NASA-CR-178,108

NASA Contractor Report 178108

ICASE REPORT NO. 86-32

ICASE

COMPUTATIONAL METHODS FOR THE CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

J. A. Burns
E. M. Cliff
R. K. Powers

Contract Nos. NAS1-17070, NAS1-18107
May 1986

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

COMPUTATIONAL METHODS FOR THE CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

J. A. Burns
Department of Mathematics
Virginia Polytechnic Institute and State University

E. M. Cliff
Aerospace and Ocean Engineering
Virginia Polytechnic Institute and State University

R. K. Powers
Institute for Computer Applications in Science and Engineering

ABSTRACT

Finite dimensional approximation schemes that work well for distributed parameter systems are often not suitable for the analysis and implementation of feedback control systems. The relationship between approximation schemes for distributed parameter systems and their application to optimal control problems is discussed. A numerical example is given.

This work was supported under the National Aeronautics and Space Administration under NASA Contract Nos. NAS1-17070 and NAS1-18107 while the first and third authors were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665. Also, work was partially supported for the first and second authors under NSF Grant No. ECS-8109245, AFOSR Grant No. 85-0287, and Lawrence Livermore National Laboratory Grant No. 7526215.

Introduction

In the past few years there has been a rapid development of computational methods for identification and control of systems governed by functional and partial differential equations. The recent literature on this topic is extensive and we will not attempt a review of the area. The purpose of this paper is to show, via a fairly simple example, that care must be taken to ensure that finite dimensional approximations of distributed parameter systems preserve important system properties (i.e., controllability, observability, stabilizability, detectability, etc.). Moreover, if the particular scheme used to construct the finite dimensional model does not take into account these system properties, then the model may not be suitable for control design and analysis.

Clearly, controllability and observability properties of the finite dimensional models depend on both the distributed parameter system and the type of approximation scheme used to construct the lumped model. It is also important to note that although a particular numerical scheme may be convergent (i.e., consistent and stable) and well suited for simulation of open-loop dynamics, the scheme may not be suitable for use in the design and analysis of feedback control systems. A finite dimensional model to be used in a LQR type design algorithm should be based on a numerical scheme that approximates both the distributed parameter system and its adjoint (see [2], [5], [7], [8]). The basic ideas can best be illustrated by a simple example.

A Hybrid Control System

Hybrid control systems are systems governed by coupled partial and ordinary differential equations. Such systems occur in large flexible structures and are typical models for boundary control problems when actuator dynamics and/or tip masses are included in the analysis. We shall focus on the simple cable-spring-mass system shown in Figure 1. The cable of length L has lineal density σ and is under tension τ. The device at the right end of the cable maintains the tension and provides no impedence to the vertical motion of the mass m. Let $y(t,x)$ denote the displacement of the cable from the x-axis and u denote an applied force acting on the mass.

The energy of the mechanical system (Kinetic plus potential) is given by

$$E(t) = \int_0^L \{\sigma y_t^2(t,x) + \tau y_x^2(t,x)\}dx + ky^2(t,L) + my_t^2(t,L). \tag{1}$$

for $0 \leq t < +\infty$. There are several possible state space models of this system. We shall consider the approach followed in [3] and formulate the problem in the state space

$$Z = L_2(0,L) \times L_2(0,L) \times \mathbb{R} \times \mathbb{R}, \tag{2}$$

If $z_1 = \text{col}(\phi_1,\phi_2,\eta_1,\eta_2)$ and $z_2 = \text{col}(\psi_1,\psi_2,\xi_1,\xi_2)$, then the inner product defined by

$$\langle z_1,z_2\rangle = \int_0^L \tau\phi_1(x)\psi_1(x)dx + \int_0^L \sigma\phi_2(x)\psi_2(x)dx + k\eta_1\xi_1 + m\eta_2\xi_2 \tag{3}$$

