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ON THE EXISTENCE OF SOLUTIONS OF AN EQUATION ARISING IN
THE THEORY OF LAMINAR FLOW IN A UNIFORMLY POROUS CHANNEL WITH
INJECTION

Ke-Gang Shih

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Ke-Gang Shih

Institute for Computer Applications in Science and Engineering

ABSTRACT

We establish the existence of concave solutions of Berman's equation which describes the laminar flow in channels with injection through porous walls. It is found that the (unique) concave solutions exist for all injection Reynolds number $R < 0$.

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I. INTRODUCTION

Studies of the transpiration- or sweat-cooling of heated surfaces such as turbine blades, rocket walls, or wing surfaces in high-speed flight by the diffusion of fluid through porous-metal combustion-chamber liners have been made within the framework of the boundary-layer flow along a porous plate with fluid injection. Laminar flow in a two-dimensional rectangular channel with porous walls was studied by Berman in 1953. He showed that the corresponding Navier-Stokes equations can be reduced to a nonlinear third order ordinary equation with two point boundary conditions and Reynolds number R based on injection-velocity. Berman also gave perturbation results for extremely small R . In later years, various asymptotic results and numerical results (e.g., Sellars (1955), Yuan and Finkelstein (1956), Terrill (1964), Raithby (1971), and Robinson (1976)) were given. Asymptotic results have of necessity been purely formal, since the existence of a solution has not been proved nor the necessary estimates obtained.

Skalak and Wang (1978) classified all possible solutions of Berman's equation, both for injection and suction. It was shown that there is only one possible solution for the injection problem. In this note we establish the existence of this solution. In Section 2, we derive the Berman's equation. In Section 3, we give some preliminary results and several important a-priori estimates of the solutions. In particular, we show that the maximum axial velocity in the channel is insensitive to the Reynolds number R . In Section 4, we use the Leray-Schauder Fixed Point Theorem to establish the existence of solutions. In Section 5, we give some numerical results which confirm our a-priori estimates and agree with the asymptotic results obtained by Yuan and Finkelstein (1955) as $R \rightarrow -\infty$.

II. DERIVATION OF BERMAN'S EQUATION

Consider the steady, incompressible, laminar flow along a two-dimensional channel with porous walls through which fluid is injected with uniform speed v_w . Take x and y to be coordinate axes parallel and perpendicular to the channel walls and assume u and v are the velocity components in the x and y directions respectively. Letting the channel width be $2h$ and introducing the dimensionless variable

$$\eta = \frac{y}{h},$$

reduce the Navier-Stokes equations to

$$(1) \quad u \frac{\partial u}{\partial x} + \frac{u}{h} \frac{\partial u}{\partial \eta} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 u}{\partial \eta^2} \right)$$

$$(2) \quad u \frac{\partial v}{\partial x} + \frac{v}{h} \frac{\partial v}{\partial \eta} = - \frac{1}{h\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2} \frac{\partial^2 v}{\partial \eta^2} \right)$$

where ρ , p , and ν are the density, pressure, and kinematic viscosity of the fluid respectively.

The continuity equation is

$$(3) \quad \frac{\partial u}{\partial x} + \frac{1}{h} \frac{\partial v}{\partial \eta} = 0.$$

With the walls of the channel at $y = \pm h$; i.e., $\eta = \pm 1$, the boundary conditions are

$$u(x, \pm 1) = 0; \quad v(x, 0) = 0,$$

(4)

$$v(x, \pm 1) = \text{constant} = v_w < 0; \quad \frac{\partial u}{\partial y}(x, 0) = 0.$$

For a two-dimensional incompressible flow, a stream function ψ exists such that

$$(5) \quad u(x, \eta) = \left(\frac{1}{h}\right) \frac{\partial \psi}{\partial \eta}$$

$$(6) \quad v(x, \eta) = - \frac{\partial \psi}{\partial x}$$

and the continuity equation, (3), is satisfied.

Since the flow is symmetrical about a plane midway between the walls, the solution will be investigated over half the channel, i.e., from the midplane to one wall.

A suitable stream function is

$$(7) \quad \psi(x, \lambda) = [h v(0) - v_w x] f(\eta)$$

where $v(0)$ is the entrance velocity at $x = 0$. Thus the velocity components u and v are given by

$$(8) \quad u(x, \eta) = h^{-1} [h v(0) - v_w x] f'(\eta),$$

$$(9) \quad v(x, \eta) = v_w f(\eta).$$

Substituting (8), (9) into (1) and (2), we have

$$(10) \quad \frac{\partial^2 p}{\partial x \partial \eta} = 0$$

$$(11) \quad f'''' + R(f'^2 - ff'') = K$$

where $R \equiv v_w h / \nu < 0$ is taken to be the injection Reynolds number of the flow and K is an integration constant, to be determined later as a function of R . The boundary conditions (4) become

$$(12) \quad f(0) = 0, \quad f''(0) = 0,$$

$$(13) \quad f'(1) = 0, \quad f(1) = 1.$$

Thus, the nonlinear two-point boundary value problem we are considering consists of (11), (12), and (13) with $R < 0$. By a solution of (11), (12), and (13) we mean a function $f \in C^3[0, 1]$ and a constant K which satisfy (11), (12), and (13).

Remark: It is easy to see that once we have a $C^3[0, 1]$ solution f , then $f \in C^k[0, 1]$, $k \geq 0$.

III. PRELIMINARY RESULTS

If we differentiate (11), we get

$$(14) \quad f^{(iv)} + R(f'f'' - ff''') = 0.$$

Lemma 15: If f is a solution of (11), (12), and (13), then $f^{(iv)}$ is increasing on $[0, 1]$.

Proof: It is sufficient to show that $f^{(iv)}(\eta) \geq 0$ on $[0, 1]$.
Differentiating (14) again, we get

$$(16) \quad f^{(v)} = -Rf^{''-2} + Rff^{(iv)}.$$

From boundary conditions (12), $f^{(iv)}(0) = 0$. (16) shows that at any point c where $f^{(iv)}(c) = 0$, $f^{(v)}(c) = -Rf^{''-2}(c) \geq 0$. Hence, $f^{(iv)}(\eta) \geq 0$ $\forall \eta \in [0, 1]$.

Remark. We have shown also that $f'f'' - ff^{(iv)} \geq 0 \quad \forall \eta \in [0, 1]$.

