NASA TECHNICAL MEMORANDUM NASA TM-88429

NASA-TM-88429 19860016598

QUASI-GENERALIZED VARIABLES

J. Baumgarte, G. P. Ostermeyer

Translation of "Quasigeneralisierte Variable," IN: Zeitschrift für angewandte Mathematik und Mechanik (ISSN 0044-2267), Vol. 65, No. 10, 1985, pp. 471-478 (A86-18713)

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, DC 20546 MAY 1986

STANDARD TITLE PAGE

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-88429		

4. Title and Subtitle	5. Report Date
QUASI-GENERALIZED VARIABLES	May 1986
	6. Performing Organization Code

7. Author(s)	8. Performing Organization Report No.
J. Baumgarte, G. P. Ostermeyer	
	10. Work Unit No.

9. Performing Organization Name and Address	11. Contract or Grant No.
The Corporate Word, inc. 1102 Arrott Bldg. Pittsburgh, PA 15222	NASW-4006
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Translation
National Aeronautics and Space Administration Washington, DC 20546	14. Sponsoring Agency Code

15. Supplementary Notes Translation of "Quasigeneralisierte Variable," IN: Zeitschrift fuer angewandte Mathematik und Mechanik (ISSN 0044-2267), Vol. 65, No. 10, 1985, pp. 471-478 (A86-18713).

16. Abstract

Numerical solution of a system of differential and algebraic equations is difficult, due to the appearance of numerical instabilities. A method is presented here which permits numerical solutions of such a system to be obtained which satisfy the algebraic constraint equations exactly without reducing the order of the differential equations. The method is demonstrated using examples from mechanics.

17. Key Words (Selected by Author(s))	18. Distribution Statement
	Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	24	

QUASI-GENERALIZED VARIABLES

J. Baumgarte, G. P. Ostermeyer

1. Introduction

/471*

Systems of mass points and fixed bodies, which are subject to certain secondary conditions, occur in analytical mechanics. These are idealized descriptions of real physical combinations between single bodies such as poles, joints, skids, cutting edges, etc.

One may proceed as follows with the mathematical description of the system. After selecting suitable coordinates q, $q^t = (q_1, \ldots, q_n)$, which may implicitly involve part of the combinations, the Lagrange function L is determined:

$$\begin{aligned} L(\dot{q}, q, t) &= T_2 + T_1 + T_0 - V \\ &= \tfrac{1}{2}\dot{q}^t M(q,t)\,\dot{q} + B(q,t)\,\dot{q} + T_0(q,t) - V(q,t) \end{aligned} \tag{1.1}$$

as are forces Q, which do not follow from L (non-potential forces)

$$Q = Q(\dot{q}, q, t), \qquad Q^t = (Q_1, \ldots, Q_n). \tag{1.2}$$

The combinations which have not yet been considered in the chosen coordinates q are the so-called holonomic equations:

$$f_\nu(q,t) = 0, \qquad \nu = 1, \ldots, s < n - r \tag{1.3}$$

and the linear non-holonomic combinations of the velocities q

$$g_\varkappa(\dot{q}, q, t) = 0, \qquad \varkappa = 1, \ldots, r < n - s. \tag{1.4}$$

---

*Numbers in the margin indicate pagination in the foreign text.

The resulting equations of motion of the observed system are now called

$$(L_{\dot{q}})^{\cdot} - L_q = Q + f_q\mu_1 + g_{\dot{q}}\mu_2 . \qquad (1.5)$$

Statements (1.3) and (1.4) for the coordinates, and the velocities, respectively, are contained herein in explicit form as so-called limits.

The Lagrangian multipliers μ_1 and μ_2 have to be calculated from secondary conditions (1.3) and (1.4). Normally, one proceeds as follows: holonomic combinations (1.3) are derived twice, non-holonomic combinations (1.4) are derived once totally in accordance with an independent variable, time t.

$$\ddot{f}_\nu = 0, \quad \dot{g}_\kappa = 0. \qquad (1.6\ a,\ b)$$

These relationships are then linear in accelerations $\ddot{q}$. If one inserts the similarly linear equations of motion in (1.6), one obtains an algebraic system of equations with Lagrange multipliers. Because of the usual independence of the several combinations, these multipliers can be clearly calculated as functions of q, $\dot{q}$, and t.

Although equations of motion (1.5) are very convenient in considering additional combinations (1.3) and (1.4), they have a major disadvantage with regard to numerical integration. The numerical solution of (1.5) exhibits characteristic errors of the combinations (1.3) and (1.4), which, practically independent of the quality of the program of integration, quickly render the solution useless. Essentially the errors in the combinations increase linearly--it should be noted that this implies a quadratically increasing error for coordinates q--and quadratically in the combinations (1.3). This pattern of errors is implicit in the calculation of the Lagrangian multipliers

using (1.6 a, b). Additional instabilities are transofered to the equations of motion.