leads to the energy norm $|z|^2 = \langle z,z\rangle$ for the system described in Figure 1. Moreover, if we identify $z_1(t,x) = y_x(t,x)$, $z_2(t,x) = y_t(t,x)$, $\mu_1(t) = y(t,L)$ and $\mu_2(t) = \dot{\mu}_1(t) = y_t(t,L)$, then the system can be described by the hybrid system (here $\alpha^2 = \tau/\sigma > 0$):

$$\frac{\partial}{\partial t}\begin{bmatrix} z_1(t,x) \\ z_2(t,x) \\ \mu_1(t) \\ \mu_2(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} z_2(t,x) \\ \alpha^2 \frac{\partial}{\partial x} z_1(t,x) \\ \mu_2(t) \\ \frac{-k}{m}\mu_1(t) - \frac{\tau}{m} z_1(t,L) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u(t) \tag{4}$$

with boundary conditions $z_2(t,0) = 0$ and $z_2(t,L) = \mu_2(t)$.

Define the operator A on the domain

$$D(A) = \{z = \mathrm{col}(\phi_1,\phi_2,\eta_1,\eta_2) \mid \phi_i \in W^{1,2}(0,L), \qquad i=1,2$$

$$\text{and } \phi_2(0) = 0,\ \phi_2(L) = \eta_2\} \tag{5}$$

by

$$A\begin{bmatrix} \phi_1 \\ \phi_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \phi_2' \\ \alpha^2 \phi_1' \\ \eta_2 \\ \frac{-k}{m}\eta_1 - \frac{\tau}{m}\phi_1(L) \end{bmatrix}, \tag{6}$$

where "prime" denotes $\frac{d}{dx}$. The operator $B:\mathbb{R} \to Z$ is defined by

$$Bu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix} u. \tag{7}$$

With A and B as above the control system in Figure 1 may be realized as

$$\dot{z}(t) = Az(t) + Bu(t) \tag{8}$$

with initial data

$$z(0) = z_0 = \mathrm{col}(\phi_1, \phi_2, \eta_1, \eta_2). \tag{9}$$

It can be shown that A generates a C_0-semigroup on Z and for suitably restricted initial data solutions to (8) - (9) are equivalent to solutions of the system

$$\begin{aligned} y_{tt}(t,x) &= \alpha^2 y_{xx}(t,x) \\ my_{tt}(t,L) &= -ky(t,L) - \tau y_x(t,L) + u(t). \end{aligned} \tag{10}$$

Note that (formally) the system (4) is the symmetric hyperbolic form of (10).

In order to define an output operator we let

$$M_k(\phi) = \frac{1}{2\varepsilon} \int_{L/k - \varepsilon}^{L/k + \varepsilon} \phi(x)dx$$

denote the mean-value operator at L/k. Define $C_\varepsilon : Z \to R^7$ by

$$C_\varepsilon \begin{bmatrix} \phi_1 \\ \phi_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} c_1 M_4(\phi_1) \\ c_1 M_2(\phi_1) \\ c_1 M_1(\phi_1) \\ c_2 M_4(\phi_2) \\ c_2 M_2(\phi_2) \\ c_2 \eta_2 \\ c_3 \eta_1 \end{bmatrix} \tag{11}$$

where $\varepsilon > 0$ is small ($\varepsilon < 10^{-3}$ is sufficient for the numerical results presented below) and c_i, $i = 1,2,3$ are positive constants. The operator C_ε acting on $z(t)$ "observes" the "average" slopes and velocities at $x = \frac{L}{4}, \frac{L}{2}$ and L and the displacement at $x = L$.

Consider the problem of minimizing

$$J(u) = \int_0^T \{\|C_\varepsilon z(t)\|^2 + R|u(t)|^2\}dt, \tag{12}$$

where $z(t)$ is the mild solution to (8) - (9) and $0 < T \leq +\infty$ is fixed.

