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OF DISTRIBUTED PARAMETER SYSTEMS

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COMPUTATIONAL METHODS FOR THE CONTROL
OF DISTRIBUTED PARAMETER SYSTEMS

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ABSTRACT

Finite dimensional approximation schemes that work well for distributed
parameter systems are often not suitable for the analysis and implementation
of feedback control systems. The relationship between approximation schemes
for distributed parameter systems and their application to optimal control
problems is discussed. A numerical example is given.
Introduction

In the past few years there has been a rapid development of computational methods for identification and control of systems governed by functional and partial differential equations. The recent literature on this topic is extensive and we will not attempt a review of the area. The purpose of this paper is to show, via a fairly simple example, that care must be taken to ensure that finite dimensional approximations of distributed parameter systems preserve important system properties (i.e., controllability, observability, stabilizability, detectability, etc.). Moreover, if the particular scheme used to construct the finite dimensional model does not take into account these system properties, then the model may not be suitable for control design and analysis.

Clearly, controllability and observability properties of the finite dimensional models depend on both the distributed parameter system and the type of approximation scheme used to construct the lumped model. It is also important to note that although a particular numerical scheme may be convergent (i.e., consistent and stable) and well suited for simulation of open-loop dynamics, the scheme may not be suitable for use in the design and analysis of feedback control systems. A finite dimensional model to be used in a LQR type design algorithm should be based on a numerical scheme that approximates both the distributed parameter system and its adjoint (see [2], [5], [7], [8]). The basic ideas can best be illustrated by a simple example.

A Hybrid Control System

Hybrid control systems are systems governed by coupled partial and ordinary differential equations. Such systems occur in large flexible
structures and are typical models for boundary control problems when actuator
dynamics and/or tip masses are included in the analysis. We shall focus on
the simple cable-spring-mass system shown in Figure 1. The cable of length $L$
has lineal density $\sigma$ and is under tension $\tau$. The device at the right end
of the cable maintains the tension and provides no impedance to the vertical
motion of the mass $m$. Let $y(t,x)$ denote the displacement of the cable from
the x-axis and $u$ denote an applied force acting on the mass.

The energy of the mechanical system (Kinetic plus potential) is given by

$$E(t) = \int_0^L \left\{ \sigma \dot{y}^2(t,x) + \tau \dot{y}^2_x(t,x) \right\} dx + ky^2(t,L) + my^2_t(t,L).$$

for $0 \leq t < +\infty$. There are several possible state space models of this
system. We shall consider the approach followed in [3] and formulate the
problem in the state space

$$Z = L^2(0,L) \times L^2(0,L) \times \mathbb{R} \times \mathbb{R},$$

If $z_1 = \text{col}(\psi_1,\psi_2,\eta_1,\eta_2)$ and $z_2 = \text{col}(\psi_1,\psi_2,\xi_1,\xi_2)$, then the inner product
defined by

$$\langle z_1, z_2 \rangle = \int_0^L \tau \psi_1(x) \psi_1(x) dx + \int_0^L \sigma \psi_2(x) \psi_2(x) dx + k \eta_1 \xi_1 + m \eta_2 \xi_2$$

leads to the energy norm $|z|^2 = \langle z, z \rangle$ for the system described in Figure
1. Moreover, if we identify $z_1(t,x) = y_x(t,x)$, $z_2(t,x) = y_t(t,x)$,
$u_1(t) = y(t,L)$ and $u_2(t) = \dot{y}_1(t) = y_t(t,L)$, then the system can be
described by the hybrid system (here $\alpha^2 = \tau/\sigma > 0$):
with boundary conditions \( z_2(t,0) = 0 \) and \( z_2(t,L) = \varphi_2(t) \).

