

**N86 - 24508**

1985

NASA/ASEE SUMMER FACULTY RESEARCH FELLOWSHIP PROGRAM

MARSHALL SPACE FLIGHT CENTER  
THE UNIVERSITY OF ALABAMA IN HUNTSVILLEA MONTE CARLO STUDY OF WEIBULL RELIABILITY ANALYSIS  
FOR SPACE SHUTTLE MAIN ENGINE COMPONENTS

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Date:	August 2, 1985
Contract Number:	NASA-NGT-01-008-021

A MONTE CARLO STUDY OF WEIBULL RELIABILITY ANALYSIS  
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ABSTRACT

The use of the Weibull failure distribution model has proven valuable in reliability analysis within the aeronautics industry. The Weibull analysis of samples with a high percentage of non-failures, or censored observations, must be undertaken using one of several large-sample approximations to the distributions of parameter estimators, since exact methods for such distributions are mathematically intractable. It is unknown whether these approximations will produce satisfactory results when used with samples, typical of the space shuttle main engine data, which contain small (fewer than 15) numbers of failures. An objective of this study was to design and implement a Monte Carlo computer simulation to assess the usefulness of the Weibull methods for such small-failure samples. In particular, under varying assumptions concerning failure distribution parameters and number of failures in the sample being analyzed, approximate 90% confidence intervals for predicted times to failure for different percentages of components are calculated by numerical methods. The confidence coefficient for these intervals is then tested by determining what percentages of intervals trap the true values of the parameters being estimated. Also, modifications to a Weibull analysis computer program incorporating methods for calculating approximate confidence intervals and including a number of options for analyzing interval data (as opposed to point data) are described.

## INTRODUCTION

The Weibull probability distribution has become a widely used lifetime distribution model since its introduction in 1951 (3). It has been found to be especially important in reliability analysis of manufactured items. Although there is a large body of literature on the distribution and its statistical properties, the distributions of many of the usual parameter estimators seem to be mathematically intractable. The mathematical difficulties encountered in analyzing these estimators are compounded when estimating with samples that include data which has been censored in nontrivial ways. The usual methods employed in this situation involve the use of some statistics whose asymptotic behaviors are understood, but whose applicability for small samples is unknown or unsure.

More particularly, Weibull analysis has proven to be quite valuable in reliability studies for aircraft engine components (1). This success has encouraged the exploration at Marshall Space Flight Center of the applicability of Weibull techniques in reliability analysis for components of the space shuttle main engine (SSME). A primary restriction for the SSME environment is a severe limitation on the test sample size. Test and flight results are collected from fewer than 30 engines. For major SSME components, data samples may contain fewer than 5 failures.

This report documents the incorporation of a number of additional capabilities into an existing Weibull analysis computer program and the results of a Monte Carlo computer simulation study to evaluate the usefulness of the Weibull methods using samples with a very small number of failures and extensive censoring. Since the censoring mechanism inherent in the SSME data is hard to analyze, it was decided to use a random censoring model, generating censoring times from a uniform probability distribution.

Section 1 of the report describes some of the statistical techniques and computer programs that are used in the SSME Weibull analysis. The methods documented in (1) were supplemented by adding computer calculations of approximate (using iterative methods) confidence intervals for several parameters of interest. These calculations are based on a likelihood ratio statistic which is asymptotically a chi-squared statistic with one degree of freedom, the basic method being taken from (2).

The assumptions built into the computer simulations are described in section 2. The simulation program and the techniques used in it are described there also. Simulation results are tabulated for various combinations of Weibull

shape parameters and the numbers of failures in the samples, in the chart of section 3. Conclusions concerning the validity of the chosen Weibull model and estimators, for the small samples characteristic of the SSME reliability analysis, are drawn from the simulation results.

In section 4, some implications of working with interval data, as opposed to point data, are explored. Modifications to the Weibull analysis computer program, with several options for handling interval data are described briefly. Finally, in the final section, some conclusions concerning the applicability of the Weibull methods to the current SSME hardware analysis are reviewed and summarized.