In order to solve this optimal control problem, one must introduce some type of approximation scheme. We shall discuss two schemes to compare their use in the solution of the above optimal control problem.

The first scheme is detailed in [3] and was used to estimate parameters in the system described in Figure 1. We shall give a brief description of the scheme. Divide the interval $[0,L]$ into $N = 2^k$ subintervals with nodes $x_i^N = iL/N$, $i = 0,1,2,\ldots,N$. Let $h_i^N(x)$ denote the unique continuous, piecewise-linear function satisfying $h_i^N(x_j^N) = \delta_{ij}$. These functions are the so-called hat or Chapeau functions. For $i = 0,1,2,\ldots,N$ define $u_i^N \varepsilon Z$ by $u_i^N = \mathrm{col}(h_i^N(\cdot),0,0,0)$, and for $i = 1,2,\ldots,N$ define $v_i^N \varepsilon Z$ by $v_i^N = \mathrm{col}(0,h_i^N(\cdot),0,h_i^N(L))$. Let v_0^N denote the vector $v_0^N = \mathrm{col}(0,0,1,0)$. Consider the finite dimensional subspace $Z^N \subseteq Z$ defined by

$$Z^N = \mathrm{span}\{u_i^N,v_i^N,\ i = 0,1,2,\ldots,N\} \tag{13}$$

and let P^N denote the orthogonal projection from Z onto Z^N. Note that the set $\{u_i^N,\ i = 0,1,\ldots,N\}$ $\{v_i^N,\ i = 0,1,\ldots,N\}$ is a basis for Z^N and $\dim Z^N = 2(N + 1)$.

Moreover, it is important to note that $D(A) \subseteq Z^N$ for each $N=2^k$ $(k=1,2,\ldots)$ and hence we may define an approximating operator A^N by

$$A^N = P^N A P^N \tag{14}$$

and approximating operators $B^N : R \to Z^N$ by $B^N = P^N B$ and $C^N : Z^N \to R^7$ by $C^N z^N = C_\varepsilon Z^N$, respectively.

Using these definitions one can show that the sequence of finite dimensional models

$$\dot{z}^N(t) = A^N z^N(t) + B^N u(t)$$
$$z^N(0) = z_0^N = P^N Z_0 \qquad (15)$$
$$y^N(t) = C^N z^N(t)$$

provides a stable and consistent approximation to the distributed parameter system (8) - (9) with output defined by (11). Moreover, since this problem is skew-adjoint, this scheme provides a stable and consistent approximation to the adjoint system.

This particular scheme was used in [3] to estimate various system parameters in the system model. It worked quite well when used for such parameter estimation problems. However, when one attempts to use system (15) to compute suboptimal gain operators by minimizing the functional

$$J^N(u) = \int_0^T \{\|C^N z^N(t)\|^2 + R|u(t)|^2\}dt, \qquad (16)$$

then a number of problems occur (both theoretical and numerical). First it is easy to see that the symmetric hyperbolic/hybrid equation (4) with boundary conditions

$$z_2(t,0) = y_t(t,0) = 0 \quad \text{and} \quad z_2(t,L) = y_t(t,L) = \mu_2(t)$$

does not take into account the (physical) constraint $y(t,0) = 0$. In particular, there is a "rigid translation" mode present in the distributed parameter model (8) - (9) that leads to a lack of controllability in the finite dimensional model (15). However, there are other problems with this particular scheme that lead to the loss of controllability/observability properties and effect the performance of numerical algorithms based on the model (15). Note that the symmetric-hyberbolic form (4) of the wave equation (10) requires that a condition of continuity be satisfied. In particular, the condition

$$y_{tx}(t,x) = y_{xt}(t,x) \tag{17}$$

that is implied by (4) is not reflected in the approximating system (15). There should be a continuity condition relating the "elements" u_i^N that approximate $y_x(t,x)$ and the "elements" v_i^N that approximate $y_t(t,x)$. The use of finite elements that satisfy such continuity conditions has been proposed for various plate and beam models in order to prevent so called "locking" elements (see [6]).

