2{}^2-\omega^2\lambda^2=(L^2-\omega^2)(1+\lambda^2)$. Consequently, /102

$$\Phi(\varkappa_\nu) = i\varepsilon_\nu|\Phi(\varkappa_\nu)|, \qquad \varepsilon_\nu = -\operatorname{sgn}\left[\omega - (-1)^\nu L\right] \tag{5.70}$$

and, for $L>\omega$, the first branch of $\Phi(\kappa_2)$ is missing. Parameters p_ν (5.68) are real, and transformation formula (5.64) contains only real quantities. We designate

$$\genfrac{}{}{0pt}{}{\gamma_1}{\gamma_2} = \frac{2}{\pi}\,\frac{\sqrt{1+\lambda^2}}{\omega + L}\,\frac{(M_2 \pm \lambda L)^2}{\omega + L(1+\lambda^2) \pm \lambda M_2}$$
$$\genfrac{}{}{0pt}{}{\delta_1}{\delta_2} = \frac{2}{\pi}\,\frac{\omega}{\omega + L}\,\frac{M_2 \pm \lambda L}{\omega + L(1+\lambda^2) \pm \lambda M_2} \qquad \genfrac{}{}{0pt}{}{a_1}{a_2} = M_2 \pm \lambda L \tag{5.71}$$

From Eq. (5.63), it is easy to obtain

$$H_{\mu c}\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix} = \left(i\,\genfrac{}{}{0pt}{}{\varepsilon_1}{\varepsilon_2}\right)^{\mu+1}\left|\Phi\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix}\right|^\mu + \cos\left(\mu\frac{\pi}{2}\right)\genfrac{}{}{0pt}{}{\gamma_1}{\gamma_2}\,\Pi_\mu\left(k, \genfrac{}{}{0pt}{}{p_1}{p_2}\right) \mp i\sin\left(\mu\frac{\pi}{2}\right)\genfrac{}{}{0pt}{}{\delta_1}{\delta_2}\,\Pi_\mu{}^\Delta\left(k, \genfrac{}{}{0pt}{}{p_1}{p_2}\right)$$
$$H_{\mu s}\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix} = -\left(i\,\genfrac{}{}{0pt}{}{\varepsilon_1}{\varepsilon_2}\right)^{\mu}\left|\Phi\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix}\right|^\mu + \sin\left(\mu\frac{\pi}{2}\right)\genfrac{}{}{0pt}{}{\gamma_1}{\gamma_2}\,\Pi_\mu\left(k, \genfrac{}{}{0pt}{}{p_1}{p_2}\right) \pm i\cos\left(\mu\frac{\pi}{2}\right)\genfrac{}{}{0pt}{}{\delta_1}{\delta_2}\,\Pi_\mu{}^\Delta\left(k, \genfrac{}{}{0pt}{}{p_1}{p_2}\right)$$

The final formulas will be: for $\mu=1, 3, 5, \ldots,$

$$\mu F_{\mu c} = -\lambda L\frac{\partial F_{\mu s}}{\partial h} + (-1)^{\frac{\mu+1}{2}}\left\{\frac{a_1\gamma_1\Pi_\mu(k_1,p_1) + a_2\gamma_2\Pi_\mu(k,p_2)}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}} - \frac{2}{\pi}\,\frac{L}{\omega + L}\,E_\mu{}^{(-1/2)}(k)\right\}$$
$$\mu F_{\mu s} = \lambda L\frac{\partial F_{\mu c}}{\partial h} + (-1)^{\frac{\mu+1}{2}}\,\frac{a_1|\Phi_1|^\mu + a_2|\Phi_2|^\mu}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}$$
$$\frac{\partial F_{\mu c}}{\partial h} = (-1)^{\frac{\mu+1}{2}}\,\frac{-|\Phi_1|^\mu + |\Phi_2|^\mu}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}} \tag{5.72}$$
$$\frac{\partial F_{\mu s}}{\partial h} = (-1)^{\frac{\mu+1}{2}}\,\frac{\gamma_1\Pi_\mu(k,p_1) - \gamma_2\Pi_\mu(k,p_2)}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}$$
$$\frac{\partial F_{\mu c}}{\partial g} = (-1)^{\frac{\mu+1}{2}}\,\frac{\delta_1\Pi_\mu^\Delta(k,p_1) + \delta_2\Pi_\mu{}^\Delta(k,p_2)}{2\sqrt{L^2(1+\lambda^2)-\omega^2}}, \qquad \frac{\partial F_{\mu s}}{\partial g} = (-1)^{\frac{\mu+1}{2}}\,\frac{\varepsilon_1|\Phi_1|^\mu - \varepsilon_2|\Phi_2|^\mu}{2\sqrt{L^2(1+\lambda^2)-\omega^2}}$$