## 1. DESCRIPTION OF STATISTICAL MODEL.

A brief description of the statistical model and methods used in the SSME reliability analysis is given in this section. For a more complete discussion of these methods and tools the reader should consult references (1) and (2). The Weibull distribution used is the two parameter Weibull whose probability density function is:

$$f(t) = \eta \beta (\eta t)^{\beta-1} \exp(-(\eta t)^\beta) \quad t > 0.$$

The parameter  $\beta$  is called the shape parameter of the distribution. For many applications  $\beta$ -values in the range .5 to 3.5 are reasonable. When  $\beta = 1$ , the Weibull reduces to the standard negative exponential distribution. The parameter  $\eta$  is called the scale parameter of the distribution and changes in  $\eta$  simply change the horizontal scale and do not alter the basic shape of  $f(t)$ . It is easily shown that for any Weibull distribution, no matter what the value of  $\beta$ ,  $\Pr(t < \eta)$  is equal to  $1 - e^{-1}$  (approximately .632). For this reason  $\eta$  is sometimes called the characteristic life of the corresponding family of Weibull distributions. Some typical Weibull probability density functions are illustrated in figure 1.1. For convenience,  $\eta$  is taken to be 1 for each curve in that figure.

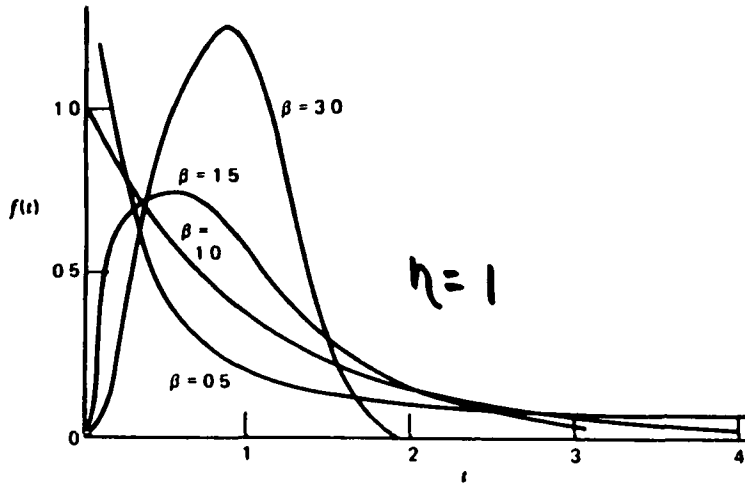


FIGURE 1.1

The basic technique in the SSME use of the Weibull distribution is to estimate the parameters  $\beta$  and  $\eta$ , and then calculate survival times for various percentages of components. For example,  $t_{.1}$  (=estimated time for 10% failures) would be calculated by solving  $F(t) = .1$ , where  $F(t)$  is the cumulative distribution function approximated by using the estimates calculated for  $\beta$  and  $\eta$ . In particular,  $F(t)$  is given by:

$$F(t) = 1 - \exp(-(\eta t)^\beta) \quad t > 0.$$

Two techniques are used to estimate  $\beta$  and  $\eta$ . A ranked regression model produces estimates for  $\beta$  and  $\eta$  and provides for a simple graphical evaluation of whether the data seems to fit the Weibull model. Graphical estimation of the  $t_p$  values can then be made. A second technique is to calculate maximum likelihood estimates, say  $\hat{\beta}$  and  $\hat{\eta}$ , for  $\beta$  and  $\eta$  and then estimate the  $t_p$  using  $F(t)$  as indicated above. Both of these techniques are included in a computer program given in (1).

It was deemed desirable to modify the above referenced program so that confidence intervals for the maximum likelihood estimators could be calculated. The type of censoring present in the data makes the calculation of exact confidence intervals impossible. Censored (or suspended) times can be observed for any values of  $t$ , and the censoring pattern is difficult to predict. The primary reason for this unpredictability is that times may be censored for a large variety of reasons, including failures of other engine

components which would terminate a test. Additionally, there is considerable modularity built into the SSME and as a consequence, times at risk for a given type of component are likely to be very variable from particular component to component. In the statistical analysis and simulation studies, type I (see (2) for a discussion of various types of censoring) censoring was assumed. Basically this assumption independently assigns to each component both a failure and a censoring time. The time observed is then the smaller of the two times.

There are two standard methods which can be used to calculate approximate confidence intervals for the Weibull parameters mentioned above. For large samples (30 or more failures),  $(\hat{\beta}, \hat{h})$  has an approximate bivariate normal distribution whose covariance matrix can be estimated in a straightforward manner. The approximations involved in this method are not generally adequate for samples with fewer than 30 failures (see (2)), and thus the method is not appropriate for the SSME analysis.