Define the operator \( A \) on the domain

\[
D(A) = \{ z = \col(\phi_1, \varphi_2, \eta_1, \eta_2) | \phi_i \in W^{1,2}(0,L), \quad i=1,2 \}
\]

and \( \phi_2(0) = 0, \phi_2(L) = \eta_2 \} \] (5)

by

\[
A = \begin{bmatrix}
\phi_1 \\
\varphi_2 \\
\eta_1 \\
\eta_2 \\
\end{bmatrix}
= \begin{bmatrix}
\check{\phi}_1 \\
\check{\varphi}_2 \\
\alpha^2 \phi_1 \\
\eta_1 \\
\eta_2 \\
\eta_2 \\
\frac{-k}{m} \eta_1 - \frac{\tau}{m} \phi_1(L)
\end{bmatrix}, \quad (6)
\]
where "prime" denotes \( \frac{d}{dx} \). The operator \( B : \mathbb{R} \rightarrow \mathbb{Z} \) is defined by

\[
Bu = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
m
\end{bmatrix} u.
\]  

(7)

With \( A \) and \( B \) as above the control system in Figure 1 may be realized as

\[
\dot{z}(t) = Az(t) + Bu(t)
\]  

(8)

with initial data

\[
z(0) = z_0 = \text{col}(\phi_1, \phi_2, \eta_1, \eta_2).
\]  

(9)

It can be shown that \( A \) generates a \( C_0 \)-semigroup on \( \mathbb{Z} \) and for suitably restricted initial data solutions to (8) - (9) are equivalent to solutions of the system

\[
y_{tt}(t,x) = \alpha^2 y_{xx}(t,x)
\]  

(10)

\[
my_{tt}(t,L) = -ky(t,L) - \tau y_x(t,L) + u(t).
\]

Note that (formally) the system (4) is the symmetric hyperbolic form of (10).

In order to define an output operator we let

\[
M_k(\phi) = \frac{1}{2\varepsilon} \int_{L/k - \varepsilon}^{L/k + \varepsilon} \phi(x) dx
\]
denote the mean-value operator at $L/k$. Define $C_\varepsilon: Z \times R^7$ by

$$
C_\varepsilon = \begin{bmatrix} 
\phi_1 \\
\phi_2 \\
\eta_1 \\
\eta_2 \\
\end{bmatrix}
= \begin{bmatrix} 
c_1M_4(\phi_1) \\
\vdots \\
c_2M_4(\phi_2) \\
\vdots \\
c_3 \eta_1 \\
\end{bmatrix}
$$

(11)

where $\varepsilon > 0$ is small ($\varepsilon < 10^{-3}$ is sufficient for the numerical results presented below) and $c_i$, $i = 1, 2, 3$ are positive constants. The operator $C_\varepsilon$ acting on $z(t)$ "observes" the "average" slopes and velocities at $x = \frac{L}{4}$, $\frac{L}{2}$ and $L$ and the displacement at $x = L$.

Consider the problem of minimizing

$$
J(u) = \int_0^T \{\|C_\varepsilon z(t)\|^2 + R|u(t)|^2\} dt,
$$

(12)

where $z(t)$ is the mild solution to (8) - (9) and $0 < T \leq +\infty$ is fixed.
In order to solve this optimal control problem, one must introduce some type of approximation scheme. We shall discuss two schemes to compare their use in the solution of the above optimal control problem.

The first scheme is detailed in [3] and was used to estimate parameters in the system described in Figure 1. We shall give a brief description of the scheme. Divide the interval \([0, L]\) into \(N = 2^k\) subintervals with nodes \(x_i = L/N, i = 0, 1, 2, \ldots, N\). Let \(h_i^N(x)\) denote the unique continuous, piecewise-linear function satisfying \(h_i^N(x) = \delta_{ij}\). These functions are the so-called hat or Chapeau functions. For \(i = 0, 1, 2, \ldots, N\) define \(u_i^N \in Z\) by \(u_i^N = \text{col}(h_i^N(\cdot), 0, 0, 0)\), and for \(i = 1, 2, \ldots, N\) define \(v_i^N \in Z\) by \(v_i^N = \text{col}(0, h_i^N(\cdot), 0, h_i^N(L))\). Let \(v_0^N\) denote the vector \(v_0^N = \text{col}(0, 0, 1, 0)\). Consider the finite dimensional subspace \(Z^N \subset Z\) defined by

\[
Z^N = \text{span\{u}_i^N, v_i^N, i = 0, 1, 2, \ldots, N\}
\] (13)

and let \(p^N\) denote the orthogonal projection from \(Z\) onto \(Z^N\). Note that the set \{\(u_i^N, i = 0, 1, \ldots, N\) \{v_i^N, i = 0, 1, \ldots, N\} is a basis for \(Z^N\) and \(\dim Z^N = 2(N + 1)\).