In order to see how these considerations can lead to "better" finite dimensional models for control, we construct a second approximation scheme. Let $f_i^N = \mathrm{col}(0,h_i^N(\cdot),0,h_i^N(L))$ and $e_i^N = \mathrm{col}(\frac{d}{dx}\, h_i^N(\cdot),0,h_i^N(L),0)$ for $i = 1,2,\ldots,N$. Define $W^N \subseteq Z$ to be the finite dimensional space

$$W^N = \mathrm{span}\{e_i^N, f_i^N,\ i = 1,2,\ldots,N\} \tag{18}$$

and note that dim $W^N = 2N$. Let $Q^N: Z \to W^N$ denote the orthogonal projection onto W^N. Since W^N does not contain $D(A)$ it is not possible to define A^N by $A^N = Q^N A Q^N$. A rigorous deriviation of the approximating operator A^N relies on the theory of sesquilinear forms and is similar to the approach used in [1] and [8]. The basic idea is to expand the state $z(t,x)$ in the basis elements e_i^N and f_i^N and use the weak form of (8) - (9) to construct the finite dimensional model. In particular, let

$$z^N(t,x) = \begin{bmatrix} z_1^N(t,x) \\ \\ z_2^N(t,x) \\ \\ \mu_1^N(t) \\ \\ \mu_2^N(t) \end{bmatrix} = \sum_{i=1}^{N} y_i^N(t)e_i^N(x) + w_i^N(t)f_i^N(x) \tag{19}$$

and substitute $z^N(t,x)$ into the weak form of (4) (see [1], [8]) to obtain a finite dimensional model for the coefficients $y_i^N(t)$ and $w_i^N(t)$. If one defines $x^N(t) \in \mathbb{R}^{2N}$ by

$$x^N(t) = \begin{bmatrix} y_1^N(t) \\ y_2^N(t) \\ \vdots \\ y_N^N(t) \\ w_1^N(t) \\ w_2^N(t) \\ \vdots \\ w_N^N(t) \end{bmatrix}, \tag{20}$$

then the resulting finite dimensional model becomes the 2N-dimensional system of the form

$$M^N \dot{x}^N(t) = F^N x^N(t) + G^N u(t)$$

$$x^N(0) = x_0^N \tag{21}$$

with output

$$y^N(t) = D^N x^N(t), \tag{22}$$

where D^N is a matrix representation of the operator C_ε restricted to W^N, and x_0^N is the vector of coefficients of the (representation of the) projection of z_0 onto W^N.

This particular scheme has a number of nice features that make it suitable for control design. First observe that in (19) one can show that

$$\frac{\partial}{\partial t} z_1^N(t,x) = \frac{\partial}{\partial x} z_2^N(t,x) \tag{23}$$

and

$$\dot{y}_i^N(t) = w_i^N(t), \quad i = 1,2,\ldots,N. \tag{24}$$

Thus, the continuity condition (17) is preserved. Moreover, the expansion (19) removes the uncontrollable "rigid translation" mode that appears in the model (15) and numerical checks show that the finite dimensional model (21) is controllable.

Control synthesis was attempted with both finite dimensional models. The numerical algorithm used for computing the (time-varying) feedback gain operator is based on Chandrasekhar factorization of the Riccati operator equation (see [4], [9]). The system parameters and initial conditions are the same as in [3]. The time interval is [0,48 sec], the output weights are $C_1 = 50.$, $C_2 = 25.$, $C_3 = 1.$ and the control weight is unity.

Several numerical experiments were attempted with the design model (15). Various values of the grid parameter N were tried, system parameters were varied and we even added viscous damping to the model. None of the results were satisfactory.