Lemma 17: Let $f'(0) = \alpha$, $f^{(iv)}(0) = \beta$. Then for $R < 0$,

$$(i) \quad \sqrt{3} \geq \alpha \geq \frac{3}{2}; \quad \beta \leq -3; \quad -3 \leq f''(1) \leq -\frac{9}{4}$$

$$(ii) \quad 0 \leq f'(\eta) \leq \alpha; \quad \beta \leq f^{(iv)}(\eta) \leq 0; \quad f''(1) \leq f''(x) \leq 0, \quad \text{on } [0, 1].$$

Proof: Let $G_4(\eta, t)$ be the Green's function of

$$(18) \quad f^{(iv)} = 0;$$

and

$$(19) \quad v(0) = v'(0) = v(1) = v'(1) = 0,$$

which is given by

$$(20) \quad G(\eta, t) = \frac{1}{12} \begin{cases} \eta(t-1)^2 [(3 - \eta^2)t - 2\eta^2], & 0 \leq \eta \leq t, \\ t(\eta - 1)^2 [(3 - t^2)\eta - 2t^2], & t \leq \eta \leq 1. \end{cases}$$

Then the solution of (14) can be written as

$$(21) \quad f(\eta) = R \int_0^1 G(\eta, t) (ff'''' - f'f'') dt + f_0(\eta),$$

where

$$(22) \quad f_0(\eta) = \frac{\eta}{2} (3 - \eta^2).$$

From (20), we can write

$$(23) \quad \alpha = \frac{3}{2} + \frac{1}{4} R \int_0^1 t(1 - t)^2 (ff'''' - f'f'') dt$$

$$(24) \quad \beta = -3 - \frac{R}{2} \int_0^1 (2 + t)(1 - t)^2 (ff'''' - f'f'') dt$$

$$(25) \quad f''(1) = -3 + \frac{R}{2} \int_0^1 t(1 - t^2) (ff'''' - f'f'') dt.$$

By the remark following (16), we conclude that

$$(26) \quad \alpha \geq \frac{3}{2}, \quad \beta \leq -3, \quad f''(1) \geq -3.$$

From Lemma 15 and equation (11), we have

$$f'''(0) = \beta = K - R\alpha^2 \leq f'''(1) = K + Rf''(1).$$

Thus,

$$(27) \quad f''(1) \leq -\alpha^2 \leq -\frac{9}{4} < 0.$$

Combining (26) and (27), we get (i). Now we will prove that $f''(\eta)$ is monotonically decreasing on $[0, 1]$.

From (20) and (21), it is easy to show that

$$(28) \quad f'''(\eta) < 0 \quad \forall \eta \in [0, 1].$$

Indeed, should $f'''(c) = 0$ at $\forall c \in [0, 1]$. Assume c is the first such point; then

$$f^{(iv)}(c) = (-R)f'f''(c) < 0,$$

which contradicts the fact that $f^{(iv)} \geq 0$ on $[0, 1]$. Hence,

$f''(\eta) < 0$ and both f'' and f' are monotonically decreasing on $[0, 1]$.

f' looks like

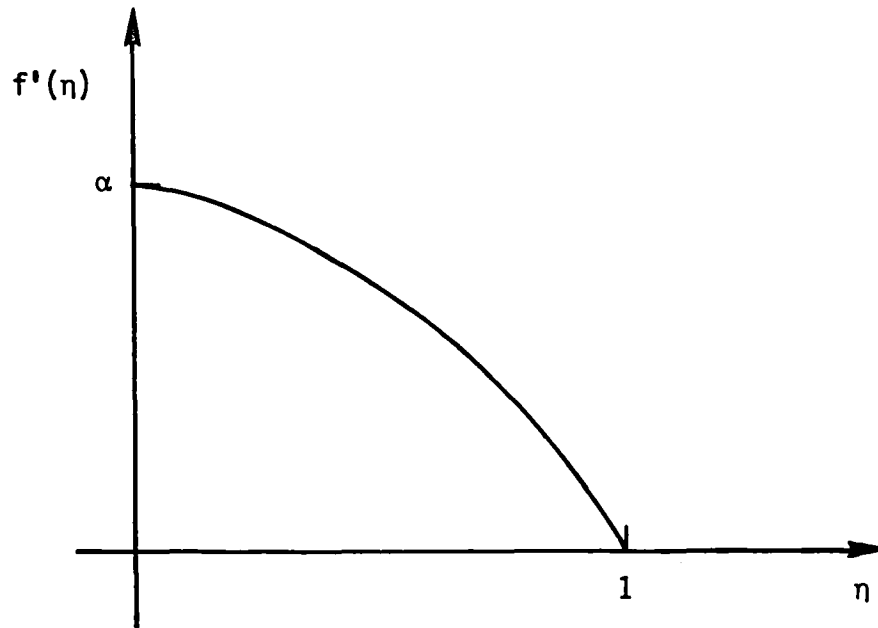


Figure 1

Similarly, f looks like

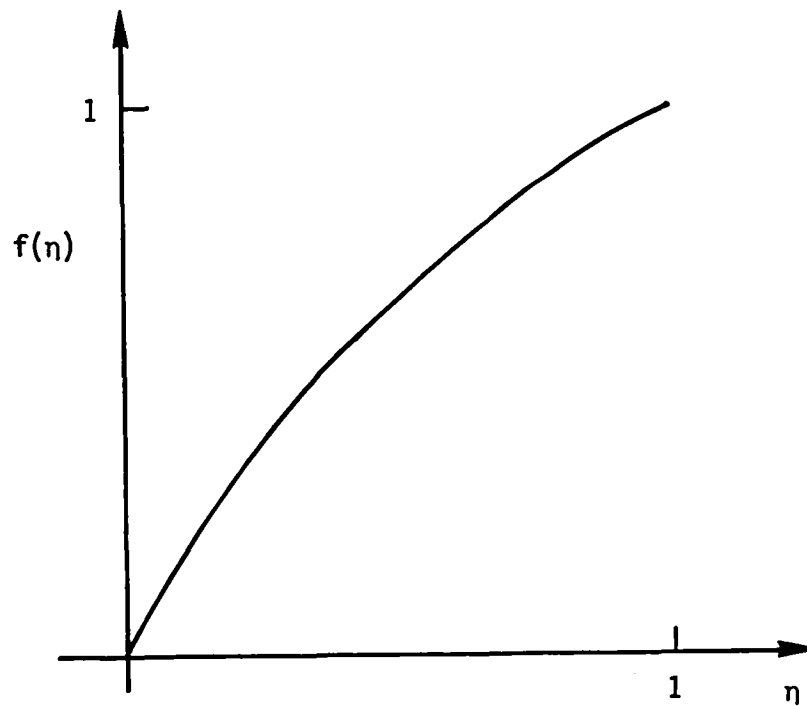


Figure 2

This proves (ii).

Now let us get a rough lower bound for β . By the integral representation (24), since $(2+t)(1-t^2) \leq 3$ for $t \in [0, 1]$, we have

$$(29) \quad \beta \geq -3 - \frac{3}{2} R \int_0^1 (ff'''' - f'f'') dt = -3 - \frac{3}{2} R(f''(1) + \alpha^2).$$

We can get other lower bounds on β , e.g., by integrating (11) once and use (12), (13) and the fact that

$$(30) \quad K = \beta + R\alpha^2$$

we get

$$(31) \quad \beta \geq -2\alpha + R(\alpha^2 - 1).$$

We prefer (29) to (31), because from the asymptotic expansions of $f(\eta)$ as $R \rightarrow -\infty$ given by Yuan and Finkelstein (1956), we have

$$f''(1) + \alpha^2 = O\left(\frac{1}{R}\right) \quad \text{as } R \rightarrow -\infty;$$

then by (29) and the fact that β is monotone decreasing with respect to $|R|$, one can expect that β is bounded by a fixed number (which turns out to be $-\frac{\pi^3}{8}$, see Section 5).