A second method which seems to be a better choice for moderately sized samples (around 20 failures) uses a likelihood ratio statistic. This method has been incorporated into the computer program being used at MSFC. Suppose the sample consists of observations  $x_1, x_2, x_3, \dots, x_m$ ,  $F =$  set of indices  $j$ , for which  $x_j$  is a failure (as opposed to a censored time) and  $r =$  number of failures in the sample. The method for calculating confidence intervals for  $t_p$  is based on the fact that under the assumption that  $t_p = t^*$ , the statistic  $S_1(t^*)$  has approximately a chi-squared distribution with one degree of freedom.  $S_1(t^*)$  is given by:

$$S_1(t^*) = -2 \log L(\bar{\beta}, \bar{n}) + 2 \log L(\hat{\beta}, \hat{h}).$$

Here  $\log L$  is the log likelihood function given by:

$$\log L(\beta, n) = r \log \beta - r \beta \log h + (\beta - 1) \sum_{j \in F} \log x_j - \sum_{i=1}^m (x_i/h)^\beta$$

Also  $\hat{\beta}$  and  $\hat{h}$  are the maximum likelihood estimators calculated from the given sample, and  $\bar{\beta}$  is calculated (approximated) by solving the following equation iteratively:

$$r/\bar{\beta} - r \log t^* + \sum_{j \in F} \log x_j + \log(1-p) \sum_{i=1}^m (x_i/t^*)^{\bar{\beta}} \log(x_i/t^*) = 0.$$

After  $\bar{\beta}$  is found then  $\bar{n} = t^* / (-\log(1-p))^{1/\bar{\beta}}$

In the computer program,  $\bar{\beta}$  is calculated using a hybrid secant/false-position method. A  $\gamma$ -confidence interval for  $t_p$  is found by finding the set of values  $t^*$  for which  $S_1 < \chi_{1,\gamma}^2$ . This is accomplished in the computer program by starting with the point estimate for  $t_p$  (based on the estimates  $\hat{\beta}$  and  $\hat{h}$ ), and testing values for  $t^*$  diverging from  $t_p$  in both directions until values are found on either side of  $t_p$  for

which  $S_1$  exceeds the appropriate critical value.

In a similar manner, a  $\gamma$ -confidence interval for  $\beta$  can be calculated by finding all values  $\beta^*$  for which  $S_2 < \chi^2_{1,\gamma}$ . Here

$$S_2(\beta^*) = -2 \log L(\beta^*, n^*) + 2 \log L(\hat{\beta}, \hat{n})$$

where  $\log L$ ,  $\hat{\beta}$  and  $\hat{n}$  are as before and

$$n^* = \left( \sum_{i=1}^m x_i \beta^* / n \right)^{1/\beta^*}$$

Representative output from the computer program is given in figure 1.2.

THE FOLLOWING ESTIMATES ARE RANKED REGRESSION ESTIMATES

```
BETA=          1.1397   ETA=          448.33
R=  0.96470R**2=  0.93865
DO YOU WISH TO DO MAXIMUM LIKELIHOOD ESTIMATION?
ANSWER Y OR N
Y
```

```
DO YOU WISH 80% OR 90% CONFIDENCE INTERVALS?
TYPE IN 80 OR 90.
```

```
90
MAXIMUM LIKELIHOOD ESTIMATES FOR THIS CASE FOLLOW
BETA=          1.3264   ETA=          440.80
```

```
90% BETA CONFIDENCE LIMITS          0.83   ...   1.93
```

PERCENTAGE OF FAILURES	ESTIMATED TIME	90% CONF. INTERVAL	
0.1	2.	0.	12.
1.0	14.	2.	41.
10.0	81.	32.	154.
20.0	142.	71.	249.
50.0	334.	201.	602.
63.2 ETA	441.	286.	859.
90.0	827.	496.	2025.

FIGURE 1.2

A primary problem with the above analysis in the SSME case is that the methods for calculating the approximate confidence intervals are based on asymptotic distribution approximations, whose validities are in question for such small samples. The problem is a serious one since the small size of the samples make us intuitively skeptical of the point estimates calculated from them and hence anxious to have some reasonably accurate confidence measure for those estimates. In order to address this problem, Monte Carlo experiments were designed and conducted to assess the accuracy, and hence usefulness, of the confidence intervals being calculated.