and for μ=0, 2, 4, . . .,

$$\mu F_{\mu c}=-\lambda L\frac{\partial F_{\mu s}}{\partial h}+(-1)^{\frac{\mu}{2}}\frac{a_1|\Phi_1|^{\mu}+a_2|\Phi_2|^{\mu}}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}\qquad(\mu\neq 0)$$

$$\mu F_{\mu s}=\lambda L\frac{\partial F_{\mu c}}{\partial h}+(-1)^{\frac{\mu}{2}}\left\{\frac{a_1\gamma_1\Pi_{\mu}(k,p_1)+a_2\gamma_2\Pi_{\mu}(k,p_2)}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}-\frac{2}{\pi}\frac{L}{(\omega+L)}E_{\mu}^{(-1/2)}(k)\right\}$$

$$\frac{\partial F_{\mu c}}{\partial h}=(-1)^{\frac{\mu}{2}}\frac{-\gamma_1\Pi_{\mu}(k,p_1)+\gamma_2\Pi_{\mu}(k,p_2)}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}\qquad(\mu\neq 0)\qquad(5.73)$$

$$\frac{\partial F_{\mu s}}{\partial h}=(-1)^{\frac{\mu}{2}}\frac{|\Phi_1|^{\mu}-|\Phi_2|^{\mu}}{2\sqrt{1+\lambda^2}\sqrt{L^2(1+\lambda^2)-\omega^2}}$$

$$\frac{\partial F_{\mu c}}{\partial g}=(-1)^{\frac{\mu}{2}}\frac{\varepsilon_1|\Phi_1|^{\mu}-\varepsilon_2|\Phi_2|^{\mu}}{2\sqrt{L^2(1+\lambda^2)-\omega^2}}$$

$$\frac{\partial F_{\mu s}}{\partial g}=(-1)^{\frac{\mu}{2}}\frac{\delta_1\Pi_{\mu}{}^{\Delta}(k,p_1)+\delta_2\Pi_{\mu}{}^{\Delta}(k,p_2)}{2\sqrt{L^2(1+\lambda^2)-\omega^2}}$$

Moreover, /103

$$\begin{matrix}\partial F_{\mu c}/\partial\omega\\ \partial F_{\mu s}/\partial\omega\end{matrix}=-\frac{L}{\omega}\begin{matrix}\partial F_{\mu c}/\partial g\\ \partial F_{\mu s}/\partial g\end{matrix}\quad(\mu=1,2,3,\ldots),\qquad\frac{\partial F_0}{\partial\omega}=-\frac{L}{\omega}\frac{\partial F_0}{\partial g}-\frac{1}{\omega\sqrt{1+\lambda^2}}\qquad(5.74)$$

In case (b),

$$M_1=\sqrt{\omega^2-L^2(1+\lambda^2)},\qquad M_2=0$$

$$\begin{matrix}\varkappa_1\\ \varkappa_2\end{matrix}=(i\lambda L\pm M_1)\sqrt{1+\lambda^2}=-\sqrt{1+\lambda^2}\begin{matrix}N_1\\ N_2\end{matrix}\qquad(5.75)$$

$$\mu\begin{matrix}F_{\mu c}\\ F_{\mu s}\end{matrix}=\mp\lambda L\begin{matrix}\partial F_{\mu s}/\partial h\\ \partial F_{\mu c}/\partial h\end{matrix}\pm\frac{1}{1+\lambda^2}\sum_{\nu=1}^{2}\frac{(-1)^{\nu}}{2M_1}\mathrm{Re}\left\{\varkappa_{\nu}\begin{matrix}H_{\mu s}\\ H_{\mu c}\end{matrix}(\varkappa_{\nu})\right\}\pm$$