On the other hand, numerical results with the new model, described by equations (19) - (22), were quite good. Shown in Figures 2 - 5 are comparisons of open-loop and closed-loop responses for both $y_x(t,x)$ and $y_t(t,x)$ at several locations along the cable. These results were obtained with grid parameter $N = 16$. Several simulations were run with the gain operator fixed at its steady-state value. To the scale shown, the graphs were not distinguishable from those presented here.

Note that both schemes are convergent to the original distributed parameter model and both finite dimensional models are sufficient for various simulation and identification studies. However, the important point is that the second scheme preserves a number of system properties that make it more suitable for control design.

References

1. H. T. Banks and K. Kunisch, The Linear Regulator Problem for Parabolic Systems, SIAM J. Control Optim., Vol. 22, No. 5 (1984), pp. 684-699.

2. H. T. Banks, I. G. Rosen and K. Ito, A Spline-based Technique for Computing Riccati Operators and Feedback Controls in Regulator Problems for Delay Equations, SIAM J. Sci. Stat. Comput., No. 5 (1984), pp. 830-855.

3. J. A. Burns and E. M. Cliff, On the Formulation of Some Distributed System Parameter Identification Problems, Proc. AIAA Symp. on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Virginia, June, 1977, pp. 87-105.

4. J. A. Burns, K. Ito and R. Powers, Chandrasekhar Equations and Computational Algorithms for Distributed Parameter Systems, 23rd IEEE CDC, Las Vegas, Nevada, 1984, pp. 262-267.

5. J. A. Burns, et. al., State-space Models and Advances Control Concepts for Large-space Structures, Final Report on AFRPL Contract F04611-84-C-0032, April, 1985.

6. M. A. Chrisfield, A Quadratic Mindlin Element Using Shear Constraints, Computers & Structures, Vol. 18, (1984), pp. 833-850.

7. J. S. Gibson, Linear-quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations, SIAM J. Control Optim., Vol. 21, (1983), pp. 95-139.

8. I. Lasiecka, Approximations of Riccati Equations for Abstract Boundary Control Problems-Applications to Hyperbolic Systems, preprint.

9. R. Powers, Chandrasekhar Equations for Distributed Parameter Systems, Ph.D Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1984.