## 2. SIMULATION SAMPLE GENERATION.

Failure times were calculated by generating a random number  $r$  from the uniform distribution on  $[0,1]$  (this was done using RAN, the built in random number function in the VAX FORTRAN), and then computing  $F^{-1}(r)$ , where  $F(t)$  is the Weibull cumulative distribution function for a given choice of parameters  $\beta$  and  $n$ . In fact,  $n$  was always taken to be 100 and  $\beta$  was varied. Sample failure times thus generated were tested, using the standard chi-squared goodness-of-fit test, against the given Weibull distribution and found to be representative of that distribution. Censoring times were generated to follow a uniform distribution.

To generate a sample of times, the number of failure times to be present in the sample was fixed. Failure times and censoring times were generated in pairs and the corresponding observed time was taken to be the minimum of the pair. Since the number of failures was predetermined for a given simulation run, the overall sample sizes varied. The censoring distribution range was varied (ranges to be used were determined by simulation) so as to achieve (approximately) the desired average sample size. An average sample size of 50 was chosen to be representative of a number of projected SSME sample sizes. Additional Monte Carlo studies, in which both sample sizes and censoring distributions will be varied, are planned.

For an occasional sample one or more of the iterative numerical methods employed in the confidence interval calculations failed to converge properly. Such samples were omitted in the subsequent Monte Carlo analysis. The number of samples for which this occurred was small (on the order of 1%), and so it is clear that ignoring them could not unduly bias the simulation results. At any rate, when such samples occur in practice, they would not be evaluated using the confidence interval calculations.

## 3. SIMULATION RESULTS.

Most of the Monte Carlo simulations that have been run are summarized in Table 3.1. Some additional isolated runs with varying sample sizes and censoring distributions have been made, but these were not systematic enough to include here. It might be noted, however, that these runs showed results consistent with the results reported here. A more thorough collection of simulation results will be reported on later.



For each of the 12 simulation runs (each column represents a run) summarized in Table 3.1, 100 samples of approximate size 50 each were used. Each run of 100 simulated samples took approximately an hour of computing time on a VAX 11/780, which explains why more extensive runs were not made.

The upper part of Table 3.1 contains the number (out of 100) of calculated approximate 90% confidence intervals which actually trapped the true parameter values being estimated. Lawless (2) has predicted that these confidence intervals would tend to be too large (and hence trap true values more than 90% of the time) for samples with small numbers of failures. Our results seem to support that prediction when the number of failures is 10 or 20, but such an effect is not obvious in the 3 or 5 failure cases. More extensive simulations will be needed to gain insight into this matter.

Eta = 100

Failures=	Beta=1.5				Beta=2.0				Beta=3.0				
	3	5	10	20	3	5	10	20	3	5	10	20	
$\hat{\beta}$	85	91	90	99	88	92	93	89	95	87	90	84	Number of Confidence Intervals Trapping Actual Values
$t_{.001}$	93	89	93	99	88	90	93	88	88	90	91	92	
$t_{.01}$	93	89	93	98	87	92	90	89	91	90	91	90	
$t_{.1}$	87	91	91	96	92	92	94	91	91	95	90	92	
$t_{.2}$	86	91	88	95	91	91	95	89	90	96	96	95	
$t_{.5}$	84	91	84	95	89	86	93	95	90	92	97	97	
$n = t_{.632}$	85	90	85	96	88	86	95	94	90	91	96	96	
$t_{.9}$	88	91	90	95	88	90	94	92	91	91	94	96	
Average $\hat{\beta}$	2.03	1.75	1.59	1.58	2.76	2.33	2.18	2.10	3.78	3.42	3.25	3.07	
$\hat{\beta}$ too large	59	55	48	56	64	58	61	52	64	57	52	50	
Average $\hat{n}$	134	107	101	101	107	101	99	99	103	101	99	100	
$\hat{n}$ too large	41	34	39	48	35	36	38	51	35	37	45	42	

Each column based on 100 samples of (approximate) size 50.