$$\pm\frac{2}{\pi}\begin{matrix}\sin\\ \cos\end{matrix}\left(\mu\frac{\pi}{2}\right)\frac{L}{\omega+L}E_{\mu}^{(-1/2)}(k)\qquad(5.76)$$

$$\begin{matrix}\partial F_{\mu c}/\partial h\\ \partial F_{\mu s}/\partial h\end{matrix}=-\frac{1}{\sqrt{1+\lambda^2}}\sum_{\nu=1}^{2}\frac{(-1)^{\nu}}{2M_1}\mathrm{Im}\left\{\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_{\nu})\right\}$$

$$\begin{matrix}\partial F_{\mu c}/\partial g\\ \partial F_{\mu s}/\partial g\end{matrix}=-\sum_{\nu=1}^{2}\frac{(-1)^{\nu}}{2M_1}\mathrm{Re}\left\{\begin{matrix}H_{\mu c}\\ H_{\mu s}\end{matrix}(\varkappa_{\nu})\right\}$$

$$\begin{matrix}\partial F_{\mu c}/\partial\omega\\ \partial F_{\mu s}/\partial\omega\end{matrix}=-\frac{L}{\omega}\begin{matrix}\partial F_{\mu c}/\partial g\\ \partial F_{\mu s}/\partial g\end{matrix},\qquad\frac{\partial F_0}{\partial\omega}=-\frac{L}{\omega}\frac{\partial F_0}{\partial g}-\frac{1}{\omega\sqrt{1+\lambda^2}}$$

$$\omega\,\Phi\begin{pmatrix}\varkappa_1\\ \varkappa_2\end{pmatrix}=(\sqrt{1+\lambda^2}-\lambda)(\pm M_1-iL\sqrt{1+\lambda^2})$$

$$\varepsilon=-1,\quad|\Phi(\varkappa_{\nu})|=\sqrt{1+\lambda^2}-\lambda\qquad(5.77)$$

$$\begin{matrix}p_1\\ p_2\end{matrix}=-\frac{2\omega}{\omega+L(1+\lambda^2)\mp i\lambda M_1}=$$

$$=\frac{2\omega}{(\omega+L)^2}\,\frac{-[\omega+L(1+\lambda^2)]\mp i\lambda M_1}{1+\lambda^2}=p^{(1)}\pm ip^{(2)}\qquad(5.78)$$

It is difficult however to separate the real and imaginary parts of functions $H_{\mu c}$ and $H_{\mu s}$ in general form (for any number μ) in Eq. (5.63) since, for a specific number μ, it is initially necessary to express $\Pi_\mu(k, p_\nu)$ through $\Pi_0(k, p_\nu)$ by means of Eq. (5.53) and to use transformation formula (5.64) with complex p_ν.

6. Appendix

In the preceding text, complete elliptical integrals of the third kind of $\Pi(k, p)$ are found, parameter p of which is complex or changes from -1 to ∞. If $p>-1$, $\Pi(k, p)$ can be expressed, based on the Legendre equation [4, pp. 402-403], by elliptical integrals of the first and second kinds of $F(k, \beta)$ and $E(k,\beta)$. With the notations

$$Q_1(k, \beta) = \frac{\pi}{2} - F(k)E(k', \beta) - E(k)F(k', \beta) + F(k)F(k', \beta) \tag{6.1}$$

$$Q_2(k, \beta) = F(k)E(k, \beta) - E(k)F(k, \beta) \qquad (k' = \sqrt{1-k^2})$$

$$\Pi(k, p) = \sqrt{\frac{p}{1+p}}\frac{Q_1(k, \beta)}{\sqrt{k^2+p}} + \frac{F(k)}{1+p}, \qquad \sin\beta = \frac{1}{\sqrt{1+p}} \qquad (p>0) \tag{6.2}$$

$$\Pi(k, p) = \begin{cases} \sqrt{\dfrac{-p}{1+p}}\dfrac{Q_2(k, \beta)}{\sqrt{k^2+p}} + F(k), & \sin\beta = \dfrac{\sqrt{-p}}{k} \quad (0<-p<k^2) \\ \sqrt{\dfrac{-p}{1+p}}\dfrac{Q_1(k, \beta)}{\sqrt{-(p+k^2)}} + F(k), & \sin\beta = \sqrt{\dfrac{1+p}{1-k^2}} \quad (k^2<-p<1) \end{cases} \tag{6.3}$$