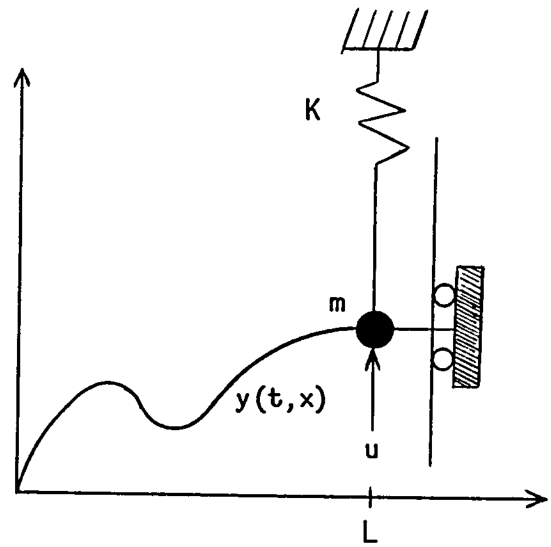

Figure 1

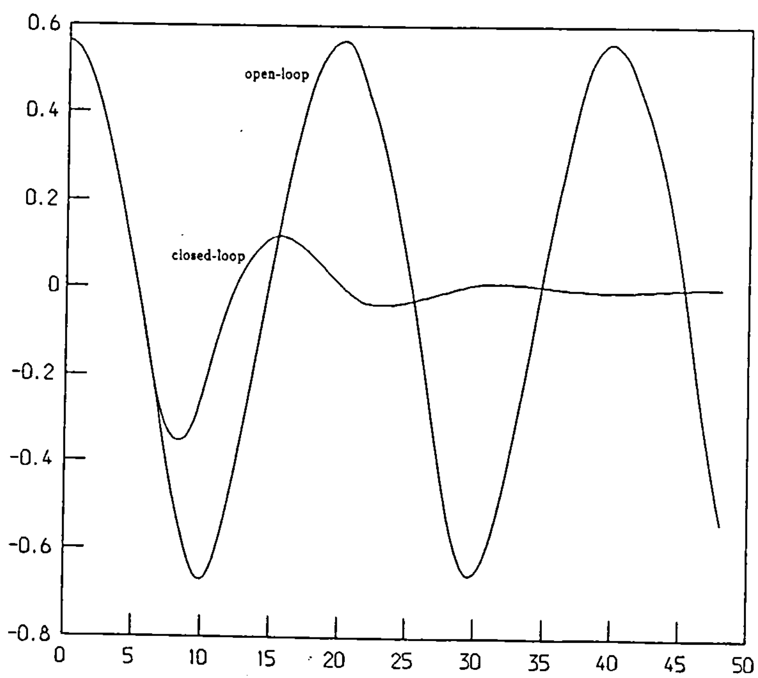

Figure 2: Time History of $y_x^{16}(t, \frac{L}{4})$

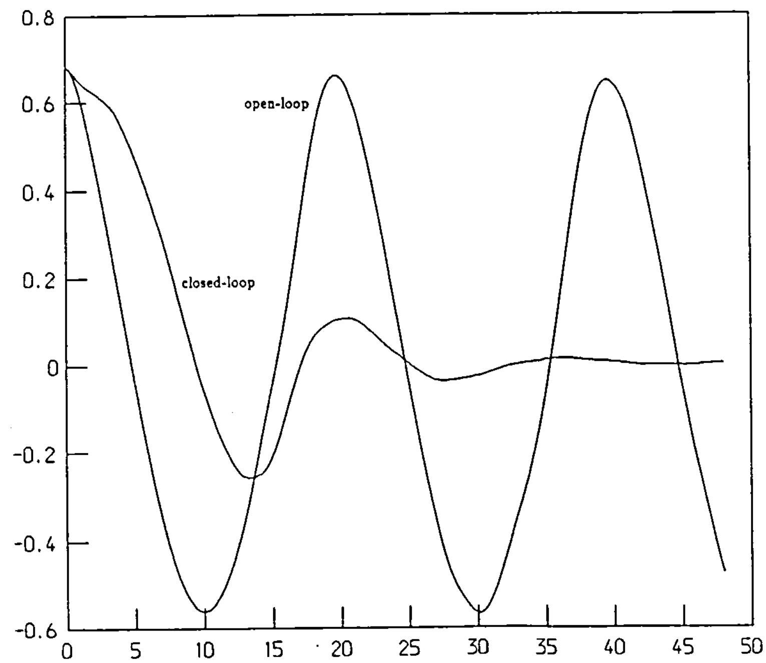

Figure 3: Time History of $y_x^{16}(t, L)$

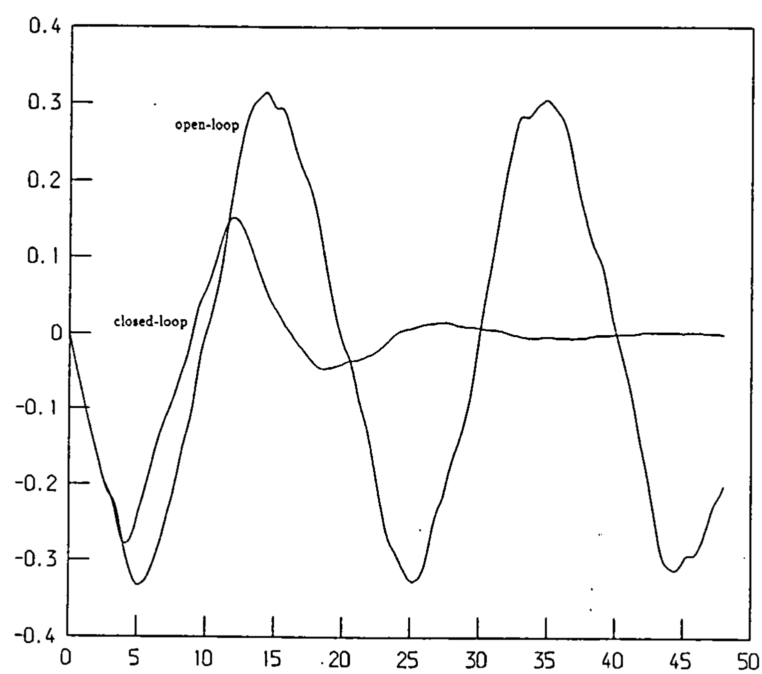

Figure 4: Time History of $y_t^{16}(t, \frac{L}{2})$

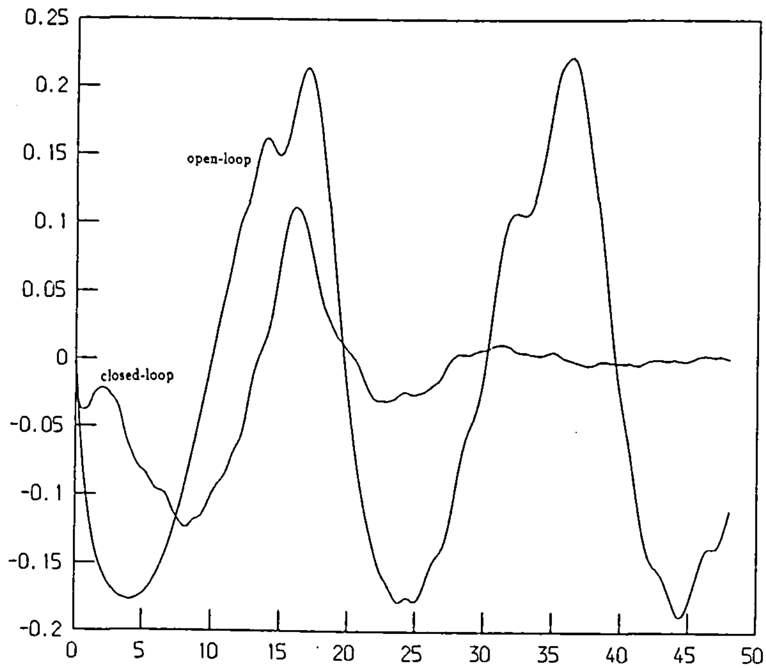

Figure 5: Time History of $y_t^{16}(t, L)$

1. Report No. NASA CR-178108 ICASE Report No. 86-32	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle COMPUTATIONAL METHODS FOR THE CONTROL OF DISTRIBUTED PARAMETER SYSTEMS		5. Report Date May 1986
		6. Performing Organization Code
7. Author(s) J. A. Burns, E. M. Cliff, R. K. Powers		8. Performing Organization Report No. 86-32
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225		10. Work Unit No.
		11. Contract or Grant No. NAS1-17070, NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Contractor Report
		14. Sponsoring Agency Code 505-31-83-01

15. Supplementary Notes

Langley Technical Monitor:
J. C. South

Final Report

Submitted to Proc. of the 24th IEEE Conf. on Decision and Control

16. Abstract

Finite dimensional approximation schemes that work well for distributed parameter systems are often not suitable for the analysis and implementation of feedback control systems. The relationship between approximation schemes for distributed parameter systems and their application to optimal control problems is discussed. A numerical example is given.

17. Key Words (Suggested by Authors(s))	18. Distribution Statement
distributed parameter systems, control, approximation	59 - Mathematical and Computer Sciences Unclassified - unlimited

19. Security Classif.(of this report)	20. Security Classif.(of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	18	A02