TABLE 3.1

In the bottom part of Table 3.1, the bias in the estimators  $\hat{\beta}$  and  $\hat{\kappa}$  is examined, and we see that for smaller samples the bias in each estimator grows larger. Both  $\hat{\beta}$  and  $\hat{\kappa}$  are biased high (in average), but in the case of  $\hat{\kappa}$ , median values are below the actual parameter values. The possibility of constructing (by simulation) a table of bias-correcting multipliers as functions of sample size,  $\hat{\beta}$ ,  $\hat{\kappa}$ , and the number of failures may be worth considering.

Although these simulation results are not extensive enough to provide a definitive conclusion, the tentative conclusion is that the approximate confidence intervals calculated using the likelihood ratio statistic are reasonably accurate even for samples with very few failures. Thus, the calculation of these intervals would seem to be a useful addition to the SSME Weibull analysis methodology.

We should note here that the calculations of the maximum likelihood estimators and the associated approximate confidence intervals are based on the assumption that our failure times are true point data, i.e. that we know the failure times exactly. Whenever failure times are known only to the extent of falling in some time interval (interval data), the above calculations do not apply. To use the above methods with interval data, we must make some assumption about the placement of failure times within failure intervals. Since much of the SSME failure data is interval data, we will pursue this issue in the next section.

#### 4. INTERVAL DATA CONSIDERATIONS.

The most conservative approach in working with interval data is to make no additional assumptions about the true failure times and use only the failure intervals in any statistical analysis. It is not difficult to modify the likelihood function to accommodate interval data, and calculate the maximum likelihood estimates for  $\beta$  and  $\kappa$  based on this function. The numerical procedures are a bit more complicated and sensitive than in the point data case; hence reasonably good beginning guesses for  $\beta$  and  $\kappa$  are required in order for the iterative methods to converge properly. The capability to calculate interval data maximum likelihood estimates for  $\beta$  and  $\kappa$  has been added to the Weibull analysis computer program being used at MSFC.

One approach which has been used in analyzing SSME interval data is to make the optimistic assumption that all failures in a failure interval occur at the right hand boundary of the interval. As might be expected, the estimates for  $\beta$  and  $\kappa$  that are calculated using the interval data method and the point data method (with the above assumption)

are quite different for most sets of data. Of course the differences in these estimates will decrease as the intervals involved get smaller, but for the interval sizes typical of current SSME data, these differences can be expected to be significant.

Chart 4.1 illustrates a range of possible assumptions about failure time placement for interval data. All of these choices have been included as options in the Weibull analysis computer program. This capability allows an exploration of the implications arising from various assumptions about the unknown failure times. In the computer calculation for choice A (interval data method), the initial "guesses" for  $\beta$  and  $n$  are gotten by calculating estimates using choice D.