Although one may achieve very good results by using stabilization strategies (see section 2) to diminish these instabilities, it is actually unsatisfactory from the analytical point of view, out of explicit consideration for combinations such as those in movement equations (1.5), to transfer instabilities into the motion equations only to eliminate them later in the numerical solution.

Theoretically these difficulties can be completely avoided, when all holonomic combinations are implicitly considered via suitably chosen generalized coordinates $\hat{q}$, and when the /472 non-holonomic combinations are eliminated through "non-holonomic velocity parameters" (Hamel [1]). While non-holonomic combinations are eliminated rather easily--non-holonomic secondary conditions are generally linear in velocity--this procedure is not always very easy or even possible for holonomic combinations, which are generally non-linear in coordinates. Therefore, the following is recommended in related literature. Holonomic combinations are completely derived once from time. Then they can be treated formally like non-holonomic combinations. The reduction using non-holonomic velocity parameters $\omega, \omega^T = (\omega_1, \dots, \omega_{n-r-s})$ then yields a system of differential equations of the order 2n - r - s

$$\dot{q} = F(q, \omega, t), \quad \omega = G(q, \omega, t). \qquad (1.9\ a,\ b)$$

In the process, the derived holonomic combinations (1.3) are thus implicitly stated using the variables ω. The combinations (1.3) in the non-derived form are now first integrals for the n equations (1.9 a).

Peculiarly, the numerical solution disturbs first integrals as much as the explicitly considered non-holonomic combinations. Here too, it is true that instabilities occur which make themselves numerically noticeable in strongly increasing errors with reference to the combinations. These instabilities are caused by the incomplete--though mathematically possible--reduction of the order of differential motion equations.

It is the goal of this work to formulate motion equations so that, even with explicitly considered combinations, not only the analytical, but also the numerical, solution is error-free regarding the secondary conditions.

2. Stabilization

The point of departure for stabilizations of combinations with motion equations of the order (1.5) or (1.9) involves (Baumbarte [2]) calculating the Lagrangian multipliers in (1.5) no longer using (1.6 a) or (1.6 b), but rather the asymptotically stable differential equations

$$\ddot{f}_\nu + 2\alpha\dot{f}_\nu + \beta^2 f_\nu = 0, \quad \beta = \alpha > 0; \qquad \dot{g}_\varkappa + \gamma g_\varkappa = 0, \quad \gamma > 0. \tag{2.1 a; b}$$

The Lagrangian multipliers μ_1 and μ_2 calculated thusly, now contain the classic portion together with expressions which disappear identically in the analytical solution, but for which regulator terms constitute the numerical solution regarding the combinations. The stabilizations (2.1 a, b) limit the errors regarding the combinations (1.3) and (1.4) in the numerical solution. From the regulating-technical point of view, (2.1 b) is a P- and (2.1 a) a PD-regulator.

Extremely precise stabilizations (Ostermeyer [3]) may be

obtained by inducing an I-member into the regulator, since then, on one hand, the temporal median of the numerical combination error disappears and, on the other hand, the coefficients in the stabilizations can be directly indicated after inducing such a substitute constant T* using the optimum criterion employed in regulator technology.

$$\ddot{f}+\frac{1}{2T^*}\left(4\dot{f}+\frac{1}{T^*}f+\frac{1}{16T^{*2}}\int_{t_0}^{t} f\,d\tau\right)=0; \qquad \dot{g}+\frac{1}{2T^*}\left(g+\frac{1}{4T^*}\int_{t_0}^{t} g\,d\tau\right)=0, \quad T^*>0. \tag{2.2a; b}$$

These stabilizations certainly increase the order of a differential equations system--with the additional differential equation

$$\dot{z}=f(x,t) \quad or \quad =g(x,\dot{x},t)$$

one obtains the integrations necessary with z using the combinations, but the substitute time-constant "T*" only accounts for the process of the "real", i.e., error-ridden, integration.

An optimum value can be given even for quantity T* , whereby essentially the salient frequencies of the then adaptive regulator, as a function of the highest occurring frequency in the mechanical system, are conducted (OSTERMEYER [4]). Thus the parameters in the stabilizations are completely determined.