IV. EXISTENCE THEOREM

Let $\{E = u \in C^2[0, 1], u(0) = u''(0) = u'(1) = 0\}$. E is a Banach space if E is given the usual sup-norm of $C^2[0, 1]$, i.e.,

$$(32) \quad \|u\|_E = \sup_{\eta \in [0, 1]} |u(\eta)| + \sup_{\eta \in [0, 1]} |u'(\eta)| + \sup_{\eta \in [0, 1]} |u''(\eta)|.$$

Our system (11), (12), and (13) is equivalent to

$$(33) \quad f(\eta) = h(\eta) + R \int_0^1 [J(\eta, t) - h(\eta)J(1, t)](f'^2 - ff'')dt,$$

and

$$(34) \quad K = -3 + 3R \int_0^1 J(1, t)(f'^2 - ff'')dt,$$

where $J(\eta, t)$ is the Green's function of $-v'''' = 0$, $v(0) = v'(0) = v'(1) = 0$ and is given by

$$(35) \quad J(\eta, t) = \begin{cases} (1-t)\eta & 0 \leq \eta \leq t, \\ -\frac{1}{2}(\eta^2 + t^2) + \eta & t \leq \eta \leq 1; \end{cases}$$

while

$$h(\eta) = 3 \int_0^1 J(\eta, s)ds.$$

Note that (33) is independent of K . Any solution f of (33) in E will automatically satisfy $f(1) = 1$, since $h(1) = 1$.

We will use the following version of the Leray-Schauder Fixed Point Theorem to prove the existence of solutions to (33):

Theorem 36 (Learay and Schauder (1934)): Assume that the operator
 $A(u, \lambda)$ ($u \in E, 0 \leq \lambda \leq 1$) are completely continuous and $A(u, 1) = Au$.
Assume that all solutions of the equation $u = A(u, \lambda)$ ($0 \leq \lambda \leq 1$) satisfy
the common a priori bound $\|u\| \leq r_0$. If the vector field $u - A(u, 0)$ has
nonzero Learay-Schauder degree on spheres $\{u \mid \|u\| = r\}$ of larger radii
 $r > r_0$, then the equation $(I - A) u = 0$ has at least one solution.

We shall define our operator $A(u, \lambda)$ in the following way. Let

$$(37) \quad Lu = u - h.$$

Then L^{-1} exists and equation (33) is equivalent to

$$(38) \quad u = A(u, \lambda) = L^{-1}[\lambda R \int_0^1 [J(\eta, t) - h(\eta)J(1, t)](u^{-2} - uu'')dt].$$

To show that $A(u, \lambda)$ is completely continuous on E , since L^{-1} is linear and bounded, it is sufficient to show that the integral operator on the right of (38) is compact from $E \rightarrow E$ for all $\lambda \in [0, 1]$. This is easily shown by noting that the operator

$$(39) \quad F(u) = u^{-2} - uu''$$

is a continuous bounded mapping from $E \rightarrow C^0[0, 1]$, i.e., F maps bounded sets of E into bounded sets of $C^0[0, 1]$, and that the operator

$$(40) \quad Bv = \int_0^1 [J(\eta, t) - h(\eta)J(1, t)]v dt$$

is compact from $C^0[0, 1] \rightarrow E$. A simple application of Arzela-Ascoli Theorem implies that the composite operator $BF(u)$ is completely continuous from E into E .

The estimates of Lemma 17 imply that all solutions of equation (38) satisfy

$$(41) \quad \|u\|_E \leq 3.$$

Thus, we may take $r = 4$ in the above theorem. Note that $u = h$ is the unique solution of $u = A(u, 0)$ and $\|u\|_E < 4$. Thus, we have proved

Theorem 42: The boundary value problem (11) - (13) has a solution for all $R < 0$.

Skala and Wang (1978) have shown that this is the only solution.

V. NUMERICAL RESULTS

Almost all numerical results reported in the past three decades on the Berman's equation were obtained as follows:

First, one makes changes of variables

$$(43) \quad f(\eta) = \frac{b}{R} g(t), \quad b\eta = t.$$

Then (11) - (13) become

$$(44) \quad g'''' + g'^2 - gg'' = k_0 \equiv \frac{Rk}{b^4},$$

$$(45) \quad g(0) = g'(0) = 0,$$

$$(46) \quad g(b) = 0; \quad bg(b) = k,$$

where b and R are treated unknown a priori. By suitable choice of $g'(0)$ and $g''(0)$ together with (45), one can solve the fourth-order equation (by differentiating (44) once) step by step until (if possible) $g(b) = 0$ for some b . Then R is determined by $R = bg(b)$. Even though this method has been popular since Terrill (1964), it is not natural in the sense that, in practice, we are given R --the Reynolds numbers, and to find the velocity fields. Since from the estimates on α , we can expect that we must have a very good control on $g'(0)$ and $g''(0)$ in order to get accurate R from Terrill's method. Previous authors did not give sufficient information about how large is b for different R .

We ran the boundary value problem (14), (12), and (13) under NOS. 2.3 at the NASA Langley Research Center by using PASVART, a commercial subroutine to solve multipoint boundary value problem by collocation and continuation. The results of f , f' , f'' for R between 0 to -8080 are given in Figs. 3 - 5.

It is interesting to note that the asymptotic results obtained by Yuan and Finkelstein (1956) as $R \rightarrow -\infty$ are in very good agreement with our numerical results as can be seen from Figs. 6 - 8.

The perturbed solution for large negative R obtained by Yuan and Finkelstein (1956) is

$$(47) \quad f(\eta) = \sin \frac{\pi}{2} \eta + \frac{1}{R} \left\{ \cos \frac{\pi}{2} \eta (1.438 + \frac{\pi^3}{16} \int_1^\eta \frac{d}{\sin \frac{\pi}{2}}) + \right. \\ \left. (0.662 + \frac{\pi^2}{8} \ln |\tan \frac{\pi}{4} \eta|) (\frac{2}{\pi} \sin \frac{\pi}{2} \eta - \eta \cos \frac{\pi}{2} \eta) - \frac{1.324}{\pi} \sin \frac{\pi}{2} \eta \right\}.$$