Moreover, it is important to note that \(D(A) \subseteq Z^N\) for each \(N=2^k\) \((k=1, 2, \ldots)\) and hence we may define an approximating operator \(A^N\) by

\[
A^N = p^N A p^N
\] (14)

and approximating operators \(B^N: \mathbb{R} \rightarrow Z^N\) by \(B^N = p^N B\) and \(C^N: Z^N \rightarrow \mathbb{R}^T\) by \(C^N Z^N = C \epsilon Z^N\), respectively.
Using these definitions one can show that the sequence of finite dimensional models

\[ \dot{z}^N(t) = A^N z^N(t) + B^N u(t) \]
\[ z^N(0) = z_0^N = P^N z_0 \]
\[ y^N(t) = C^N z^N(t) \] (15)

provides a stable and consistent approximation to the distributed parameter system (8) - (9) with output defined by (11). Moreover, since this problem is skew-adjoint, this scheme provides a stable and consistent approximation to the adjoint system.

This particular scheme was used in [3] to estimate various system parameters in the system model. It worked quite well when used for such parameter estimation problems. However, when one attempts to use system (15) to compute suboptimal gain operators by minimizing the functional

\[ J^N(u) = \int_0^T \{ \| C^N z^N(t) \|^2 + R|u(t)|^2 \} dt, \] (16)

then a number of problems occur (both theoretical and numerical). First it is easy to see that the symmetric hyperbolic/hybrid equation (4) with boundary conditions

\[ z_2(t,0) = y_t(t,0) = 0 \text{ and } z_2(t,L) = y_t(t,L) = \mu_2(t) \]

does not take into account the (physical) constraint \( y(t,0) = 0 \). In particular, there is a "rigid translation" mode present in the distributed
parameter model (8) - (9) that leads to a lack of controllability in the finite dimensional model (15). However, there are other problems with this particular scheme that lead to the loss of controllability/observability properties and effect the performance of numerical algorithms based on the model (15). Note that the symmetric-hyperbolic form (4) of the wave equation (10) requires that a condition of continuity be satisfied. In particular, the condition

\[ y_{tx}(t,x) = y_{xt}(t,x) \] (17)

that is implied by (4) is not reflected in the approximating system (15). There should be a continuity condition relating the "elements" \( u_i^N \) that approximate \( y_x(t,x) \) and the "elements" \( v_i^N \) that approximate \( y_t(t,x) \). The use of finite elements that satisfy such continuity conditions has been proposed for various plate and beam models in order to prevent so called "locking" elements (see [6]).

In order to see how these considerations can lead to "better" finite dimensional models for control, we construct a second approximation scheme. Let \( f_i^N = \text{col}(0, h_i^N(\cdot), 0, h_i^N(0)) \) and \( e_i^N = \text{col}(\frac{d}{dx} h_i^N(\cdot), 0, h_i^N(0), 0) \) for \( i = 1, 2, \ldots, N \). Define \( W_N \subseteq Z \) to be the finite dimensional space

\[ W_N = \text{span}\{ e_i^N, f_i^N, i = 1, 2, \ldots, N \} \] (18)

and note that \( \dim W_N = 2N \). Let \( Q^N: Z \to W_N \) denote the orthogonal projection onto \( W_N \). Since \( W_N \) does not contain \( D(A) \) it is not possible to define \( A_N \) by \( A_N = Q^N A Q^N \). A rigorous deriviation of the approximating operator \( A_N \) relies on the theory of sesquilinear forms and is similar to the approach used
in [1] and [8]. The basic idea is to expand the state $z(t,x)$ in the basis elements $\psi_i^N$ and $\phi_1^N$ and use the weak form of (8) - (9) to construct the finite dimensional model. In particular, let