Even first integrals can be stabilized with the integration of a given mechanical system (BAUMGARTE [5]). The integrals $I_\nu(x,\dot{x},t)=0$ are hereby interpreted as non-linear non-holonomic combinations, just as in the velocities. From the Gaussian principle of the smallest limit, one can formally obtain the motion equation with the "limit"

$$Z=\sigma I_{\dot{x}}. \tag{2.3}$$

The usual calculation of the Lagrangian multiplicator from

$$\dot{I} = 0 \tag{2.4}$$

yields, of course, $\sigma \equiv 0$, since, according to the definition, the motion equations $\dot{I} = 0$ are identically realized. However, if the multiplier σ is calculated from $\dot{I}_\nu$ in the stabilized form (2.1 b), (2.2 b), one obtains an algebraic expression for σ. This expression is a pure regulator-term and disappears identically upon the exact solution.

The stabilizations are a simple aid for controlling numerical instabilities with great accuracy and little effort.

But some questions still remain (see [4]). Although the procedure of stabilization seems very plausible, it has not yet been determined in a general fashion whether and in what sense the numerical solution of (1.5) with stabilized combinations is altogether better than the numerical solution of (1.5) without stabilization, if one disregards the component of the total numerical error, which makes itself noticeable in the disturbance of the combinations.

3. Quasi-Generalized Variables /473

A differential equation system is given in the variables $x = x(t)$ with $x^T = (x_1, \ldots, x_n)$:

$$\dot{x} = F(x, t), \quad t \in I = [t_0, t_1], \quad \text{with } x(t = t_0) = x_0 \in R^n. \tag{3.1), (3.2}$$

Apply an additional combination to this system

$$f(x, t) = 0, \quad \text{with } f(x_0, t_0) = 0. \tag{3.3}$$

There are two classic means in mechanics to obtain a solution to this problem.

A. Reduction

Here the generalized variable $y(t), y^T = (y_1, \dots, y_{n-1})$, generalized on n - 1, which covers the subspace given by (3. 3), is transformed. With

$$x = x(y, t), \qquad x_y \cdot f_x \equiv 0 \tag{3.4a}$$

one obtains the differential equations

$$x_y^T \dot{y} + x_t = F(x(y, t), t), \quad or \quad \dot{y} = (x_y \cdot x_y^T)^{-1} x_y (F - x_t). \tag{3.4b}$$

B. Lagrangian Multiplier Technique

Here, the minimum of the quadratic form Z

$$Z = \tfrac{1}{2} (\dot{x} - F)^T (\dot{x} - F) \tag{3.5}$$

with regard to the quantity $\dot{x}$ is sought under the secondary condition (3.3). Once (3.3) has been totally derived according to time, then one obtains a linear combination for $\dot{x}$. With the Lagrangian multiplier μ, one obtains from

$$\frac{\partial}{\partial \dot{x}} (Z - \mu \dot{f}) = 0 \tag{3.6}$$

the differential equation system

$$\dot{x} = F + \mu f_x, \qquad f(x, t) = 0. \tag{3.7 a, b}$$

If one inserts (3.7a) into $\dot{f} = 0$, one has an algebraic equation for calculating μ. The integration of the n differential equations

$$\dot{x} = F - \frac{f_x F + f_t}{f_x^T \cdot f_x} f_x \tag{3.8}$$

yields the solution to the problem.

Procedure A yields numerical solutions which conform exactly to the combinations. The disadvantage of this procedure is that, under certain circumstances, the search for generalized variables is not very easy.

Procedure B has the advantage that one simply obtains the resulting differential equation system. The numeric instability of the combination is disadvantageous.

Principle procedure C, submitted here--introduction of quasi-generalized variables, see below--no longer considers a differential equation system with the actual involved variables x and y, but rather introduces auxiliary variables $\bar{x}, \bar{x}^T = (\bar{x}_1, \ldots, \bar{x}_n)$, which are unrelated, but which contain the involved related variables x (i.e. x with $f(x,t) \equiv 0$) in a simple fashion. The differential equation systems pertaining to $\bar{x}$ should be easily constructed and formed in such a way that, after numerically solving the differential equations in $\bar{x}$, one may transform from (3.8) into the numerical error-free solution of x regarding the combination.