Thus,

$$\alpha = \frac{\pi}{2} + O(\frac{1}{R}),$$

$$f''(1) = -\frac{\pi^2}{4} + O(\frac{1}{R}), \quad \text{as } R \rightarrow -\infty.$$

$$\beta = -\frac{\pi^3}{8} + O(\frac{1}{R}).$$

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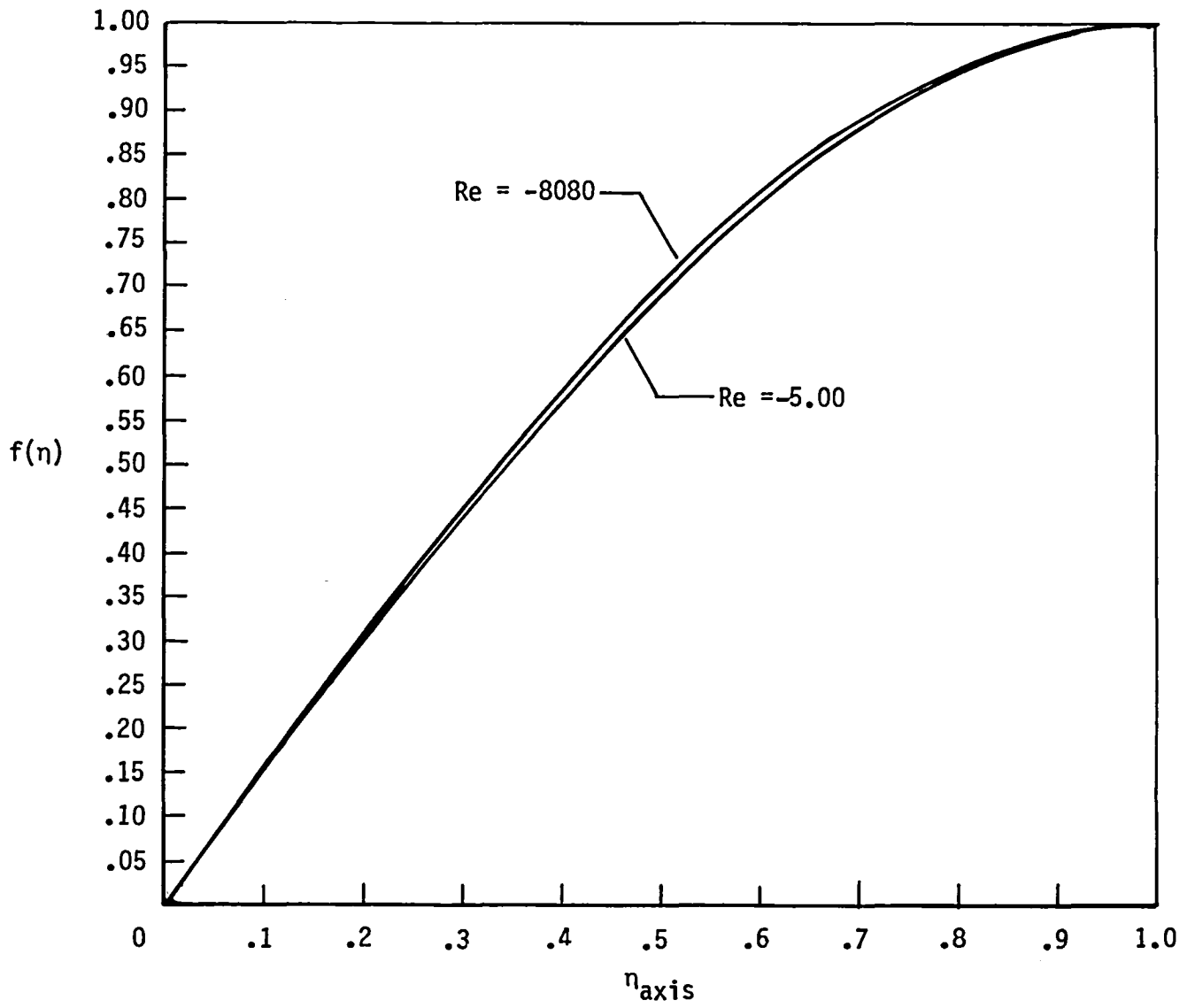