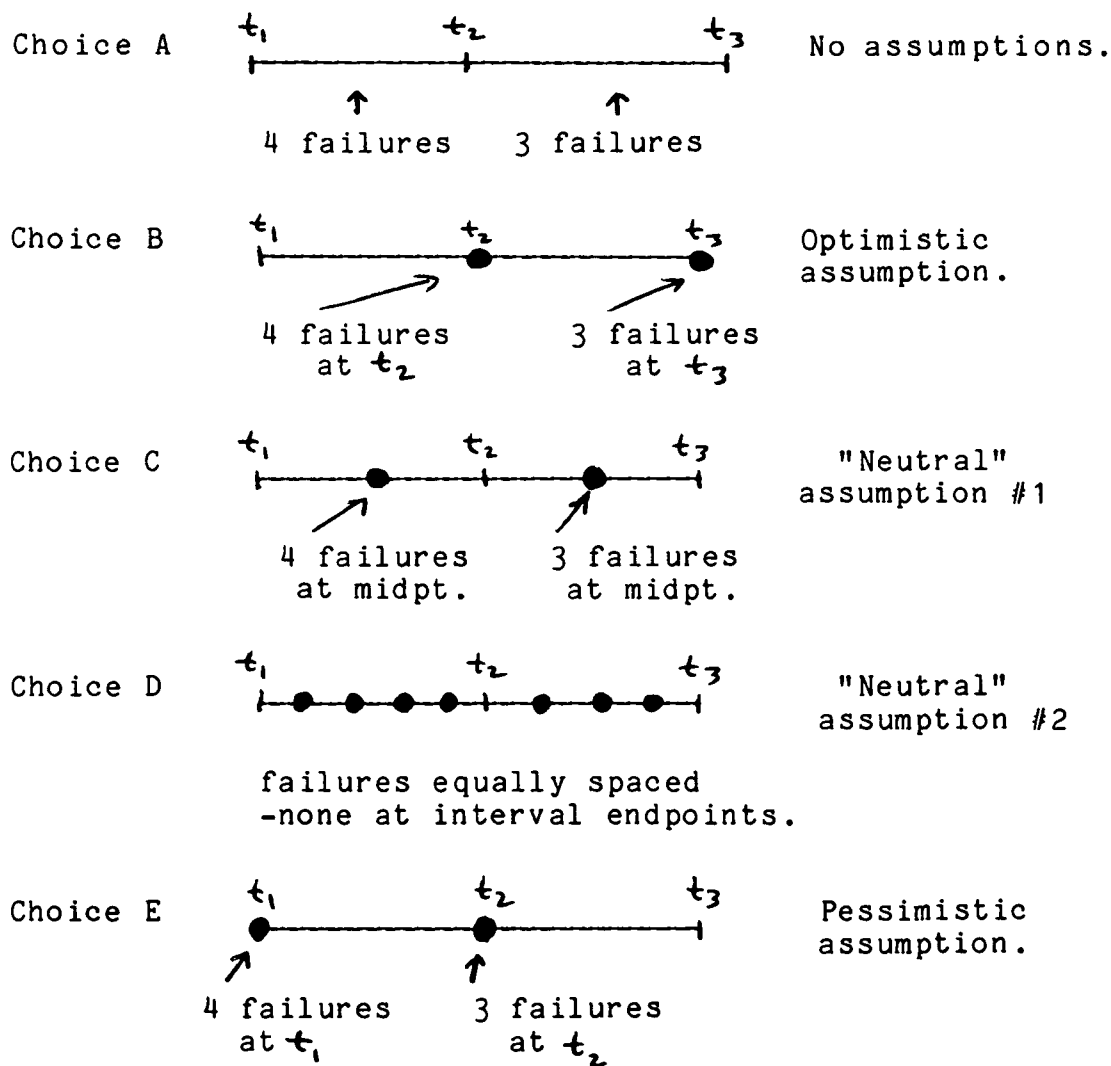


CHART 4.1

## CONCLUSIONS

The primary conclusion of this study is that Weibull methods can be of positive value in SSME hardware reliability analysis. However, caution should be exercised in the use of these methods with small-failure samples and interval data.

When using point data, it is important to consider confidence intervals because point estimates can be quite erroneous and misleading, especially when samples contain very small numbers of failures. One of the primary results of this study was to implement computer methods for calculating approximate confidence intervals from such samples and to verify that these approximate confidence intervals are reasonably accurate.

The use of the Weibayes analysis method discussed in (1) is especially risky for current SSME use because of the lack of a significantly large SSME failure data base. In fact, making a priori estimates of betas from the existing SSME failure data base is not much better than simple guessing.

When analyzing interval data, it is important to realize that any assumptions made about the placement of the failure times within failure intervals is likely to have more impact upon the statistical predictions made from that data, than the choice of a particular statistical method, or even model, is likely to have. This is not to suggest that such assumptions should never be made, but rather to point out that when they are made, careful consideration and study should be given the choice of an assumption. Additions to the Weibull analysis computer program allow explorations of the implications of various of these assumptions.

Finally, no matter what assumptions are chosen, it is imperative to remember that the predictive power of any model is limited by the integrity and informational content of the data used as input to that model. The SSME data is typically interval data with a small number of failures, and no statistical model or method will be capable of extracting extremely dependable and accurate predictions from it.

In summary, the modified Weibull analysis computer program now provides a range of capabilities and options for treating estimating methods and data assumptions. These capabilities enhance the chances that the Weibull model can be used to advantage in SSME hardware reliability analysis. When applied and interpreted with the caution dictated by the considerations outlined here, the model should provide an important additional tool for SSME data analysis.

## REFERENCES

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2. Lawless, G.F., Statistical Models and Methods for Lifetime Data, J. Wiley, New York, NY, 1982 (see especially chapter 4).
3. Weibull, W., "A Statistical Distribution Function of Wide Applicability," Journal of Applied Mechanics, 1951, pp 293-297.

Note: Computer program listings (both Weibull analysis and simulation programs) are available from author.