The point of departure is the formulation of the problem in the form (3.7) or (3.8). If $\lambda = \lambda(t)$ is an arbitrary (continuous) function of time, it is valid that:

$$\dot{x} = F + \dot{\lambda} f_x + (\mu - \dot{\lambda}) f_x . \tag{3.9}$$

With the transformation

$$\bar{x} = x - \lambda f_x \tag{3.10}$$

one obtains further

$$\dot{\bar{x}} = F - \lambda(f_x)^{\cdot} + (\mu - \dot{\lambda})\, f_x \,. \tag{3.11}$$

If one executes the differentiation of the term (f_x) and re-inserts (3.7), (3.11) is simplified to

$$\dot{\bar{x}} = (I - \lambda f_{xx})\,(F + \mu f_x) - \dot{\lambda} f_x - \lambda f_{xt} \,. \tag{3.12}$$

Since on this premise, the analytical solution x of (3.7) or (3.9) of the combination conforms exactly to f(x,t) = 0, the following is valid using (3.10)

$$f(x, t) = f(x(\bar{x}, \lambda), t) \equiv 0 \,. \tag{3.13}$$

Using (3.10), the right side of (3.12) can be completely formulated in the new variable $\bar{x}$. According to the construction, $\bar{x}$ for $\lambda \neq 0$ no longer lies on the hypersurface given by the combination.

Until now λ can be set arbitrarily. The solution of /474
(3.12) in the new variable $\bar{x}$ contains the first integral (3.13) $f(\bar{x}, \lambda, t) = 0$. How should λ be suitably chosen? It is possible, here for instance, to set $\dot{\lambda} \equiv 0$. Then, $\bar{x} \equiv x + c \cdot f_x$ with the set constant c is valid for the analytical solution. If one is to insert the constant c for λ into the equation resulting from (3.10)

$$\dot{\bar{x}} = (I - \lambda f_{xx})\,(F + \mu f_x) - \lambda f_{xt} \tag{3.14}$$

the calculated solution for x obtained from the numerical solution for $\bar{x}$ in (3.14) using (3.10) disturbs the combination as much as the numerical solution of the starting system (3.7) or (3.8). Up to now one has only executed a coordinate transformation which leaves the same numerical difficulties as before, because the numerical solution of (3.14) itself disturbs

its first integral (3.13). But only the exact satisfaction of (3.13) with the integration of (3.14) guarantees that the re-transformation of arbitrary values for $\bar{x}$ to the starting coordinates x excludes errors in the combination f(x,t) = 0 or x.

The first integral (3.13) can be used to eliminate and in (3.12). This reduction (of the arbitrary choice of λ) has the result that (3.13) is implicitly considered in (3.12) and that thus (3.13) can no longer be disturbed by the numerical solution of (3.12). Furthermore, using simple calculation, one finds the identity

$$\lambda \equiv \mu, \tag{3.15}$$

so that the motion equations now have the following form

$$\ddot{\bar{x}} = (I - \lambda f_{xx})\, F - \lambda(\dot{\lambda} f_{xx} \cdot f_x + f_{xt}), \quad \lambda = \lambda(\bar{x}, t) \tag{3.16 a, b}$$

The analytical expression for $\dot{\lambda}$ can now be calculated from the total derivation of (3.16 b) to t, or, for instance, from the total derivation of (3.13) to t. (This is equivalent to the transformation of μ in (3.7), (3.8) into $\bar{x}, \lambda$). The first method presumes that λ can be indicated explicitly from the implicit system of equations (3.10), (3.13). The second method yields an explicit expression for $\dot{\lambda}$ even when λ is numerically iteratively established (see sections 4, 5).

This choice of procedure for λ corresponds in principle to the possibility to partially integrate the Lagrangian multiplier μ in (3.7a). (See Baumgarte, Ostermeyer [6]).

In this sense, n variables $\bar{x}$ are quasi-generalized variables, since the differential equations (3.16) formulated by them no longer contain the classic "limit" term μf_x and since they implicitly contain the relevant solution x, which exactly satisfies the combination after the construction.

The last point is given inasmuch as the first integral (3.13) regarding the variable $\bar{x}$ through the reduction regarding the freedom in the choice of λ is also exactly numerically satisfying.

C. Procedure for Induction of Quasi-Generalized Variables

The new variable $\bar{x}, \bar{x}^T = (\bar{x}_1, \ldots, \bar{x}_n)$ on n is here transformed with

$$\bar{x} = x - \lambda f_x\,, \qquad f(x,t) = 0\,, \qquad x = x(\bar{x}, \lambda, t)\,, \qquad \lambda = \lambda(\bar{x}, t)\,. \tag{3.17}$$

The differential equation system is

$$\dot{\bar{x}} = (I - \lambda f_{xx})\,F - \lambda(\dot{\lambda} f_{xx} f_x + f_{xt})\,. \tag{3.18}$$