$$z^N(t,x) = \begin{bmatrix} z_1^N(t,x) \\ z_2^N(t,x) \\ \mu_1^N(t) \\ \mu_2^N(t) \end{bmatrix} = \sum_{i=1}^{N} y_1^N(t) \psi_i^N(x) + w_1^N(t) \phi_1^N(x) \quad (19)$$

and substitute $z^N(t,x)$ into the weak form of (4) (see [1], [8]) to obtain a finite dimensional model for the coefficients $y_1^N(t)$ and $w_1^N(t)$. If one defines $x^N(t) \in \mathbb{R}^N$ by

$$x^N(t) = \begin{bmatrix} y_1^N(t) \\ y_2^N(t) \\ \vdots \\ y_N^N(t) \\ w_1^N(t) \\ w_2^N(t) \\ \vdots \\ w_N^N(t) \end{bmatrix}, \quad (20)$$
then the resulting finite dimensional model becomes the 2N-dimensional system of the form

\[ M^N \dot{x}^N(t) = F^N x^N(t) + G^N u(t) \quad (21) \]

\[ x^N(0) = x_0^N \]

with output

\[ y^N(t) = D^N x^N(t), \quad (22) \]

where \( D^N \) is a matrix representation of the operator \( C_\varepsilon \) restricted to \( W^N \), and \( x_0^N \) is the vector of coefficients of the (representation of the) projection of \( z_0 \) onto \( W^N \).

This particular scheme has a number of nice features that make it suitable for control design. First observe that in (19) one can show that

\[ \frac{\partial}{\partial t} z_1^N(t,x) = \frac{\partial}{\partial x} z_2^N(t,x) \quad (23) \]

and

\[ y_1^N(t) = \omega_1^N(t), \quad i = 1,2,\ldots,N. \quad (24) \]

Thus, the continuity condition (17) is preserved. Moreover, the expansion (19) removes the uncontrollable "rigid translation" mode that appears in the model (15) and numerical checks show that the finite dimensional model (21) is controllable.
Control synthesis was attempted with both finite dimensional models. The numerical algorithm used for computing the (time-varying) feedback gain operator is based on Chandrasekhar factorization of the Riccati operator equation (see [4], [9]). The system parameters and initial conditions are the same as in [3]. The time interval is [0, 48 sec], the output weights are $C_1 = 50$, $C_2 = 25$, $C_3 = 1$, and the control weight is unity.

Several numerical experiments were attempted with the design model (15). Various values of the grid parameter $N$ were tried, system parameters were varied and we even added viscous damping to the model. None of the results were satisfactory.

On the other hand, numerical results with the new model, described by equations (19) - (22), were quite good. Shown in Figures 2 - 5 are comparisons of open-loop and closed-loop responses for both $y_x(t, x)$ and $y_t(t, x)$ at several locations along the cable. These results were obtained with grid parameter $N = 16$. Several simulations were run with the gain operator fixed at its steady-state value. To the scale shown, the graphs were not distinguishable from those presented here.

Note that both schemes are convergent to the original distributed parameter model and both finite dimensional models are sufficient for various simulation and identification studies. However, the important point is that the second scheme preserves a number of system properties that make it more suitable for control design.


Figure 1

Figure 2: Time History of $y_x^{16}(t, \frac{L}{4})$
Figure 3: Time History of $y_{x}^{16}(t, L)$

Figure 4: Time History of $y_{t}^{16}(t, \frac{L}{2})$
Figure 5: Time History of $y_{16}(t, L)$
Finite dimensional approximation schemes that work well for distributed parameter systems are often not suitable for the analysis and implementation of feedback control systems. The relationship between approximation schemes for distributed parameter systems and their application to optimal control problems is discussed. A numerical example is given.