According to Eq. (3.45), in the intermediate case ($p=-k^2$), it will be /104

$$\Pi(k, -k^2) = \int_0^{\pi/2} \frac{d\varphi}{(1-k^2\sin^2\varphi)^{3/2}} = \frac{E(k)}{1-k^2} \tag{6.4}$$

It is easy to see that $Q_2(0, \beta)=0$, $Q_2(1, \beta)=\infty$,

$$Q_1(k, \beta) = \begin{cases} {}^1/_2\pi - \beta & (k=1) \\ {}^1/_2\pi(1-\sin\beta) & (k=0) \end{cases} \qquad 0 \leqslant Q_1(k, \beta) \leqslant \pi/2 \tag{6.5}$$

The case of a complex p can be reduced to the case of a real one by means of the transformation found by the author. The results will be presented briefly here.

For transformations, in place of integral

$$\Pi(k, p, \alpha) = \int_0^\alpha \frac{d\varphi}{(1 + p\sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}} \qquad (k^2 < 1) \tag{6.6}$$

it is more convenient to consider integral

$$J(\varepsilon, \alpha) = \int_0^\alpha \frac{d\varphi}{(\varepsilon - \sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}} = \frac{1}{\varepsilon}\Pi(k, -\varepsilon^{-1}, \alpha) \tag{6.7}$$

The derivative with respect to ϕ of function

$$L(\varphi) = \begin{cases} \dfrac{1}{\sqrt{u}} \operatorname{arctg} \dfrac{\sin\varphi\cos\varphi\sqrt{u}}{(v + \sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}} & (u > 0) \\[2ex] \dfrac{1}{2\sqrt{-u}} \ln \dfrac{\sqrt{-u}\sin\varphi\cos\varphi + (v + \sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}}{\sqrt{-u}\sin\varphi\cos\varphi - (v + \sin^2\varphi)\sqrt{1 - k^2\sin^2\varphi}} & (u < 0) \end{cases} \tag{6.8}$$

can be presented in this form

$$\frac{dL}{\partial\varphi} = \left\{1 - \frac{Q(t)}{k^2R(t)}\right\}\frac{1}{\sqrt{1 - k^2t}}, \qquad t = \sin^2\varphi$$

where Q(t) is a polynomial of the second degree,

$$R(t) = t^3 + \frac{2k^2v + u - 1}{k^2}t^2 + \frac{k^2v^2 - 2v - u}{k^2}t - \frac{v^2}{k^2} \tag{6.9}$$

With the corresponding numbers A, B and E,

$$R(t) = (t^2 + At + B)(t - E) = t^3 + (A - E)t^2 + (B - AE)t - BE \tag{6.10}$$

The roots of polynomial R(t) will be

$$\begin{matrix}\varepsilon_1 \\ \varepsilon_2\end{matrix} = -\frac{A}{2} \pm i\sqrt{S}, \qquad S = B - \frac{A^2}{4}, \qquad \varepsilon_3 = E \tag{6.11}$$

and it is easy to establish such a connection between three integrals of the Eq. (6.7) type,

$$k^2L(\alpha) = k^2F(k, \alpha) + \sum_{\nu=1}^{3} W_\nu^* J(\varepsilon_\nu, \alpha), \qquad W_\nu = \frac{Q(\varepsilon_\nu)}{R'(\varepsilon_\nu)} \tag{6.12}$$

If numbers A and B are fixed, number E and parameters u and v of function (6.8) are subject to determination. Comparison of Eq. (6.9) and (6.10) gives the three necessary equations and exclusion of u and E leads to a quadratic equation for v. As a result,

$$\begin{matrix} v_1 \\ v_2 \end{matrix} = \frac{(1-k^2)B \pm \sqrt{K}}{1+A+k^2B}, \quad k^2E = \frac{v^2}{B}, \quad u = 1 + k^2(A-E) - 2k^2v \tag{6.13}$$

$$K = B(1+A+B)(1+k^2A+k^4B) \tag{6.14}$$

Let roots ε_1 and ε_2 (6.11) always be complex; in this case, S>0, and it can be shown that K>0 and the numbers of Eq. (6.13) are real, $E_1 > E_2 \geqslant 1$, /105