These are n first-order differential equations in $\bar{x}$. The relevant solution x is established after the (numerical) integration of (3.18), x is error-free regarding the combination f(x,t). The starting quantities of $\bar{x}$ are calculated from x_0 using (3.17), and calculated using an arbitrary value $\lambda(t_0)$. For the choice $\lambda(t_0) = 0$ it is true that

$$\bar{x}(t = t_0) \equiv x(t = t_0)\,. \tag{3.19}$$

4. Treatment of First Integrals

If, for a differential equation system

$$\dot{x} = F(x,t) \tag{4.1}$$

the combination f(x,t) = 0 is a first integral (f is then a so-called inner connection), then $f_x\dot{x} + f_t = f_x F + f_t \equiv 0$ is valid, and the Lagrangian multiplier is $\dot{\lambda} \equiv 0$ with the tranformation into quasi-generalized variables. The differential equation system (4.1) is transformed with $\bar{x} = x + \lambda f_x$ into

$$\dot{\bar{x}} = (I - \lambda f_{xx})\, F - \lambda f_{xt}\,. \tag{4.2}$$

Example 1: The Quaternion Differential Equation

While describing the position of a fixed body or a gyroscope, with a given turning vector

$$\omega = \omega(t)\,, \qquad \omega^T = (\omega_1, \omega_2, \omega_3)$$

singularities occur in Euler's angles ("frame-locking"). In order to avoid these difficulties, other position coordinates are often introduced.

The turning quaternions $x, x^T = (x_1, x_2, x_3, x_4)$. constitute one possibility. The related differential equation is /475

$$\dot{x} = \tfrac{1}{2}\,\Omega x \tag{4.3}$$

with

$$\Omega = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{4.4}$$

The quaternions satisfy the first integral

$$f(x) = x^2 - 1 = 0\,. \tag{4.5}$$

Numerical integration of the system (4.3) disturbs the combination (4.5) linearly in t. The procedure C (see section 3) leads, using the transformation

$$\bar{x} = x - \lambda f_x = x - 2\lambda x\,, \qquad x = \bar{x}/(1 - 2\lambda) \tag{4.6}$$

to the new differential equation system

$$\dot{\bar{x}} = \tfrac{1}{2}(I - 2\lambda I)\,\Omega \cdot \frac{\bar{x}}{(1-2\lambda)} \quad \text{bzw.} \quad \dot{\bar{x}} = \tfrac{1}{2}\,\Omega\bar{x}\,. \tag{4.7}$$

The differential equation system in the quasi-generalized variables (4.7) has the same form as the starting system.

For the re-transformation into the starting variable, λ is needed. The insertion of (4.6) into (4.5) yields:

$$\lambda = \tfrac{1}{2}(1 - |\bar{x}|)\,. \tag{4.8}$$

Therefore

$$x = \bar{x}/|\bar{x}|\,. \tag{4.9}$$

The rule used to divide quaternions by their norm, after every step of integration, to take the next step of integration with the new starting values for the quaternions x, has been known for a long time as a numerical variant of the integration of the differential equation (4.3). (4.9), however, signifies more: when dealing with the problem (4.3), formulated in the quasi-generalized variables (4.7), one may not interfere in the integration. One only has to realize the depiction onto the starting quaternions at those points of time when the solution is relevant. The resulting x vector is, independent of these points of time, a numerically correct solution of (4.3), which exactly satisfies its inner combinations.

Example 2: The Unperturbed Kepler Problem

With $x = (x_1, x_2, x_3)$ and r := $|x|$, the motion equations are

$$\ddot{x} + \frac{K^2}{r^3}x = 0\,, \quad K^2 = \text{parameter of gravitation} \tag{4.10}$$

The analytical solution to the unperturbed Kepler problem can be given immediately. In practice, one will often work with a mildly perturbed Kepler problem (see Stiefel, Scheifele [8]).

However, the numerical difficulties with integration of Kepler equations may be examined with particular ease in the unperturbed problem.

An essential difficulty with the integration of (4.10) is the upset of the energy integral

$$f = \tfrac{1}{2}\dot{x}^2 - \frac{K^2}{r} - E = 0, \quad E = const. < 0. \tag{4.11}$$

Since the energy determines the frequency of revolution ω, Kepler's orbit will be only mildly perturbed with numerical errors in (4.11)--the Kepler problem is stable in orbit--but the error in calculating the location of the satellite in orbit increases quadratically with time.

The consideration of (4.11) using quasi-generalizeed variables in procedure C can be executed when (4.10) is written as a first-order differential equation system.

$$\dot{x} = v, \quad \dot{v} = -\frac{K^2}{r^3}x \tag{4.12 a, b}$$

with

$$f(x, v) = \tfrac{1}{2}v^2 - \frac{K^2}{\sqrt{x^2}} = E. \tag{4.12 c}$$

The transformation could be executed with regard to x and v. Consequently, one has a form of the gradient procedure.