Figure 3

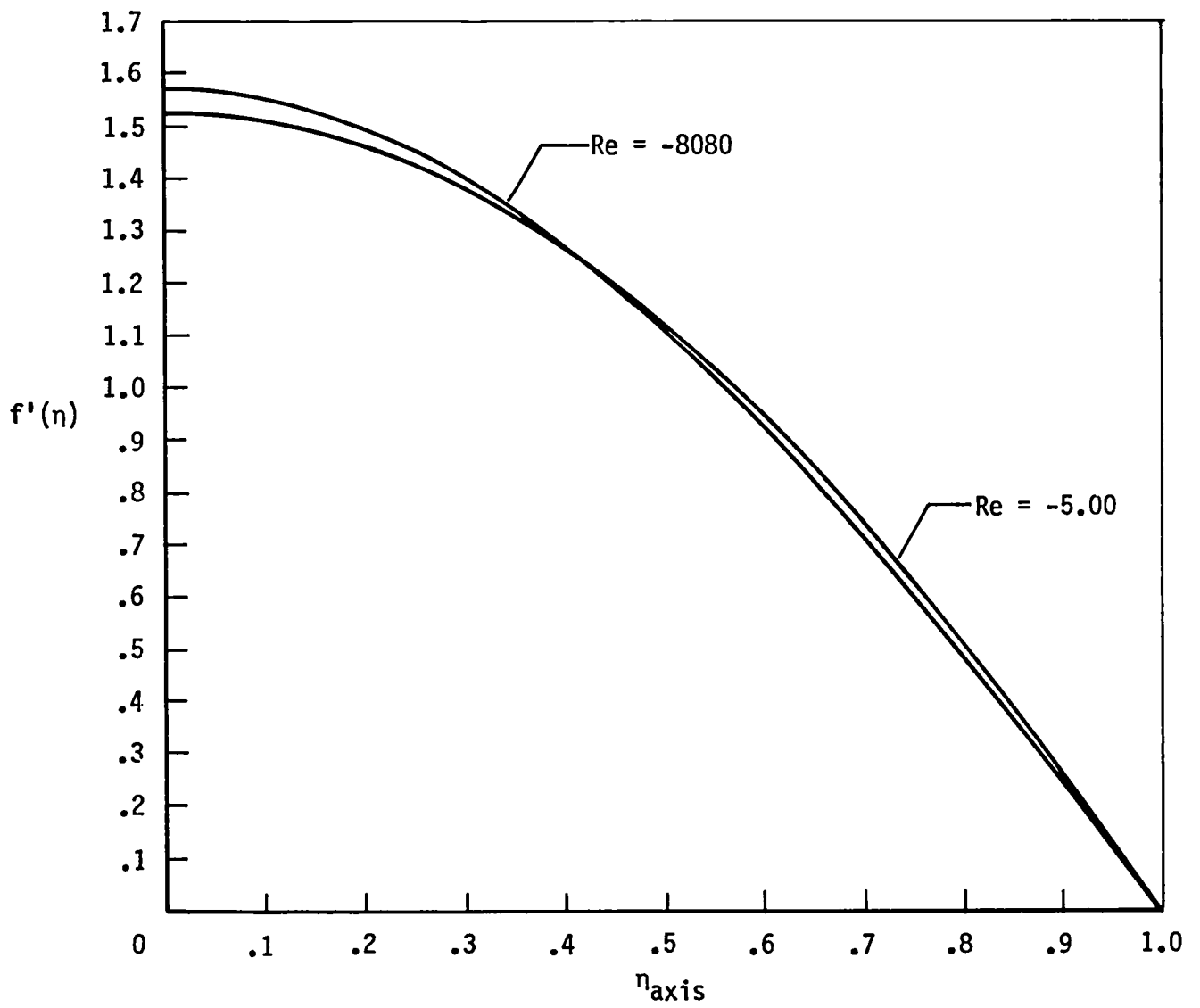


Figure 4

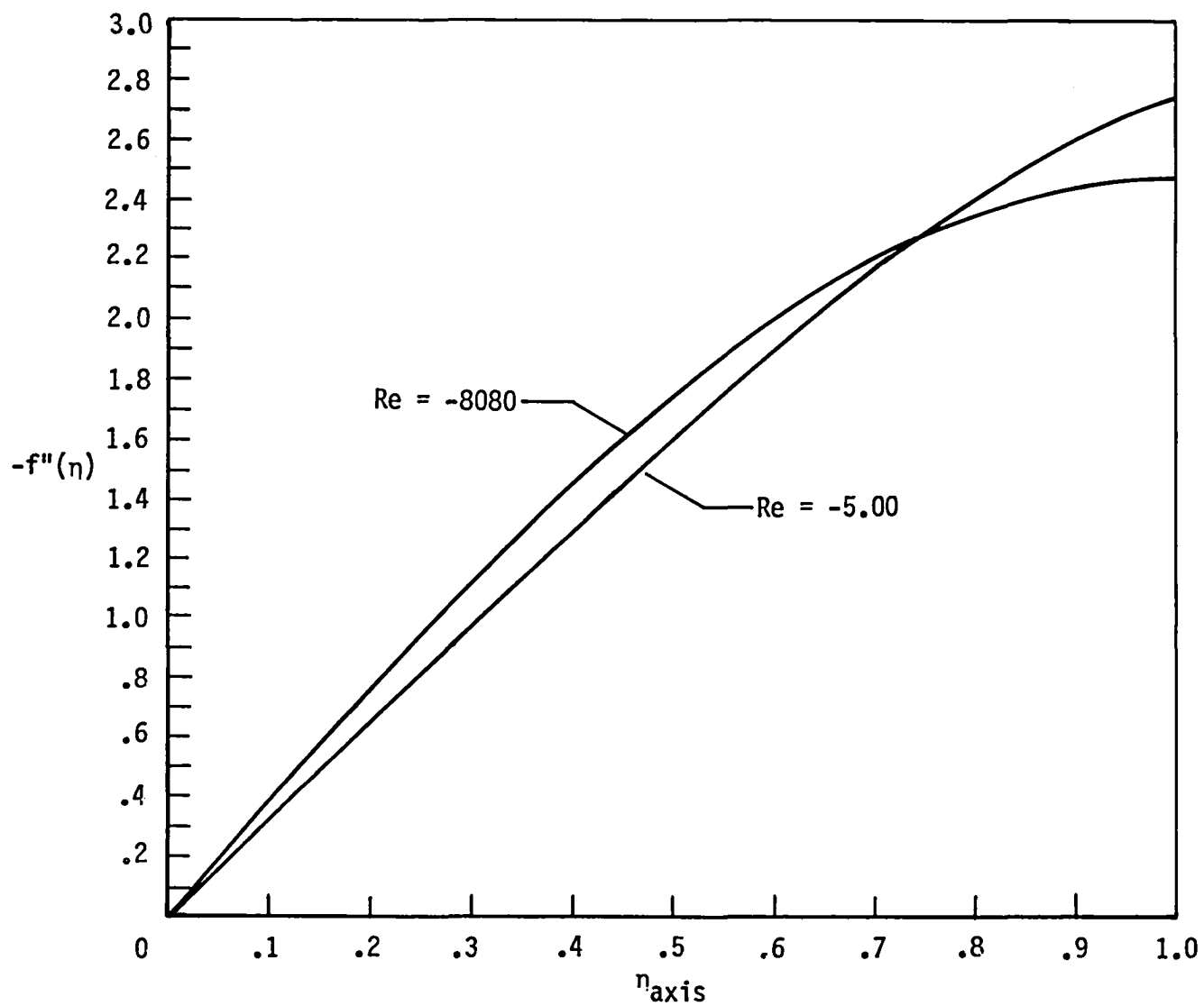


Figure 5