$$W_3 = Q(E) / R'(E),$$
$$R'(E) = B + AE + E^2, \quad Q(E) = (2-E)k^2E^2 - E + v(1 - 2E + k^2E^2) \tag{6.15}$$

$$\begin{matrix} W_1 \\ W_2 \end{matrix} = M \pm iN, \quad M = \frac{qr + \varkappa\rho}{r^2+\rho^2}, \quad N = \frac{\varkappa r - q\rho}{r^2+\rho^2} \tag{6.16}$$

$$q = {}^1/_2 A\,[1 - k^2B + (2+A)k^2A + v(2+k^2A)] - (2+A)k^2B + v(1-k^2B)$$
$$\varkappa = -[1 - k^2B + (2+A)k^2A + v(2+k^2A)]\sqrt{S} \tag{6.17}$$
$$r = -2S, \quad \rho = -(A+2E)\sqrt{S}, \quad r^2 + \rho^2 = R'(E)\,4S$$

Further, with S>0,

$$J(\varepsilon_1, \alpha) = J^{(1)}(\alpha) + iJ^{(2)}(\alpha), \quad J(\varepsilon_2, \alpha) = J^{(1)}(\alpha) - iJ^{(2)}(\alpha) \tag{6.18}$$

and the real parts of Eq. (6.12) form the equation

$$MJ^{(1)}(\alpha) - NJ^{(2)}(\alpha) = {}^1/_2\{k^2L(\alpha) - k^2F(k,\alpha) - W_3J(E,\alpha)\} = U(\alpha) \tag{6.19}$$

By taking two roots v_1 and v_2 (6.13) and numbers M, N and U corresponding to them, two equations of the (6.19) type can be written. The solution of such a system of equations will be

$$J^{(1)} = (-U_1N_2 + U_2N_1)/\Delta, \quad J^{(2)} = (U_2M_1 - U_1M_2)/\Delta, \quad \Delta = -M_1N_2 + M_2N_1 \tag{6.20}$$

Since the integral of $J(\varepsilon, \alpha)$ of the (6.7) type with complex ε, analytical function ε or A and B (for $k^2<1$ even with $\alpha=\pi/2$), quantities $J^{(1)}$ and $J^{(2)}$ exist in the case of $\Delta=0$ (if it is possible). In any case, with $\Delta=0$ (as well as with $1+A+k^2B=0$), one of the subsequently indicated transformations of $J(\varepsilon, \alpha)$, of which the first changes only ε, can be used first.

Thus, for $\varepsilon=a+ib$, by finding the numbers

$$A = -2a, \quad B = a^2 + b^2 \tag{6.21}$$

$J(\varepsilon, \alpha)$ can be presented in the form of a linear combination of integrals $J(E_1)$ and $J(E_2)$ with real parameters $E_1>E_2>1$. In the case of complete integrals ($\alpha=\pi/2$), functions $L_1(\alpha)$ and $L_2(\alpha)$ are absent, and integrals $J(E)$ can be calculated simply by the above specified equations (6.1)-(6.4).

By differentiating over ϕ the function

$$\theta(\lambda, \varphi)=\begin{cases} \dfrac{1}{\sqrt{-\lambda}} \operatorname{arc\,tg} \dfrac{\sqrt{-\lambda} \sin\varphi \cos\varphi}{\sqrt{1-k^2\sin^2\varphi}} & (\lambda<0) \\ \dfrac{1}{2\sqrt{\lambda}} \ln \dfrac{\sqrt{\lambda}\sin\varphi\cos\varphi+\sqrt{1-k^2\sin^2\varphi}}{\sqrt{\lambda}\sin\varphi\cos\varphi-\sqrt{1-k^2\sin^2\varphi}} & (\lambda>0) \end{cases} \tag{6.22}$$

it can be determined that

$$\frac{d\theta}{d\varphi}=\frac{1-2t+k^2t}{1-(k^2+\lambda)t+\lambda t^2} \frac{1}{\sqrt{1-k^2t}} \qquad (t=\sin^2\varphi)$$