However, the principles of mechanics indicate that--(4.11) being understood as a non-holonomic combination--the combination is only considered in the equations (4.12 b), meaning that only the quantities v have to be transformed into the quasi-generalized variable $\overline{v}$.

Using /476

$$\overline{v} = v - \lambda v, \quad v = \overline{v}/(1 - \lambda) \tag{4.13}$$

can be calculated from (4.12 c), using the abbreviation V for the potential

$$V = -K^2/|x| \tag{4.14}$$

the multiplier λ into

$$\lambda = 1 - |\overline{v}|/\sqrt{2(E-V)} \tag{4.15}$$

and thus the transformed motion equations into

$$\dot{x} = \frac{\overline{v}}{|\overline{v}|}\cdot\sqrt{2(E-V)}, \qquad \dot{\overline{v}} = -\frac{|\overline{v}|}{\sqrt{2(E-V)}}\cdot\frac{K^2}{r^3}x\,. \tag{4.16 a, b}$$

Re-transformation into the variables (x, v) yields solutions to (4.12) which exactly satisfy the energy integral.

Example 3: The Perturbed Kepler Problem

Generally, one always has to work with a mildly perturbed problem:

$$\ddot{x} + \frac{K^2}{r^3}x = \varepsilon Q(x, \dot{x}, t)\,. \tag{4.17}$$

If the forces εQ are potential, the energy integral (4.12 c), with additional potential terms, continues to be valid. The transformed equations keep the structure (4.16). If the terms εQ are non-potential forces, the energy integral is no longer valid. With expansion, however, of the motion equations, the transformation can even be executed here. Using

$$\dot{x} = v\,, \qquad \dot{v} = -V_x + \varepsilon Q\,, \qquad \dot{E} = \varepsilon Q v \tag{4.18 a, b, c}$$

one again induces a first integral:

$$f(x, v) = \tfrac{1}{2}v^2 + V(x) - E \equiv 0\,. \tag{4.18 d}$$

As the errors in the additional scalar differential equation (4.18 c) are significantly smaller, using the numercial interpretation, than the errors in the system (4.18) with regard to the integral (4.18d), the transformation into quasi-generalized variables yields, even here after re-transformation, significantly better results. The transformed system reads

$$\dot{x} = \frac{\overline{v}}{|\overline{v}|}\sqrt{2(E-V)}, \qquad \dot{\overline{v}} = \frac{|\overline{v}|}{\sqrt{2(E-V)}}(-V_x + \varepsilon Q), \qquad \dot{E} = \varepsilon Q \frac{\overline{v}}{|\overline{v}|}\sqrt{2(E-V)}. \tag{4.19}$$

Example 4: The Symmetric Gyroscope

With the principle moments of inertia $I_1 = I_2, I_3$, which belong to the center of gravity (fulcrum), and with the components of the angular velocity with regard to the solid principle axis of inertia, Euler's equations read

$$I_1\dot{\omega}_1 = K\omega_2, \qquad I_1\dot{\omega}_2 = -K\omega_1, \qquad I_3\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \omega_3^0, \qquad K = \frac{I_1 - I_3}{I_3}\omega_3^0. \tag{4.20 a, b, c}$$

The system contains the first integral

$$f(\omega) = \tfrac{1}{2} I_1(\omega_1^2 + \omega_2^2) - \widetilde{E} = 0.$$

Using the transformation

$$\omega_1 = \frac{\overline{\omega}_1}{(1-\lambda I_1)}, \qquad \omega_2 = \frac{\overline{\omega}_2}{(1-\lambda I_1)}$$

follows, from (4.20), the system

$$I_1\dot{\overline{\omega}}_1 = K\overline{\omega}_2, \qquad I_1\dot{\overline{\omega}}_2 = -K\overline{\omega}_1, \qquad I_3\dot{\omega}_3 = 0. \tag{4.21}$$

Here again, one obtains the starting system, formulated with the new variables. The re-transformation reads

$$\omega_i = \sqrt{\frac{2E}{I_1(\overline{\omega}_1^2 + \overline{\omega}_2^2)}}\,\overline{\omega}_i, \qquad i = 1, 2. \tag{4.22}$$

Concerning the practically important problem of the perturbed gyroscope, one can proceed similarly to deliberations of the perturbed Kepler problem. With the vector of perturbation $N = (N_1, N_2, N_3)^T$, the system now reads

$$I_1\dot{\omega}_1 = K\omega_2 + \varepsilon N_1, \quad I_1\dot{\omega}_2 = -K\omega_1 + \varepsilon N_2, \quad K = \frac{I_1 - I_3}{I_3}\omega_3, \quad \omega_3 \neq \text{const.}, \tag{4.23}$$
$$I_3\dot{\omega}_3 = +\varepsilon N_3, \quad \tilde{E} = \varepsilon(N_1\omega_1 + N_2\omega_2).$$