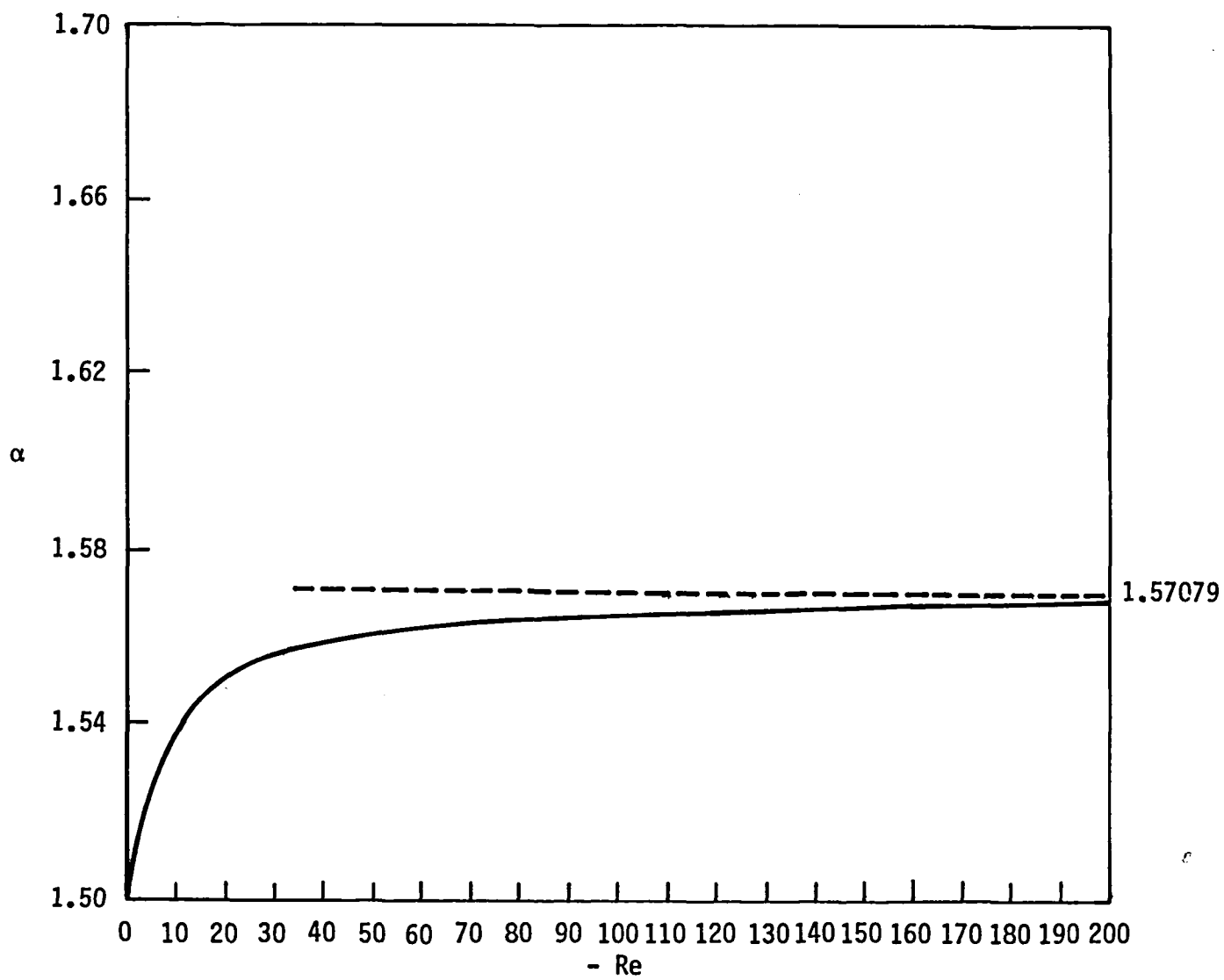


Figure 6.a

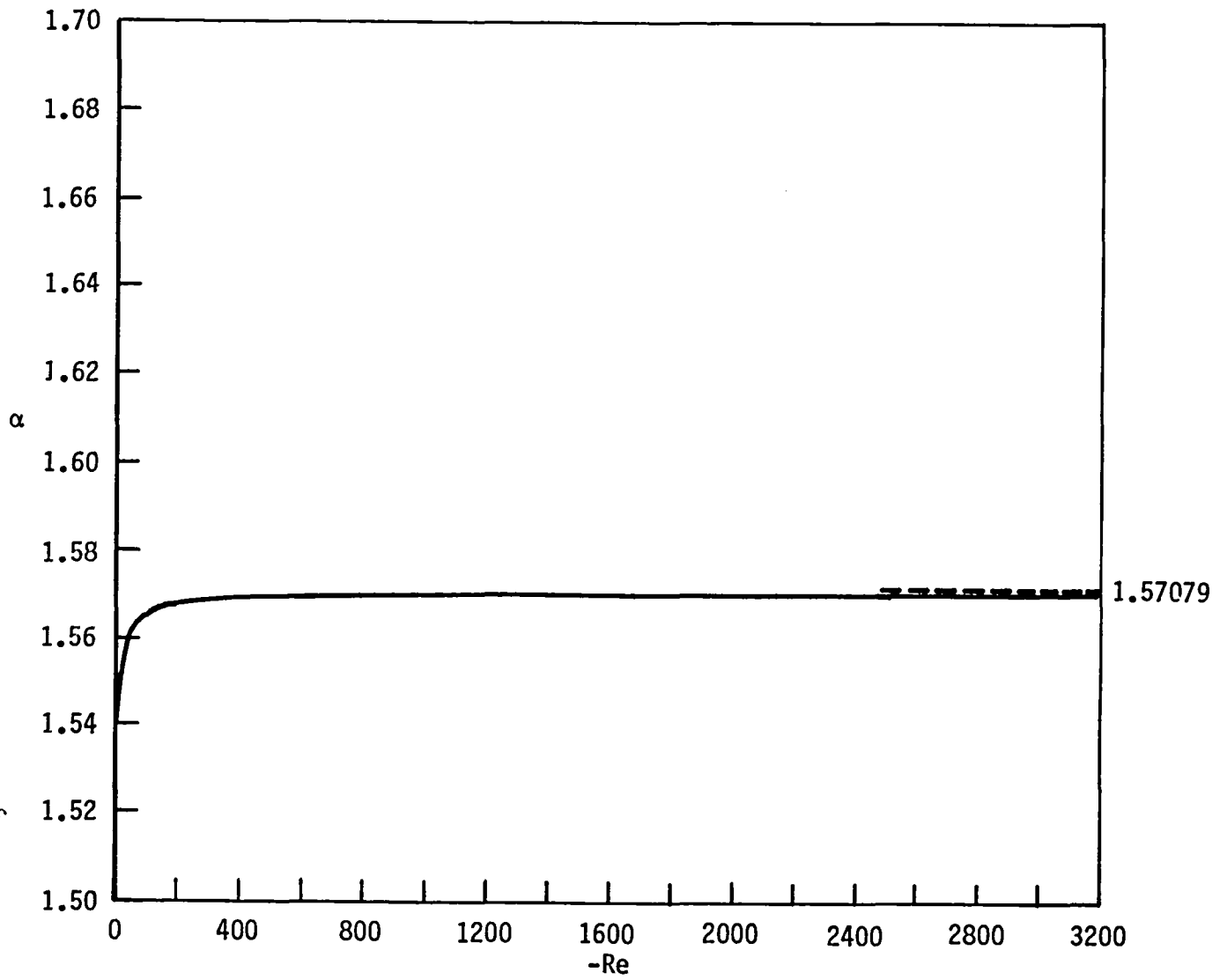


Figure 6.b

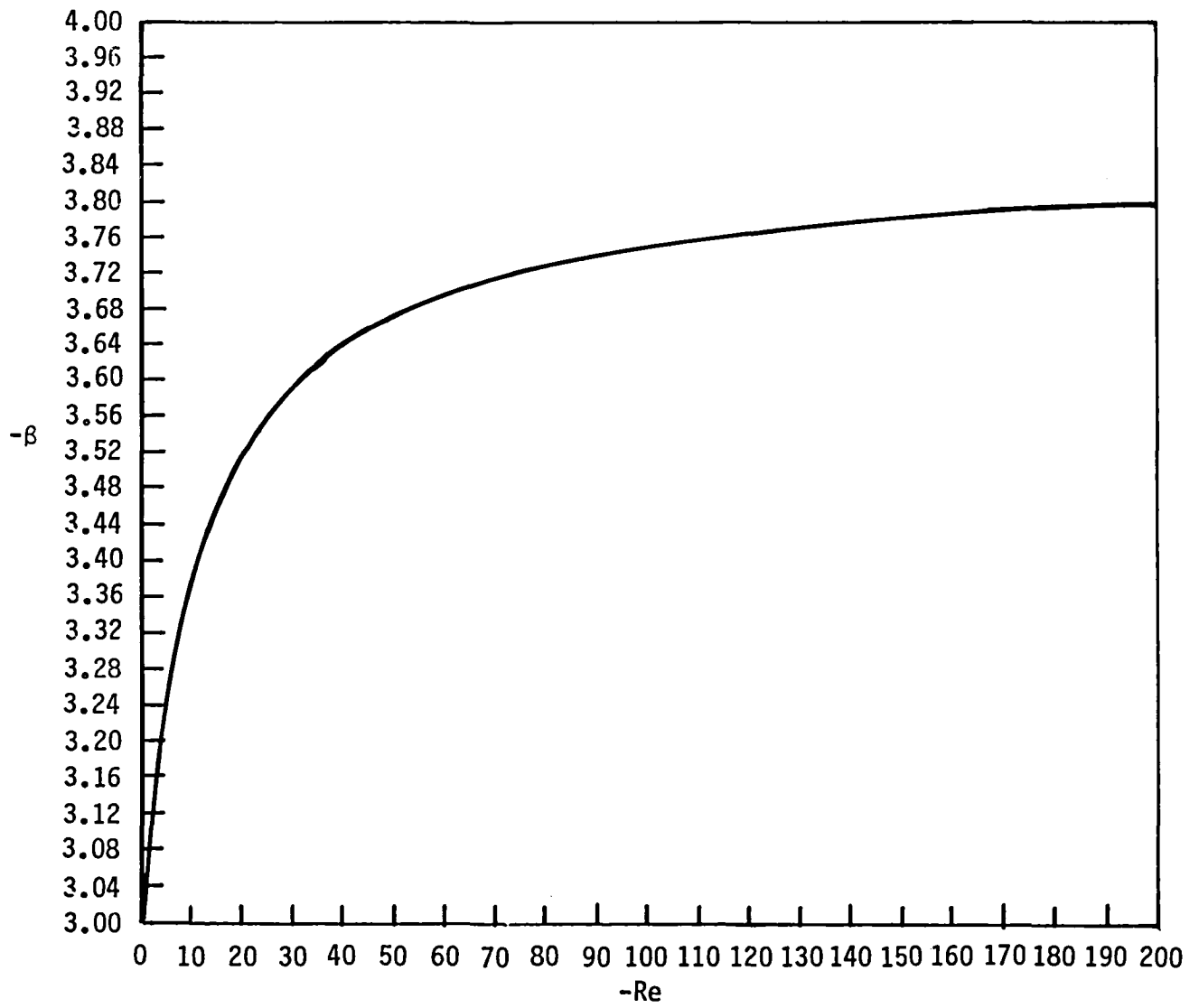