and the equation

$$\frac{1}{ab}\theta\left(\frac{1}{ab}, \alpha\right)=k^2F(k, \alpha)+(1-k^2a)J(a, \alpha)+(1-k^2b)J(b, \alpha) \tag{6.23}$$

can be reached, where a and b are connected by the relationships

$$b=\frac{1-a}{1-k^2a}, \qquad a=\frac{1-b}{1-k^2b} \tag{6.24}$$

From this, with notations $p=-1/a$, $p_*=-1/b$, it follows that

$$pp_*\theta(pp_*, \alpha)=k^2F(k, \alpha)-(p+k^2)\Pi(k, p, \alpha)-(p_*+k^2)\Pi(k, p_*, \alpha) \tag{6.25}$$

$$p_*=-\frac{p+k^2}{p+1}, \qquad p=-\frac{p_*+k^2}{p_*+1} \tag{6.26}$$

We call this transformation symmetrical linear-fractional. For /106
$k^2<-p<1$, there will be $0<p_*<\infty$, and the transformation proves to be extremely useful with $p\approx-1$. With $\alpha=\pi/2$, function θ (6.22) is absent.

We point out still another important transformation. We introduce variable ϕ_1, with the use of the Landen substitution (proposed by him for transformation of elliptical integrals of the first and second kinds)

$$\tan\varphi_1 = \frac{\sin 2\varphi}{k_1 + \cos 2\varphi}, \quad k_1 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{k^2}{(1+\sqrt{1-k^2})^2}, \quad k^2 = \frac{4k_1}{(1+k_1)^2}$$

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{1+k_1}{2}\frac{d\varphi_1}{\sqrt{1-k_1^2\sin^2\varphi_1}}, \quad \frac{1+k_1}{2} = \frac{1}{1+\sqrt{1-k^2}}$$

By converting the first equality of (6.27) to a quadratic equation for cos 2ϕ, we obtain

$$\cos 2\varphi = -k_1\sin^2\varphi_1 + \cos\varphi_1\sqrt{1-k_1^2\sin^2\varphi_1} = 1 - 2\sin^2\varphi \tag{6.28}$$

The root was selected here which corresponds to $0<\phi<\pi/2$. Actually, $\phi<\phi_1<2\phi$, $\phi_1=0$ with $\phi=0$, $\phi_1=\pi$ with $\phi=\pi/2$. It can now be shown that

$$\int_0^\alpha \frac{d\varphi}{(\varepsilon-\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} = \frac{1+k_1}{(1+k_1)^2-4k_1\varepsilon}\left\{\int_0^{\alpha_1}\frac{\cos\varphi_1 d\varphi_1}{\varepsilon_1-\sin^2\varphi_1} + \right.$$

$$\left. + (1-2\varepsilon+\varepsilon_1 k_1)\int_0^{\alpha_1}\frac{d\varphi_1}{(\varepsilon_1-\sin^2\varphi_1)\sqrt{1-k_1^2\sin^2\varphi_1}} - k_1\int_0^{\alpha_1}\frac{d\varphi_1}{\sqrt{1-k_1^2\sin^2\varphi_1}}\right\} \tag{6.29}$$

$$\varepsilon_1 = \frac{4\varepsilon(1-\varepsilon)}{(1+k_1)^2-4k_1\varepsilon} = \frac{\varepsilon(1-\varepsilon)}{1-k^2\varepsilon}(1+\sqrt{1-k^2})^2, \quad \mathrm{tg}\,\alpha_1 = \frac{\sin 2\alpha}{k_1+\cos 2\alpha} \tag{6.30}$$

Let a, b be real numbers, $\varepsilon=a+ib$; parameter ε_1 will be real only under the condition

$$(1-2a)(1-k^2a) + k^2[a(1-a)+b^2] = 0$$

By changing to integral (6.6), the following relationships can be obtained:

$$\Pi(k,p,\alpha) = \frac{1+k_1}{4}\frac{p}{p+1}\left\{\left(1+\frac{2}{p}-\frac{k_1}{p_1}\right)\Pi(k_1,p_1,\alpha_1) + \frac{k_1}{p_1}F(k_1,\alpha_1) + \right. \tag{6.31}$$