The integral is

$$f = \tfrac{1}{2} I_1(\omega_1^2 + \omega_2^2) - \tilde{E} = 0. \tag{4.24}$$

The transformation into quasi-generalized variables $\bar{\omega}_1$, $\bar{\omega}_2$ leaves the structure of system (4.23) unchanged, as above. Re-transformation (4.22) into the values ω_1, ω_2 after numerical integration of (4.23), is error-free after construction with regard to (4.24). /477

5. Concerning Numerics with Non-Explicitly Realizable Transformation into Quasi-Generalized Variables

In all the previously examined examples, the transformation could be explicitly executed. The non-symmetric gyroscope constitutes an example with which this no longer seems possible. For purposes of simplification, let us examine it here in its unperturbed form. The perturbed case can easily be examined as demonstrated above.

Example 5: The Non-Symmetrical Gyroscope

With the terms of the previous example, Euler's equations here read

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2. \quad (4.22\ a,\ b,\ c)$$

The system contains two first integrals:

$$f_1 = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) - C_1 = 0\,, \qquad f_2 = \tfrac{1}{2}(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2) - C_2 = 0\,. \quad (4.23)\,,\ (4.24)$$

Here C_1 and C_2 are constants for the unperturbed case. The specification for transformation reads

$$\bar{\omega}_i = \omega_i - \lambda_1 I_i\omega_i - \lambda_2 I_i^2\omega_i\,, \qquad i = 1, 2, 3\,. \quad (4.25)$$

The motion equation with quasi-generalized variables reads

$$I_1\dot{\bar{\omega}}_1 = \frac{(1 - \lambda_1 I_1 - \lambda_2 I_1^2)(I_2 - I_3)\,\bar{\omega}_2\bar{\omega}_3}{(1 - \lambda_1 I_2 - \lambda_2 I_2^2)(1 - \lambda_1 I_3 - \lambda_2 I_3^2)}\,. \quad (4.26)$$

The remaining equations can be obtained via cyclical permutations.

In (4.26), however, the Lagrangian multipliers λ_1, λ_2 are still to be calculated using (4.23) and (4.24). This calculation can be performed numerically. One must consider, however, that with every new call-up of the differential equation (4.26), through the numerical routine of integration, the actual multipliers

$$f_1(\bar{\omega}_i, \lambda_1, \lambda_2) = 0\,, \qquad f_2(\bar{\omega}_i, \lambda_1, \lambda_2) = 0\,, \qquad \lambda_1 = \lambda_1(\bar{\omega}_i)\,, \qquad \lambda_2 = \lambda_2(\bar{\omega}_i) \quad (4.27)$$

should be established via an iteration procedure.

It may also occur that not only the Lagrangian multipliers λ, but even the initial variables x as a function of the quasi-generalized variables $\bar{x}$ cannot be explicitly established. In this case, the (n + m) equations

$$\bar{x} - x - f_x\lambda = 0\,, \qquad f(x, t) = 0\,, \qquad f^T = (f_1, \ldots, f_m)\,, \qquad \lambda^T = (\lambda_1, \ldots, \lambda_m) \quad (4.28\ a,\ b)$$

have to be solved iteratively into x and λ to establish the differential equation system

$$\dot{\bar{x}} = F(\bar{x}, x, \lambda, t) \tag{4.29}$$

for the integration routine.

To avoid difficulties while searching for the zero digit, as well as long iteration periods, the following procedure has proved effective.

The starting quantities of the quasi-generalized variables are defined with only the exception of a free choice of starting quantities of λ. For purposes of simplification, $\lambda(t = t_0) = 0$ is given (see (3.19)). If the transformations into x and λ following from (4.28) can be explicitly executed, it will be evident that the Lagrangian multipliers increase, in general, linearly when using the numerical integration of (4.29), and even quadratically, when using the quasi-generalized variables for holonomic combinations (see section 6).

If one now evaluates the system (4.28) during and after the first step of integration Δt, the desired solution vector $x(t_0 + \Delta t)$ is present. At the beginning of the next step of integration, the vector value λ can again be designated zero, $\lambda(t_0 + \Delta t) = 0$. With the vector of solution $x(t_0 + \Delta t_0)$, one obtains, using (4.28), the initial quantities $\bar{x}(t_0 + \Delta t)$ for the next step of integration.