Figure 7.a

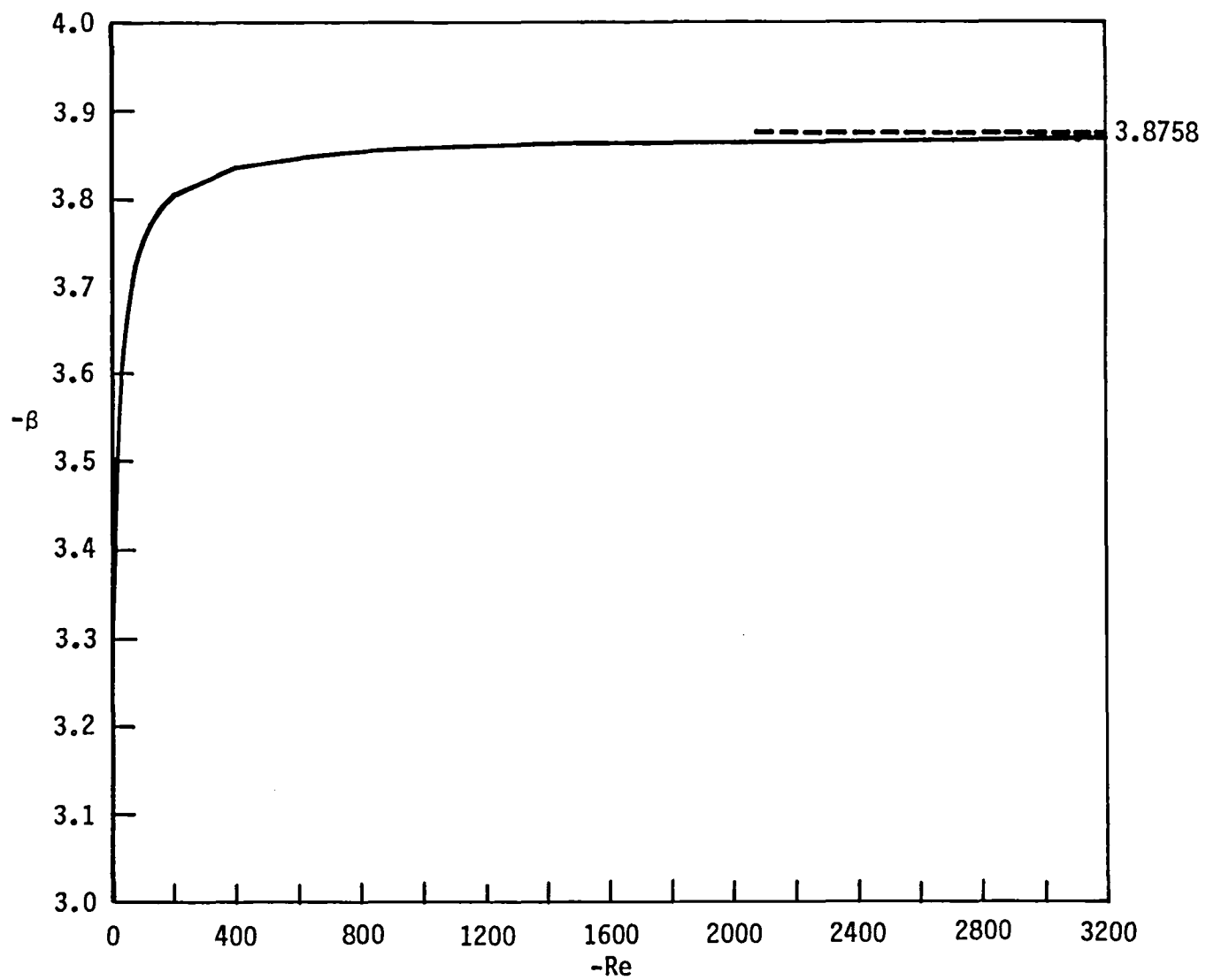


Figure 7.b

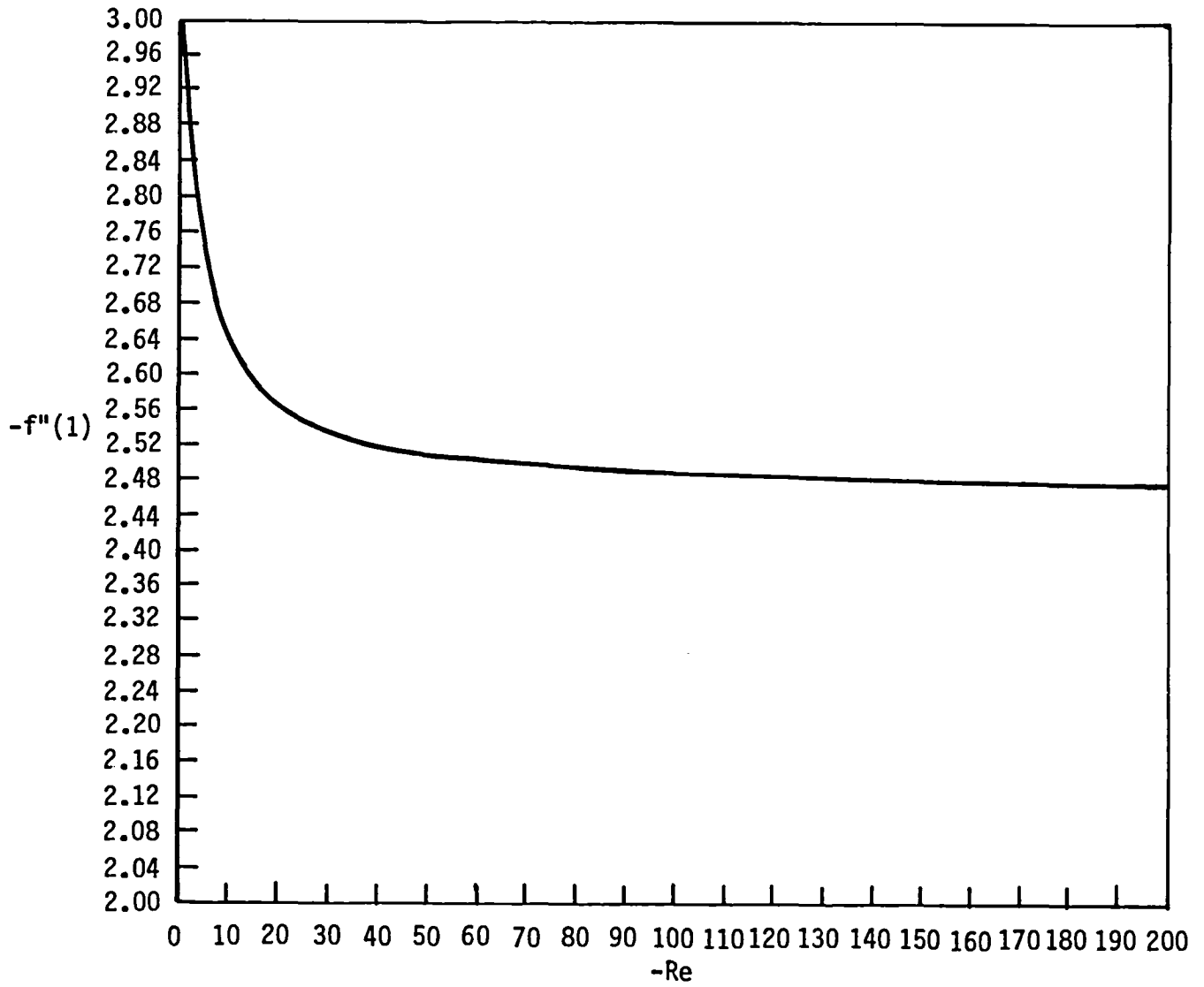