$$\left. + \int_0^{\alpha_1}\frac{\cos\varphi_1 d\varphi_1}{1+p_1\sin^2\varphi_1}\right\}, \quad k_1 = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}, \quad p_1 = \frac{1-2k_1\gamma+k_1^2}{\gamma^2-1}, \quad \gamma = -\left(1+\frac{2}{p}\right)$$

$$\Pi(k,p,\alpha) = \frac{1}{2(1+\sqrt{1-k^2})}\left\{\frac{k^2}{p+k^2}F(k_1,\alpha_1) + \frac{p^2+2p+k^2}{(p+1)(p+k^2)}\Pi(k_1,p_1,\alpha_1) + \right.$$

$$\left. + \frac{p}{p+1}L(p_1,\alpha_1)\right\}, \quad p_1 = \frac{p}{p+1}\frac{p+k^2}{(1+\sqrt{1-k^2})^2} \tag{6.32}$$

$$L(q,\vartheta) = \int_0^\vartheta \frac{\cos\vartheta d\vartheta}{1+q\sin^2\vartheta} = \frac{1}{i2\sqrt{q}}\ln\frac{1+i\sqrt{q}\sin\vartheta}{1-i\sqrt{q}\sin\vartheta} \tag{6.33}$$

Since $k_1<k$, by using transformation (6.32) n times, an extremely small module k_n can be reached, and it can be assumed with great accuracy that

$$\Pi(k_n, p_n, \alpha_n) \approx \int_0^{\alpha_n} \frac{d\varphi_n}{1 + p_n \sin^2\varphi_n} = \frac{1}{i2\sqrt{1+p_n}} \ln \frac{1 + i\sqrt{1+p_n}\,\mathrm{tg}\,\alpha_n}{1 - i\sqrt{1+p_n}\,\mathrm{tg}\,\alpha_n} \quad (6.34)$$

Modules k_n decrease extremely rapidly (for example, $k_3^2<10^{-6}$ for $k^2=0.6$ and $k_4^2<10^{-8}$ for $k^2=0.99$), and the successive approximations method proves to be quite effective. /107

For complete integrals ($\alpha=\pi/2$), we have

$$\alpha_n = 2\alpha_{n-1}, \quad L(p_n, \pi) = 0, \quad 2F(k) = \pi(1+k_1)(1+k_2)\ldots$$

$$\Pi(k, p) = \frac{1}{1+\sqrt{1-k^2}} \left\{ \frac{k^2}{p+k^2} F(k_1) + \frac{p^2+2p+k^2}{(p+1)(p+k^2)} \Pi(k_1, p_1) \right\} \quad (6.35)$$

It can be shown that, for real numbers g and h,

$$\int_0^{\pi/2} \frac{d\vartheta}{1+g\sin^2\vartheta} = \begin{cases} 1/2\,\pi(1+g)^{-1/2} & (1+g>0) \\ 0 & (1+g<0) \end{cases}$$

$$\int_0^{\pi/2} \frac{d\vartheta}{1+(g+ih)\sin^2\vartheta} = \frac{N_1 - i\,\mathrm{sgn}\,h\,N_2}{\sqrt{(1+g)^2+h^2}} \frac{\pi}{2} \qquad (h \neq 0)$$

$$\begin{matrix} N_1 \\ N_2 \end{matrix} = \left\{ \frac{\sqrt{(1+g)^2+h^2} \pm (1+g)}{2} \right\}^{1/2}$$

In generalizing elliptical integrals of all kinds, the following function should be introduced,

$$\Pi_m^{(n)}(k, p, \alpha) = \int_0^{\alpha} \frac{\cos 2m\varphi}{1+p\sin^2\varphi} (1 - k^2\sin^2\varphi)^n d\varphi \quad (6.36)$$

Since

$$\cos(2m-2)\varphi + \cos(2m+2)\varphi = 2\cos 2m\varphi(1 - 2\sin^2\varphi) \quad (6.37)$$

then,
$$1/2\{\Pi_{m+1}^{(n)}(k,p,\alpha) + \Pi_{m-1}^{(n)}(k,p,\alpha)\} = (1+2/p)\,\Pi_m^{(n)}(k,p,\alpha) - (2/p)\,E_m^{(n)}(k,\alpha)$$

where $E_m^{(n)}(k, \alpha)$ is a function introduced above (in Section 3), which generalizes integrals of the first and second kinds.

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