If one proceeds in such a way through every step of integration, vector λ remains very small. Numerical experiments have indicated that, using this procedure to evaluate equations (4.28), only one or two steps of a Newtonian procedure are needed to obtain very high accuracies--for instance, to obtain the factor 10^5 or 10^{10} times better than the stabilization in section 2. Using our integrational procedure, RKF 7 [8], the differential equation system is evaluated at a total of 13 times per step of integration. Having executed each time the iteration of (4.28), the resulting calculation time was slightly

longer, by a factor of 1.5 than the integration of the starting differential equation system.

6. Treatment of Holonomic Connections

The holonomic combination is f(x,t) = 0. If one examines it in derived form like a non-holonomic combination using non-holonomic velocity parameters, one obtains the system (1.9). f(x,t) = 0 is then a first integral of equations (1.9) and can be treated with the procedures in sections 4 and 5.

If f(x,t) = 0 is examined using a limit in the motion equations (see (1.5)), these read: /478

$$M(x,t)\ddot{x} = F(x,\dot{x},t) + \mu f_x.$$

or, as a system of the first order

$$\dot{x} = M^{-1}v\,, \qquad \dot{v} = F(x,vt) + \mu f_x\,. \tag{6.1 a, b}$$

In this instance, transformation into a quasi-generalized variable is also possible. Using

$$\bar{x} := x - \lambda f_x \tag{6.2}$$

one again obtains, using the secondary condition

$$x = x(\bar{x},\lambda,t)\,, \qquad \lambda = \lambda(\bar{x},t)\,. \tag{6.3 a, b}$$

In Baumgarte, Ostermeyer [7] it is shown that conversion into quasi-generalized variables $\bar{x}$ converts the equation (6.1) into the following system

$$\dot{\bar{x}} = M^{-1}(\bar{v} + \lambda(f_x)^{\cdot})\,, \qquad \dot{\bar{v}} = F + \dot{\lambda}(f_x)^{\cdot} \tag{6.4 a, b}$$

using $\bar{v} = v + \dot{\lambda}f_x$, $(f_x)^{\cdot} = f_{xx}(M^{-1}\bar{v} - \dot{\lambda}f_x) + f_{xt}$.

It is given that $\ddot{\lambda} \equiv \mu$. In this instance, the function x, established via numerical solution of (6.4) and (6.3), is also a solution to initial system (6.1) which is error-free with regard to the combinations f(x,t) = 0. In the system (6.4), λ can be calculated either by total derivation of λ or from the derived combination $\dot{f} = 0$. Since the derived combination is linear in its velocities, transformations into v can always be explicitly executed:

$$v = v(\bar{v}, \dot{\lambda}, x, t) = \bar{v} - \lambda f_x, \qquad \dot{\lambda} = \dot{\lambda}(\bar{v}, x, t) = \frac{f_x \bar{v} + f_t}{f_x \cdot f_x}.$$

If this is substituted into (6.4), only the calculation of $x(\bar{x}, \lambda t)$ and $\lambda = \lambda(\bar{x}, t)$ remains. If this is not explicitly possible, one may proceed numerically in this instance, as demonstrated in section 5.

REFERENCES

1. HAMEL, G.; Theoretische Mechanik (Theoretical Mechanics), Springer Verlag, (1967).

2. BAUMGARTE, J.; Stabilization of Constraints and Integrals of Motion in Dynamical Systems, Comp. Meth. Appl. Mech. and Eng., 1, (1972), 1-16.

3. OSTERMEYER, G.P.; The Stabilization of Constraints and first Integrals as control problems and their consequences, ZAMM* 65, (1985), 3, 185-187.

4. OSTERMEYER, G.P.; On a Theory of BAUMGARTE's Stabilization Procedures, in preparation.

5. BAUMGARTE, J.; Asymptotic Stabilization of Integrals for Common Differential Equations of the First Order, ZAMM 53, (1973) 701-704.

6. BAUMGARTE, J. and OSTERMEYER, G.P.; On the Numerically Exact Consideration of Internal and External Constraints, Gammtagung 1984, appeared in ZAMM

7. BAUMGARTE, J. and OSTERMEYER, G.P.; Elaborations on the D'Alembert Principle, in preparation.

8. STIEFEL, E.L. and SCHEIFELE, G.; Linear and Regular Celestial Mechanics, Springer Verlag, Berlin, 1971.

Submitted July 23, 1984

Address: Prof. Dr.-Ing. J. BAUMGARTE, and Dr.-Ing. G.P. OSTERMEYER, Mechanikzentrum, Technische Universitaet, D-3300 Braunschweig, Post Box 3329, Federal Republic of Germany

* ZAMM = Zeitschrift fuer angewandte Mathematik und Mechanik (Journal of Applied Mathematics and Mechanics)

End of Document