Figure 8.a

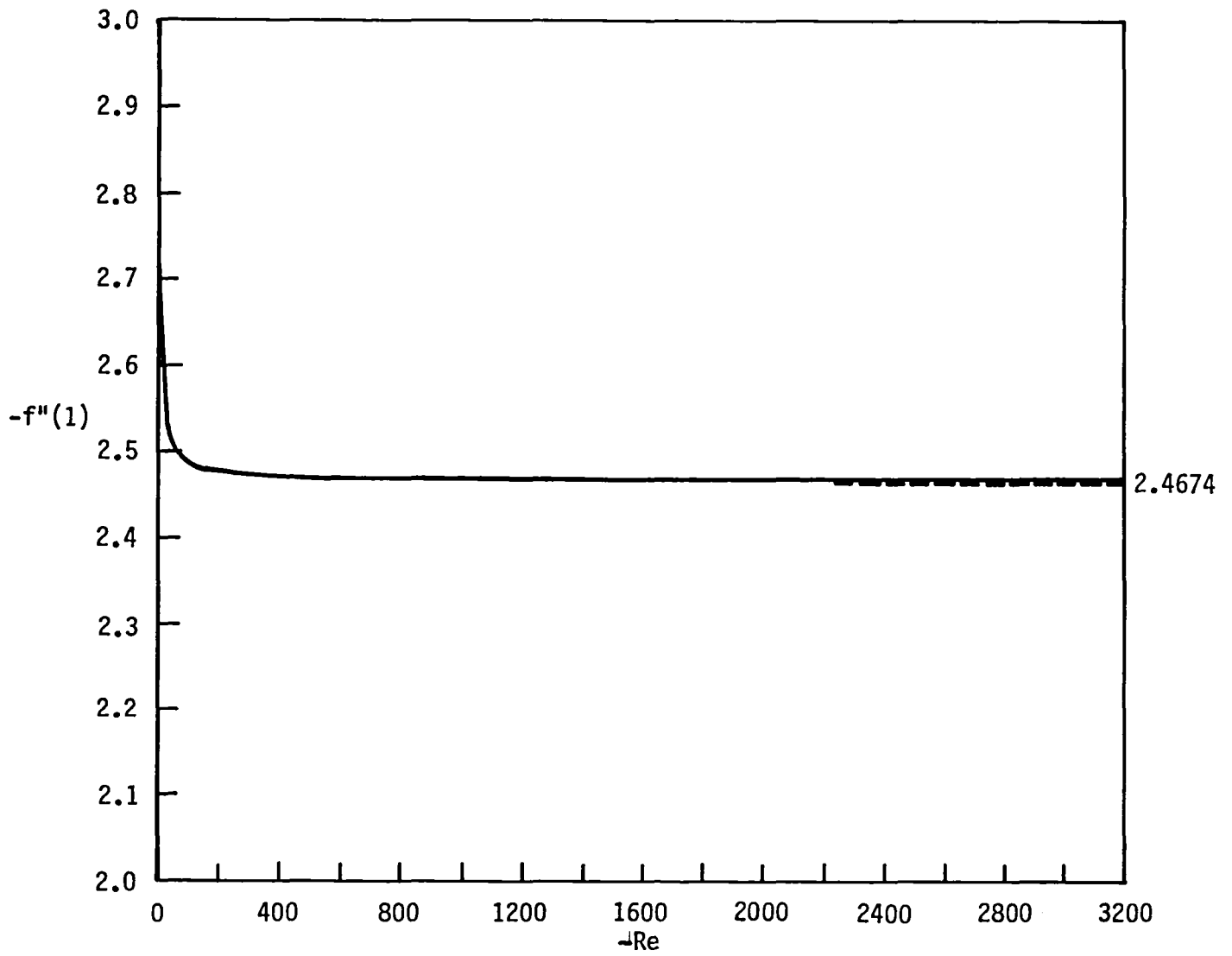


Figure 8.b

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